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WEAKLY PERIODIC GIBBS MEASURES OF THE ISING MODEL WITH AN EXTERNAL FIELD ON THE CAYLEY TREE

M. M. Rahmatullaev*

We study weakly periodic Gibbs measures of the Ising model with an external field on the Cayley tree. We prove that under some conditions on the model parameters, there exist at least two weakly periodic Gibbs measures for the antiferromagnetic Ising model with an external field.

Keywords: Cayley tree, Gibbs measure, Ising model with external field, weakly periodic measure

1. Introduction

One of the main problems emerging when studying a Hamiltonian is to describe all limit Gibbs measures corresponding to this Hamiltonian. It is known that the set of such measures for the Ising model constitutes a nonempty, convex, compact subset in the set of all probability measures. The problem of describing all elements of this subset is still far from completion. For an Ising model with a zero external field, translation-invariant (see, e.g., [1]–[4]), periodic [1], [5], and continuum sets of nonperiodic [1], [5] Gibbs measures for the Ising model on the Cayley tree were described. Translation-invariant and periodic Gibbs measures for the Ising model with an external field were analyzed in [1], [2], [6], [7].

To extend the set of Gibbs measures, the notion of periodic Gibbs measures was generalized to that of weakly periodic Gibbs measures in [8]–[11], where the existence of such new measures was proved for the Ising model on the Cayley tree. Under some conditions on the parameters of some invariant sets, weakly periodic (nonperiodic) Gibbs measures for the Ising model on the Cayley tree were found in [8] and [9]. But weakly periodic Gibbs measures for Ising models with external fields have not yet been studied.

Here, we consider the Ising model with an external field and prove that weakly periodic (nonperiodic) Gibbs measures exist under some conditions on the model parameters.

The paper is organized as follows. We give necessary definitions and formulate the problem in Sec. 2 and devote Sec. 3 to studying weakly periodic Gibbs measures corresponding to normal divisors of index two.

2. Definitions and the problem setting

Let $\tau^k = (V, L), k \geq 1$, be the Cayley tree of order k, i.e., an infinite tree graph every vertex of which is incident to exactly $k+1$ edges. Here, V is the set of vertices, and L is the set of edges of the tree τ^k. It is known that τ^k can be represented as G_k, the free product of $k+1$ cyclic groups of the second order. For an arbitrary point $x^0 \in V$, we set

$$W_n = \{x \in V \mid d(x^0, x) = n\}, \qquad V_n = \bigcup_{m=0}^{n} W_m, \qquad L_n = \{\langle x, y\rangle \in L \mid x, y \in V_n\},$$

*Institute of Mathematics, National University of Uzbekistan, Tashkent, Uzbekistan,
e-mail: mrahmatullaev@rambler.ru.

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where $d(x, y)$ is the distance between the vertices x and y in the Cayley tree, i.e., the number of edges in the shortest path joining the vertices x and y.

Let $\Phi = \{-1, 1\}$, and let $\sigma \in \Omega = \Phi^V$ be a configuration, i.e.,

$$\sigma = \{\sigma(x) \in \Phi,\ x \in V\}.$$

Let $A \subset V$. We let Ω_A denote the space of configurations defined on the set A and taking values in $\Phi = \{-1, 1\}$.

We consider the Hamiltonian of the Ising model with an external field,

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \sigma(x)\sigma(y) - \lambda \sum_{x\in V} \sigma(x), \tag{1}$$

where $J, \lambda \in \mathbb{R}$ and $\langle x, y\rangle$ are nearest neighbors.

Let $h_x \in \mathbb{R}$, $x \in V$. For every n, we then define a measure μ_n on Ω_{V_n} setting

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\biggl\{-\beta H(\sigma_n) + \sum_{x\in W_n} h_x \sigma(x)\biggr\}, \tag{2}$$

where $\beta = 1/T$ (T is temperature, $T > 0$), $\sigma_n = \{\sigma(x),\ x \in V_n\} \in \Omega_{V_n}$, Z_n^{-1} is the normalizing factor, and

$$H(\sigma_n) = -J \sum_{\langle x,y\rangle \in L_n} \sigma(x)\sigma(y) - \lambda \sum_{x\in V_n} \sigma(x).$$

The compatibility condition for the measures $\mu_n(\sigma_n)$, $n \geq 1$, is

$$\sum_{\sigma^{(n)}} \mu_n(\sigma_{n-1}, \sigma^{(n)}) = \mu_{n-1}(\sigma_{n-1}), \tag{3}$$

where $\sigma^{(n)} = \{\sigma(x),\ x \in W_n\}$.

Let μ_n, $n \geq 1$, be a sequence of measures on the sets Ω_{V_n} that satisfy compatibility condition (3). By the Kolmogorov theorem, we then have a unique limit measure μ on $\Omega_V = \Omega$ (called the limit Gibbs measure) such that $\mu(\sigma_n) = \mu_n(\sigma_n)$ for every $n = 1, 2, \dots$. It is known that measures (2) satisfy condition (3) if and only if the set $h = \{h_x,\ x \in G_k\}$ of quantities satisfies the condition

$$h_x = \lambda\beta + \sum_{y\in S(x)} f(h_y, \theta), \tag{4}$$

where $S(x)$ is the set of children of the point $x \in V$ (see [1]). Here,

$$f(x, \theta) = \operatorname{arctanh}(\theta \tanh x), \qquad \theta = \tanh(J\beta).$$

Let $G_k/\widehat{G}_k = \{H_1, \dots, H_r\}$ be the quotient group, where $\widehat{G}_k$ is a normal divisor of index $r \geq 1$.

Definition 1. We call a set $h = \{h_x, x \in G_k\}$ of quantities a $\widehat{G}_k$*-periodic set* if $h_{xy} = h_x$ for all $x \in G_k$ and $y \in \widehat{G}_k$. We call a G_k-periodic measure a *translation-invariant* measure.

For $x \in G_k$, we introduce the notation $x_\downarrow = \{y \in G_k \mid \langle x, y\rangle\} \setminus S(x)$.

Definition 2. We call the set $h = \{h_x, x \in G_k\}$ of quantities a $\widehat{G}_k$-*weakly periodic set* if $h_x = h_{ij}$ for $x \in H_i$ and $x_\downarrow \in H_j$ and any $x \in G_k$.

We note that a weakly periodic set h coincides with the standard periodic set (see Definition 1) if the value h_x is independent of $x_\downarrow$.

Definition 3. We call a measure μ $\widehat{G}_k$-(*weakly*) *periodic* if it corresponds to a $\widehat{G}_k$-(weakly) periodic set h of quantities.

In this paper, we study weakly periodic Gibbs measures and demonstrate that such measures exist for the Ising model with an external field.

3. Weakly periodic measures

The difficulty in the problem of describing weakly periodic Gibbs measures depends on the structure and index of the normal divisor with respect to which periodicity is required. It was proved in [12] that there are no normal divisors with an odd index differing from unity for the group G_k. We therefore consider normal divisors with even indices. Here, we restrict ourself to the case of index two.

We describe $\bar{G}_k$-weakly periodic Gibbs measures for any normal divisor of $\bar{G}_k$ of index two. We note that any normal divisor of the group G_k of index two has the form

$$H_A = \left\{ x \in G_k \;\middle|\; \sum_{i \in A} w_x(a_i) \text{ is even} \right\},$$

where $\varnothing \neq A \subseteq N_k = \{1, 2, \ldots, k+1\}$ and $w_x(a_i)$ is the number of letters in the word $x \in G_k$ [1].

Let the set $A \subset \{1, 2, \ldots, k+1\}$, and let H_A be the corresponding normal divisor of index two. We note that in the case $|A| = k+1$ (where $|A|$ is the cardinality of a set A), i.e., in the case where $A = N_k$, the notion of weak periodicity coincides with the notion of the standard periodicity. We therefore consider a set $A \subset N_k$ such that $A \neq N_k$. By virtue of (4), a G_k-weakly periodic set h is then

$$h_x = \begin{cases} h_1, & x \in H_A, \quad x_\downarrow \in H_A, \\ h_2, & x \in H_A, \quad x_\downarrow \in G_k \setminus H_A, \\ h_3, & x \in G_k \setminus H_A, \; x_\downarrow \in H_A, \\ h_4, & x \in G_k \setminus H_A, \; x_\downarrow \in G_k \setminus H_A, \end{cases} \tag{5}$$

where h_i, $i = \overline{1,4}$, satisfy the system of equations

$$\begin{aligned} h_1 &= \lambda\beta + |A| f(h_3, \theta) + (k - |A|) f(h_1, \theta), \\ h_2 &= \lambda\beta + (|A| - 1) f(h_3, \theta) + (k + 1 - |A|) f(h_1, \theta), \\ h_3 &= \lambda\beta + (|A| - 1) f(h_2, \theta) + (k + 1 - |A|) f(h_4, \theta), \\ h_4 &= \lambda\beta + |A| f(h_2, \theta) + (k - |A|) f(h_4, \theta). \end{aligned} \tag{6}$$

We now consider the map $W\colon \mathbb{R}^4 \to \mathbb{R}^4$ determined by system (6) such that system (6) is the equation $h = W(h)$. The map W has the invariant subsets

$$I_1 = \{h \in \mathbb{R}^4 \colon h_1 = h_2 = h_3 = h_4\}, \qquad I_2 = \{h \in \mathbb{R}^4 \colon h_1 = h_4,\; h_2 = h_3\}. \tag{7}$$

Theorem 1. *We have the following statements*:

1. *For an Ising model with external fields, all H_A-weakly periodic Gibbs measures on the sets I_1 and I_2 are translation-invariant.*

2. *For $|A| = k$ and $\theta > 0$, all H_A-weakly periodic Gibbs measures are translation-invariant.*

Proof. 1. It suffices to demonstrate that system of equations (6) has a unique solution $h_1 = h_2 = h_3 = h_4$. The proof of the theorem for the invariant subset I_1 is obvious. We now prove the theorem for the invariant subset I_2.

Using the formula

$$f(h,\theta) = \operatorname{arctanh}(\theta \tanh h) = \frac{1}{2} \log \frac{(1+\theta)e^{2h} + (1-\theta)}{(1-\theta)e^{2h} + (1+\theta)}$$

and introducing the notation $\alpha = (1-\theta)/(1+\theta)$ and $z_i = e^{2h_i}$, $i = \overline{1,4}$, instead of (6), we obtain the system of equations

$$\begin{aligned} z_1 &= e^{2\lambda\beta}\left(\frac{z_3+\alpha}{\alpha z_3+1}\right)^{|A|}\left(\frac{z_1+\alpha}{\alpha z_1+1}\right)^{(k-|A|)}, \\ z_2 &= e^{2\lambda\beta}\left(\frac{z_3+\alpha}{\alpha z_3+1}\right)^{|A|-1}\left(\frac{z_1+\alpha}{\alpha z_1+1}\right)^{(k+1-|A|)}, \\ z_3 &= e^{2\lambda\beta}\left(\frac{z_2+\alpha}{\alpha z_2+1}\right)^{|A|-1}\left(\frac{z_4+\alpha}{\alpha z_4+1}\right)^{(k+1-|A|)}, \\ z_4 &= e^{2\lambda\beta}\left(\frac{z_2+\alpha}{\alpha z_2+1}\right)^{|A|}\left(\frac{z_4+\alpha}{\alpha z_4+1}\right)^{(k-|A|)}. \end{aligned} \tag{8}$$

Straightforward but cumbersome algebra brings this system to the form

$$\begin{aligned} z_1 - z_2 &= A_1(z_3 - z_1), \\ z_1 - z_3 &= A_2(z_1 - z_4) + B_2(z_3 - z_4) + C_2(z_3 - z_2), \\ z_1 - z_4 &= A_3(z_1 - z_4) + B_3(z_3 - z_2), \\ z_2 - z_3 &= A_4(z_3 - z_2) + B_4(z_1 - z_4), \\ z_2 - z_4 &= A_5(z_3 - z_2) + B_5(z_1 - z_2) + C_5(z_1 - z_4), \\ z_3 - z_4 &= A_6(z_4 - z_2), \end{aligned} \tag{9}$$

where

$$\begin{aligned} A_i &= (1-\alpha^2)\widetilde{A}_i(z_1, z_2, z_3, z_4), \\ B_i &= (1-\alpha^2)\widetilde{B}_i(z_1, z_2, z_3, z_4), \\ C_i &= (1-\alpha^2)\widetilde{C}_i(z_1, z_2, z_3, z_4), \end{aligned}$$

and $\widetilde{A}_i$, $\widetilde{B}_i$, and $\widetilde{C}_i$ are positive for all $i = \overline{1,6}$.

For the invariant subset I_2, we have $h_2 = h_3$, whence the equality $z_1 - z_2 = A_1(z_3 - z_1)$ implies that $z_1 = z_2$ for $\alpha < 1$.

In the antiferromagnetic case, i.e., for $\alpha \in (1, +\infty)$, we obtain $A_i, B_i, C_i < 0$ for all i. We then have $h_2 = h_3$, i.e., $z_2 = z_3$, on the invariant subset I_2, and by virtue of the relation $z_2 - z_4 = A_3(z_1 - z_4)$, we therefore obtain $z_1 = z_4$. Hence, for all $\alpha \in (0, +\infty)$, we have $z_1 = z_2$, whence $z_1 = z_2 = z_3 = z_4$ on the subset I_2.

2. From (6) in the case $|A| = k$, we obtain

$$\begin{aligned} h_2 &= \lambda\beta + (k-1)f(h_3,\theta) + f(\lambda\beta + kf(h_3,\theta),\theta), \\ h_3 &= \lambda\beta + (k-1)f(h_2,\theta) + f(\lambda\beta + kf(h_2,\theta),\theta). \end{aligned} \tag{10}$$

We now prove that this system has only solutions with $h_2 = h_3$. Let $h_2 > h_3$. From (10), we then have

$$h_2 - h_3 = (k-1)\big(f(h_3,\theta) - f(h_2,\theta)\big) + f(\lambda\beta + kf(h_3,\theta),\theta) - f(\lambda\beta + kf(h_2,\theta),\theta). \tag{11}$$

It is easy to see that the function f increases monotonically for $\theta > 0$. Hence, equality (11) fails because its left-hand side is positive while its right-hand side is negative. Equality (11) also fails for $h_2 < h_3$, and therefore $h_2 = h_3$, which results in translation-invariant solutions of system (6). The theorem is proved.

We next consider an antiferromagnetic Ising model with an external field, i.e., the case $\alpha > 1$ ($\theta < 0$). We introduce the notation

$$a = e^{2\lambda\beta}, \qquad \varphi(x) = \frac{x+\alpha}{\alpha x + 1}.$$

It is known [1], [6], [7] that in this case, we have a unique translation-invariant Gibbs measure corresponding to the unique solution of the equation

$$z = a\varphi^k(z).$$

We let z_* denote this solution.

Assuming that $|A| = k$, we can write system of equations (8) in the form

$$\begin{aligned} z_1 &= a\varphi^k(z_3), & z_2 &= a\varphi^{k-1}(z_3)\varphi(z_1), \\ z_3 &= a\varphi^{k-1}(z_2)\varphi(z_4), & z_4 &= a\varphi^k(z_2). \end{aligned} \tag{12}$$

Solving system (12) reduces to analyzing the system of equations

$$\begin{aligned} z_2 &= a\varphi^{k-1}(z_3)\varphi(a(\varphi^k(z_3))), \\ z_3 &= a\varphi^{k-1}(z_2)\varphi(a(\varphi^k(z_2))). \end{aligned} \tag{13}$$

Introducing the notation

$$\psi(z) = a\varphi^{k-1}(z)\varphi(a(\varphi^k(z))), \tag{14}$$

we reduce system of equations (13) to the form

$$z_2 = \psi(z_3), \qquad z_3 = \psi(z_2). \tag{15}$$

The number of solutions of this system coincides with the number of solutions of the equation $\psi(\psi(z)) = z$.

Lemma 1. *Let $\gamma\colon [0,1] \to [0,1]$ be a continuous function with a fixed point $\xi \in (0,1)$. Assuming that the function γ is differentiable at ξ and that $\gamma'(\xi) < -1$, we have values x_0 and x_1 such that the inequalities $0 \le x_0 < \xi < x_1 \le 1$ hold and $\gamma(x_0) = x_1$ and $\gamma(x_1) = x_0$.*

Proof. This lemma is proved in [13].

The following statements hold for function (14): this function is defined on $\mathbb{R}_+$, it is bounded and differentiable, and $\psi(z_*) = z_*$. By virtue of Lemma 1 for $\psi'(z_*) < -1$, system of equations (15) has three solutions: (z_*, z_*), (z_0, z_1), and (z_1, z_0), where $\psi(z_0) = z_1$ and $\psi(z_1) = z_0$. The inequality $\psi'(z_*) < -1$ is equivalent to the inequality

$$k\frac{(1-\alpha^2)^2 z_*^{2(k-1)/k}}{(\alpha z_* + 1)^4} + b(k-1)\frac{(1-\alpha^2) z_*^{(k-1)/k}}{(\alpha z_* + 1)^2} + b^2 < 0, \tag{16}$$

where $b = \sqrt[k]{a}$. Hence, $(b - b_1)(b - b_2) < 0$, where

$$\begin{aligned} b_1 &= \frac{(k-1-\sqrt{k^2-6k+1}\,)(\alpha^2-1) z_*^{(k-1)/k}}{2(\alpha z_* + 1)^2}, \\ b_2 &= \frac{(k-1+\sqrt{k^2-6k+1}\,)(\alpha^2-1) z_*^{(k-1)/k}}{2(\alpha z_* + 1)^2}. \end{aligned} \tag{17}$$

We have thus proved the following theorem.

Theorem 2. *For $|A| \ge 6$ and $\lambda \in (\lambda_1, \lambda_2)$, where $\lambda_{1,2} = (k/2\beta)\log b_{1,2}$ and the quantities $b_{1,2}$ are defined in* (17), *at least two H_A-weakly periodic* (*nonperiodic*) *Gibbs measures exist for the antiferromagnetic Ising model with an external field.*

The existence of at least two weakly periodic (nonperiodic) Gibbs measures for the Ising model was proved in [14], and Theorem 2 generalizes this result to the case of the Ising model with an external field. Indeed, if we take the Ising model with a zero external field, i.e., with $a = 1$, then inequality (16) becomes

$$k\frac{(1-\alpha)^2}{(1+\alpha)^2} + (k+1)\frac{(1-\alpha)}{(1+\alpha)} + 1 < 0.$$

Hence, $\alpha \in (\alpha_1, \alpha_2)$, where $\alpha_{1,2} = (k - 1 \pm \sqrt{k^2 - 6k + 1}\,)/2$, i.e., we reproduce the result in [14].

Remark 1. The H_A-weakly periodic Gibbs measures obtained in Theorem 2 are new and open a possibility to describe a continuum of nonperiodic Gibbs measures differing from those previously known.

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