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EXACT TWO-SOLITON SOLUTIONS AND TWO-PERIODIC SOLUTIONS OF THE PERTURBED mKdV EQUATION WITH VARIABLE COEFFICIENTS

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We discuss the Darboux transformation method for a modified Korteweg–de Vries equation with variable coefficients and perturbing terms in detail based on the general form of the Darboux transformations for some nonlinear evolution equations solvable by the Ablowitz–Kaup–Newell–Segur inverse scattering method. We use this method to generate families of two-soliton solutions and two-periodic solutions.

Keywords: Darboux transformation, perturbed mKdV equation, two-soliton solution, two-periodic solution

1. Introduction

It is commonly acknowledged that the most powerful perturbation technique is based on the inverse scattering transformation (IST). This technique requires that the unperturbed equation be exactly solvable by the IST, which restricts the range of applications but allows solving the most sophisticated dynamical problems. Equations exactly integrable by the IST have many remarkable properties, such as the Darboux transformation, the Painlevé property, and the Hirota bilinear form. Some of these properties, for example, the Hirota bilinear form and the Darboux transformation [1], [2], have been used as bases for obtaining exact multisoliton solutions.

The perturbed modified Korteweg–de Vries (mKdV) equation

$$p_t + 6p^2 p_x + p_{xxx} = \epsilon f(p), \tag{1}$$

where ϵ is a small perturbation coefficient, is encountered in the theory of quasi-one-dimensional solids and in liquid-crystal hydrodynamics [3], [4]. Being a perturbation of the mKdV equation, the perturbation-induced effects are interesting mainly because they represent physical phenomena that cannot be encompassed by exactly integrable models.

Here, we consider a perturbed mKdV equation

$$p_t + a(6p^2 p_x + p_{xxx}) = \epsilon f_1(p), \tag{2}$$

where $a = a(t)$, $\epsilon = \epsilon(t)$, and $f_1(p) = (p_{xxxx} + 10p^2 p_{xx} + 10pp_x^2 + 6p^5)_x$. This equation is called the perturbed mKdV equation with variable coefficients; the perturbing term $\epsilon(t)f_1(p)$ with a time-dependent $\epsilon(t)$ of either sign is also physically meaningful [3]. Especially, if $a = 1$ and ϵ is a common constant rather

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than a perturbing coefficient, then Eq. (2) is called the fifth-order mKdV equation. We note that many exact solutions of the fifth-order KdV equation were constructed [5]–[11], and significant analytic results for the perturbed KdV equation were obtained [12]–[16], but there has been very little research into the fifth-order mKdV equation or the perturbed mKdV equation. Only series reduction solutions for the perturbed mKdV equation with a weak fourth-order dispersion and weak dissipation have been derived [17].

2. Darboux transformation of the perturbed mKdV equation with variable coefficients

Some Darboux transformations of the 2×2 Ablowitz–Kaup–Newell–Segur (AKNS) system were derived in [18]. The researchers noted that a class of nonlinear evolution equations

$$p_t - \frac{1}{4}\left(\left(\frac{\alpha_{n,x}}{p}\right)_x + 4\alpha_n p\right)_x = 0 \tag{3}$$

are the compatibility condition for the AKNS system

$$\Phi_x = U\Phi = \begin{pmatrix} \lambda & p \\ -p & -\lambda \end{pmatrix} \Phi, \tag{4a}$$

$$\Phi_t = V\Phi = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi, \tag{4b}$$

where λ is the spectral parameter, A, B, and C are scalar functions of α_i, $i = 0, 1, \ldots, n$, and U and V must satisfy the equation

$$U_t - V_x + [U, V] = 0. \tag{5}$$

If $\Phi = \Phi(x, t, \lambda) = \binom{\varphi_1}{\varphi_2}$ is the general solution of system (4), then the map

$$p' = p + 4\lambda_1 \frac{\varphi_1 \varphi_2}{\varphi_1^2 + \varphi_2^2}\bigg|_{\lambda=\lambda_1} \tag{6}$$

is called a Darboux transformation, and p' is a new solution generated from the old solution p. Furthermore,

$$\overline{\Phi}(x, t, \lambda) = \begin{pmatrix} \bar{\varphi}_1 \\ \bar{\varphi}_2 \end{pmatrix} = \begin{pmatrix} \lambda - \lambda_1 \dfrac{\varphi_1^2 - \varphi_2^2}{\varphi_1^2 + \varphi_2^2}\bigg|_{\lambda=\lambda_1} & -2\lambda_1 \dfrac{\varphi_1 \varphi_2}{\varphi_1^2 + \varphi_2^2}\bigg|_{\lambda=\lambda_1} \\ -2\lambda_1 \dfrac{\varphi_1 \varphi_2}{\varphi_1^2 + \varphi_2^2}\bigg|_{\lambda=\lambda_1} & \lambda + \lambda_1 \dfrac{\varphi_1^2 - \varphi_2^2}{\varphi_1^2 + \varphi_2^2}\bigg|_{\lambda=\lambda_1} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \tag{7}$$

is the general solution of the AKNS system for p', and

$$p'' = p' + 4\lambda_2 \frac{\bar{\varphi}_1 \bar{\varphi}_2}{\bar{\varphi}_1^2 + \bar{\varphi}_2^2}\bigg|_{\lambda=\lambda_2} \tag{8}$$

becomes a new solution based on p'.

We can derive some relations from (5):

$$\begin{aligned} &A_x = p(B + C), \\ &p_t - B_x - 2pA + 2\lambda B = 0, \\ &p_t + C_x + 2pA + 2\lambda C = 0. \end{aligned} \tag{9}$$

With (9), we further obtain

$$B + C = \frac{A_x}{p}, \qquad B - C = \frac{(B+C)_x + 4pA}{2\lambda}. \tag{10}$$

If we set $n = 2$ and $A = \sum_{j=0}^{2} \alpha_j \lambda^{5-2j}$, then from (10), we obtain

$$B = \sum_{j=0}^{2} \left\{ \frac{\alpha_{j,x}}{2p} \lambda^{5-2j} + \left[\alpha_j p + \left(\frac{\alpha_{j,x}}{4p} \right)_x \right] \lambda^{4-2j} \right\},$$

$$C = \sum_{j=0}^{2} \left\{ \frac{\alpha_{j,x}}{2p} \lambda^{5-2j} - \left[\alpha_j p + \left(\frac{\alpha_{j,x}}{4p} \right)_x \right] \lambda^{4-2j} \right\},$$

where $\alpha_{j,x} = \partial \alpha_j / \partial x$. Substituting A, B, and C in (9) and setting the coefficients of λ^j, $j = 0, 1, 2$ to zero, we obtain the set of differential equations

$$\alpha_{0,x} = 0, \qquad \alpha_{1,x} = \alpha_0 p p_x \tag{11}$$

and

$$\alpha_{2,x} = \frac{1}{4} p \left[\left(\frac{\alpha_{1,x}}{p} \right)_x + 4\alpha_1 p \right]_x. \tag{12}$$

Taking $\alpha_0 = 16\epsilon$ and $\alpha_1 = 8\epsilon p^2 - 4a$ as a pair of particular solutions of (11) and substituting α_1 in (12), we obtain

$$\alpha_{2,x} = 4p(\epsilon p_{xxx} + 6\epsilon p^2 p_x - a p_x). \tag{13}$$

This is a solvable ordinary differential equation, and its particular solution can be taken in the form

$$\alpha_2 = 4\epsilon p p_{xx} - 2\epsilon p_x^2 + 6\epsilon p^4 - 2ap^2. \tag{14}$$

Substituting (13) in (3), we finally obtain Eq. (2), and substituting α_j and $\alpha_{j,x}$, $j = 0, 1, 2$, in A, B, and C, we also obtain system (4b) in the form

$$\Phi_t = \begin{pmatrix} 16\epsilon\lambda^5 + \lambda^3(8\epsilon p^2 - 4a) + \lambda Q & 16\lambda^4 \epsilon p + 8\lambda^3 \epsilon p_x + I \\ -16\lambda^4 \epsilon p + 8\lambda^3 \epsilon p_x + J & -16\epsilon\lambda^5 - \lambda^3(8\epsilon p^2 - 4a) - \lambda Q \end{pmatrix} \Phi, \tag{15}$$

where

$$Q = 4\epsilon p p_{xx} - 2\epsilon p_x^2 + 6\epsilon p^4 - 2ap^2,$$

$$I = 4\lambda^2(\epsilon p_{xx} + 2\epsilon p^3 - ap) + 2\lambda(\epsilon p_{xxx} + 6\epsilon p^2 p_x - a p_x) + \\ + \epsilon p_{xxxx} + 10\epsilon p^2 p_{xx} - a p_{xx} + 10\epsilon p p_x^2 + 6\epsilon p^5 - 2ap^3,$$

$$J = -4\lambda^2(\epsilon p_{xx} + 2\epsilon p^3 - ap) + 2\lambda(\epsilon p_{xxx} + 6\epsilon p^2 p_x - a p_x) - \\ - \epsilon p_{xxxx} - 10\epsilon p^2 p_{xx} + a p_{xx} - 10\epsilon p p_x^2 - 6\epsilon p^5 + 2ap^3.$$

3. Exact solutions of the perturbed mKdV equation with variable coefficients

We take $p = p_0$, where p_0 is an arbitrary constant, as the initial solution of Eq. (2). From (4a) and (15), we obtain the related AKNS system

$$\begin{aligned} \Phi_x &= \begin{pmatrix} \lambda & p_0 \\ -p_0 & -\lambda \end{pmatrix} \Phi, \\ \Phi_t &= [16\epsilon\lambda^4 + 4\lambda^2(2\epsilon p_0^2 - a) + 6\epsilon p_0^4 - 2ap_0^2] \begin{pmatrix} \lambda & p_0 \\ -p_0 & -\lambda \end{pmatrix} \Phi. \end{aligned} \tag{16}$$

Obviously,

$$\Phi_t = [16\epsilon\lambda^4 + 4\lambda^2(2\epsilon p_0^2 - a) + 6\epsilon p_0^4 - 2ap_0^2]\Phi_x.$$

If we set

$$\xi = x + \int [16\epsilon\lambda^4 + 4\lambda^2(2\epsilon p_0^2 - a) + 6\epsilon p_0^4 - 2ap_0^2]\, dt + c_0,$$

where c_0 is an arbitrary constant, then system (16) and the system

$$\Phi_\xi = \begin{pmatrix} \lambda & p_0 \\ -p_0 & -\lambda \end{pmatrix} \Phi \tag{17}$$

have exactly the same solutions. By the eigenvalue method, we obtain two different general solutions of system (17),

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} c_1(\lambda + \sqrt{\lambda^2 - p_0^2})e^{\eta} - c_2 p_0 e^{-\eta} \\ -c_1 p_0 e^{\eta} + c_2(\lambda + \sqrt{\lambda^2 - p_0^2})e^{-\eta} \end{pmatrix}, \quad |\lambda| > |p_0| \geq 0, \tag{18}$$

and also

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} c_1(\lambda\cos\theta - \sqrt{p_0^2 - \lambda^2}\sin\theta) - c_2 p_0\cos\theta \\ -c_1 p_0\cos\theta + c_2(\lambda\cos\theta + \sqrt{p_0^2 - \lambda^2}\sin\theta) \end{pmatrix}, \quad 0 \leq |\lambda| < |p_0|, \tag{19}$$

where $\eta = \eta(\lambda) = \sqrt{\lambda^2 - p_0^2}\xi$, $\theta = \theta(\lambda) = \sqrt{p_0^2 - \lambda^2}\xi$, and c_1 and c_2 are arbitrary constants.

The next step is to construct new solutions. For simplicity, we set $\eta_i = \eta(\lambda_i)$ and $\theta_i = \theta(\lambda_i)$, where $i = 1, 2$. From (18), we obtain

$$\begin{aligned} \varphi_1\varphi_2 &= (\lambda + \sqrt{\lambda^2 - p_0^2})[2\lambda c_1 c_2 - p_0(c_1^2 e^{2\eta} + c_2^2 e^{-2\eta})] = \\ &= \begin{cases} (\lambda + \sqrt{\lambda^2 - p_0^2})(\lambda - p_0\cosh 2\eta), & c_1 = c_2 = \dfrac{1}{\sqrt{2}}, \\ (\lambda + \sqrt{\lambda^2 - p_0^2})(-\lambda - p_0\cosh 2\eta), & c_1 = -c_2 = \dfrac{1}{\sqrt{2}}, \end{cases} \end{aligned} \tag{20}$$

$$\begin{aligned} \varphi_1^2 + \varphi_2^2 &= 2(\lambda + \sqrt{\lambda^2 - p_0^2})[\lambda(c_1^2 e^{2\eta} + c_2^2 e^{-2\eta}) - 2p_0 c_1 c_2] = \\ &= \begin{cases} 2(\lambda + \sqrt{\lambda^2 - p_0^2})(\lambda\cosh 2\eta - p_0), & c_1 = c_2 = \dfrac{1}{\sqrt{2}}, \\ 2(\lambda + \sqrt{\lambda^2 - p_0^2})(\lambda\cosh 2\eta + p_0), & c_1 = -c_2 = \dfrac{1}{\sqrt{2}}, \end{cases} \end{aligned} \tag{21}$$

and

$$\varphi_1^2 - \varphi_2^2 = 2\sqrt{\lambda^2 - p_0^2}(\lambda + \sqrt{\lambda^2 - p_0^2})(c_1^2 e^{2\eta} - c_2^2 e^{-2\eta}) =$$
$$= 2\sqrt{\lambda^2 - p_0^2}(\lambda + \sqrt{\lambda^2 - p_0^2}) \sinh 2\eta, \qquad c_1 = \pm c_2 = \frac{1}{\sqrt{2}}. \tag{22}$$

Substituting $p = p_0$ and (20) and (21) in (6), we obtain new soliton solutions of Eq. (2):

$$p_1' = \frac{2\lambda_1^2 - p_0^2 - \lambda_1 p_0 \cosh 2\eta_1}{\lambda_1 \cosh 2\eta_1 - p_0}$$

and

$$p_2' = \frac{p_0^2 - 2\lambda_1^2 - \lambda_1 p_0 \cosh 2\eta_1}{\lambda_1 \cosh 2\eta_1 + p_0}.$$

Similarly, from (19), we obtain

$$\phi_1\phi_2 = [c_1c_2(\lambda^2 + p_0^2) - \lambda p_0(c_1^2 + c_2^2)] \cos^2\theta +$$
$$+ c_1c_2(\lambda^2 - p_0^2)\sin^2\theta + p_0\sqrt{p_0^2 - \lambda^2}(c_1^2 - c_2^2)\cos\theta \sin\theta =$$
$$= \begin{cases} \frac{1}{2}(\lambda - p_0)(\lambda - p_0 \cos 2\theta), & c_1 = c_2 = \frac{1}{\sqrt{2}}, \\ -\frac{1}{2}(\lambda + p_0)(\lambda + p_0 \cos 2\theta), & c_1 = -c_2 = \frac{1}{\sqrt{2}}, \end{cases} \tag{23}$$

$$\phi_1^2 + \phi_2^2 = [(c_1^2 + c_2^2)(\lambda^2 + p_0^2) - 4\lambda p_0 c_1 c_2]\cos^2\theta + (p_0^2 - \lambda^2)\sin^2\theta =$$
$$= \begin{cases} (p_0 - \lambda)(p_0 - \lambda\cos 2\theta), & c_1 = c_2 = \frac{1}{\sqrt{2}}, \\ (p_0 + \lambda)(p_0 + \lambda\cos 2\theta), & c_1 = -c_2 = \frac{1}{\sqrt{2}}, \end{cases} \tag{24}$$

and

$$\phi_1^2 - \phi_2^2 = (c_1^2 - c_2^2)(\lambda^2 - p_0^2)\cos 2\theta + [2c_1c_2p_0 - \lambda(c_1^2 + c_2^2)]\sqrt{p_0^2 - \lambda^2}\sin 2\theta =$$
$$= \begin{cases} -\sqrt{p_0^2 - \lambda^2}(\lambda - p_0)\sin 2\theta, & c_1 = c_2 = \frac{1}{\sqrt{2}}, \\ -\sqrt{p_0^2 - \lambda^2}(\lambda + p_0)\sin 2\theta, & c_1 = -c_2 = \frac{1}{\sqrt{2}}. \end{cases} \tag{25}$$

Substituting $p = p_0$ and (23) and (24) in (6), we obtain new periodic solutions of Eq. (2):

$$p_3' = \frac{2\lambda_1^2 - p_0^2 - \lambda_1 p_0 \cos 2\theta_1}{\lambda_1 \cos 2\theta_1 - p_0}$$

and

$$p_4' = \frac{p_0^2 - 2\lambda_1^2 - \lambda_1 p_0 \cos 2\theta_1}{\lambda_1 \cos 2\theta_1 + p_0}.$$

Multiple wave solutions of nonlinear evolution equations are generally very complicated, and we therefore first construct the related formal solution. For convenience, we set $\varphi_i' = \varphi_i|_{\lambda=\lambda_1}$ and $\varphi_i'' = \varphi_i|_{\lambda=\lambda_2}$, where $i = 1, 2$.

With (7) taken into account, the formal solution of the AKNS system associated with p' is given by

$$\begin{pmatrix}\bar\varphi_1\\ \bar\varphi_2\end{pmatrix} = \frac{1}{\varphi_1'^2+\varphi_2'^2}\begin{pmatrix}\lambda\varphi_1(\varphi_1'^2+\varphi_2'^2)-\lambda_1\varphi_1(\varphi_1'^2-\varphi_2'^2)-2\lambda_1\varphi_1\varphi_2\varphi_2'\\ \lambda\varphi_2(\varphi_1'^2+\varphi_2'^2)+\lambda_1\varphi_2(\varphi_1'^2-\varphi_2'^2)-2\lambda_1\varphi_1\varphi_2\varphi_1'\end{pmatrix}, \tag{26}$$

which implies

$$\begin{aligned}
&4\lambda_2\frac{\bar\varphi_1\bar\varphi_2}{\bar\varphi_1^2+\bar\varphi_2^2}\bigg|_{\lambda=\lambda_2} = \\
&\quad = \frac{4\lambda_2}{\varphi_1'^2+\varphi_2'^2}\times \\
&\qquad \times\left\{\frac{(\lambda_2^2-\lambda_1^2)\varphi_1''\varphi_2''(\varphi_1'^2+\varphi_2'^2)^2+8\lambda_1^2\varphi_1'^2\varphi_2'^2\varphi_1''\varphi_2''}{(\lambda_2^2+\lambda_1^2)(\varphi_1'^2+\varphi_2'^2)(\varphi_1''^2+\varphi_2''^2)-2\lambda_1\lambda_2(\varphi_1'^2-\varphi_2'^2)(\varphi_1''^2-\varphi_2''^2)-8\lambda_1\lambda_2\varphi_1'\varphi_2'\varphi_1''\varphi_2''}+\right.\\
&\qquad \left.+\frac{2\lambda_1^2\varphi_1'\varphi_2'(\varphi_1'^2-\varphi_2'^2)(\varphi_1''^2-\varphi_2''^2)-2\lambda_1\lambda_2\varphi_1'\varphi_2'(\varphi_1'^2+\varphi_2'^2)(\varphi_1''^2+\varphi_2''^2)}{(\lambda_2^2+\lambda_1^2)(\varphi_1'^2+\varphi_2'^2)(\varphi_1''^2+\varphi_2''^2)-2\lambda_1\lambda_2(\varphi_1'^2-\varphi_2'^2)(\varphi_1''^2-\varphi_2''^2)-8\lambda_1\lambda_2\varphi_1'\varphi_2'\varphi_1''\varphi_2''}\right\}.
\end{aligned} \tag{27}$$

At the same time, combining (6) and (8) leads to

$$p'' = p + 4\lambda_1\frac{\varphi_1'\varphi_2'}{\varphi_1'^2+\varphi_2'^2} + 4\lambda_2\frac{\bar\varphi_1\bar\varphi_2}{\bar\varphi_1^2+\bar\varphi_2^2}\bigg|_{\lambda=\lambda_2}.$$

Substituting (27) in this formula, we obtain

$$p'' = p + \frac{4(\lambda_1^2-\lambda_2^2)[\lambda_1\varphi_1'\varphi_2'(\varphi_1''^2+\varphi_2''^2)-\lambda_2\varphi_1''\varphi_2''(\varphi_1'^2+\varphi_2'^2)]}{(\lambda_1^2+\lambda_2^2)(\varphi_1'^2+\varphi_2'^2)(\varphi_1''^2+\varphi_2''^2)-2\lambda_1\lambda_2(\varphi_1'^2-\varphi_2'^2)(\varphi_1''^2-\varphi_2''^2)-8\lambda_1\lambda_2\varphi_1'\varphi_2'\varphi_1''\varphi_2''}. \tag{28}$$

Again substituting $p = p_0$ and (20)–(22) in (28), we finally obtain the two-soliton solutions of Eq. (2)

$$\begin{aligned}
p_1'' &= p_0 + \\
&+\frac{2(\lambda_1^2-\lambda_2^2)[\lambda_1(\lambda_1-p_0\cosh 2\eta_1)(\lambda_2\cosh 2\eta_2-p_0)-\lambda_2(\lambda_1\cosh 2\eta_1-p_0)(\lambda_2-p_0\cosh 2\eta_2)]}{\mu(\lambda_1\cosh 2\eta_1-p_0)(\lambda_2\cosh 2\eta_2-p_0)-\gamma k\sinh 2\eta_1\sinh 2\eta_2-\gamma(\lambda_1-p_0\cosh 2\eta_1)(\lambda_2-p_0\cosh 2\eta_2)},\\
p_2'' &= p_0 + \\
&+\frac{2(\lambda_1^2-\lambda_2^2)[\lambda_1(\lambda_1-p_0\cosh 2\eta_1)(\lambda_2\cosh 2\eta_2+p_0)+\lambda_2(\lambda_1\cosh 2\eta_1-p_0)(\lambda_2+p_0\cosh 2\eta_2)]}{\mu(\lambda_1\cosh 2\eta_1-p_0)(\lambda_2\cosh 2\eta_2+p_0)-\gamma k\sinh 2\eta_1\sinh 2\eta_2+\gamma(\lambda_1-p_0\cosh 2\eta_1)(\lambda_2+p_0\cosh 2\eta_2)},\\
p_3'' &= p_0 - \\
&-\frac{2(\lambda_1^2-\lambda_2^2)[\lambda_1(\lambda_1+p_0\cosh 2\eta_1)(\lambda_2\cosh 2\eta_2-p_0)+\lambda_2(\lambda_1\cosh 2\eta_1+p_0)(\lambda_2-p_0\cosh 2\eta_2)]}{\mu(\lambda_1\cosh 2\eta_1+p_0)(\lambda_2\cosh 2\eta_2-p_0)-\gamma k\sinh 2\eta_1\sinh 2\eta_2+\gamma(\lambda_1+p_0\cosh 2\eta_1)(\lambda_2-p_0\cosh 2\eta_2)},\\
p_4'' &= p_0 - \\
&-\frac{2(\lambda_1^2-\lambda_2^2)[\lambda_1(\lambda_1+p_0\cosh 2\eta_1)(\lambda_2\cosh 2\eta_2+p_0)-\lambda_2(\lambda_1\cosh 2\eta_1+p_0)(\lambda_2+p_0\cosh 2\eta_2)]}{\mu(\lambda_1\cosh 2\eta_1+p_0)(\lambda_2\cosh 2\eta_2+p_0)-\gamma k\sinh 2\eta_1\sinh 2\eta_2-\gamma(\lambda_1+p_0\cosh 2\eta_1)(\lambda_2+p_0\cosh 2\eta_2)},
\end{aligned}$$

where $\mu = \lambda_1^2+\lambda_2^2$, $\gamma = 2\lambda_1\lambda_2$, and $k = \sqrt{\lambda_1^2-p_0^2}\sqrt{\lambda_2^2-p_0^2}$, $|\lambda_i| > |p_0| \geq 0$, $i = 1, 2$.

Because (28) is applicable in the case where $0 \le |\lambda| < |p_0|$, similarly, using (23)–(25) and (28), we can obtain two-periodic solutions of Eq. (2):

$$p_5'' = p_0 + \frac{2(\lambda_1^2 - \lambda_2^2)[\lambda_1(\lambda_1 - p_0 \cos 2\theta_1)(\lambda_2 \cos 2\theta_2 - p_0) - \lambda_2(\lambda_1 \cos 2\theta_1 - p_0)(\lambda_2 - p_0 \cos 2\theta_2)]}{\mu(\lambda_1 \cos 2\theta_1 - p_0)(\lambda_2 \cos 2\theta_2 - p_0) - \gamma s \sin 2\theta_1 \sin 2\theta_2 - \gamma(\lambda_1 - p_0 \cos 2\theta_1)(\lambda_2 - p_0 \cos 2\theta_2)},$$

$$p_6'' = p_0 + \frac{2(\lambda_1^2 - \lambda_2^2)[\lambda_1(\lambda_1 - p_0 \cos 2\theta_1)(\lambda_2 \cos 2\theta_2 + p_0) + \lambda_2(\lambda_1 \cos 2\theta_1 - p_0)(\lambda_2 + p_0 \cos 2\theta_2)]}{\mu(\lambda_1 \cos 2\theta_1 - p_0)(\lambda_2 \cos 2\theta_2 + p_0) - \gamma s \sin 2\theta_1 \sin 2\theta_2 + \gamma(\lambda_1 - p_0 \cos 2\theta_1)(\lambda_2 + p_0 \cos 2\theta_2)},$$

$$p_7'' = p_0 - \frac{2(\lambda_1^2 - \lambda_2^2)[\lambda_1(\lambda_1 + p_0 \cos 2\theta_1)(\lambda_2 \cos 2\theta_2 - p_0) + \lambda_2(\lambda_1 \cos 2\theta_1 + p_0)(\lambda_2 - p_0 \cos 2\theta_2)]}{\mu(\lambda_1 \cos 2\theta_1 + p_0)(\lambda_2 \cos 2\theta_2 - p_0) - \gamma s \sin 2\theta_1 \sin 2\theta_2 + \gamma(\lambda_1 + p_0 \cos 2\theta_1)(\lambda_2 - p_0 \cos 2\theta_2)},$$

and also

$$p_8'' = p_0 - \frac{2(\lambda_1^2 - \lambda_2^2)[\lambda_1(\lambda_1 + p_0 \cos 2\theta_1)(\lambda_2 \cos 2\theta_2 + p_0) - \lambda_2(\lambda_1 \cos 2\theta_1 + p_0)(\lambda_2 + p_0 \cos 2\theta_2)]}{\mu(\lambda_1 \cos 2\theta_1 + p_0)(\lambda_2 \cos 2\theta_2 + p_0) - \gamma s \sin 2\theta_1 \sin 2\theta_2 - \gamma(\lambda_1 + p_0 \cos 2\theta_1)(\lambda_2 + p_0 \cos 2\theta_2)},$$

where $s = \sqrt{p_0^2 - \lambda_1^2}\sqrt{p_0^2 - \lambda_2^2}$, $0 \le |\lambda_i| < |p_0|$, $i = 1, 2$.

4. Conclusion

In $p_1' - p_4'$ and $p_1'' - p_8''$, the amplitudes are independent of the perturbation coefficient ϵ, and only the wave velocities depend on ϵ. This shows that the influence of these small perturbations, including the dissipative p_{xxxxx} and the higher dispersion $p^5 p_x$, is subtler for these solutions: they do not destroy or change the shapes of the traveling solutions, but they may render a collision of solitons inelastic because quasilinear waves are emitted.

In particular, if $p_0 = 0$, then we can write the two-soliton solution p_1'' as

$$p_*'' = \frac{2(\lambda_1^2 - \lambda_2^2)(\lambda_1 \cosh \zeta_2 - \lambda_2 \cosh \zeta_1)}{(\lambda_1^2 + \lambda_2^2) \cosh \zeta_1 \cosh \zeta_2 - 2\lambda_1 \lambda_2 (1 + \sinh \zeta_1 \sinh \zeta_2)},$$

where

$$\zeta_i = 2\eta_i|_{p_0=0} = 2\lambda_i x + \int (32\epsilon \lambda_i^5 - 8a\lambda_i^3)\, dt + 2\lambda_i c_0, \quad i = 1, 2.$$

On one hand, in the case where $a = 1$ and $\epsilon = 0$, p_*'' is just the known two-soliton solution of the mKdV equation [18]

$$p_t + 6p^2 p_x + p_{xxx} = 0.$$

On the other hand, if a and ϵ are constants, then $\zeta_i = 2\lambda_i x + (32\epsilon\lambda_i^5 - 8a\lambda_i^3)t + 2\lambda_i c_0$, and p_*'' becomes a prototype example for multisoliton solutions describing purely elastic interactions between individual solitons. This can reveal an important property that these waves interact such that their identities are asymptotically preserved in the limit. We set $h(\lambda_i) = 32\epsilon\lambda_i^5 - 8a\lambda_i^3$ and assume that $h(\lambda_2) > h(\lambda_1)$, $\lambda_2 > \lambda_1 > 0$. Because $\zeta_2 = (\lambda_2/\lambda_1)\zeta_1 + 2\lambda_2[h(\lambda_2) - h(\lambda_1)]t$, for a fixed ζ_1, we obtain a pair of asymptotic soliton solutions

$$p_*'' \sim -2\lambda_1 \operatorname{sech}(\zeta_1 \mp v_0), \quad t \to \pm\infty, \tag{29}$$

with $v_0 = \tanh^{-1}(2\lambda_1\lambda_2/(\lambda_1^2 + \lambda_2^2))$. Similarly, for a fixed ζ_2, we obtain another pair of asymptotic soliton solutions

$$p_*'' \sim 2\lambda_2 \operatorname{sech}(\zeta_2 \pm v_0), \quad t \to \pm\infty. \tag{30}$$

Using p_*'' given by (29) and by (30), we can easily show that the only effect of the interaction of two solitary waves is a phase shift and the total phase shift experienced by a soliton is exactly equal to the sum of the partial shifts resulting from separate collisions with each of the solitons involved. In addition, we note that the phase shift is not changed by any small perturbing terms and has nothing to do with the coefficient a.

It is known that the mKdV equation arises as an approximate equation that holds in a certain asymptotic sense. Taking additional physical factors into account, we can obtain different kinds of small perturbations of the mKdV equation. Differential system (2) is so detailed that it can well represent the corresponding physical situations, but analyzing it mathematically may be too difficult because of the higher nonlinear dispersion terms and the higher spatial dispersion terms. Nevertheless, based on [18], we discussed the AKNS system of Eq. (2) in detail and constructed some exact double-wave solutions using the Darboux transformation method.

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