Theoretical and Mathematical Physics, **184**(1): 973–985 (2015)

COMPLETE CLASSIFICATION OF SPHERICALLY SYMMETRIC STATIC SPACE–TIMES VIA NOETHER SYMMETRIES

F. Ali,* T. Feroze,* and S. Ali†

We provide a complete classification of spherically symmetric static space–times by their Noether symmetries. We obtain the determining equations for the Noether symmetries using the usual Lagrangian of a general spherically symmetric static space–time and integrate them in each considered case. In particular, we find that spherically symmetric static space–times are categorized into six distinct classes corresponding to the Noether algebras of dimensions 5, 6, 7, 9, 11, *and* 17. *Using Noether's theorem, we also find the first integrals corresponding to each symmetry. Moreover, we obtain some new spherically symmetric static solutions.*

Keywords: Noether symmetry, static space–time

1. Introduction

The first exact solution of the Einstein field equations was obtained by Schwarzschild and is a spherically symmetric static solution. For a spherically symmetric space–time, there are exactly three rotational Killing vector fields preserving the metric, which gives $SO(3)$ as the isometry group of symmetries of these space–times. The interest in studying spherically symmetric space–times is explained by the help it gives in understanding the phenomena of gravitational collapse and black holes, widely known in the literature. For example, the Schwarzschild solution is a nontrivial exact spherically symmetric solution of the Einstein field equations and describes the gravitational field exterior to a static, spherical, uncharged massive body without angular momentum and isolated from all other bodies.

Seeking spherically symmetric space–times is an important task, and because they are significant in understanding the dynamics around black holes, classifying them according to their physical properties is crucial. It would therefore be interesting to find the general form of these space–times along with a detailed characterization of the first integrals of the corresponding geodesic equations. Moreover, the quantities that remain invariant along the geodesics carry significant physical information. Plane, cylindrically, and spherically symmetric space–times were classified with respect to their Killing vectors, homotheties, Ricci collineations, and curvature collineations in [1]–[7].

In [8] and also [9], a connection was found between the symmetries of the geodesic equations and the Killing vectors of the underlying spaces by comparing the Lie algebras of the two sets of symmetries. The Noether symmetries, being a superset of the Killing vectors, give more information about the invariants/conserved quantities. The Noether symmetries for different space–time metrics were obtained in a

*School of Natural Sciences, National University of Sciences and Technology, Islamabad, Pakistan, e-mail: tferoze@sns.nust.edu.pk.

†School of Electrical Engineering and Computer Science, National University of Sciences and Technology, Islamabad, Pakistan.

Prepared from an English manuscript submitted by the authors; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 184, No. 1, pp. 92–105, July, 2015. Original article submitted December 15, 2014; revised January 28, 2015.

series of papers [10]–[12]. Finding a connection between the Noether symmetries and the Killing vectors was the basic goal in those papers. To establish this connection, the authors of [10]–[12] found the Lie algebra of the Noether symmetries and then compared this algebra with the algebra of isometries. Ali et al. [13] recently used the symmetry method approach to completely classify the Lagrangians of plane-symmetric static space–times using Noether symmetries and also presented the first integrals/conserved quantities in each case.

In differential geometry, a Lagrangian that yields the geodesic equations (in the form of the Euler–Lagrange equations) is introduced. The Lagrangian can be obtained directly from the line element [14]

$$ds^2 = g_{ab}(x^c)\, dx^a\, dx^b,$$

namely,

$$L = g_{ab}(x^c)\dot{x}^a\dot{x}^b.$$

Here, we employ symmetry methods to completely classify spherically symmetric static space–times according to the Noether symmetries. We then use the famous Noether theorem to write the first integrals for each space–time. We thus recover all known solutions of the Einstein field equations and the Noether symmetries together with their first integrals.

The plan of the paper is as follows. In Sec. 2, we give the basic definitions and describe the structure of Noether symmetries. In Sec. 3, we write the determining equations for spherically symmetric static space–times, which is a system of nineteen linear partial differential equations (PDEs). We obtain different values of the metric coefficient, namely, ν and μ, while integrating the PDEs. This yields a complete classification of spherically symmetric static space–times by Noether symmetries. The corresponding first integrals are also given in the same section. Section 4 contains the conclusion.

2. Preliminaries

A symmetry

$$\mathbf{X} = \xi\frac{\partial}{\partial s} + \eta^i\frac{\partial}{\partial x^i}, \quad i = 1, 2, \ldots, n,$$

is a Noether symmetry if it leaves the action

$$W = \int L(s, x^i(s), \dot{x}^i(s))\, ds \tag{1}$$

invariant up to some gauge function A [14], i.e., under the transformation

$$\tilde{s} = s + \epsilon\xi(s, x^i),$$
$$\tilde{x}^i = x^i + \epsilon\eta^i(s, x^i),$$

the action given by (1) becomes

$$\widetilde{W} = \int L(\tilde{s}, \tilde{x}^i(s), \dot{\tilde{x}}^i(s))\, ds,$$

where L is the Lagrangian, s is the independent variable, x^i are the dependent variables, and $\dot{x}^i$ are their derivatives with respect to s. The variation up to the gauge function is

$$\widetilde{W} - W = \int \mathbf{D}A(s, x^i(s))\, ds, \tag{2}$$

where $\mathbf{D}$ is the standard total derivative operator given by

$$\mathbf{D} = \frac{\partial}{\partial s} + \dot{x}^i \frac{\partial}{\partial x^i}.$$

Equation (2) can also be written as

$$\mathbf{X}^{(1)} L + \mathbf{D}(\xi) L = \mathbf{D} A, \tag{3}$$

where

$$\mathbf{X}^{(1)} = \mathbf{X} + \eta^i_{,s} \frac{\partial}{\partial x^i}$$

is the first-order prolonged generator. The coefficients ξ and η^i of the Noether symmetry are functions of s and x^i. The coefficients $\eta^i_{,s}$ of the prolonged operator $\mathbf{X}^{(1)}$ are functions of s, $x^i(s)$, and $\dot{x}^i(s)$ and are defined as

$$\eta^i_{,s} = \mathbf{D}(\eta^i) - \dot{x}^i \mathbf{D}(\xi),$$

where x^i refers to the space of dependent variables.

Using the same identification, we state the famous **Noether theorem**: If $\mathbf{X}$ is a Noether symmetry of a given Lagrangian L with respect to the gauge function A, then the quantity

$$I = A - \left(\xi L + (\eta^i - \xi \dot{x}^i) \frac{\partial L}{\partial \dot{x}^i} \right),$$

called a first integral, is annihilated by the total derivative operator, i.e., $\mathbf{D} I = 0$. In other words, corresponding to each Noether symmetry, there is a conservation law/first integral [15], for example, the generators of time translation $\partial/\partial t$ and rotation $\partial/\partial\theta$ respectively give conservation of energy and angular momentum.

The general form of a spherically symmetric static space–time is [3]

$$ds^2 = e^{\nu(r)} dt^2 - e^{\mu(r)} dr^2 - e^{\lambda(r)} d\Omega^2, \tag{4}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$ and both ν and μ are arbitrary functions of the radial coordinate r. (In particular, for spherically symmetric space–times, the dependent variables are (t, r, θ, ϕ) for x^i, $i = 0, 1, 2, 3$.) It can be seen that $e^{\lambda(r)}$ can have one of two forms: β^2 or r^2, where β is some constant [2] (we absorb β^2 in the definition of $d\Omega^2$).

We write the determining equations using the corresponding Lagrangian of the above space–time and study the complete integrability of those equations in each case. The usual Lagrangian L for a general spherically symmetric static space–time is

$$L = e^{\nu(r)} \dot{t}^2 - e^{\mu(r)} \dot{r}^2 - e^{\lambda(r)} (\dot{\theta}^2 + \sin^2\theta \, \dot{\phi}^2), \tag{5}$$

where the dot denotes differentiation with respect to the arc-length parameter s. For this Lagrangian, the Noether symmetry generator becomes

$$\mathbf{X}^{(1)} = \xi \frac{\partial}{\partial s} + \eta^0 \frac{\partial}{\partial t} + \eta^1 \frac{\partial}{\partial r} + \eta^2 \frac{\partial}{\partial \theta} + \eta^3 \frac{\partial}{\partial \phi} + \eta^0_{,s} \frac{\partial}{\partial \dot{t}} + \eta^1_{,s} \frac{\partial}{\partial \dot{r}} + \eta^2_{,s} \frac{\partial}{\partial \dot{\theta}} + \eta^3_{,s} \frac{\partial}{\partial \dot{\phi}},$$

and the operator $\mathbf{D}$ becomes

$$\mathbf{D} = \frac{\partial}{\partial s} + \dot{t} \frac{\partial}{\partial t} + \dot{r} \frac{\partial}{\partial r} + \dot{\theta} \frac{\partial}{\partial \theta} + \dot{\phi} \frac{\partial}{\partial \phi}.$$

3. Classification results and computational remarks

Substituting Lagrangian (5) in (3) and comparing the coefficients of all monomials, we obtain the system of 19 linear PDEs

$$
\begin{aligned}
&\xi_t = 0, \qquad \xi_r = 0, \qquad \xi_\theta = 0, \qquad \xi_\phi = 0,\\
&A_s = 0, \qquad A_t - 2e^{\nu(r)}\eta^0_s = 0, \qquad A_r + 2\eta^1_s = 0,\\
&A_\theta + 2e^{\mu(r)}\eta^2_s = 0, \qquad A_\phi + 2e^{\mu(r)}\eta^3_s = 0,\\
&\xi_s - \mu_r\eta^1 - 2\eta^1_r = 0, \qquad \xi_s - \nu_r\eta^1 - 2\eta^0_t = 0,\\
&\xi_s - \frac{2}{r}\eta^1 - 2\eta^2_\theta = 0, \qquad \xi_s - \frac{2}{r}\eta^1 - 2\cot\theta\eta^2 - 2\eta^3_\phi = 0,\\
&\eta^2_\phi + \sin^2\theta\eta^3_\theta = 0, \qquad e^{\nu(r)}\eta^0_r - e^{\mu(r)}\eta^1_t = 0,\\
&e^{\mu(r)}\eta^1_\theta + r^2\eta^2_r = 0, \qquad e^{\nu(r)}\eta^0_\theta - r^2\eta^2_t = 0,\\
&e^{\nu(r)}\eta^0_\phi - r^2\sin^2\theta\eta^3_t = 0, \qquad e^{\mu(r)}\eta^1_\phi + r^2\sin^2\theta\eta^3_r = 0.
\end{aligned}
\tag{6}
$$

We intend to classify all spherically symmetric static space–times with respect to their Noether symmetries by finding the solutions of above system of PDEs.

In solving system (6), we used the computer algebra system `Maple`-17 and split out the cases with the remarkable algorithm `rifsimp`, which is essentially an extension of the Gaussian elimination and Groebner basis algorithms for simplifying overdetermined systems of polynomially nonlinear PDEs or ODEs and inequalities by transforming them into a convenient form.

In what follows, we list spherically symmetric static space–times, their Noether symmetries, and the corresponding first integrals. We also present the Noether algebra of Noether symmetries in the cases that are unknown in the literature.

For solving system of PDEs (6), we note that the first equation simply implies that ξ can only be a function of the arc-length parameter, i.e., $\xi = \xi(s)$. To distinguish Killing vector fields from Noether symmetries, we use different letters: Noether symmetries that are not Killing vector fields are denoted by $\mathbf{Y}$. We also note that a static space–time always admits a timelike Killing vector field. Moreover, Lagrangian (5) is not explicitly dependent on t; therefore, a timelike Killing vector field appears as a Noether symmetry in each case. Furthermore, Lagrangian (5) is spherically symmetric; therefore, the Lie algebra of Killing vector fields, $so(3)$, corresponding to the Lie group $SO(3)$ is intrinsically admitted by each space–time. It is also important to note here that a static space–time always admits a timelike Killing vector field. Hence,

$$
\begin{aligned}
&\mathbf{X}_0 = \frac{\partial}{\partial t}, \qquad \mathbf{Y}_0 = \frac{\partial}{\partial s},\\
&\mathbf{X}_1 = \frac{\partial}{\partial \phi}, \qquad \mathbf{X}_2 = \cos\phi\,\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\,\frac{\partial}{\partial\phi}, \qquad \mathbf{X}_3 = \sin\phi\,\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\,\frac{\partial}{\partial\phi}
\end{aligned}
$$

form a basis of a minimal five-dimensional Noether algebra, in which $\mathbf{Y}_0$ is not a Killing vector field of spherically symmetric space–time (4). The Noether algebra of these five Noether symmetries is

$$
\begin{aligned}
&[\mathbf{X}_1, \mathbf{X}_2] = -\mathbf{X}_3, \qquad [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2, \qquad [\mathbf{X}_2, \mathbf{X}_3] = -\mathbf{X}_1,\\
&[\mathbf{X}_i, \mathbf{X}_0] = 0, \qquad [\mathbf{X}_i, \mathbf{Y}_0] = 0 \quad \text{otherwise}
\end{aligned}
$$

and is identified with the associated group $SO(3) \times \mathbb{R}^2$.

3.1. Spherically symmetric static space–times with five Noether symmetries. Some examples of space–times that admit the minimum set of Noether symmetries (five symmetries) and appeared during the calculations are given in Table 1.

Table 1

No.	$\nu(r)$	$\mu(r)$
1.	$\log\left(\frac{r}{\alpha}\right)^2$	arbitrary
2.	$\log\left(1-\left(\frac{r}{\alpha}\right)^2\right)$	arbitrary
3.	$\log\left(\frac{r}{\alpha}\right)^2$	$-\log\left(1-\left(\frac{r}{\alpha}\right)^2\right)$
4.	arbitrary	$-\log\left(1-\left(\frac{r}{\alpha}\right)^2\right)$
5.	$\log\left(1-\frac{\alpha}{r}\right)$	$-\log\left(1-\frac{\alpha}{r}\right)$

Forms of μ and ν.

The Noether symmetries and corresponding first integrals with a constant value of the gauge function, $A = \text{const}$, are listed in Table 2.

Table 2

Gen	First integrals
$\mathbf{X}_0$	$\phi_0 = e^{\nu(r)}\dot{t}$
$\mathbf{X}_1$	$\phi_1 = r^2 \sin^2\theta\dot{\phi}$
$\mathbf{X}_2$	$\phi_2 = r^2(\cos\phi\dot{\theta} - \cot\theta\sin\phi\dot{\phi})$
$\mathbf{X}_3$	$\phi_3 = r^2(\sin\phi\dot{\theta} + \cot\theta\cos\phi\dot{\phi})$
$\mathbf{Y}_0$	$\phi_4 = e^{\nu(r)}\dot{t}^2 - e^{\mu(r)}\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$

First integrals.

3.2. Spherically symmetric static space–times with six Noether symmetries. There are two distinct classes of spherically symmetric static space–times admitting six Noether symmetries. In particular, we obtain

$$ds^2 = \left(\frac{r}{a}\right)^{\alpha} dt^2 - dr^2 - r^2\, d\Omega^2, \quad \alpha \neq 0, 2,$$

which in addition to the minimum five-dimensional Noether algebra also admits a Noether symmetry corresponding to the scaling transformation $(s, t, r) \to (\lambda s, \lambda^p t, \lambda^{1/2} r)$ and given by

$$\mathbf{Y}_1 = s\frac{\partial}{\partial s} + pt\frac{\partial}{\partial t} + \frac{r}{2}\frac{\partial}{\partial r}, \qquad p = \frac{2-\alpha}{4},$$

forming a six-dimensional Noether algebra. This induces a scale-invariant spherically symmetric static space–time. The corresponding first integral is

$$\phi_6 = s\left(\left(\frac{r}{a}\right)^{\alpha}\dot{t}^2 - \dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right) + \frac{\alpha-2}{2}\left(\frac{r}{a}\right)^2 t\dot{t} + r\dot{r}. \tag{7}$$

The other space–time with a six-dimensional Noether algebra is given by

$$ds^2 = dt^2 - e^{\mu(r)}dr^2 - r^2 d\Omega^2, \quad \mu(r) \neq \log\left(1 - \frac{r^2}{b^2}\right)^{-1}, \quad \mu(r) \neq \text{const},$$

with an additional Noether symmetry relative to a nontrivial gauge term,

$$\mathbf{Y}_1 = s\frac{\partial}{\partial t}, \qquad A = 2t.$$

The first integral corresponding to $\mathbf{Y}_1$ is $\phi_6 = t - s\dot{t}$.

3.3. Spherically symmetric static space–times with seven Noether symmetries. There arise five space–times with seven Noether symmetries in which four cases contain the group of six Killing vector fields and one case contains only the minimum group of Killing vectors while the other two symmetries are Noether symmetries. We discuss them separately.

The four metrics are given by

$$ds^2 = e^{r/b}dt^2 - dr^2 - d\Omega^2, \qquad b \neq 0, \tag{8}$$

$$ds^2 = \sec^2\frac{r}{a}\,dt^2 - \sec^2\frac{r}{a}\,dr^2 - d\Omega^2, \qquad a \neq 0, \tag{9}$$

$$ds^2 = \left(1 - \frac{r^2}{b^2}\right)dt^2 - \left(1 - \frac{r^2}{b^2}\right)^{-1}dr^2 - d\Omega^2, \quad b \neq 0, \tag{10}$$

$$ds^2 = \frac{\alpha^2}{r^2}\,dt^2 - \frac{\alpha^2}{r^2}\,dr^2 - d\Omega^2, \qquad \alpha \neq 0. \tag{11}$$

Together with the minimum set of symmetries, they respectively contain two additional symmetries:

$$\mathbf{X}_{4,1} = t\frac{\partial}{\partial r} - b\left(e^{-r/b} + \frac{t^2}{4b^2}\right)\frac{\partial}{\partial t}, \qquad \mathbf{X}_{5,1} = \frac{\partial}{\partial r} - \frac{t}{2b}\frac{\partial}{\partial t},$$

$$\mathbf{X}_{4,2} = \sin\frac{r}{a}\cos\frac{t}{a}\frac{\partial}{\partial t} + \sin\frac{t}{a}\cos\frac{r}{a}\frac{\partial}{\partial r}, \qquad \mathbf{X}_{5,2} = \cos\frac{t}{a}\cos\frac{r}{a}\frac{\partial}{\partial r} - \sin\frac{r}{a}\sin\frac{t}{a}\frac{\partial}{\partial t},$$

$$\mathbf{X}_{4,3} = -\frac{rbe^{t/b}}{\sqrt{r^2-b^2}}\frac{\partial}{\partial t} + \sqrt{r^2-b^2}\,e^{t/b}\frac{\partial}{\partial r}, \qquad \mathbf{X}_{5,3} = \frac{rbe^{-t/b}}{\sqrt{r^2-b^2}}\frac{\partial}{\partial t} + \sqrt{r^2-b^2}\,e^{-t/b}\frac{\partial}{\partial r},$$

$$\mathbf{X}_{4,4} = \frac{t^2+r^2}{2}\frac{\partial}{\partial t} + rt\frac{\partial}{\partial r}, \qquad \mathbf{X}_{5,4} = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r},$$

where the new subscript refers to different cases, which we consider hereafter.

The corresponding first integrals are listed in Table 3.

Table 3

Gen	First integrals
$\mathbf{X}_{4,1}$	$\phi_5 = b\left(1 + \dfrac{t^2 e^{r/b}}{4b^2}\right)\dot{t} + t\dot{r}$
$\mathbf{X}_{5,1}$	$\phi_6 = \dfrac{te^{r/b}}{b}\dot{t} + 2\dot{r}$
$\mathbf{X}_{4,2}$	$\phi_5 = \sec^2\dfrac{r}{a}\left(\dot{t}\sin\dfrac{r}{a}\cos\dfrac{t}{a} - \dot{r}\sin\dfrac{t}{a}\cos\dfrac{r}{a}\right)$
$\mathbf{X}_{5,2}$	$\phi_6 = -\sec^2\dfrac{r}{a}\left(\dot{t}\sin\dfrac{r}{a}\sin\dfrac{t}{a} + \dot{r}\cos\dfrac{t}{a}\cos\dfrac{r}{a}\right)$
$\mathbf{X}_{4,3}$	$\phi_5 = e^{t/b}\left(\dfrac{r\dot{t}\sqrt{r^2-b^2}}{b} + \dfrac{b^2\dot{r}}{\sqrt{r^2-b^2}}\right)$
$\mathbf{X}_{5,3}$	$\phi_6 = e^{t/b}\left(-\dfrac{r\dot{t}\sqrt{r^2-b^2}}{b} + \dfrac{b^2\dot{r}}{\sqrt{r^2-b^2}}\right)$
$\mathbf{X}_{4,4}$	$\phi_5 = \dfrac{(t^2+r^2)\dot{t}}{r^2} + \dfrac{t\dot{r}}{r}$
$\mathbf{X}_{5,4}$	$\phi_6 = 2\left(\dfrac{t\dot{t}}{r^2} - \dot{r}r\right)$
$\mathbf{Y}_1$	$\phi_5 = sL - r\dot{r} + 2s\dot{r}^2$
$\mathbf{Y}_2$	$\phi_6 = 2s^2L + 2(s\dot{r}^2 - sr\dot{r}) + r^2$

First integrals of seven Noether symmetries

The Lie algebra of the symmetries of metric (9) is

$$[\mathbf{X}_1, \mathbf{X}_2] = -\mathbf{X}_3, \qquad [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2, \qquad [\mathbf{X}_2, \mathbf{X}_3] = -\mathbf{X}_1,$$

$$[\mathbf{X}_0, \mathbf{X}_{4,2}] = \frac{1}{a}\mathbf{X}_{5,2}, \qquad [\mathbf{X}_0, \mathbf{X}_{5,2}] = -\frac{1}{a}\mathbf{X}_{4,2},$$

$$[\mathbf{X}_i, \mathbf{X}_j] = 0, \qquad [\mathbf{X}_i, \mathbf{Y}_0] = 0 \quad \text{otherwise.}$$

The Lie algebra of the symmetries of metric (10) is

$$[\mathbf{X}_1, \mathbf{X}_2] = -\mathbf{X}_3, \qquad [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2, \qquad [\mathbf{X}_2, \mathbf{X}_3] = -\mathbf{X}_1,$$

$$[\mathbf{X}_0, \mathbf{X}_{4,4}] = \mathbf{X}_{5,4}, \qquad [\mathbf{X}_{4,4}, \mathbf{X}_{5,4}] = -\mathbf{X}_{4,4},$$

$$[\mathbf{X}_0, \mathbf{X}_{5,4}] = \mathbf{X}_0, \qquad [\mathbf{X}_i, \mathbf{X}_j] = 0, \quad [\mathbf{X}_i, \mathbf{Y}_0] = 0 \quad \text{otherwise.}$$

The space–time

$$ds^2 = \left(\frac{r}{a}\right)^2 dt^2 - dr^2 - r^2\, d\Omega^2$$

contains two nontrivial Noether symmetries

$$\mathbf{Y}_1 = s\frac{\partial}{\partial s} + \frac{r}{2}\frac{\partial}{\partial r}, \qquad \mathbf{Y}_2 = \frac{s^2}{2}\frac{\partial}{\partial s} + \frac{rs}{2}\frac{\partial}{\partial r}, \qquad A = -\frac{r^2}{2},$$

whose first integrals are also given in Table 3.

3.4. Spherically symmetric static space–times with nine Noether symmetries. This section contains some well-known and important space–times. Here, we have five different cases of space–times in which three contain two additional Noether symmetries and one case contains one additional Noether symmetry with all others being Killing vector fields.

We obtained the following results (space–times) with nine Noether symmetries:

$$ds^2 = dt^2 - dr^2 - d\Omega^2, \tag{12}$$

$$ds^2 = \frac{\beta^2}{r^2}\,dt^2 - \frac{\beta^4}{r^4}\,dr^2 - d\Omega^2, \qquad \beta \neq 0, \tag{13}$$

$$ds^2 = \left(1 + \frac{r}{b}\right)^2 dt^2 - dr^2 - d\Omega^2, \quad b \neq 0, \tag{14}$$

$$ds^2 = dt^2 - \frac{dr^2}{1 - r^2/b^2} - r^2\,d\Omega^2, \quad b \neq 0. \tag{15}$$

The first three space–times correspond to the famous Bertotti-Robinson-like solutions of the Einstein field equations, which describe a universe with a uniform magnetic field, and the last case is the Einstein universe.

We first list the Killing vector fields that are also Noether symmetries:

$$\mathbf{X}_{4,1} = r\frac{\partial}{\partial t} + t\frac{\partial}{\partial r}, \qquad \mathbf{X}_{5,1} = \frac{\partial}{\partial r},$$

$$\mathbf{X}_{4,2} = -\beta r e^{-t/\beta}\frac{\partial}{\partial t} + r^2 e^{-t/\beta}\frac{\partial}{\partial r}, \qquad \mathbf{X}_{5,2} = \beta r e^{t/\beta}\frac{\partial}{\partial t} + r^2 e^{t/\beta}\frac{\partial}{\partial r},$$

$$\mathbf{X}_{4,3} = \frac{b}{b+r}e^{-t/b}\frac{\partial}{\partial t} + e^{-t/b}\frac{\partial}{\partial r},$$

$$\mathbf{X}_{5,3} = -\frac{b}{b+r}e^{t/b}\frac{\partial}{\partial t} + e^{t/b}\frac{\partial}{\partial r},$$

$$\mathbf{X}_{4,4} = \sqrt{b^2 - r^2}\sin\phi\sin\theta\frac{\partial}{\partial r} - \frac{\sqrt{b^2 - r^2}}{r}\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\sqrt{b^2 - r^2}}{r\sin\theta}\cos\phi\frac{\partial}{\partial\phi},$$

$$\mathbf{X}_{5,4} = \sqrt{b^2 - r^2}\cos\phi\sin\theta\frac{\partial}{\partial r} - \frac{\sqrt{b^2 - r^2}}{r}\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sqrt{b^2 - r^2}}{r\sin\theta}\sin\phi\frac{\partial}{\partial\phi},$$

$$\mathbf{X}_{6,4} = \sqrt{b^2 - r^2}\cos\theta\frac{\partial}{\partial r} - \frac{\sqrt{b^2 - r^2}}{r}\sin\theta\frac{\partial}{\partial\theta},$$

and the Noether symmetries corresponding to nontrivial gauge terms are

$$\mathbf{Y}_{1,1} = s\frac{\partial}{\partial t}, \qquad A_{1,1} = 2t,$$

$$\mathbf{Y}_{2,1} = s\frac{\partial}{\partial r}, \qquad A_{2,1} = -2r,$$

$$\mathbf{Y}_{1,2} = -\frac{rse^{-t/\beta}}{\beta^3}\frac{\partial}{\partial t} + \frac{r^2 se^{-t/\beta}}{\beta^4}\frac{\partial}{\partial r}, \qquad A_{1,2} = \frac{2e^{-t/\beta}}{r},$$

$$\mathbf{Y}_{2,2} = \frac{rse^{t/\beta}}{\beta^3}\frac{\partial}{\partial t} + \frac{r^2 se^{t/\beta}}{\beta^4}\frac{\partial}{\partial r}, \qquad A_{1,2} = \frac{2e^{t/\beta}}{r},$$

$$\mathbf{Y}_{1,3} = -\frac{bs}{2(b+r)}e^{-t/b}\frac{\partial}{\partial t} - \frac{s}{2}e^{-t/b}\frac{\partial}{\partial r}, \qquad A_{1,3} = (b+r)e^{-t/b},$$

$$\mathbf{Y}_{2,3} = \frac{bs}{2(b+r)} e^{t/b} \frac{\partial}{\partial t} - \frac{s}{2} e^{t/b} \frac{\partial}{\partial r}, \qquad A_{2,3} = (b+r)e^{t/b},$$

$$\mathbf{Y}_{1,4} = s\frac{\partial}{\partial t}, \qquad A = 2t.$$

The Lie algebra of symmetries of metric (13) above is

$$[\mathbf{X}_1, \mathbf{X}_2] = -\mathbf{X}_3, \qquad [\mathbf{X}_1, \mathbf{X}_3] = \mathbf{X}_2, \qquad [\mathbf{X}_2, \mathbf{X}_3] = -\mathbf{X}_1,$$

$$[\mathbf{X}_0, \mathbf{X}_{4,2}] = \frac{1}{\alpha}\mathbf{X}_{4,2}, \qquad [\mathbf{X}_{4,2}, \mathbf{X}_{5,2}] = -\mathbf{X}_{4,2}, \qquad [\mathbf{X}_0, \mathbf{X}_{5,2}] = \frac{1}{\alpha}\mathbf{X}_{5,2},$$

$$[\mathbf{X}_0, \mathbf{Y}_{1,2}] = -\frac{1}{\alpha}\mathbf{Y}_{1,2}, \qquad [\mathbf{X}_0, \mathbf{Y}_{2,2}] = \frac{1}{\alpha}\mathbf{Y}_{2,2},$$

$$[\mathbf{Y}_0, \mathbf{Y}_{1,2}] = \frac{1}{\alpha^4}\mathbf{X}_{4,2}, \qquad [\mathbf{Y}_0, \mathbf{Y}_{2,2}] = \frac{1}{\alpha^4}\mathbf{X}_{5,2},$$

$$[\mathbf{X}_i, \mathbf{X}_j] = 0, \qquad [\mathbf{X}_i, \mathbf{Y}_0] = 0, \qquad [\mathbf{Y}_i, \mathbf{Y}_j] = 0 \qquad \text{otherwise.}$$

The first integrals for the nine Noether symmetries are given in Table 4.

Table 4

Gen	First integrals
$\mathbf{X}_{4,1},\ \mathbf{X}_{5,1}$	$\phi_5 = t\dot{r} - r\dot{t}, \quad \phi_6 = \dot{r}$
$\mathbf{Y}_{1,1},\ \mathbf{Y}_{2,1}$	$\phi_7 = 2(t - s\dot{t}), \quad \phi_8 = r - s\dot{r}$
$\mathbf{X}_{4,2},\ \mathbf{X}_{5,2}$	$\phi_5 = 2e^{-t/\beta}\beta^3\left(\frac{\dot{t}}{r} + \frac{\beta\dot{r}}{r^2}\right), \quad \phi_6 = 2e^{t/\beta}\beta^3\left(-\frac{\dot{t}}{r} + \frac{\beta\dot{r}}{r^2}\right)$
$\mathbf{Y}_{1,2},\ \mathbf{Y}_{2,2}$	$\phi_7 = 2se^{-t/\beta}\left(\frac{\dot{t}}{r\beta} + \frac{\dot{r}}{r^2}\right) + \frac{2e^{-t/\beta}}{r}, \quad \phi_8 = 2se^{t/\beta}\left(\frac{-\dot{t}}{\beta r} + \frac{\dot{r}}{r^2}\right) + \frac{2e^{t/\beta}}{r}$
$\mathbf{X}_{4,3},\ \mathbf{X}_{5,3}$	$\phi_5 = e^{-t/b}\left(-\frac{\dot{t}(b+r)}{b} + \dot{r}\right), \quad \phi_6 = e^{t/b}\left(\frac{\dot{t}(b+r)}{b} + \dot{r}\right)$
$\mathbf{Y}_{1,3},\ \mathbf{Y}_{2,3}$	$\phi_7 = e^{-t/b}\left(\frac{s\dot{t}(b+r)}{b} + s\dot{r} + (b+r)\right),$ $\phi_8 = e^{t/b}\left(\frac{s\dot{t}(b+r)}{b} - s\dot{r} + (b+r)\right)$
$\mathbf{X}_{4,4}$	$\phi_5 = \frac{b^2\dot{r}\sin\phi\sin\theta}{\sqrt{b^2-r^2}} - r\dot{\theta}\sqrt{b^2-r^2}\cos\theta\sin\phi + r\dot{\phi}\sqrt{b^2-r^2}\sin\theta\cos\phi$
$\mathbf{X}_{5,4}$	$\phi_6 = \frac{b^2\dot{r}\cos\phi\sin\theta}{\sqrt{b^2-r^2}} - r\dot{\theta}\sqrt{b^2-r^2}\cos\theta\cos\phi + r\dot{\phi}\sqrt{b^2-r^2}\sin\theta\sin\phi$
$\mathbf{X}_{6,4},\ \mathbf{Y}_{1,4}$	$\phi_7 = \frac{b^2\dot{r}\cos\theta}{\sqrt{b^2-r^2}} - r\dot{\theta}\sqrt{b^2-r^2}\sin\theta, \quad \phi_8 = t - s\dot{t}$

First integrals of nine Noether symmetries.

3.5. Spherically symmetric static space–times with eleven Noether symmetries. It turns out that the famous de Sitter metric

$$ds^2 = \left(1 - \frac{r^2}{b^2}\right)dt^2 - \frac{dr^2}{1 - r^2/b^2} - r^2\, d\Omega^2 \tag{16}$$

is the only case with eleven Noether symmetries. All except $\mathbf{Y}_0$ are Killing vectors. Together with the minimum set of Noether symmetries, for the metric given by (16), we have the Noether symmetries

$$\begin{aligned}
\mathbf{X}_4 =& \frac{br\sin\phi\sin\theta\cos(t/b)}{\sqrt{b^2-r^2}}\frac{\partial}{\partial t} + \\
&+ \sin\left(\frac{t}{b}\right)\sqrt{b^2-r^2}\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + r\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right),\\
\mathbf{X}_5 =& \frac{br\cos\phi\sin\theta\cos(t/b)}{\sqrt{b^2-r^2}}\frac{\partial}{\partial t} + \\
&+ \sin\left(\frac{t}{b}\right)\sqrt{b^2-r^2}\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right),\\
\mathbf{X}_6 =& -\frac{br\sin\phi\sin\theta\sin(t/b)}{\sqrt{b^2-r^2}}\frac{\partial}{\partial t} + \\
&+ \cos\left(\frac{t}{b}\right)\sqrt{b^2-r^2}\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + r\cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right),\\
\mathbf{X}_7 =& -\frac{br\cos\phi\sin\theta\sin(t/b)}{\sqrt{b^2-r^2}}\frac{\partial}{\partial t} + \\
&+ \cos\left(\frac{t}{b}\right)\sqrt{b^2-r^2}\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi}\right),\\
\mathbf{X}_8 =& \frac{br\cos\theta\cos(t/b)}{\sqrt{b^2-r^2}}\frac{\partial}{\partial t} + \sin\left(\frac{t}{b}\right)\sqrt{b^2-r^2}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right),\\
\mathbf{X}_9 =& \frac{-br\cos\theta\sin(t/b)}{\sqrt{b^2-r^2}}\frac{\partial}{\partial t} + \cos\left(\frac{t}{b}\right)\sqrt{b^2-r^2}\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right).
\end{aligned}$$

The first integrals are given in Table 5.

3.6. Spherically symmetric static space–times with seventeen Noether symmetries. The famous Minkowski metric, which represents a flat space–time, admits seventeen Noether symmetries. The list of all symmetries was given in [11], and the first integrals in Cartesian coordinates were mentioned in [13].

4. Conclusion

We have given a complete list of spherically symmetric static space–times. It can be seen that the Lagrangian of spherically symmetric static space–times can give 5, 6, 7, 9, 11, or 17 Noether symmetries. A few examples of space–times with the minimum (i.e., five) Noether symmetries are given in Table 1. Briefly, there are respectively two, four, and five cases in which the Lagrangian of the space–times give six, seven, and nine Noether symmetries (including the Bertotti–Robinson and the Einstein metrics). There is only one case (which is the famous de Sitter space–time) of 11 Noether symmetries, and there is one case (Minkowski space–time) of 17 Noether symmetries. Just like the Lagrangian of plane symmetric static space–times,

Table 5

Gen	First integrals
$\mathbf{X}_5$	$\phi_5 = -\frac{r}{b}\sqrt{b^2-r^2}\sin\phi\sin\theta\cos\left(\frac{t}{b}\right)\dot{t} + \frac{b^2}{\sqrt{b^2-r^2}}\sin\phi\sin\theta\sin\left(\frac{t}{b}\right)\dot{r} + {}$ $+ r\sqrt{b^2-r^2}\cos\theta\sin\phi\sin\left(\frac{t}{b}\right)\dot{\theta} + r\sqrt{b^2-r^2}\sin\theta\cos\phi\sin\left(\frac{t}{b}\right)\dot{\phi}$
$\mathbf{X}_6$	$\phi_6 = -\frac{r}{b}\sqrt{b^2-r^2}\cos\phi\sin\theta\cos\left(\frac{t}{b}\right)\dot{t} + \frac{b^2}{\sqrt{b^2-r^2}}\cos\phi\sin\theta\sin\left(\frac{t}{b}\right)\dot{r} + {}$ $+ r\sqrt{b^2-r^2}\cos\theta\cos\phi\sin\left(\frac{t}{b}\right)\dot{\theta} - r\sqrt{b^2-r^2}\sin\theta\sin\phi\sin\left(\frac{t}{b}\right)\dot{\phi}$
$\mathbf{X}_7$	$\phi_8 = \frac{r}{b}\sqrt{b^2-r^2}\sin\phi\sin\theta\sin\left(\frac{t}{b}\right)\dot{t} + \frac{b^2}{\sqrt{b^2-r^2}}\sin\phi\sin\theta\cos\left(\frac{t}{b}\right)\dot{r} + {}$ $+ r\sqrt{b^2-r^2}\cos\theta\sin\phi\cos\left(\frac{t}{b}\right)\dot{\theta} + r\sqrt{b^2-r^2}\sin\theta\cos\phi\cos\left(\frac{t}{b}\right)\dot{\phi}$
$\mathbf{X}_8$	$\phi_9 = -\frac{r}{b}\sqrt{b^2-r^2}\cos\phi\sin\theta\cos\left(\frac{t}{b}\right)\dot{t} + \frac{b^2}{\sqrt{b^2-r^2}}\cos\phi\sin\theta\sin\left(\frac{t}{b}\right)\dot{r} + {}$ $+ r\sqrt{b^2-r^2}\cos\theta\cos\phi\sin\left(\frac{t}{b}\right)\dot{\theta} - r\sqrt{b^2-r^2}\sin\theta\sin\phi\sin\left(\frac{t}{b}\right)\dot{\phi}$
$\mathbf{X}_9$	$\phi_7 = -\frac{r}{b}\sqrt{b^2-r^2}\cos\theta\cos\left(\frac{t}{b}\right)\dot{t} + \frac{b^2}{\sqrt{b^2-r^2}}\cos\theta\sin\left(\frac{t}{b}\right)\dot{r} - {}$ $- r\sqrt{b^2-r^2}\sin\theta\sin\left(\frac{t}{b}\right)\dot{\theta}$
$\mathbf{X}_{10}$	$\phi_{10} = \frac{r}{b}\sqrt{b^2-r^2}\cos\theta\sin\left(\frac{t}{b}\right)\dot{t} + \frac{b^2}{\sqrt{b^2-r^2}}\cos\theta\cos\left(\frac{t}{b}\right)\dot{r} - {}$ $- r\sqrt{b^2-r^2}\sin\theta\cos\left(\frac{t}{b}\right)\dot{\theta}$

First integrals for eleven Noether symmetries.

the minimum number of Noether symmetries for spherically symmetric static space–times is five, and the maximum number of Noether symmetries is 17, while the minimum number of isometries is four, and the maximum number of isometries is 10. This completes the classification of spherically symmetric static space–times by their Noether symmetries. A similar work on cylindrically symmetric static space–times is in progress.

There are three new classes that we have not seen in the literature, and we must mention them briefly. We also list the nonzero components of the Riemannian tensors, Ricci tensors, and Ricci scalar.

The first class is

$$ds^2 = \sec^2\frac{r}{a}(dt^2 - dr^2) - d\Omega^2,$$

in which we have seven Noether symmetries and six Killing vectors. The Ricci scalar and nonzero components of the Ricci and Riemann tensors are

$$R_{\text{scalar}} = \frac{2(a^2-1)}{a^2},$$

$$R_{00} = -\frac{\sec^2(r/a)}{a^2}, \qquad R_{11} = \frac{\sec^2(r/a)}{a^2}, \qquad R_{22} = -1, \qquad R_{33} = -\sin^2\theta,$$

$$R_{0101} = -\frac{\sec^4(r/a)}{a^2}, \qquad R_{2323} = -\sin^2\theta.$$

The second class is

$$ds^2 = \left(\frac{\alpha}{r}\right)^2 (dt^2 - dr^2) - d\Omega^2,$$

in which we have seven Noether symmetries and six Killing vectors. We note that this metric becomes

$$r^2 ds^2 = \alpha^2(dt^2 - dr^2) - r^2 d\Omega^2, \qquad ds^2 = \frac{1}{r^2} d\tilde{s}^2,$$

where $d\tilde{s}$ is the standard Minkowski metric. Therefore, (7) represents a conformal Minkowski metric with the conformal factor $1/r^2$. The Ricci scalar and nonzero components of the Ricci and Riemann tensors are

$$R_{\text{scalar}} = \frac{\alpha^2 - 1}{\alpha^2},$$

$$R_{00} = -\frac{1}{r^2}, \qquad R_{11} = \frac{1}{r^2}, \qquad R_{22} = -1, \qquad R_{33} = -\sin^2\theta,$$

$$R_{0101} = -\frac{\alpha^2}{r^4}, \qquad R_{2323} = -\sin^2\theta.$$

The third class is

$$ds^2 = \left(\frac{\beta}{r}\right)^2 dt^2 - \left(\frac{\beta}{r}\right)^4 dr^2 - d\Omega^2,$$

in which we have nine Noether symmetries. The Ricci scalar and nonzero components of the Ricci and Riemann tensors are

$$R_{\text{scalar}} = 2,$$

$$R_{22} = -1, \qquad R_{33} = -\sin^2\theta,$$

$$R_{2323} = -\sin^2\theta.$$

Acknowledgments. The authors are very grateful to Mr. Helmut Ruhland and the unknown referee for their kind concern and valuable suggestions.

REFERENCES

1. G. S. Hall, *Symmetries and Curvature Structure in General Relativity*, World Scientific, Singapore (2004).
2. A. R. Kashif, "Curvature collineations of some spacetimes and their physical interpretation," Doctoral dissertation, Quaid-e-Azam University, Islamabad (2003).
3. H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge Univ. Press, Cambridge (2003).
4. A. Qadir and M. Ziad, *Il Nuovo Cimento B Ser.* 11, **110**, 317–334 (1995).
5. B. O. J. Tupper, A. J. Keane, and J. C. Carot, *Class. Q. Grav.*, **29**, 145016 (2012); arXiv:1206.6508v1 [gr-qc] (2012).
6. J. M. Foyster and C. B. G. McIntosh, *Bull. Austral. Math. Soc.*, **8**, 187–190 (1973).

7. D. Ahmad and M. Ziad, *J. Math. Phys.*, **38**, 2547–2552 (1997).
8. A. V. Aminova and N. A.-M. Aminov, *Tensor*, n.s., **62**, 65–86 (2000).
9. T. Feroze, F. M. Mahomed, and A. Qadir, *Nonlinear Dynam.*, **45**, 65–74 (2006).
10. A. H. Bokhari, A. H. Kara, A. R. Kashif, and F. D. Zaman, *Internat. J. Theoret. Phys.*, **45**, 1029–1039 (2006).
11. A. H. Bokhari, A. H. Kara, A. R. Kashif, and F. D. Zaman, *Internat. J. Theoret. Phys.*, **46**, 2795–2800 (2007).
12. A. H. Bokhari and A. H. Kara, *Gen. Rel. Grav.*, **39**, 2053–2059 (2007).
13. F. Ali and T. Feroze, *Internat. J. Theoret. Phys.*, **52**, 3329–3342 (2013).
14. H. Stephani, *Differential Equations: Their Solution Using Symmetries* (M. MacCallum, ed.), Cambridge Univ. Press, Cambridge (1990).
15. E. Noether, *Transport Theory Statist. Phys.*, **1**, 186–207 (1971); arXiv:physics/0503066v1 (2005).