Theoretical and Mathematical Physics, **184**(1): 961–972 (2015)

SOLUTIONS OF THE SINE-GORDON EQUATION WITH A VARIABLE AMPLITUDE

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We propose methods for constructing functionally invariant solutions $u(x,y,z,t)$ of the sine-Gordon equation with a variable amplitude in 3+1 dimensions. We find solutions $u(x,y,z,t)$ in the form of arbitrary functions depending on either one ($\alpha(x,y,z,t)$) or two ($\alpha(x,y,z,t),\beta(x,y,z,t)$) specially constructed functions. Solutions $f(\alpha)$ and $f(\alpha,\beta)$ relate to the class of functionally invariant solutions, and the functions $\alpha(x,y,z,t)$ and $\beta(x,y,z,t)$ are called the ansatzes. The ansatzes (α,β) are defined as the roots of either algebraic or mixed (algebraic and first-order partial differential) equations. The equations defining the ansatzes also contain arbitrary functions depending on (α,β). The proposed methods allow finding $u(x,y,z,t)$ for a particular, but wide, class of both regular and singular amplitudes and can be easily generalized to the case of a space with any number of dimensions.

Keywords: sine-Gordon equation, wave equation, eikonal equation, functionally invariant solution, ansatz

1. Introduction

The sine-Gordon (SG) equation

$$u_{xx}+u_{yy}+u_{zz}-\frac{u_{tt}}{v^2}=p_0\sin u \tag{1}$$

(where $p_0=\text{const}$ and the subscript means the derivative with respect to the corresponding variable) appears in many branches of modern science. Equation (1) describes the motion of dislocations in solids [1], the deformation of a nonlinear crystal lattice [2], fluxon propagation in Josephson transmission lines [3], spin orientation in ferromagnets [4], the orientation structure of liquid crystals [5], propagation of resonant ultrashort optical pulses [6], and "commensurability–noncommensurability" phase transitions [7]. It also arises in simulations of the processes in the earth's crust [8], in the description of surface metrics [9], in molecular biology [10], in field theory models, and in elementary particle physics [11], [12].

Currently, effective methods have been developed for solving the SG equation with a constant amplitude, but methods for solving the SG equation with variable amplitude are practically absent. This essentially restricts the application sphere of the SG equation because many physical phenomena and technological processes are described by the SG equation with a variable amplitude,

$$u_{xx}+u_{yy}+u_{zz}-\frac{u_{tt}}{v^2}=p(x,y,z,t)\sin u. \tag{2}$$

For example, in the mechanics of a nonlinear crystal lattice, the constant-amplitude case describes the deformation of the ideal crystal lattice by a field of uniform stresses. The deformation of a real lattice with structural defects (dislocations, disclinations, pore seeds, splits, etc.), even by uniform stresses and

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Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 184, No. 1, pp. 79–91, July, 2015. Original article submitted November 20, 2014; revised February 24, 2015.

all the more by a field of nonuniform stresses, is described by the SG equation with a variable amplitude. In the mechanics of liquid crystals, the case $p = \text{const}$ simulates the deformation of the long axes by an electromagnetic field under the condition that the orientational continuum has no defects and the electromagnetic field is uniform. Otherwise, $p(x, y, z, t) \neq \text{const}$. In the surface theory, the SG equation with $p(x, y, z, t) = \text{const}$ describes the metrics of Chebyshev nets on a surface of constant curvature. If the surface curvature varies, then $p(x, y, z, t) \neq \text{const}$. In simulating wave propagation in the earth's crust [13], the SG equation with constant amplitude describes the case where the mechanical properties of the earth's crust are uniform, which is of course a strong idealization. It is clear from the above examples that the application sphere of the SG equation would be essentially extended if solutions of the SG equation with a variable amplitude were found.

In describing real media, structural defects are usually simulated by functions with appropriate singularities (point, line, surface, etc.). The singularities can be either stationary or movable, i.e., propagating with some velocity. They can differ by the rate of growth (logarithmic, power-law, etc.). Media with nonuniformities described by regular functions without singularities are well known; moreover, the nonuniformities have a regular geometric spatial arrangement in many cases. From the above, it is clear that a mathematical model constructed based on the SG equation would be relevant if the amplitude $p(x, y, z, t)$ took structural features of real media into account.

Below, we present a method for solving the SG equation with a variable amplitude. This method is based on the following statements, which are proved by direct calculation.

Proposition 1. *If a function $\varphi(x, y, z, t)$ simultaneously satisfies the equations*

$$\begin{aligned} &\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \frac{\varphi_{tt}}{v^2} = 0, \\ &\varphi_x^2 + \varphi_y^2 + \varphi_z^2 - \frac{\varphi_t^2}{v^2} = p(x, y, z, t), \end{aligned} \tag{3}$$

then

$$u = 4 \arctan e^{\varphi(x,y,z,t)} \tag{4}$$

is a solution of Eq. (2).

Proof. Indeed, if we substitute (4) in (2) and take the identity

$$\sin(4 \arctan e^{\varphi}) = \frac{4e^{\varphi}}{1 + e^{2\varphi}} - \frac{8e^{3\varphi}}{(1 + e^{2\varphi})^2} \tag{5}$$

into account, then Eq. (2) is

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \frac{\varphi_{tt}}{v^2} = \left(\varphi_x^2 + \varphi_y^2 + \varphi_z^2 - \frac{\varphi_t^2}{v^2} - p(x, y, z, t)\right) \tanh \varphi. \tag{6}$$

The validity of Proposition 1 follows from (6).

Proposition 2. *If a function $\varphi(x, y, z, t)$ simultaneously satisfies the equations*

$$\begin{aligned} &\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \frac{\varphi_{tt}}{v^2} = p(x, y, z, t), \\ &\varphi_x^2 + \varphi_y^2 + \varphi_z^2 - \frac{\varphi_t^2}{v^2} = 0, \end{aligned} \tag{7}$$

then

$$u = 2 \arctan e^{\varphi(x,y,z,t)} \tag{8}$$

is a solution of Eq. (2).

Proof. This proposition can be proved similarly to Proposition 1. If we substitute (8) in (2) and take the relation

$$\sin(2\arctan e^{\varphi}) = \frac{2e^{\varphi}}{1+e^{2\varphi}} \tag{9}$$

into account, then Eq. (2) is

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \frac{\varphi_{tt}}{v^2} - p(x,y,z,t) = \left(\varphi_x^2 + \varphi_y^2 + \varphi_z^2 - \frac{\varphi_t^2}{v^2}\right)\tanh\varphi. \tag{10}$$

The validity of Proposition 2 is obvious from (10).

The obtained results reduce the solution of Eq. (2) to finding a function $\varphi(x,y,z,t)$ that simultaneously satisfies either system of equations (3) or (7). The function $\varphi(x,y,z,t)$ can be found using the methods for constructing functionally invariant solutions of differential equations [14]–[25].

2. Functionally independent solutions of the SG equation with a variable amplitude

A solution of differential equations is said to be functionally invariant if it has the form of an arbitrary function depending on an ansatz. The ansatz is a solution of one or several equations. Equations can be algebraic or differential or a mixed type. There are functionally invariant solutions depending on two or more ansatzes.

The idea of the existence of functionally invariant solutions was suggested by Jacobi. He noted that for the Laplace equation, the argument of an arbitrary function arising in the solution of a partial differential equation must also satisfy the characteristic equation [26]. Forsyth [15] found functionally invariant solutions of the Laplace equation, of the wave equation in a space of arbitrary dimensions, and of the Helmholtz equation in the three-dimensional space. In studying electromagnetic waves, Bateman [14] fundamentally and consistently developed the Jacobi idea as applied to the wave equation. Soboleff and Smirnoff [16]–[19] successfully used the method to construct functionally invariant solutions of the wave equation to solve problems of diffraction and sound wave propagation in uniform and layered solid media. Erugin [20] made a large contribution to developing the theory of this method. We found functionally invariant solutions of the SG equation with a constant amplitude in [22].

2.1. Solution with one ansatz. In constructing functionally invariant solutions of partial differential equations, the basic problem is to find the ansatz. It can usually be chosen from solutions of special equations.

Let the ansatz τ be a root of the algebraic equation

$$x\xi(\tau) + y\eta(\tau) + z\zeta(\tau) - v^2 t\tau = \frac{s^2+q^2}{2}, \tag{11}$$

where

$$s^2 = x^2+y^2+z^2-v^2t^2, \qquad q^2 = \xi^2(\tau)+\eta^2(\tau)+\zeta^2(\tau)-v^2\tau^2, \tag{12}$$

and $\xi(\tau)$, $\eta(\tau)$, and $\zeta(\tau)$ are arbitrary functions of τ. Equation (11) implicitly defines the dependence of τ on the time and space coordinates. From it, we find the partial derivatives of the ansatz $\tau(x,y,z,t)$ using the rule for differentiating implicit functions, and taking the relations

$$\xi_\tau\tau_x + \eta_\tau\tau_y + \zeta_\tau\tau_z + \tau_t = 1, \qquad \nu_x\tau_x + \nu_y\tau_y + \nu_z\tau_z - \frac{\nu_t\tau_t}{v^2} = 1 \tag{13}$$

into account where

$$\nu = \xi_\tau(x-\xi) + \eta_\tau(y-\eta) + \zeta_\tau(z-\zeta) - v^2(t-\tau), \tag{14}$$

we find that the ansatz τ satisfies the equations

$$\begin{aligned} &\tau_x^2+\tau_y^2+\tau_z^2-\frac{\tau_t^2}{v^2}=0, \\ &\tau_{xx}+\tau_{yy}+\tau_{zz}-\frac{\tau_{tt}}{v^2}=p(x,y,z,t), \qquad p(x,y,z,t)=\frac{2}{\nu}. \end{aligned} \tag{15}$$

According to Eq. (15) and Proposition 2, an arbitrary function depending on τ, $\varphi(x,y,z,t)=f(\tau)$, satisfies Eqs. (7), and the function

$$u=2\arctan e^{f(\tau)} \tag{16}$$

is consequently a solution of SG equation (2) with the amplitude

$$p(x,y,z,t)=\frac{2}{\nu}f_\tau. \tag{17}$$

The function $\varphi=f(\tau)/\nu$ satisfies Eqs. (3) if the amplitude has the form

$$p(x,y,z,t)=\frac{f^2}{\nu^4}(2W+\xi_\tau^2+\eta_\tau^2+\zeta_\tau^2-v^2)-2\frac{ff_\tau}{\nu^3} \tag{18}$$

and

$$W=\xi_{\tau\tau}(x-\xi)+\eta_{\tau\tau}(y-\eta)+\zeta_{\tau\tau}(z-\zeta)-\xi_\tau^2-\eta_\tau^2-\zeta_\tau^2+v^2. \tag{19}$$

In proving this assertion, in addition to Eqs. (13)—(15), we must use

$$\begin{aligned} &\nu_x^2+\nu_y^2+\nu_z^2-\frac{\nu_t^2}{v^2}=2W+\xi_\tau^2+\eta_\tau^2+\zeta_\tau^2-v^2, \\ &\nu_{xx}+\nu_{yy}+\nu_{zz}-\frac{\nu_{tt}}{v^2}=\frac{2}{\nu}[2W+\xi_\tau^2+\eta_\tau^2+\zeta_\tau^2-v^2] \end{aligned} \tag{20}$$

satisfied by the function ν.

According to Proposition 1, the function

$$u=4\arctan e^{f(\tau)/\nu}$$

is a solution of the SG equation with amplitude (18). The obtained solutions allow finding a large number of particular solutions by specifying arbitrary functions. We note two elementary solutions:

$$\begin{aligned} 1.\quad &\xi=0, \qquad \eta=0, \qquad \zeta=0, \\ &\tau=t\mp\frac{R}{v}, \qquad \nu=\mp vR, \qquad R=\sqrt{x^2+y^2+z^2}, \end{aligned} \tag{21}$$

$$\begin{aligned} 2.\quad &\xi=x_1\tau, \qquad \eta=x_2\tau, \qquad \zeta=x_3\tau, \qquad x_1^2+x_2^2+x_3^2=v^2, \\ &\tau=\frac{s^2}{2\nu}, \qquad \nu=xx_1+yx_2+zx_3-v^2t, \qquad s^2=x^2+y^2+z^2-v^2t^2. \end{aligned} \tag{22}$$

In Fig. 1, we present plots of the amplitude and solution (21) of the SG equation at different times for $v=1$, $z=0$, and $f(\tau)=\sin\tau$. The solution $u(x,y,0,t)$ has the form of a standing wave.

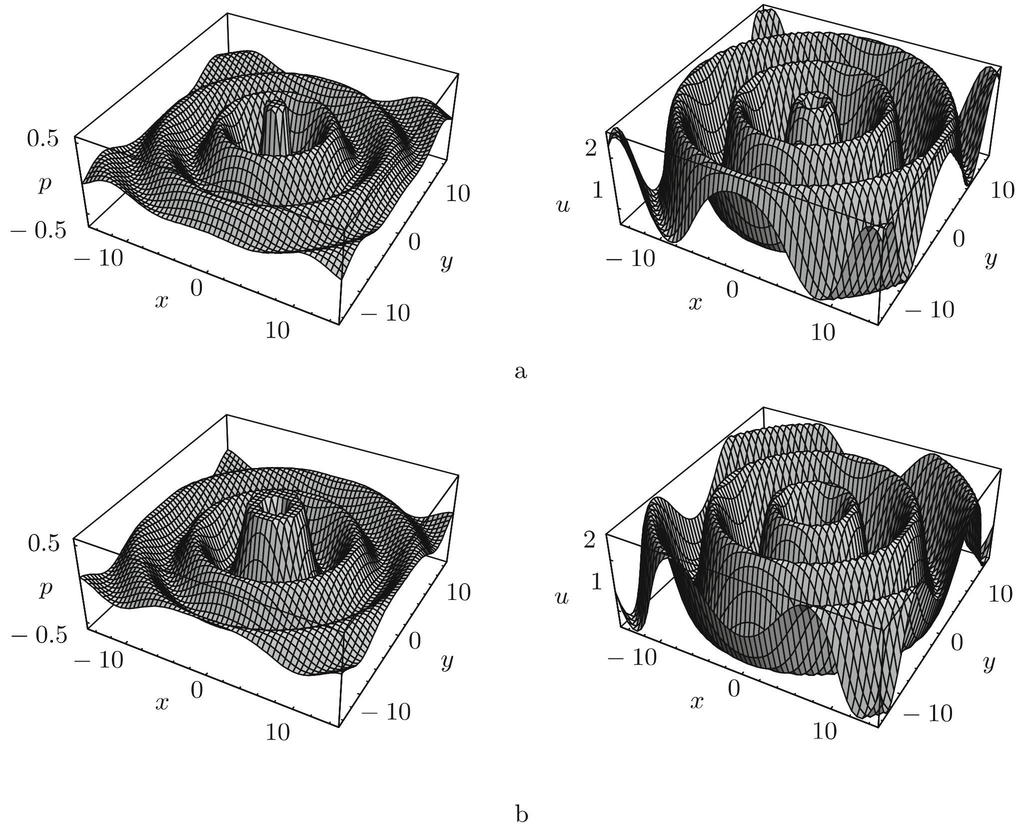

Fig. 1. The amplitude p and solution $u(x, y, 0, t)$ corresponding to (21) at (a) $t = 0$ and (b) $t = 10$.

Functionally invariant solutions of the SG equation can be constructed if the ansatz $\tau(x, y, z, t)$ is chosen from solutions of the system of equations

$$x\xi(\alpha, \beta, \tau) + y\eta(\alpha, \beta, \tau) + z\zeta(\alpha, \beta, \tau) - v^2 t\theta(\alpha, \beta, \tau) = \frac{s^2 + q^2}{2}, \tag{23}$$

$$x\xi_\alpha + y\eta_\alpha + z\zeta_\alpha - v^2 t\theta_\alpha = \frac{(q^2)_\alpha}{2}, \tag{24}$$

$$x\xi_\beta + y\eta_\beta + z\zeta_\beta - v^2 t\theta_\beta = \frac{(q^2)_\beta}{2}, \tag{25}$$

where $q^2 = \xi^2 + \eta^2 + \zeta^2 - v^2\theta^2$.

Equation (23) is algebraic. It coincides with (11) in form, but differs from it in that the functions ξ, ν, ζ, and θ depend on three arguments α, β, and τ rather than the one argument τ. Equations (24) and (25) are first-order partial differential equations. We can find the partial derivatives of the ansatz τ from (23)–(25) and prove that it satisfies Eqs. (7) with the function

$$p(x, y, z, t) = \frac{1}{\nu}[3 - (\xi_x + \eta_y + \zeta_z + \theta_t)]. \tag{26}$$

Consequently, the function $u = 2 \arctan e^{f(\tau)}$ is a solution of SG equation (2) with amplitude (26).

From the above solution, we can obtain interesting elementary solutions:

$$1. \quad \xi = v\tau\cos\alpha\cos\beta, \quad \eta = v\tau\cos\alpha\sin\beta, \quad \zeta = v\tau\sin\alpha, \quad \theta = \tau,$$
$$\tau = \frac{1}{2}\left(t + \frac{R}{v}\right), \quad p(x,y,z,t) = \frac{f_\tau}{vR}, \tag{27}$$

$$2. \quad \xi = v\tau, \quad \eta = \zeta = 0\tau, \quad \theta = \tau,$$
$$\tau = \frac{s^2}{2v(x-vt)}, \quad p(x,y,z,t) = \frac{2}{v(x-vt)}f_\tau, \tag{28}$$

$$3. \quad \xi = \tau\cos\alpha, \quad \eta = \tau\sin\alpha, \quad \zeta = \tau\sinh\beta, \quad \theta = \frac{\tau}{v}\cosh\beta,$$
$$\tau = \frac{1}{2}[\sqrt{x^2+y^2} + \sqrt{v^2t^2-z^2}], \quad p(x,y,z,t) = \frac{1}{2}\left[\frac{1}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{v^2t^2-z^2}}\right]f_\tau, \tag{29}$$

$$4. \quad \xi = \tau\cos\alpha\sinh\beta, \quad \eta = \tau\sin\alpha\sinh\beta, \quad \zeta = \tau, \quad \theta = \frac{\tau}{v}\cosh\beta,$$
$$\tau = \frac{1}{2}[z + \sqrt{v^2t^2-x^2-y^2}], \quad p(x,y,z,t) = \frac{-f_\tau}{\sqrt{v^2t^2-x^2-y^2}}. \tag{30}$$

Plots of the amplitude and solution (28) of the SG equation at $v = 1$ and $z = 0$ in the case $f(\tau) = \tau$ are shown in Fig. 2. The solution $u(x, y, 0, t)$ represents a superposition of two types of perturbations. The first one is of the kink type. The break plane moves with the velocity v. The second is of the soliton type. The transverse sizes of the second solution increase with the velocity v, and the height reaches its maximum value $u = \pi$. In time, the top of the second solution approaches the plane.

2.2. Solutions with two ansatzes. There are functionally invariant solutions of Eq. (2) with two ansatzes. Let the ansatz $\alpha(x, y, z, t)$ be the root of the equation

$$xl(\alpha) + ym(\alpha) + zn(\alpha) - v^2tw(\alpha) + g(\alpha) = 0 \tag{31}$$

and the ansatz $\beta(x, y, z, t)$ be

$$\beta(x,y,z,t) = xl_\alpha + ym_\alpha + zn_\alpha - v^2tw_\alpha + g_\alpha. \tag{32}$$

Here, $l(\alpha)$, $m(\alpha)$, $n(\alpha)$, $w(\alpha)$, and $g(\alpha)$ are arbitrary functions of α related by the condition

$$l^2 + m^2 + n^2 = v^2w^2. \tag{33}$$

From (31) and (32), we can find the partial derivatives of the ansatzes (α, β) and verify that an arbitrary function of (α, β),

$$\varphi = F(\alpha, \beta), \tag{34}$$

satisfies the system of equations

$$\varphi_x^2 + \varphi_y^2 + \varphi_z^2 - \frac{\varphi_t^2}{v^2} = F_\beta^2 C^2(\alpha), \tag{35}$$

$$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} - \frac{\varphi_{tt}}{v^2} = \left(F_{\beta\beta} + \frac{2}{\beta}F_\beta\right)C^2(\alpha), \tag{36}$$

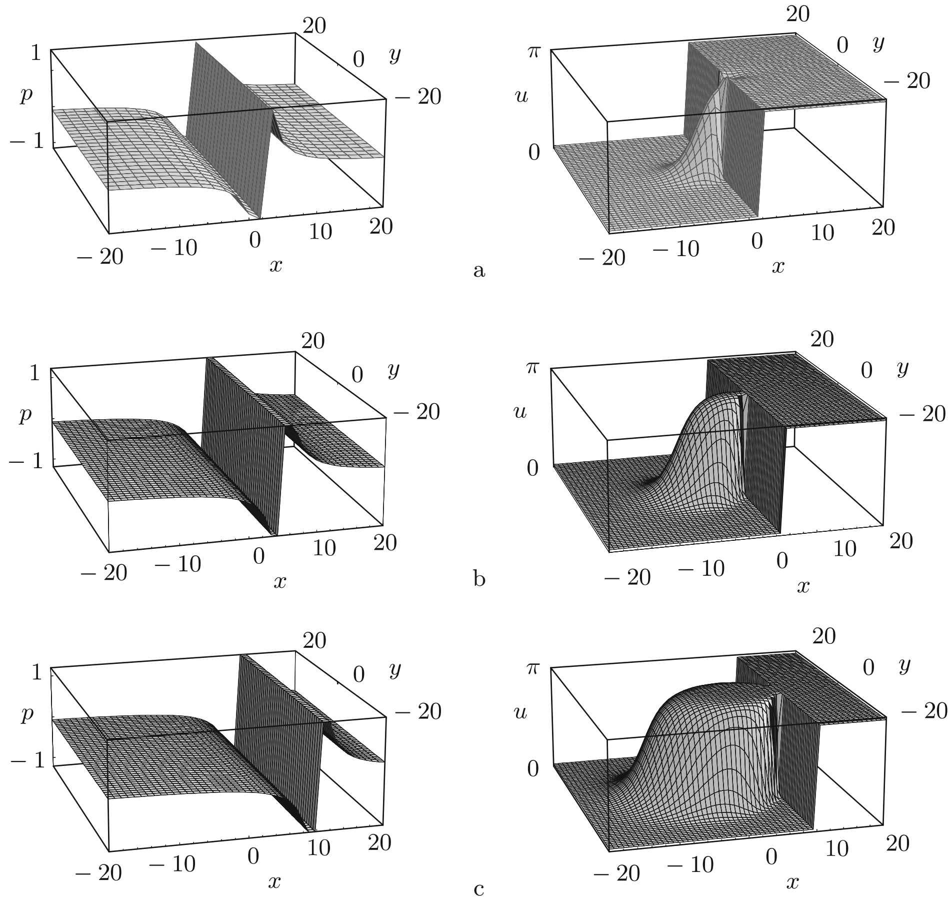

Fig. 2. The amplitude p and solution $u(x, y, 0, t)$ corresponding to (28) for (a) $t = 2$, (b) $t = 5$, and (c) $t = 10$.

where

$$C^2(\alpha) = l_\alpha^2 + m_\alpha^2 + n_\alpha^2 - v^2 w_\alpha^2.$$

Function (34) is a solution of the homogeneous wave equation if

$$F = A(\alpha) + \frac{B(\alpha)}{\beta}, \tag{37}$$

where $A(\alpha)$ and $B(\alpha)$ are arbitrary functions of α. Substituting (37) in (35), we obtain

$$\varphi_x^2 + \varphi_y^2 + \varphi_z^2 - \frac{\varphi_t^2}{v^2} = \frac{B^2(\alpha)C^2(\alpha)}{\beta^4}. \tag{38}$$

Hence, the function

$$u = 4\arctan[e^{A(\alpha)+B(\alpha)/\beta}] \tag{39}$$

is a solution of SG equation (2) with the amplitude

$$p(x, y, z, t) = \frac{B^2(\alpha)C^2(\beta)}{\beta^4}. \tag{40}$$

We give a simple example of this solution. Let $l = \cos\alpha$, $m = \sin\alpha$, $n = 0$, $vw = 1$, and $g = 0$. Then the ansatz has the form

$$\begin{aligned} &\alpha = -\gamma + (-1)^k \arcsin\frac{vt}{\rho}, \quad k = 0, \pm 1, \ldots, \\ &\tan\gamma = \frac{x}{y}, \qquad \rho = \sqrt{x^2+y^2}, \qquad \beta^2 = \rho^2 - v^2t^2, \end{aligned} \tag{41}$$

and the amplitude is

$$p(x,y,z,t) = \frac{B^2(\alpha)}{\beta^4}. \tag{42}$$

The solutions of Eqs. (7) can be obtained if we take the roots of the equations

$$\begin{aligned} &x\xi(\alpha,\beta) + y\eta(\alpha,\beta) + z\zeta(\alpha,\beta) - v^2t\tau(\alpha,\beta) = \frac{s^2+q^2}{2}, \\ &xl(\alpha,\beta) + ym(\alpha,\beta) + zn(\alpha,\beta) - v^2tw(\alpha,\beta) = g(\alpha,\beta) \end{aligned} \tag{43}$$

for the ansatzes (α,β). Here,

$$g(\alpha,\beta) = l\xi + m\eta + n\zeta - v^2w\tau,$$

and arbitrary functions l, m, n, and w of (α,β) are related by the conditions

$$l^2 + m^2 + n^2 = v^2w^2, \qquad l\xi_\beta + m\eta_\beta + n\zeta_\beta = v^2w\tau_\beta.$$

The quantities (s^2, q^2) are defined in Eq. (12).

From Eqs. (43), we find the partial derivatives of (α,β) and verify that the first-order derivatives satisfy the homogeneous equations

$$\begin{aligned} &\alpha_x^2 + \alpha_y^2 + \alpha_z^2 - \frac{\alpha_t^2}{v^2} = 0, \\ &\beta_x^2 + \beta_y^2 + \beta_z^2 - \frac{\beta_t^2}{v^2} = 0, \\ &\alpha_x\beta_x + \alpha_y\beta_y + \alpha_z\beta_z - \frac{\alpha_t\beta_t}{v^2} = 0, \end{aligned} \tag{44}$$

and the second-order derivatives satisfy the nonhomogeneous equations

$$\begin{aligned} &\alpha_{xx} + \alpha_{yy} + \alpha_{zz} - \frac{\alpha_{tt}}{v^2} = \frac{2}{\Delta}S, \\ &\beta_{xx} + \beta_{yy} + \beta_{zz} - \frac{\beta_{tt}}{v^2} = \frac{2}{\Delta}(\lambda - R), \end{aligned} \tag{45}$$

where

$$\begin{aligned} &\Delta = PS - RQ, \\ &P = \xi_\alpha(x-\xi) + \eta_\alpha(y-\eta) + \zeta_\alpha(z-\zeta) - v^2\tau_\alpha(t-\tau), \\ &Q = \xi_\beta(x-\xi) + \eta_\beta(y-\eta) + \zeta_\beta(z-\zeta) - v^2\tau_\beta(t-\tau), \\ &S = -l_\beta(x-\xi) - m_\beta(y-\eta) - n_\beta(z-\zeta) + v^2w_\beta(t-\tau), \\ &R = -l_\alpha(x-\xi) - m_\alpha(y-\eta) - n_\alpha(z-\zeta) + v^2w_\alpha(t-\tau) + \lambda, \\ &\lambda = \xi_\alpha l + \eta_\alpha m + \zeta_\alpha n - v^2w\tau_\alpha. \end{aligned} \tag{46}$$

The obtained results allow claiming that a function φ of form (34) satisfies Eqs. (3) if

$$p(x, y, z, t) = \frac{2N}{\Delta}, \tag{47}$$

where

$$N = \frac{\partial(l, f)}{\partial(\alpha, \beta)}(x - \xi) + \frac{\partial(m, f)}{\partial(\alpha, \beta)}(y - \eta) + \frac{\partial(n, f)}{\partial(\alpha, \beta)}(z - \zeta) - v^2 \frac{\partial(w, f)}{\partial(\alpha, \beta)}(t - \tau). \tag{48}$$

Here, we use the notation for the Jacobi determinant

$$\frac{\partial(\varphi, \psi)}{\partial(\alpha, \beta)} = \frac{\partial\varphi}{\partial\alpha}\frac{\partial\psi}{\partial\beta} - \frac{\partial\varphi}{\partial\beta}\frac{\partial\psi}{\partial\alpha}. \tag{49}$$

Consequently, the function

$$u = 2\arctan[e^{F(\alpha,\beta)}] \tag{50}$$

is a solution of SG equation (2) with amplitude (47).

The following simplest ansatzes might be of interest in simulating physical processes and in solving problems in mechanics:

1. $\alpha = x \pm iy, \qquad \beta = z \pm vt,$
2. $\alpha = x + iz\sqrt{1 + c^2} + icvt, \qquad \beta = y + cz + vt\sqrt{1 + c^2},$

where c is an arbitrary constant.

If we restrict ourself to real solutions, then the function

$$\varphi = \frac{1}{2}[F(\alpha, \beta) + \overline{F}(\bar{\alpha}, \beta)] \tag{51}$$

is a solution of Eqs. (7) if

$$p(x, y, z, t) = F_\alpha \overline{F}_{\bar{\alpha}}, \tag{52}$$

and

$$u = 4\arctan[e^{(F(\alpha,\beta)+\overline{F}(\bar{\alpha},\beta))/2}]. \tag{53}$$

Here, the bar over a symbol denotes complex conjugation.

Plots of the amplitude and solution (53) of the SG equation for $v = 1$ and $z = 0$ in the case $F(\alpha, \beta) = \beta/\cosh\alpha$ are shown in Fig. 3. The solution $u(x, y, 0, t)$ has the form of perturbations periodic in the axis Oy, which change their sizes and shapes in time; the height reaches its maximal value $u = 2\pi$ and eventually approaches the plane.

The obtained solutions of Eq. (2) are given by functional relations (4) and (8). We can construct functionally invariant solutions of the SG equation with a variable amplitude having another functional relation. We seek solutions of Eq. (2) in either the form

$$u = 2\arcsin\left[\sqrt{1 - \nu_0^2}\frac{\mathrm{sn}(\varphi, \nu_0)}{\mathrm{dn}(\varphi, \nu_0)}\right] \tag{54}$$

or the form

$$u = \pi + 2\mathrm{am}(\varphi, \nu_0). \tag{55}$$

Here, $\mathrm{sn}(\varphi, \nu_0)$ and $\mathrm{dn}(\varphi, \nu_0)$ are the Jacobi elliptic functions, $\mathrm{am}(\varphi, \nu_0)$ is the Jacobi amplitude, $\nu_0 = \mathrm{const}$ is the modulus, and $\varphi = \varphi(x, y, z, t)$ is the argument. Substituting either (54) or (55) in (2), we verify

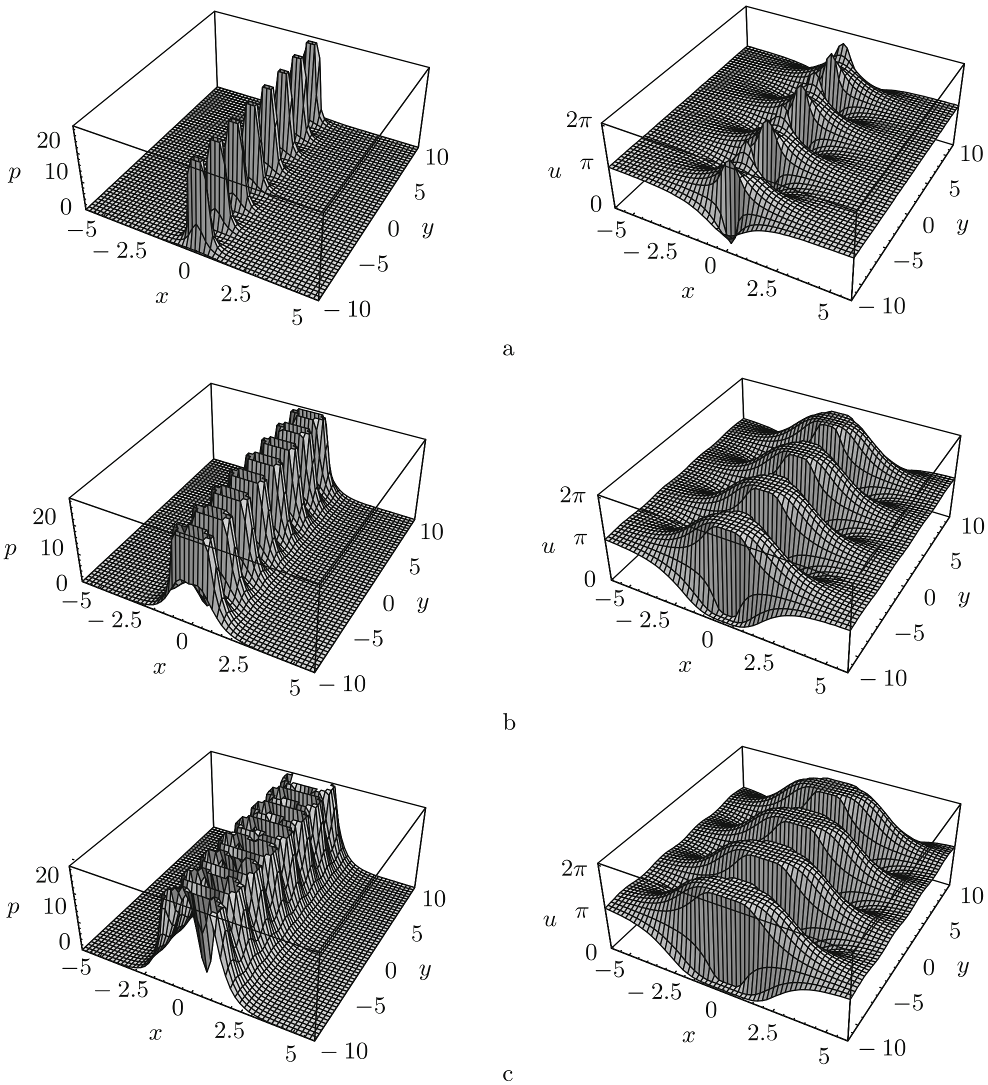

Fig. 3. The amplitude p and solution $u(x, y, 0, t)$ corresponding to (53) at (a) $t = 1$, (b) $t = 5$, and (c) $t = 10$.

that (54) and (55) are solutions of the SG equation with the amplitude $\nu_0^2 p(x, y, z, t)$ if $\varphi(x, y, z, t)$ satisfies Eqs. (3).

We also note that if the function $\varphi(x, y, z, t)$ satisfies either Eq. (3) or (7), then the function

$$V = e^{\varphi(x,y,z,t)} \tag{56}$$

is a solution of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 V}{\partial t^2} = p_{(}x, y, z, t)V. \tag{57}$$

Therefore, the solutions of Eqs. (3) obtained above can be used in Eqs. (54)–(56) to obtain new solutions of both Eq. (2) (in form (54) or (55)) and Eq. (57).

3. Conclusion

We have proposed methods for constructing functionally invariant solutions of the SG equation with a variable amplitude in 3+1 dimensions. They can be simply generalized to the case of an arbitrary-dimensional space. Moreover, the greater the dimensionality of the space is, the more possible types there are of equations whose roots can be ansatzes of functionally invariant solutions [20]. As the dimensionality increases, the number of arbitrary functions arising in the equations defining the ansatz also increases. This increases the variety of solutions of the SG equation with a variable amplitude.

The proposed methods allow obtaining solutions for amplitudes with a special form. But the obtained solutions form a rather wide class of functions. This follows from the solution method itself and also from the method for constructing it. Solutions have the form of arbitrary functions depending on either one or two ansatzes. The ansatzes can be found either from one equation or from a system of equations containing arbitrary functions. And although they must satisfy certain conditions (different for different methods), there remains a freedom in defining the ansatz (ansatzes).

It can be expected that the proposed approach allows properly describing physical processes in media with a real structure and finding solutions of particular engineering problems. The freedom in choosing the ansatz will suffice to satisfy the required initial and boundary conditions.

Acknowledgments. This work is supported by the Russian Foundation for Basic Research (Grant No. 13-01-00224_a).

REFERENCES

1. J. Frenkel and T. Kontorova, *Acad. Sci. USSR J. Phys.*, **1**, 137–149 (1939).
2. E. L. Aero and A. N. Bulygin, *Mech. Solids*, **42**, 807–822 (2007).
3. P. Guéret, *IEEE Trans. Magnetics*, **11**, 751–754 (1975).
4. R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equations*, Acad. Press, London (1982).
5. P. G. de Gennes, *The Physics of Liquid Crystals*, Clarendon, Oxford (1974).
6. K. Lonngren and A. C. Scott, eds., *Solitons in Action*, Acad. Press, New York (1978).
7. W. L. McMillan, *Phys. Rev. B*, **14**, 1496–1502 (1976).
8. V. G. Bykov, *Nonlinear Wave Processes in Geologic Media* [in Russian], Dal'nauka, Vladivostok (2000).
9. P. L. Chebyshev, *Uspekhi Mat. Nauk*, **1**, No. 2(12), 38–42 (1946).
10. A. S. Davydov, *Solitons in Bioenergetics* [in Russian], Naukova Dumka, Kiev (1986).
11. L. A. Takhtadzhyan and L. D. Faddeev, *Hamiltonian Approach in the Theory of Solitons* [in Russian], Nauka, Moscow (1986); English transl.: *Hamiltonian Methods in the Theory of Solitons*, Springer, Berlin (1987).
12. L. A. Takhtadzhyan and L. D. Faddeev, *Theor. Math. Phys.*, **21**, 1046–1057 (1974).
13. I. A. Garagash, V. N. Nikolaevskiy, *Computational Continuum Mechanics*, **2**, 44–66 (2009).
14. H. Bateman, *Electrical and Optical Wave Motion*, Cambridge Univ. Press, London (1914).
15. A. R. Forsyth, *Messenger Math.*, **27**, 99–118 (1898).
16. V. Smirnoff and S. Soboleff, "Sur une méthode nouvelle dans le problème plan des vibrations élastiques," in: *Trudy seismologichecskogo in-ta* [Works of the Seismological Institute], No. 20, Acad. Sci. USSR, Leningrad (1932).
17. V. Smirnoff and S. Soboleff, *C. R. Acad. Sci. Paris*, **194**, 1437–1439 (1932).
18. S. Sobolev, "Functionally invariant solutions of wave equation," in: *Travaux Inst. Physico-Math. Stekloff*, Vol. 5, Acad. Sci. USSR, Leningrad (1934), pp. 259–264.
19. S. L. Sobolev, *Selected Works* [in Russian], Vol. 1, *Equations of Mathematical Physics*: *Computational Mathematics and Cubature Formulas*, Sobolev Inst. Math., Siberian Branch, Russ. Acad. Sci., Novosibirsk (2003); *Op. cit.*, Vol. 2, *Functional Analysis*: *Partial Differential Equations*, Sobolev Inst. Math., Siberian Branch, Russ. Acad. Sci., Novosibirsk (2006).

20. N. P. Erugin, *Uchenye zap. Leningr. un-ta.*, **15**, 101–134 (1948).
21. M. M. Smirnov, *Dokl. AN SSSR*, **67**, 977–980 (1949).
22. E. L. Aero, A. N. Bulygin, and Yu. V. Pavlov, *Theor. Math. Phys.*, **158**, 313–319 (2009).
23. E. L. Aero, A. N. Bulygin, and Yu. V. Pavlov, *Nelineinyi Mir*, **7**, 513–517 (2009).
24. E. L. Aero, A. N. Bulygin, and Yu. V. Pavlov, *Differ. Equ.*, **47**, 1442–1452 (2011).
25. E. L. Aero, A. N. Bulygin, and Yu. V. Pavlov, *Appl. Math. Comput.*, **223**, 160–166 (2013).
26. C. G. J. Jacobi, *J. Reine Angew. Math.*, **36**, 113–134 (1848); "Über eine particuläre Lösung der partiellen Differentialgleichung $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$," in: *C. G. J. Jacobi's Gesammelte Werke* (C. G. J. Jacobi and C. W. Borchardt, eds.), Vol. 2, Verlag von G. Reimer, Berlin (1882), pp. 191–216.