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ONE-DIMENSIONAL TWO-COMPONENT BOSE GAS AND THE ALGEBRAIC BETHE ANSATZ

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We apply the nested algebraic Bethe ansatz to a model of a one-dimensional two-component Bose gas with a δ-function repulsive interaction. Using a lattice approximation of the L-operator, we find the Bethe vectors of the model in the continuum limit. We also obtain a series representation for the monodromy matrix of the model in terms of Bose fields. This representation allows studying an asymptotic expansion of the monodromy matrix over the spectral parameter.

Keywords: Bethe ansatz, monodromy matrix, Bethe vector

1. Introduction

We consider a model of a one-dimensional two-component Bose gas with a δ-function repulsive interaction (TCBG model). This model is a generalization of the Lieb–Liniger model [1], [2] (quantum nonlinear Schrödinger equation) in which Bose fields have two internal degrees of freedom (colors). This model was solved by Yang in [3], where the eigenvectors and the spectrum of the Hamiltonian were found. The general approach for solving the model with n internal degrees of freedom (multicomponent Bose gas) was given in [4] (also see [5], [6]). The nested algebraic Bethe ansatz was applied to this model in [7], [8]. Our main goal here is to create a base for calculating form factors of local operators in this model in the framework of the nested algebraic Bethe ansatz.

The algebraic Bethe ansatz is an effective method for finding the spectra of quantum Hamiltonians. But from the standpoint of calculating form factors of local operators, applying this method encounters some difficulties. The main problem is to embed the local operators of the model under consideration into the algebra of monodromy matrix elements. This problem is solvable in some cases [9], [10]. But constructing such a solution requires expressing the monodromy matrix $T(u)$ of the model in terms of the R-matrix. This is not the case with the TCBG model. On the other hand, representations for form factors of local operators and correlation functions in terms of multiple integrals of the product of the wave functions can be easily obtained in the framework of the traditional approach. But evaluating those multiple integrals faces serious technical difficulties, and they have so far been computed only in some relatively simple special cases [11].

A method for calculating form factors of local operators in models with the $GL(3)$ symmetry was recently developed in [12]. This method is based on the nested algebraic Bethe ansatz and deals with partial zero modes of the monodromy matrix elements $T_{ij}(u)$ [13] in a composite model [14]. Most of the tools in this approach can be used directly in the TCBG model, but some of them should be slightly modified. In particular, the definition of zero modes should be adjusted. We solve these problems here.

We consider a lattice approximation of the TCBG model. Using the L-operator obtained in [7], [8] we construct a monodromy matrix and the Bethe vectors. We show that these vectors have a correct continuum

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limit. We also obtain an explicit series representation for the monodromy matrix in terms of local Bose fields. Using this representation, we can derive an asymptotic expansion of the monodromy matrix over the spectral parameter. We thus find the zero modes.

The paper is organized as follows. In Sec. 2, we describe a general scheme of the algebraic Bethe ansatz. We define the Bethe vectors of $GL(3)$-invariant models and give their representation in a multicomposite model. Section 3 is devoted to a brief description of the TCBG model. In Sec. 4, we give a lattice approximation of the TCBG model in the framework of the nested algebraic Bethe ansatz. In Sec. 5, we consider the continuum limit of the Bethe vectors of the lattice model. In Sec. 6, we obtain a series representation of the TCBG monodromy matrix. Using this representation, we find an antimorphism between the Bose fields in Sec. 7 and the zero modes of the monodromy matrix elements in Sec. 8. In conclusion, we discuss some further applications of the obtained results.

2. Algebraic Bethe ansatz

In this section, we describe an abstract scheme of the algebraic Bethe ansatz, which is applicable to a wide class of quantum integrable models [15]–[17]. The key objects of the algebraic Bethe ansatz are a monodromy matrix and an R-matrix. The models considered below are described by the $GL(3)$-invariant R-matrix [18], [19] acting in the tensor product $V_1 \otimes V_2$ of two auxiliary spaces $V_k \sim \mathbb{C}^3$, $k = 1, 2$:

$$R(x, y) = \mathbf{I} + g(x, y)\mathbf{P}, \qquad g(x, y) = \frac{c}{x - y}. \tag{2.1}$$

In this definition, $\mathbf{I}$ is the identity matrix in $V_1 \otimes V_2$, $\mathbf{P}$ is the permutation matrix exchanging V_1 and V_2, and c is a constant.

The monodromy matrix $T(w)$ satisfies the algebra

$$R_{12}(w_1, w_2)T_1(w_1)T_2(w_2) = T_2(w_2)T_1(w_1)R_{12}(w_1, w_2). \tag{2.2}$$

Equation (2.2) holds in the tensor product $V_1 \otimes V_2 \otimes \mathcal{H}$, where $\mathcal{H}$ is the Hilbert space of the Hamiltonian of the considered model. The matrices $T_k(w)$ act nontrivially in $V_k \otimes \mathcal{H}$. We assume that the space $\mathcal{H}$ has a pseudovacuum vector $|0\rangle$. Similarly, the dual space $\mathcal{H}^*$ has a dual pseudovacuum vector $\langle 0|$. These vectors are annihilated by the operators $T_{ij}(w)$, where $i > j$ for $|0\rangle$ and $i < j$ for $\langle 0|$. Both vectors are simultaneously eigenvectors of the diagonal elements of the monodromy matrix,

$$T_{ii}(w)|0\rangle = \lambda_i(w)|0\rangle, \qquad \langle 0|T_{ii}(w) = \lambda_i(w)\langle 0|, \quad i = 1, 2, 3, \tag{2.3}$$

where $\lambda_i(w)$ are some scalar functions. In the framework of the general scheme of the algebraic Bethe ansatz, $\lambda_i(w)$ remain free functional parameters. In fact, we can always normalize the monodromy matrix $T(w) \to \lambda_2^{-1}(w)T(w)$ such that we deal with only the ratios

$$r_1(w) = \frac{\lambda_1(w)}{\lambda_2(w)}, \qquad r_3(w) = \frac{\lambda_3(w)}{\lambda_2(w)}. \tag{2.4}$$

Below, we assume that $\lambda_2(w) = 1$.

The trace in the auxiliary space $V \sim \mathbb{C}^3$ of the monodromy matrix $\operatorname{tr} T(w)$ is called the transfer matrix. It is the generating functional of the Hamiltonian and all integrals of motion of the model.

2.1. Notation. We use the same notation and conventions as in [20], [21]. In addition to the function $g(x,y)$, we also introduce the function

$$f(x,y) = 1 + g(x,y) = \frac{x-y+c}{x-y}. \tag{2.5}$$

Sets of variables are denoted by a bar, $\bar w$, $\bar u$, $\bar v$, etc. Individual elements of the sets are denoted by subscripts, w_j, u_k, etc. The expression $\bar u_i$, for example, means $\bar u \setminus u_i$. We also consider partitions of sets into disjoint subsets and let the symbol $\Rightarrow$ denote them. Subsets are denoted by superscripts in parenthesis, $\bar u^{(j)}$. For example, $\bar u \Rightarrow \{\bar u^{(1)}, \bar u^{(2)}\}$ means that the set $\bar u$ is divided into two disjoint subsets $\bar u^{(1)}$ and $\bar u^{(2)}$ such that $\bar u^{(1)} \cap \bar u^{(2)} = \varnothing$ and $\{\bar u^{(1)}, \bar u^{(2)}\} = \bar u$.

To avoid excessively cumbersome formulas, we use a shorthand notation for products of operators or functions depending on one or two variables: if the arguments of the operators T_{ij} or the functions r_k given by (2.4) are sets of variables, then the product should be taken over the indicated set. For example,

$$T_{ij}(\bar u) = \prod_{u_k \in \bar u} T_{ij}(u_k), \qquad r_3(\bar u^{(1)}) = \prod_{u_j \in \bar u^{(1)}} r_3(u_j). \tag{2.6}$$

A similar convention is applied to the products of the functions $f(x,y)$:

$$f(z,\bar w_i) = \prod_{\substack{w_j \in \bar w,\\ w_j \neq w_i}} f(z,w_j), \qquad f(\bar u,\bar v) = \prod_{u_j \in \bar u} \prod_{v_k \in \bar v} f(u_j,v_k). \tag{2.7}$$

2.2. Bethe vectors. The eigenvectors of the transfer matrix are called *on-shell* Bethe vectors (or simply *on-shell* vectors). To find them, generic Bethe vectors should be constructed first. In the framework of the algebraic Bethe ansatz, generic Bethe vectors are polynomials in the operators T_{ij} with $i<j$ applied to the pseudovacuum vector. We let $\mathbb{B}_{a,b}(\bar u;\bar v)$ denote them, stressing that they are parameterized by two sets of complex parameters $\bar u = \{u_1,\dots,u_a\}$ and $\bar v = \{v_1,\dots,v_b\}$ with $a,b = 0,1,\dots$. Different representations for Bethe vectors were found in [22]–[25]. Here, we give one of the representations obtained in [25]:

$$\mathbb{B}_{a,b}(\bar u;\bar v) = \sum \frac{\mathsf{K}_n(\bar v^{(1)}|\bar u^{(1)})}{f(\bar v,\bar u)} f(\bar v^{(2)},\bar v^{(1)}) f(\bar u^{(1)},\bar u^{(2)})\, T_{13}(\bar v^{(1)}) T_{23}(\bar v^{(2)}) T_{12}(\bar u^{(2)})|0\rangle. \tag{2.8}$$

Here, the sums are taken over partitions of the sets $\bar u \Rightarrow \{\bar u^{(1)}, \bar u^{(2)}\}$ and $\bar v \Rightarrow \{\bar v^{(1)}, \bar v^{(2)}\}$ with $0 \le \#\,\bar u^{(1)} = \#\,\bar v^{(1)} = n \le \min(a,b)$. We recall that $T_{13}(\bar u^{(1)})$ (and similar expressions) means the product of the operators $T_{13}(u)$ over the subset $\bar u^{(1)}$. Finally, $\mathsf{K}_n(\bar v^{(1)}|\bar u^{(1)})$ is the the partition function of the six-vertex model with domain-wall boundary conditions [26]. Its explicit representation was found in [27]:

$$\mathsf{K}_n(\bar x|\bar y) = \biggl(\prod_{1\le k<j\le n} g(x_j,x_k) g(y_k,y_j)\biggr) \frac{f(\bar x,\bar y)}{g(\bar x,\bar y)} \det_n \biggl(\frac{g^2(x_j,y_k)}{f(x_j,y_k)}\biggr). \tag{2.9}$$

In particular, $\mathsf{K}_1(x|y) = g(x,y)$.

A generic Bethe vector becomes on-shell if the parameters $\bar u$ and $\bar v$ satisfy a system of Bethe equations:

$$\begin{aligned} r_1(u_i) &= \frac{f(u_i,\bar u_i)}{f(\bar u_i,u_i)} f(\bar v,u_i), \quad i = 1,\dots,a, \\ r_3(v_j) &= \frac{f(\bar v_j,v_j)}{f(v_j,\bar v_j)} f(v_j,\bar u), \quad j = 1,\dots,b. \end{aligned} \tag{2.10}$$

We recall that $\bar u_i = \bar u \setminus u_i$ and $\bar v_j = \bar v \setminus v_j$.

2.3. Multicomposite model. The properties of local operators in the framework of the algebraic Bethe ansatz can be studied using a composite model [14]. We suppose that we have a lattice quantum model with N sites. The monodromy matrix $T(u)$ is then a product of local L-operators,

$$T(u) = L_N(u) \cdots L_1(u). \tag{2.11}$$

We fix an arbitrary site m, $1 \le m \le N$. Then (2.11) can be written as

$$T(u) = T^{(2)}(u)T^{(1)}(u), \tag{2.12}$$

where

$$T^{(1)}(u) = L_m(u) \cdots L_1(u), \qquad T^{(2)}(u) = L_N(u) \cdots L_{m+1}(u). \tag{2.13}$$

Representation (2.12) defines a composite model. In the framework of the composite model, the original matrix $T(u)$ is called the total monodromy matrix, and the matrices $T^{(2)}(u)$ and $T^{(1)}(u)$ are called partial monodromy matrices. The elements of the partial monodromy matrices $T^{(1)}(u)$ and $T^{(2)}(u)$ act in the spaces $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ associated with the respective lattice intervals $[1, m]$ and $[m+1, N]$. The elements of the total monodromy matrix act in the state space $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$.

In the framework of the algebraic Bethe ansatz, it is assumed that $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ have pseudovacuum vectors $|0\rangle^{(k)}$, $k = 1, 2$, such that $|0\rangle = |0\rangle^{(1)} \otimes |0\rangle^{(2)}$. These vectors have properties analogous to (2.3):

$$T^{(k)}_{ij}(u)|0\rangle^{(k)} = 0, \quad i > j, \qquad T^{(k)}_{ii}(u)|0\rangle^{(k)} = \lambda^{(k)}_i(u)|0\rangle^{(k)}, \quad k = 1, 2. \tag{2.14}$$

Similarly to (2.4), we introduce the ratios

$$r^{(k)}_1(w) = \frac{\lambda^{(k)}_1(w)}{\lambda^{(k)}_2(w)}, \qquad r^{(k)}_3(w) = \frac{\lambda^{(k)}_3(w)}{\lambda^{(k)}_2(w)}, \quad k = 1, 2. \tag{2.15}$$

Because of the normalization $\lambda_2(u) = 1$, we can always set $\lambda^{(k)}_2(u) = 1$. Below, we also extend convention (2.6) to products of functions (2.15).

For each partial monodromy matrix $T^{(k)}(u)$, we can construct the corresponding partial Bethe vectors $\mathbb{B}^{(k)}_{a,b}(\bar u; \bar v)$. They are given by Eq. (2.8), where we should replace all $T_{ij}(u)$ with $T^{(k)}_{ij}(u)$ and $|0\rangle$ with $|0\rangle^{(k)}$. The main problem considered in the framework of the composite model is to express the total Bethe vectors $\mathbb{B}_{a,b}(\bar u; \bar v)$ in terms of partial Bethe vectors $\mathbb{B}^{(k)}_{a,b}(\bar u; \bar v)$. This problem was solved in [14] for $GL(2)$-based models. The more general case of $GL(N)$-invariant models was considered in [22], [28]. The particular case of $GL(3)$-invariant models was studied in [29], where the representation

$$\mathbb{B}_{a,b}(\bar u; \bar v) = \sum r^{(2)}_1(\bar u^{(1)}) r^{(1)}_3(\bar v^{(2)}) \times \\ \times \frac{f(\bar u^{(2)}, \bar u^{(1)}) f(\bar v^{(2)}, \bar v^{(1)})}{f(\bar v^{(2)}, \bar u^{(1)})} \mathbb{B}^{(1)}_{a_1,b_1}(\bar u^{(1)}; \bar v^{(1)}) \mathbb{B}^{(2)}_{a_2,b_2}(\bar u^{(2)}; \bar v^{(2)}) \tag{2.16}$$

was found. Here, the sum is taken over all possible partitions $\bar u \Rightarrow \{\bar u^{(1)}, \bar u^{(2)}\}$ and $\bar v \Rightarrow \{\bar v^{(1)}, \bar v^{(2)}\}$. The cardinalities of the subsets are shown by the subscripts on the partial Bethe vectors.

Similarly, we can define a multicomposite model in which the original interval is divided into $M > 2$ intervals,

$$T(u) = T^{(M)}(u) \cdots T^{(1)}(u). \tag{2.17}$$

For each of these intervals, we can define partial Bethe vectors $\mathbb{B}^{(j)}_{a_j,b_j}$. The total Bethe vector can then be expressed in terms of the partial Bethe vectors as

$$\mathbb{B}_{a,b}(\bar u;\bar v)=\sum\Biggl\{\prod_{1\le k<j\le M}\biggl[r_1^{(j)}(\bar u^{(k)})r_3^{(k)}(\bar v^{(j)})\frac{f(\bar u^{(j)},\bar u^{(k)})f(\bar v^{(j)},\bar v^{(k)})}{f(\bar v^{(j)},\bar u^{(k)})}\biggr]\prod_{j=1}^{M}\mathbb{B}^{(j)}_{a_j,b_j}(\bar u^{(j)};\bar v^{(j)})\Biggr\}. \tag{2.18}$$

Here, the functions $r_1^{(j)}(u)$ and $r_3^{(j)}(v)$ are vacuum eigenvalues of the respective operators $T_{11}^{(j)}(u)$ and $T_{33}^{(j)}(v)$. The sum in (2.18) is taken over all possible partitions

$$\begin{aligned}&\bar u\Rightarrow\{\bar u^{(1)},\dots,\bar u^{(M)}\}, \qquad \#\bar u^{(j)}=a_j, \qquad a_1+\dots+a_M=a,\\ &\bar v\Rightarrow\{\bar v^{(1)},\dots,\bar v^{(M)}\}, \qquad \#\bar v^{(j)}=b_j, \qquad b_1+\dots+b_M=b.\end{aligned} \tag{2.19}$$

It is important that the number M of partial monodromy matrices is not related to the cardinalities of the Bethe parameters a and b. In particular, we can have $M>a$ and $M>b$. In that case, some of the numbers a_j and b_j are equal to zero, i.e., the corresponding subsets are empty.

Equation (2.18) can be easily proved by induction on M. Indeed, assuming that it holds for $M-1$ partial monodromy matrices, we apply (2.16) to the partial Bethe vector $\mathbb{B}^{(M-1)}_{a_{M-1},b_{M-1}}(\bar u^{(M-1)};\bar v^{(M-1)})$. This immediately gives (2.18) for M partial monodromy matrices.

In the particular cases $a=0$ or $b=0$, we reproduce the known formulas for the Bethe vectors in the $GL(2)$-invariant multicomposite model [30], [31]. For instance,

$$\mathbb{B}_{a,0}(\bar u,\varnothing)\equiv\mathbb{B}_a(\bar u)=\sum\prod_{1\le k<j\le M}\{r_1^{(j)}(\bar u^{(k)})f(\bar u^{(j)},\bar u^{(k)}\}\prod_{j=1}^{M}\mathbb{B}^{(j)}_{a_j}(\bar u^{(j)}). \tag{2.20}$$

The multicomposite model is a convenient way to express Bethe vectors in terms of local operators. In the next section, we discuss the method in more detail.

2.4. Bethe vectors in the $SU(2)$ XXX chain. As a first application of the multicomposite model, we construct the Bethe vectors of the $SU(2)$ inhomogeneous XXX chain. This result is used in Sec. 3 to describe the Bethe vectors of the TCBG model.

We consider an inhomogeneous XXX chain consisting of M sites. This model has a 2×2 monodromy matrix $T^{(xxx)}(u)$, and the Bethe vectors are therefore parameterized by only one set of Bethe parameters, for example, $\bar u$. Correspondingly, Eq. (2.20) should be used to consider the multicomposite model.

The monodromy matrix is defined as a product of local L-operators,

$$T^{(xxx)}(u)=L_M^{(xxx)}(u-\xi_M)\cdots L_1^{(xxx)}(u-\xi_1), \tag{2.21}$$

where ξ_k are inhomogeneities and

$$L_n^{(xxx)}(u)=\frac{1}{u}\begin{pmatrix}u+\dfrac{c}{2}(1+\sigma_n^z) & c\sigma_n^-\\ c\sigma_n^+ & u+\dfrac{c}{2}(1-\sigma_n^z)\end{pmatrix}. \tag{2.22}$$

Here, σ_n^z and $\sigma_n^\pm$ are spin-1/2 operators acting at the nth site of the chain. They are given by the standard Pauli matrices acting in the nth copy of the tensor product $(\mathbb{C}^2)^{\otimes M}$. The pseudovacuum vector is the state with all spins up:

$$|\tilde 0\rangle=\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)_M\otimes\cdots\otimes\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)_1. \tag{2.23}$$

Bethe vectors with a spins down and $M-a$ spins up have the form

$$\mathbb{B}_a^{(xxx)}(\bar{u}) = \sum_{M\ge j_a>\dots>j_1\ge 1} \Omega^{(a,M)}_{j_1,\dots,j_a}(\bar{u};\bar{\xi}) \prod_{m=1}^{a} \sigma^-_{j_m} |\tilde{0}\rangle. \tag{2.24}$$

where $\Omega^{(a,M)}_{j_1,\dots,j_a}(\bar{u};\bar{\xi})$ are coefficients depending on the Bethe parameters $\bar{u}$ and inhomogeneities $\bar{\xi}$. We find these coefficients explicitly.

We consider a multicomposite model with M partial monodromy matrices $T^{(j)}$. This means that each $T^{(j)}$ coincides with the L-operator $L_j(u-\xi_j)$. Each partial Bethe vector $\mathbb{B}^{(j)}_{a_j}(\bar{u}^{(j)})$ in (2.20) then corresponds to the jth site of the chain, and because of (2.22), we hence obtain

$$\mathbb{B}^{(j)}_{a_j}(\bar{u}^{(j)}) = g(\bar{u}^{(j)},\xi_j)(\sigma^-_j)^{a_j}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)_j. \tag{2.25}$$

Obviously, $\mathbb{B}^{(j)}_{a_j}$ vanishes if $a_j>1$, because $(\sigma^-_j)^2=0$. We therefore conclude that $a_j\le 1$ and the subsets $\bar{u}^{(j)}$ either are empty or contain exactly one element. Let the subsets $\bar{u}^{(j_k)}$, $k=1,\dots,a$, corresponding to the lattice sites $j_1,\dots,j_a$ contain one element u_k and the other subsets be empty. Then the sum over partitions of the set $\bar{u}$ becomes the sum over permutations in $\bar{u}$ and the sum over the lattice sites $j_1,\dots,j_a$ with the restriction $j_a>\dots>j_1$.

It is easy to see that

$$\frac{u-\xi_j+\frac{c}{2}(1+\sigma^z_j)}{u-\xi_j}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)_j = f(u,\xi_j)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)_j, \qquad \frac{u-\xi_j+\frac{c}{2}(1-\sigma^z_j)}{u-\xi_j}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)_j = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)_j \tag{2.26}$$

and therefore

$$r_1^{(j)}(u) = f(u,\xi_j). \tag{2.27}$$

Then Eq. (2.20) becomes

$$\mathbb{B}_a^{(xxx)}(\bar{u}) = \operatorname*{Sym}_{\bar{u}} \prod_{1\le k<j\le a} f(u_j,u_k)\,\times \\ \times \sum_{M\ge j_a>\dots>j_1\ge 1}\prod_{k=1}^{a}\Bigg[\Bigg(\prod_{m=j_k+1}^{M} f(u_k,\xi_m)\Bigg) g(u_k,\xi_{j_k})\sigma^-_{j_k}\Bigg]|\tilde{0}\rangle, \tag{2.28}$$

where Sym denotes symmetrization (i.e., the sum over permutations) over the set indicated by the subscript. The symmetrization in (2.28) acts on all the expressions depending on $\bar{u}$. Comparing (2.28) with (2.24), we can see that

$$\Omega^{(a,M)}_{j_1,\dots,j_a}(\bar{u};\bar{\xi}) = \operatorname*{Sym}_{\bar{u}} \prod_{1\le k<j\le a} f(u_j,u_k)\prod_{k=1}^{a}\Bigg[\Bigg(\prod_{m=j_k+1}^{M} f(u_k,\xi_m)\Bigg) g(u_k,\xi_{j_k})\Bigg]. \tag{2.29}$$

In the homogeneous limit $\xi_k=c/2$, this expression coincides with the amplitude of the Bethe vector in the coordinate Bethe ansatz representation (see [6]).

3. Two-component Bose gas

We consider the TCBG model on a finite interval $[0,L]$ with periodic boundary conditions. In the secondary quantized form, the Hamiltonian is

$$H = \int_0^L (\partial_x\Psi^\dagger_\alpha\partial_x\Psi_\alpha + \varkappa\Psi^\dagger_\alpha\Psi^\dagger_\beta\Psi_\beta\Psi_\alpha)\,dx, \tag{3.1}$$

where $\varkappa > 0$ is a coupling constant, $\alpha, \beta = 1, 2$, and summation over repeated subscripts is assumed. The Bose fields $\Psi_\alpha(x)$ and $\Psi^\dagger_\alpha(x)$ satisfy the canonical commutation relations

$$[\Psi_\alpha(x), \Psi^\dagger_\beta(y)] = \delta_{\alpha\beta}\delta(x-y). \tag{3.2}$$

The coupling constant $\varkappa$ is related to the constant c in (2.1) by $\varkappa = ic$.

The basis in the Fock space of the model is constructed by acting with the operators $\Psi^\dagger_\alpha(x)$ on the Fock vacuum $|0\rangle$ as

$$\Psi_\alpha(x)|0\rangle = 0, \qquad \langle 0|\Psi^\dagger_\alpha(x) = 0, \qquad \langle 0|0\rangle = 1. \tag{3.3}$$

We note that pseudovacuum vector (2.3) in the case of the TCBG model coincides with the Fock vacuum $|0\rangle$, and we therefore use the same symbol for them.

The spectral problem for the TCBG model was solved in [3] (also see [4], [6]). The Hamiltonian eigenvectors can be found in two steps. Using the terminology of the algebraic Bethe ansatz, we can say that a generic Bethe vector $\mathbb{B}_{a,b}(\bar u;\bar v)$ should be constructed at the first stage. In the TCBG model, Bethe vectors exist for $a \le b$. They have the form[1]

$$\mathbb{B}_{a,b}(\bar u;\bar v) = \sum_{b\ge k_a>\dots>k_1\ge 1} \int_{\mathcal{D}} dz_1\cdots dz_b\, \chi_{k_1,\dots,k_a}(z_1,\dots,z_b|\bar u,\bar v)\, \times \\ \times \prod_{m=1}^{a} \Psi^\dagger_1(z_{k_m}) \prod_{\substack{\ell=1,\\ \ell\notin\{k_1,\dots,k_m\}}}^{b} \Psi^\dagger_2(z_\ell)|0\rangle. \tag{3.4}$$

Here, the integration domain is $\mathcal{D} = \{L > z_b > \dots > z_1 > 0\}$. In this domain, the wave function $\chi_{k_1,\dots,k_a}(z_1,\dots,z_b|\bar u,\bar v)$ has the form

$$\chi_{k_1,\dots,k_a}(z_1,\dots,z_b|\bar u,\bar v) = \operatorname*{Sym}_{\bar v} \Omega^{(a,b)}_{k_1,\dots,k_a}(\bar u;\bar v + c) \prod_{b\ge j>k\ge 1} f(v_j,v_k) \prod_{k=1}^{b} e^{iz_k v_k}\Big|_{c=-i\varkappa}, \tag{3.5}$$

where the coefficients $\Omega^{(a,b)}_{k_1,\dots,k_a}(\bar u;\bar v + c)$ are given by (2.29).

Generic Bethe vector (3.4) becomes an eigenvector of Hamiltonian (3.1) if the parameters $\bar u$ and $\bar v$ satisfy system of Bethe equations (2.10). In the TCBG model, it has the form [3]

$$\begin{aligned} e^{iLv_j} &= \prod_{\substack{k=1,\\ k\ne j}}^{b} \frac{v_j - v_k + i\varkappa}{v_j - v_k - i\varkappa} \prod_{\ell=1}^{a} \frac{u_\ell - v_j + i\varkappa}{u_\ell - v_j}, \quad j = 1,\dots,b, \\ 1 &= \prod_{\substack{\ell=1,\\ \ell\ne j}}^{a} \frac{u_i - u_\ell - i\varkappa}{u_i - u_\ell + i\varkappa} \prod_{k=1}^{b} \frac{v_k - u_i - i\varkappa}{v_k - u_i}, \qquad i = 1,\dots,a. \end{aligned} \tag{3.6}$$

Comparing this system with (2.10), we conclude that $r_1(u) = 1$ and $r_3(u) = e^{iLu}$ in the TCBG model.

[1] Here and hereafter, we neglect the normalization of eigenvectors in all formulas for them.

4. Lattice two-component Bose gas

Quantum systems described by $GL(3)$-invariant R-matrix (2.1) were considered in [8], where a prototype of a lattice L-operator of the TCBG model was found. It has the form

$$L^{(a)}(u) = u\mathbf{1} + p, \tag{4.1}$$

where $\mathbf{1}$ is the unit matrix and

$$p = \begin{pmatrix} a_1^\dagger a_1 & a_1^\dagger a_2 & i a_1^\dagger \sqrt{m+\rho} \\ a_2^\dagger a_1 & a_2^\dagger a_2 & i a_2^\dagger \sqrt{m+\rho} \\ i\sqrt{m+\rho}\, a_1 & i\sqrt{m+\rho}\, a_2 & -m-\rho \end{pmatrix}. \tag{4.2}$$

Here, m is an arbitrary complex number and $\rho = a_1^\dagger a_1 + a_2^\dagger a_2$. The operators a_k and $a_k^\dagger$, $k = 1, 2$, act in a Fock space with the Fock vacuum $|0\rangle$: $a_k|0\rangle = 0$. They have the standard commutation relations of the Heisenberg algebra, $[a_i, a_k^\dagger] = \delta_{ik}$.

At $c = -1$, L-operator (4.1) satisfies algebra (2.2) with R-matrix (2.1). Based on L-operator (4.1), we can construct a quantum system of discrete bosons. To obtain a continuous quantum system, we should make several transforms of (4.1). First, we introduce the operators

$$\psi_k = \Delta^{-1/2} a_k, \qquad \psi_k^\dagger = \Delta^{-1/2} a_k^\dagger, \quad k = 1, 2, \tag{4.3}$$

such that

$$[\psi_j, \psi_k^\dagger] = \frac{\delta_{jk}}{\Delta}. \tag{4.4}$$

In these formulas, Δ is a lattice interval. Setting $m = 4/\varkappa\Delta$, we introduce a new L-operator as

$$L(u) = \frac{\varkappa\Delta}{2}\, L^{(a)}\left(\frac{u + 2i/\Delta}{i\varkappa}\right) \cdot J, \tag{4.5}$$

where $J = \mathrm{diag}(1, 1, -1)$. Obviously $L(u)$ satisfies RTT-relation (2.2) with R-matrix (2.1) at $c = -i\varkappa$.

The last transformation is to make N copies L_n, $n = 1, \dots, N$, of L-operator (4.5) by changing $\psi_k \to \psi_k(n)$ and $\psi_k^\dagger \to \psi_k^\dagger(n)$ with

$$[\psi_j(n), \psi_k^\dagger(m)] = \frac{\delta_{jk}\delta_{nm}}{\Delta}. \tag{4.6}$$

The operators $\psi_k(n)$ and $\psi_k^\dagger(n)$ are lattice approximations of the Bose fields $\Psi_k(x)$ and $\Psi_k^\dagger(x)$. Indeed, we divide the interval $[0, L]$ into N sites of length Δ. Setting $x_n = n\Delta$ and

$$\psi_k(n) = \frac{1}{\Delta}\int_{x_{n-1}}^{x_n} \Psi_k(x)\, dx, \qquad \psi_k^\dagger(n) = \frac{1}{\Delta}\int_{x_{n-1}}^{x_n} \Psi_k^\dagger(x)\, dx, \tag{4.7}$$

we reproduce commutation relations (4.6). On the other hand, in the limit $\Delta \to 0$, operators (4.7) obviously become the Bose fields $\Psi_k(x)$ and $\Psi_k^\dagger(x)$.[2]

We can now standardly define the monodromy matrix:

$$T(u) = L_N(u) \cdots L_1(u), \tag{4.8}$$

[2]Here and hereafter, limits of operator-valued expressions should be understood in the weak sense.

where

$$L_n(u) = \frac{1}{\mathcal{N}} \begin{pmatrix} 1 - \frac{iu\Delta}{2} + \frac{\varkappa\Delta^2}{2}\psi_1^\dagger(n)\psi_1(n) & \frac{\varkappa\Delta^2}{2}\psi_1^\dagger(n)\psi_2(n) & -i\Delta\psi_1^\dagger(n)Q_n \\ \frac{\varkappa\Delta^2}{2}\psi_2^\dagger(n)\psi_1(n) & 1 - \frac{iu\Delta}{2} + \frac{\varkappa\Delta^2}{2}\psi_2^\dagger(n)\psi_2(n) & -i\Delta\psi_2^\dagger(n)Q_n \\ i\Delta Q_n\psi_1(n) & i\Delta Q_n\psi_2(n) & 1 + \frac{iu\Delta}{2} + \frac{\varkappa\Delta^2}{2}\hat\rho_n \end{pmatrix} \tag{4.9}$$

and

$$\begin{aligned} &\mathcal{N} = 1 - \frac{iu\Delta}{2}, \qquad Q_n = \left(\varkappa + \frac{\varkappa^2\Delta^2}{4}\hat\rho_n\right)^{1/2}, \\ &\hat\rho_n = \psi_1^\dagger(n)\psi_1(n) + \psi_2^\dagger(n)\psi_2(n). \end{aligned} \tag{4.10}$$

The normalization factor $\mathcal{N}$ in (4.9) is used to satisfy the condition $\lambda_2(u) = 1$.

Remark 1. We write the number of lattice site n as the argument of the operators ψ_i and $\psi_i^\dagger$. This number is traditionally written as a subscript of ψ_i and $\psi_i^\dagger$, but this is inconvenient in the case of the TCBG model.

The L-operator (4.9) is a natural generalization of a 2×2 L-operator $\tilde L_n(u)$ found in [32] for the lattice model of one-component bosons:

$$\tilde L_n(u) = \frac{1}{\mathcal{N}} \begin{pmatrix} 1 - \frac{iu\Delta}{2} + \frac{\varkappa\Delta^2}{2}\psi^\dagger(n)\psi(n) & -i\Delta\psi^\dagger(n)Q_n \\ i\Delta Q_n\psi(n) & 1 + \frac{iu\Delta}{2} + \frac{\varkappa\Delta^2}{2}\psi^\dagger(n)\psi(n) \end{pmatrix}. \tag{4.11}$$

It is easy to see that L-operator (4.11) is the lower-right 2×2 minor of matrix (4.9) with the identification $\psi_1(n) \equiv 0$, $\psi_2(n) \equiv \psi(n)$. It was shown by different methods in [33]–[35] that L-operator (4.11) in the continuum limit $\Delta \to 0$ describes a model of one-dimensional bosons with a δ-function interaction. We must solve an analogous problem: to verify that the model with monodromy matrix (4.8) and L-operator (4.9) in the continuum limit does describe the TCBG model. For this, we find the Bethe vectors of lattice model (4.8) and show that they coincide with states (3.5) in the continuum limit.

We indicate several properties of L-operator (4.9). It is easy to see that

$$\begin{aligned} &\big(L_n(u)\big)_{11}|0\rangle = \big(L_n(u)\big)_{22}|0\rangle = |0\rangle, \qquad \big(L_n(u)\big)_{33}|0\rangle = r_0(u)\,|0\rangle, \\ &\big(L_n(u)\big)_{12}|0\rangle = 0, \qquad \langle 0|\big(L_n(u)\big)_{21} = 0, \end{aligned} \tag{4.12}$$

where

$$r_0(u) = \frac{1 + iu\Delta/2}{1 - iu\Delta/2}. \tag{4.13}$$

From these properties, we easily obtain

$$\begin{aligned} &r_1(u) = 1, \qquad r_3(u) = r_0^N(u), \\ &T_{12}(u)|0\rangle = 0, \qquad \langle 0|T_{21}(u) = 0. \end{aligned} \tag{4.14}$$

We note that the condition $r_1(u) = 1$ in fact implies the actions of $T_{12}(u)$ and $T_{21}(u)$ in the second line of (4.14). Indeed, from RTT-relation (2.2), we have

$$[T_{21}(v), T_{12}(u)] = g(v,u)\big(T_{11}(u)T_{22}(v) - T_{11}(v)T_{22}(u)\big). \tag{4.15}$$

Applying this equation, for example, to the vector $|0\rangle$ and using the fact that $r_1(u)=1$, we obtain

$$[T_{21}(v),T_{12}(u)]|0\rangle = T_{21}(v)T_{12}(u)|0\rangle = \\ = g(v,u)\big(T_{11}(u)T_{22}(v)-T_{11}(v)T_{22}(u)\big)|0\rangle = \big(r_1(u)-r_1(v)\big)|0\rangle = 0. \tag{4.16}$$

Similarly, acting with (4.15) on $\langle 0|$, we obtain $\langle 0|T_{21}(u)=0$.

The property $T_{12}(u)|0\rangle = 0$ leads to a simplification of the explicit formula for Bethe vector (2.8). Obviously, in this case, we should consider only partitions of the set $\bar u$ such that $\bar u^{(2)}=\varnothing$ and $\bar u^{(1)}=\bar u$. Then (2.8) becomes

$$\mathbb{B}_{a,b}(\bar u;\bar v) = \sum \frac{\mathsf{K}_a(\bar v^{(1)}|\bar u)}{f(\bar v,\bar u)} f(\bar v^{(2)},\bar v^{(1)})T_{13}(\bar v^{(1)})T_{23}(\bar v^{(2)})|0\rangle. \tag{4.17}$$

Here, the sum is taken over partitions of only one set $\bar v \Rightarrow \{\bar v^{(1)},\bar v^{(2)}\}$ with the restriction $\#\,\bar v^{(1)}=a$. The last restriction can obviously be satisfied if and only if $a\le b$. Hence, if $a>b$, then $\mathbb{B}_{a,b}(\bar u;\bar v)=0$. In particular,

$$\mathbb{B}_{0,1}(\varnothing;v) = T_{23}(v)|0\rangle, \qquad \mathbb{B}_{1,1}(u;v) = \frac{g(v,u)}{f(v,u)}\,T_{13}(v)|0\rangle. \tag{4.18}$$

To conclude this section, we give two formulas concerning the continuum limit $\Delta\to 0$. The first formula gives the limit of powers of the function $r_0(u)$:

$$\lim_{\Delta\to 0} r_0^n(u) = \lim_{\Delta\to 0}\left(\frac{1+iu\Delta/2}{1-iu\Delta/2}\right)^{x_n/\Delta} = e^{iux_n}. \tag{4.19}$$

The second formula describes a typical procedure for taking the continuum limit of sums over lattice sites. Let $\Phi(x)$ be an integrable function on the interval $[0,L]$. Then

$$\Delta\sum_{j=1}^{N}\Phi(x_j)\psi^\dagger(j) = \sum_{j=1}^{N}\Phi(x_j)\int_{x_{j-1}}^{x_j}\Psi_k^\dagger(x)\,dx \to \int_0^L \Phi(x)\Psi_k^\dagger(x)\,dx, \quad \Delta\to 0, \tag{4.20}$$

and we recall that all limits of operator-valued expressions are understood in the weak sense. We can hence formulate a general rule: a sum over lattice sites multiplied by Δ becomes an integral in the continuum limit. It is easy to see that if we have an m-fold sum over lattice sites multiplied by Δ^m, then it becomes an m-fold integral in the continuum limit.

5. Bethe vectors in terms of local operators

We consider a multicomposite model with total monodromy matrix (4.8). Let the number M of the partial monodromy matrices coincide with the number N of lattice sites. Then every partial monodromy matrix $T^{(n)}(u)$ is the L-operator $L_n(u)$ given by (4.9). Correspondingly, we have

$$r_1^{(k)}(u)=1, \qquad r_3^{(k)}(v)=r_0(v). \tag{5.1}$$

The formula for total Bethe vector (2.18) becomes

$$\mathbb{B}_{a,b}(\bar u;\bar v) = \sum\prod_{j=1}^{N} r_0^{j-1}(\bar v^{(j)}) \prod_{1\le k<j\le N}\frac{f(\bar u^{(j)},\bar u^{(k)})f(\bar v^{(j)},\bar v^{(k)})}{f(\bar v^{(j)},\bar u^{(k)})}\prod_{j=1}^{N}\mathbb{B}^{(j)}_{a_j,b_j}(\bar u^{(j)};\bar v^{(j)}). \tag{5.2}$$

This is the main formula that we use. But before applying this formula to the TCBG model, it is useful to see how it works in the simpler example of the one-component Bose gas.

5.1. One-component Bose gas. The L-operator of the one-component Bose gas is given by (4.11), but to construct Bethe vectors, we only need to know this L-operator up to terms of the order Δ:

$$\tilde{L}_n(u) = \begin{pmatrix} 1 - \dfrac{iu\Delta}{2} & -i\Delta\sqrt{\varkappa}\psi^\dagger(n) \\ i\Delta\sqrt{\varkappa}\psi(n) & 1 + \dfrac{iu\Delta}{2} \end{pmatrix} + O(\Delta^2). \tag{5.3}$$

We recall that we here set $\psi_2(n) \equiv \psi(n)$ and $\psi_1(n) \equiv 0$ and similarly for $\psi_k^\dagger(n)$. In the continuum limit, these operators become the respective Bose fields $\Psi(x)$ and $\Psi^\dagger(x)$.

Bethe vectors of the one-component Bose gas correspond to the particular case of $\mathbb{B}_{a,b}(\bar{u};\bar{v})$ at $a = 0$ and $\bar{u} = \varnothing$. Then formula (5.2) becomes

$$\mathbb{B}_{0,b}(\varnothing;\bar{v}) \equiv \mathbb{B}_b(\bar{v}) = \sum \prod_{j=1}^{N} r_0^{j-1}(\bar{v}^{(j)}) \prod_{1\le k<j\le N} f(\bar{v}^{(j)},\bar{v}^{(k)}) \prod_{j=1}^{N} \mathbb{B}_{b_j}^{(j)}(\bar{v}^{(j)}). \tag{5.4}$$

A partial Bethe vector at the site j is

$$\mathbb{B}_{b_j}^{(j)}(\bar{v}^{(j)}) = (-i\Delta\sqrt{\varkappa}\psi^\dagger(j))^{b_j}|0\rangle, \tag{5.5}$$

where corrections of the order $O(\Delta^{b_j+1})$ are neglected.

Remark 2. We recall that the total pseudovacuum vector $|0\rangle$ in the multicomposite model is equal to the tensor product of the partial pseudovacuum vectors $|0\rangle^{(j)}$, $j = 1,\dots,N$. But in the case of the one-component Bose gas, we can assume that all operators $\psi(j)$ and $\psi^\dagger(j)$ act in the same Fock space. Obviously, because $\psi(j)$ and $\psi^\dagger(k)$ commute for $j \ne k$, such a formulation is equivalent to the original one. In the case of the TCBG model, we use the same treatment of the multicomposite model.

We consider an example $b = 2$. We then have two possibilities:

- There exists one b_j such that $b_j = 2$, and all other $b_\ell = 0$. Then the subset $\bar{v}^{(j)}$ coincides with the original set $\{v_1, v_2\}$, and all other subsets $\bar{v}^{(\ell)}$ are empty.
- There exist two $b_j = 1$ and $b_k = 1$, and all other $b_\ell = 0$. Then the subsets $\bar{v}^{(j)}$ and $\bar{v}^{(k)}$ consist of one element (e.g., $\bar{v}^{(j)} = v_2$ and $\bar{v}^{(k)} = v_1$ or vice versa). All other subsets $\bar{v}^{(\ell)}$ are empty.

We consider the first case. We let $\mathbb{B}_{2,\varnothing}$ denote the corresponding contribution to the Bethe vector. Then

$$\mathbb{B}_{2,\varnothing} = -\varkappa\Delta^2 \sum_{j=1}^{N} (r_0(v_1)r_0(v_2))^{j-1}\,(\psi^\dagger(j))^2|0\rangle, \tag{5.6}$$

and because of (4.19), we obtain

$$\mathbb{B}_{2,\varnothing} = -\varkappa\Delta^2 \sum_{j=1}^{N} e^{ix_j(v_1+v_2)}(\psi^\dagger(j))^2|0\rangle. \tag{5.7}$$

This sum goes to zero because it has the coefficient Δ^2. Indeed, because of (4.20), we have

$$\Delta^2 \sum_{j=1}^{N} e^{ix_j(v_1+v_2)}(\psi^\dagger(j))^2|0\rangle \to \Delta \int_0^L e^{ix(v_1+v_2)}(\Psi^\dagger(x))^2\,dx|0\rangle \longrightarrow 0, \quad \Delta \to 0. \tag{5.8}$$

It remains to consider the second case. We let $\mathbb{B}_{1,1,\varnothing}$ denote the corresponding contribution to the Bethe vector. Then

$$\mathbb{B}_{1,1,\varnothing} = -\varkappa\Delta^2 \operatorname*{Sym}_{\bar v} \sum_{1\le k<j\le N} r_0^{j-1}(v_2) r_0^{k-1}(v_1) f(v_2,v_1)\psi^\dagger(j)\psi^\dagger(k)|0\rangle \tag{5.9}$$

or, because of (4.19),

$$\mathbb{B}_{1,1,\varnothing} = -\varkappa\Delta^2 \operatorname*{Sym}_{\bar v} \sum_{1\le k<j\le N} e^{ix_k v_1 + ix_j v_2}\; f(v_2,v_1)\psi^\dagger(j)\psi^\dagger(k)|0\rangle. \tag{5.10}$$

This time, we again have the coefficient Δ^2, but the sum is double. The limit is therefore finite,

$$\lim_{\Delta\to 0} \mathbb{B}_{1,1,\varnothing} = \mathbb{B}_2(\bar v) = -\varkappa \operatorname*{Sym}_{\bar v} f(v_2,v_1) \int_0^L dx_2 \int_0^{x_2} e^{ix_1v_1+ix_2v_2}\,\Psi^\dagger(x_2)\Psi^\dagger(x_1)|0\rangle\, dx_1. \tag{5.11}$$

It is clear from (5.4) and (5.5) that for general b, the Bethe vector $\mathbb{B}_b(\bar v)$ is proportional to Δ^b. This coefficient should be compensated in the continuum limit. The only possible way to obtain such a compensation is to have a b-fold sum over the lattice sites. Then Δ^b times the b-fold sum gives a b-fold integral. Hence, we should consider only such partitions of the set $\bar v = \{v_1,\dots,v_b\}$ that reduce to b-fold sums over lattice sites. Obviously, these are partitions such that there are exactly b nonempty subsets in them. In this case, each such subset contains only one variable. Hence, we in fact treat the case already considered in Sec. 2.4. Therefore, the sum over partitions reduces to the sum over lattice sites and the sum over permutations, i.e., to the symmetrization over $\bar v$.

For general b, we thus obtain

$$\mathbb{B}_b(\bar v) = (-i\sqrt{\varkappa}\Delta)^b \operatorname*{Sym}_{\bar v} \prod_{b\ge j>k\ge 1} f(v_j,v_k) \sum_{j_b>\cdots>j_1}^{N} \prod_{k=1}^{b} (r_0^{j_k-1}(v_k)\psi^\dagger(j_k))|0\rangle \tag{5.12}$$

or, partially taking the continuum limit,

$$\mathbb{B}_b(\bar v) = (-i\sqrt{\varkappa}\Delta)^b \operatorname*{Sym}_{\bar v} \prod_{b\ge j>k\ge 1} f(v_j,v_k) \sum_{j_b>\cdots>j_1}^{N} \prod_{k=1}^{b} (e^{ix_{j_k}v_k}\psi^\dagger(j_k))|0\rangle. \tag{5.13}$$

This b-fold sum over the lattice sites passes into a b-fold integral, and we finally obtain

$$\lim_{\Delta\to 0}\mathbb{B}_b(\bar v) = (-i\sqrt{\varkappa})^b \operatorname*{Sym}_{\bar v} \prod_{b\ge j>k\ge 1} f(v_j,v_k) \int_{\mathcal{D}} \prod_{k=1}^{b} (e^{ix_k v_k}\Psi^\dagger(x_k))|0\rangle\, dx_1\cdots dx_b, \tag{5.14}$$

where $\mathcal{D} = \{L > x_b > \cdots > x_1 > 0\}$. Representation (5.14) coincides with the well-known result for the Bethe vectors in the coordinate Bethe ansatz [1], [2], [31]. We have thus constructed Bethe vectors in terms of the local Bose field $\Psi^\dagger(x)$ starting from lattice L-operator (5.3).

5.2. Two-component Bose gas. The infinitesimal lattice L-operator of the TCBG model has the form [7]

$$L_n(u) = \begin{pmatrix} 1-\dfrac{iu\Delta}{2} & 0 & -i\Delta\sqrt{\varkappa}\psi_1^\dagger(n) \\ 0 & 1-\dfrac{iu\Delta}{2} & -i\Delta\sqrt{\varkappa}\psi_2^\dagger(n) \\ i\Delta\sqrt{\varkappa}\psi_1(n) & i\Delta\sqrt{\varkappa}\psi_2(n) & 1+\dfrac{iu\Delta}{2} \end{pmatrix} + O(\Delta^2). \tag{5.15}$$

We again consider a multicomposite model with the number of partial monodromy matrices $T^{(n)}(u)$ equal to the number of lattice sites. Each $T^{(n)}(u)$ of such a model then coincides with L-operator (5.15). We first find how the Bethe vector depends on Δ. In the TCBG model, Bethe vectors are given by (4.17). It is easy to see that the total number of creation operators T_{13} and T_{23} in (4.17) is b. In the case of the partial Bethe vectors $\mathbb{B}^{(j)}_{a_j,b_j}(\bar u^{(j)};\bar v^{(j)})$, we have

$$T^{(j)}_{13}(w) = -i\Delta\sqrt{\varkappa}\psi^\dagger_1(j), \qquad T^{(j)}_{23}(w) = -i\Delta\sqrt{\varkappa}\psi^\dagger_2(j). \tag{5.16}$$

Therefore,

$$\mathbb{B}^{(j)}_{a_j,b_j}(\bar u^{(j)};\bar v^{(j)}) \sim \Delta^{b_j}, \quad \text{consequently} \quad \mathbb{B}_{a,b}(\bar u;\bar v) \sim \Delta^{b}. \tag{5.17}$$

The Bethe vectors of the multicomposite TCBG model are given by (5.2). Using the same arguments as in the case of the one-component Bose gas, we conclude that we should consider only partitions of the set $\bar v$ with exactly b nonempty subsets consisting of one element. The sum over such partitions of the set $\bar v$ then becomes the sum over permutations of $\bar v$ and a b-fold sum over the lattice sites.

We now consider what happens with the partitions of the set $\bar u$. In every partial Bethe vector, $b_j \ge a_j$. As shown above, all b_j are equal to either zero or one. If $b_j = 0$, then $a_j = 0$. But if $b_j = 1$, then either $a_j = 1$ or $a_j = 0$. We obtain a partial Bethe vector of the form $B^{(j)}_{1,1}$ in the first case and of the form $B^{(j)}_{0,1}$ in the second case. But because all nonempty subsets $\bar u^{(j)}$ consist of exactly one element, the sum over partitions of the set $\bar u$ also becomes the sum over permutations in $\bar u$ and the sum over the lattice sites where $a_j = 1$.

The sum in (5.2) is therefore organized as follows. First, we should choose a set J consisting of b numbers $J = \{j_1,\dots,j_b\}$. These are the numbers of the lattice sites where $b_{j_k} = 1$. In all other sites, $b_j = 0$. We assume that the subset $\bar v^{(j_k)}$ consists of one element v_k. Symmetrizing over $\bar v$ and summing over all possible j_k with the restriction $j_b > \dots > j_1$, we thus reproduce the sum over partitions of the set $\bar v$. More precisely, we reproduce only such partitions that eventually contribute to the continuum limit.

Up to this point, everything is exactly as in the case of one-component bosons. We should now take the partitions of the set $\bar u$ into account. For this, from the set $J = \{j_1,\dots,j_b\}$, we should choose a subset K of numbers consisting of a elements: $K = \{j_{k_1},\dots,j_{k_a}\}$, $K \subset J$. These are the numbers of the lattice sites where $a_{j_{k_m}} = 1$. In all other sites, $a_j = 0$. We assume that the subset $\bar u^{(j_{k_m})}$ consists of one element u_m. Symmetrizing over $\bar u$ and summing over all possible j_{k_m} with the restriction $j_{k_a} > \dots > j_{k_1}$, we reproduce the sum over partitions of the set $\bar u$.

Summarizing all the above, we rewrite (5.2) as

$$\begin{aligned}\mathbb{B}_{a,b}(\bar u;\bar v) = \operatorname*{Sym}_{\bar v,\bar u} \prod_{b\ge j>k\ge 1} f(v_j,v_k) \prod_{a\ge j>k\ge 1} f(u_j,u_k)\,\times \\ \times \sum_{j_b>\dots>j_1}^{N} \sum_{\substack{j_{k_a}>\dots>j_{k_1},\\ j_{k_m}\in J}} \prod_{m=1}^{a}\prod_{\ell=k_m+1}^{b} f^{-1}(v_\ell,u_m) \prod_{k=1}^{b} r_0^{j_k-1}(v_k)\,\times \\ \times \prod_{m=1}^{a} \mathbb{B}^{(j_{k_m})}_{1,1}(u_m;v_{k_m}) \prod_{j_\ell\in J\setminus K} \mathbb{B}^{(j_\ell)}_{0,1}(\varnothing;v_\ell).\end{aligned} \tag{5.18}$$

Because of (4.18) and (5.15), we find

$$\mathbb{B}^{(j)}_{0,1}(\varnothing;v) = -i\Delta\sqrt{\varkappa}\psi^\dagger_2(j)|0\rangle, \qquad \mathbb{B}^{(j)}_{1,1}(u;v) = -i\Delta\sqrt{\varkappa}\,\frac{g(v,u)}{f(v,u)}\,\psi^\dagger_1(j)|0\rangle, \tag{5.19}$$

and using (4.19), we obtain

$$\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-i\Delta\sqrt{\varkappa})^b \operatorname*{Sym}_{\bar{v},\bar{u}} \prod_{b\geq j>k\geq 1} f(v_j,v_k) \prod_{a\geq j>k\geq 1} f(u_j,u_k) \times$$

$$\times \sum_{j_b>\cdots>j_1}^{N} \sum_{\substack{j_{k_a}>\cdots>j_{k_1},\\ j_{k_m}\in J}} \prod_{m=1}^{a} \prod_{\ell=k_m+1}^{b} f^{-1}(v_\ell,u_m) \prod_{k=1}^{b} e^{ix_k v_k} \times$$

$$\times \prod_{m=1}^{a} \frac{g(v_{k_m},u_m)}{f(v_{k_m},u_m)} \psi_1^\dagger(j_{k_m}) \prod_{j_\ell\in J\setminus K} \psi_2^\dagger(j_\ell)|0\rangle. \tag{5.20}$$

Using the obvious properties of the functions $g(x,y)$ and $f(x,y)$

$$f(x,y+c) = \frac{1}{f(y,x)}, \qquad g(x,y+c) = -\frac{g(y,x)}{f(y,x)}, \tag{5.21}$$

we see that

$$\operatorname*{Sym}_{\bar{u}} \prod_{a\geq j>k\geq 1} f(u_j,u_k) \prod_{m=1}^{a} \left\{ \frac{g(v_{k_m},u_m)}{f(v_{k_m},u_m)} \prod_{\ell=k_m+1}^{b} \frac{1}{f(v_\ell,u_m)} \right\} = (-1)^a \Omega^{(a,b)}_{k_1,\dots,k_a}(\bar{u};\bar{v}+c),$$

where the coefficients $\Omega^{(a,b)}_{k_1,\dots,k_a}$ are given by (2.29).

Hence, (5.20) becomes

$$\mathbb{B}_{a,b}(\bar{u};\bar{v}) = (-1)^a(-i\Delta\sqrt{\varkappa})^b \operatorname*{Sym}_{\bar{v}} \prod_{b\geq j>k\geq 1} f(v_j,v_k) \times$$

$$\times \sum_{j_b>\cdots>j_1}^{N} \sum_{\substack{j_{k_a}>\cdots>j_{k_1},\\ j_{k_m}\in J}} \prod_{k=1}^{b} e^{ix_k v_k} \Omega^{(a,b)}_{k_1,\dots,k_a}(\bar{u};\bar{v}+c) \times$$

$$\times \prod_{m=1}^{a} \psi_1^\dagger(j_{k_m}) \prod_{j_\ell\in J\setminus K} \psi_2^\dagger(j_\ell)|0\rangle, \tag{5.22}$$

and it becomes clear that in the continuum limit, we obtain (3.4) up to a normalization factor.

6. Representation of the monodromy matrix in terms of Bose fields

In this section, we derive explicit representations of the monodromy matrix elements $T_{ij}(u)$ in terms of the Bose fields. These representations have the form of a formal power series in the coupling constant $\varkappa$. But we note that these series, in a weak sense, are truncated on an arbitrary Bethe vector.

We represent infinitesimal L-operator (5.15) as a 2×2 block matrix:

$$L_n(u) = \begin{pmatrix} a & b_n \\ c_n & d \end{pmatrix} + O(\Delta^2). \tag{6.1}$$

Here, $d = 1 + iu\Delta/2$, and a is a 2×2 matrix $a = (1 - iu\Delta/2)\cdot \mathbf{1}$, where $\mathbf{1}$ is the 2×2 identity matrix. A two-component column vector b_n and two-component row vector c_n are

$$b_n = -i\Delta\sqrt{\varkappa} \begin{pmatrix} \psi_1^\dagger(n) \\ \psi_2^\dagger(n) \end{pmatrix}, \qquad c_n = i\Delta\sqrt{\varkappa} \begin{pmatrix} \psi_1(n) & \psi_2(n) \end{pmatrix}. \tag{6.2}$$

It is convenient to separate the diagonal and antidiagonal parts of L-operator (6.1) as[3]

$$L_n(u) = \Lambda(u) + W_n, \qquad \Lambda(u) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \qquad W_n = \begin{pmatrix} 0 & b_n \\ c_n & 0 \end{pmatrix}. \tag{6.3}$$

Representation (6.3) should now be substituted in (4.8) and then expanded in a series in W_n. Because the antidiagonal part W_n is proportional to $\sqrt{\varkappa}$, the monodromy matrix $T(u)$ becomes a polynomial in $\sqrt{\varkappa}$ and becomes an infinite power series in the continuum limit:

$$T(u) = \sum_{n=0}^{\infty} \varkappa^{n/2} T_n(u), \tag{6.4}$$

where

$$\varkappa^{n/2} T_n(u) = \left(1 - \frac{iu\Delta}{2}\right)^{-N} \sum_{N \geq k_n > \cdots > k_1 \geq 1} \Lambda^{N-k_n} W_{k_n} \Lambda^{k_n - k_{n-1} - 1} \cdots \Lambda^{k_2 - k_1 - 1} W_{k_1} \Lambda^{k_1 - 1}. \tag{6.5}$$

It is clear from this representation that the diagonal blocks of the monodromy matrix are series in integer powers of $\varkappa$ and the antidiagonal blocks are series in half-integer powers of $\varkappa$.

Let

$$\widetilde{W}_{k_i} = \Lambda^{-k_i} W_{k_i} \Lambda^{k_i - 1} = \begin{pmatrix} 0 & \tilde{b}_{k_i} \\ \tilde{c}_{k_i} & 0 \end{pmatrix}, \tag{6.6}$$

where

$$\tilde{b}_{k_i} = \frac{b_{k_i}}{1 + iu\Delta/2} \left(\frac{1 + iu\Delta/2}{1 - iu\Delta/2}\right)^{k_i}, \qquad \tilde{c}_{k_i} = \frac{c_{k_i}}{1 - iu\Delta/2} \left(\frac{1 - iu\Delta/2}{1 + iu\Delta/2}\right)^{k_i}. \tag{6.7}$$

Then Eq. (6.5) becomes

$$\varkappa^{n/2} T_n(u) = \left(1 - \frac{iu\Delta}{2}\right)^{-N} \Lambda^N \sum_{N \geq k_n > \cdots > k_1 \geq 1} \widetilde{W}_{k_n} \widetilde{W}_{k_{n-1}} \cdots \widetilde{W}_{k_1}. \tag{6.8}$$

Partially taking the continuum limit via (4.19), we obtain

$$\lim_{\Delta \to 0} \left(1 - \frac{iu\Delta}{2}\right)^{-N} \Lambda^N = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & e^{iuL} \end{pmatrix} \tag{6.9}$$

$\tilde{b}_{k_i} = b_{k_i} e^{iux_{k_i}}$, and $\tilde{c}_{k_i} = c_{k_i} e^{-iux_{k_i}}$.

It is convenient to study the operators $T_n(u)$ separately for even and odd n. Let $n = 2\ell$. The product of two matrices $\widetilde{W}_{k_i}$ and $\widetilde{W}_{k_{i-1}}$ gives a block-diagonal matrix

$$\widetilde{W}_{k_i} \widetilde{W}_{k_i - 1} = \begin{pmatrix} \tilde{b}_{k_i} \tilde{c}_{k_{i-1}} & 0 \\ 0 & \tilde{c}_{k_i} \tilde{b}_{k_{i-1}} \end{pmatrix}. \tag{6.10}$$

Hence, we obtain

$$\varkappa^{\ell} T_{2\ell}(u) = \begin{pmatrix} A_\ell(u) & 0 \\ 0 & D_\ell(u) \end{pmatrix}, \tag{6.11}$$

[3]Here and hereafter, we omit the terms $O(\Delta^2)$ because they do not contribute to the continuum limit.

where

$$A_\ell(u) = \sum_{N\geq k_{2\ell}>\cdots>k_1\geq 1} \tilde{b}_{k_{2\ell}}\tilde{c}_{k_{2\ell-1}}\tilde{b}_{k_{2\ell-2}}\tilde{c}_{k_{2\ell-3}}\cdots\tilde{b}_{k_2}\tilde{c}_{k_1}$$
$$D_\ell(u) = e^{iuL}\sum_{N\geq k_{2\ell}>\cdots>k_1\geq 1} \tilde{c}_{k_{2\ell}}\tilde{b}_{k_{2\ell-1}}\tilde{c}_{k_{2\ell-2}}\tilde{b}_{k_{2\ell-3}}\cdots\tilde{c}_{k_2}\tilde{b}_{k_1}. \tag{6.12}$$

We note that all operators in these products commute because they are from different lattice sites. Therefore,

$$\tilde{b}_{k_{2\ell}}\tilde{c}_{k_{2\ell-1}}\tilde{b}_{k_{2\ell-2}}\tilde{c}_{k_{2\ell-3}}\cdots\tilde{b}_{k_2}\tilde{c}_{k_1} = :\tilde{b}_{k_{2\ell}}\tilde{c}_{k_{2\ell-1}}\tilde{b}_{k_{2\ell-2}}\tilde{c}_{k_{2\ell-3}}\cdots\tilde{b}_{k_2}\tilde{c}_{k_1}:,$$
$$\tilde{c}_{k_{2\ell}}\tilde{b}_{k_{2\ell-1}}\tilde{c}_{k_{2\ell-2}}\tilde{b}_{k_{2\ell-3}}\cdots\tilde{c}_{k_2}\tilde{b}_{k_1} = :\tilde{c}_{k_{2\ell}}\tilde{b}_{k_{2\ell-1}}\tilde{c}_{k_{2\ell-2}}\tilde{b}_{k_{2\ell-3}}\cdots\tilde{c}_{k_2}\tilde{b}_{k_1}:, \tag{6.13}$$

where the symbol $:\cdot:$ denotes normal ordering. Obviously,

$$\tilde{c}_{k_{2i}}\tilde{b}_{k_{2i-1}} = \varkappa\Delta^2 e^{iu(x_{k_{2i-1}}-x_{k_{2i}})}:\big(\psi_1^\dagger(k_{2i-1})\psi_1(k_{2i}) + \psi_2^\dagger(k_{2i-1})\psi_2(k_{2i})\big):. \tag{6.14}$$

Hence, we find

$$A_\ell(u) = \varkappa^\ell\Delta^{2\ell}\sum_{N\geq k_{2\ell}>\cdots>k_1\geq 1}\prod_{i=1}^{\ell} e^{iu(x_{k_{2i}}-x_{k_{2i-1}})}\times$$
$$\times:\prod_{i=1}^{\ell-1}\big(\psi_1^\dagger(k_{2i})\psi_1(k_{2i+1}) + \psi_2^\dagger(k_{2i})\psi_2(k_{2i+1})\big)\begin{pmatrix}\psi_1^\dagger(k_{2\ell})\psi_1(k_1) & \psi_1^\dagger(k_{2\ell})\psi_2(k_1)\\ \psi_2^\dagger(k_{2\ell})\psi_1(k_1) & \psi_2^\dagger(k_{2\ell})\psi_2(k_1)\end{pmatrix}:,$$
$$D_\ell(u) = e^{iuL}\varkappa^\ell\Delta^{2\ell}\sum_{N\geq k_{2\ell}>\cdots>k_1\geq 1}\prod_{i=1}^{\ell} e^{-iu(x_{k_{2i}}-x_{k_{2i-1}})}\times$$
$$\times:\prod_{i=1}^{\ell}\big(\psi_1^\dagger(k_{2i-1})\psi_1(k_{2i}) + \psi_2^\dagger(k_{2i-1})\psi_2(k_{2i})\big):.$$

It remains to replace the sums over k_i with integrals via (4.20). It is convenient to set $x_{k_{2i}} = z_i$ and $x_{k_{2i-1}} = y_i$. Then

$$A_\ell(u) = \varkappa^\ell\int_0^L\prod_{i=1}^{\ell}\{e^{iu(z_i-y_i)}\,dz_i\,dy_i\}\Theta_\ell(\bar{z},\bar{y})\times$$
$$\times:\prod_{i=1}^{\ell-1}\big(\Psi_1^\dagger(z_i)\Psi_1(y_{i+1}) + \Psi_2^\dagger(z_i)\Psi_2(y_{i+1})\big)\begin{pmatrix}\Psi_1^\dagger(z_\ell)\Psi_1(y_1) & \Psi_1^\dagger(z_\ell)\Psi_2(y_1)\\ \Psi_2^\dagger(z_\ell)\Psi_1(y_1) & \Psi_2^\dagger(z_\ell)\Psi_2(y_1)\end{pmatrix}:, \tag{6.15}$$
$$D_\ell(u) = e^{iuL}\varkappa^\ell\int_0^L\prod_{i=1}^{\ell}\Big\{e^{-iu(z_i-y_i)}\,dz_i\,dy_i\Big\}\Theta_\ell(\bar{z},\bar{y})\times$$
$$\times:\prod_{i=1}^{\ell}\big(\Psi_1^\dagger(y_i)\Psi_1(z_i) + \Psi_2^\dagger(y_i)\Psi_2(z_i)\big):, \tag{6.16}$$

where

$$\Theta_\ell(\bar{z},\bar{y}) = \theta(z_\ell - y_\ell)\prod_{i=1}^{\ell-1}\theta(y_{i+1}-z_i)\theta(z_i-y_i). \tag{6.17}$$

The antidiagonal blocks of the monodromy matrix can be found in exactly the same manner. Setting $n = 2\ell + 1$ in (6.8), we find

$$\varkappa^{\ell+1/2} T_{2\ell+1}(u) = \begin{pmatrix} 0 & B_\ell(u) \\ C_\ell(u) & 0 \end{pmatrix}, \tag{6.18}$$

where

$$C_\ell(u) = ie^{iuL}\varkappa^{\ell+1/2}\int_0^L \prod_{i=1}^{\ell}\{e^{iu(z_i-y_i)}\,dz_i\,dy_i\}e^{-iuy_{\ell+1}}\theta(y_{\ell+1}-z_\ell)\Theta_\ell(\bar z,\bar y)\,dy_{\ell+1}\times$$

$$\times :\prod_{i=1}^{\ell}\big(\Psi_1^\dagger(z_i)\Psi_1(y_{i+1}) + \Psi_2^\dagger(z_i)\Psi_2(y_{i+1})\big)\cdot\Big(\Psi_1(y_1) \quad \Psi_2(y_1)\Big):,$$

$$B_\ell(u) = \; -i\varkappa^{\ell+1/2}\int_0^L \prod_{i=1}^{\ell}\{e^{-iu(z_i-y_i)}\,dz_i\,dy_i\}e^{iuy_{\ell+1}}\theta(y_{\ell+1}-z_\ell)\Theta_\ell(\bar z,\bar y)\,dy_{\ell+1}\times$$

$$\times :\prod_{i=1}^{\ell}\big(\Psi_1^\dagger(y_i)\Psi_1(z_i) + \Psi_2^\dagger(y_i)\Psi_2(z_i)\big)\cdot\begin{pmatrix}\Psi_1^\dagger(y_{\ell+1}) \\ \Psi_2^\dagger(y_{\ell+1})\end{pmatrix}:.$$

We have thus obtained an explicit series representation for the monodromy matrix elements $T_{ij}(u)$ in terms of the local Bose fields. This series is formal, and we do not study the problem of its convergence. But it is easy to see that if we introduce a vector

$$|\Phi_{a,b}\rangle = \int_0^L \Phi_{a,b}(x_1,\dots,x_a;y_1,\dots,y_b)\prod_{i=1}^{a}\Psi_1^\dagger(x_i)\prod_{j=1}^{b}\Psi_2^\dagger(y_j)|0\rangle\,dx_1,\dots,\,dx_a\,dy_1,\dots,\,dy_b,$$

where $\Phi_{a,b}(x_1,\dots,x_a;y_1,\dots,y_b)$ is a continuous function within the integration domain, then the action of any $T_{ij}(u)$ on $|\Phi_{a,b}\rangle$ becomes a finite sum.

7. Maps of fields

Because the R-matrix is invariant under transposition with respect to both spaces, the map

$$\phi(T_{jk}(u)) = T_{kj}(u) \tag{7.1}$$

defines an antimorphism of algebra (2.2) (see [21]). This map is a very convenient tool in studying form factors because it allows relating form factors of different operators. In the case of the TCBG model, antimorphism (7.1) agrees with the map of the Bose fields

$$\phi(\Psi_i(x)) = -\Psi_i^\dagger(L-x), \qquad \phi(\Psi_i^\dagger(x)) = -\Psi_i(L-x). \tag{7.2}$$

Indeed, for example, we consider how map (7.2) acts on the matrix elements $T_{jk}(u)$ for $j,k = 1,2$. By Eqs. (6.11) and (6.15), we have

$$T_{jk}(u) = \sum_{\ell=0}^{\infty}\varkappa^\ell (T_{2\ell})_{jk}(u), \quad j,k = 1,2, \tag{7.3}$$

where

$$(T_{2\ell})_{jk}(u) = \int_0^L \prod_{n=1}^{\ell} \{e^{iu(z_n-y_n)}\,dz_n\,dy_n\}\Theta_\ell(\bar z,\bar y)\times$$

$$\times :\Psi_j^\dagger(z_\ell)\Psi_k(y_1)\prod_{n=1}^{\ell-1}\Big(\sum_{s=1}^{2}\Psi_s^\dagger(z_n)\Psi_s(y_{n+1})\Big):. \tag{7.4}$$

We recall that because of the factor $\Theta_\ell(\bar z,\bar y)$, the integration in (7.4) is over the domain $z_\ell > y_\ell > z_{\ell-1} > \cdots > z_1 > y_1$. All the operators in (7.4) therefore commute with each other, and the normal ordering is in fact unnecessary. Acting on (7.4) with ϕ as in (7.2), we obtain

$$\phi\big((T_{2\ell})_{jk}(u)\big) = \int_0^L \prod_{n=1}^{\ell} \{e^{iu(z_n-y_n)}\,dz_n\,dy_n\}\Theta_\ell(\bar z,\bar y)\times$$

$$\times :\Psi_k^\dagger(L-y_1)\Psi_j(L-z_\ell)\prod_{n=1}^{\ell-1}\Big(\sum_{s=1}^{2}\Psi_s^\dagger(L-y_{n+1})\Psi_s(L-z_n)\Big):. \tag{7.5}$$

It now suffices to change the integration variables $z_n \to L - y_{\ell+1-n}$ and $y_n \to L - z_{\ell+1-n}$. We then have

$$\Theta_\ell(\bar z,\bar y)\Big|_{\substack{z_n\to L-y_{\ell+1-n},\\ y_n\to L-z_{\ell+1-n}}} = \prod_{n=1}^{\ell-1}\theta(y_{\ell-n+1}-z_{\ell-n})\prod_{n=1}^{\ell}\theta(z_{\ell-n+1}-y_{\ell-n+1}) =$$

$$= \prod_{i=1}^{\ell-1}\theta(y_{i+1}-z_i)\prod_{i=1}^{\ell}\theta(z_i-y_i) = \Theta_\ell(\bar z,\bar y). \tag{7.6}$$

It is also easy to see that

$$\prod_{n=1}^{\ell-1}\Big(\sum_{s=1}^{2}\Psi_s^\dagger(L-y_{n+1})\Psi_s(L-z_n)\Big)\Big|_{\substack{z_n\to L-y_{\ell+1-n},\\ y_n\to L-z_{\ell+1-n}}} = \prod_{n=1}^{\ell-1}\Big(\sum_{s=1}^{2}\Psi_s^\dagger(z_{\ell-n})\Psi_s(y_{\ell-n+1})\Big) =$$

$$= \prod_{n=1}^{\ell-1}\Big(\sum_{s=1}^{2}\Psi_s^\dagger(z_n)\Psi_s(y_{n+1})\Big). \tag{7.7}$$

Hence, we obtain

$$\phi\big(\big(T_{2\ell}\big)_{jk}(u)\big) = \int_0^L \prod_{n=1}^{\ell} \{e^{iu(z_n-y_n)}\,dz_n\,dy_n\}\Theta_\ell(\bar z,\bar y)\times$$

$$\times :\Psi_k^\dagger(y_1)\Psi_j(z_\ell)\prod_{n=1}^{\ell-1}\Big(\sum_{s=1}^{2}\Psi_s^\dagger(z_n)\Psi_s(y_{n+1})\Big): = (T_{2\ell})_{kj}(u). \tag{7.8}$$

Similarly, using the explicit representations for other operators $T_{jk}(u)$, we can prove that (7.2) implies (7.1).

8. Zero modes

A method for calculating the form factors of local operators in $GL(3)$-invariant models was developed in [12]. This method is based on using partial zero modes of the monodromy matrix elements $T_{ij}(u)$ in the composite model consisting of two partial monodromy matrices (2.12). Although this approach is applicable to a wide class of integrable models, it should be slightly modified in the case of the TCBG model. The point is that it was assumed in [12] that the monodromy matrix $T(u)$ passes into the identity operator as $|u| \to \infty$. This restriction is not very important, but it leads to minor changes in the case of the TCBG model.

We note that a monodromy matrix $T^{(a)}(u)$ constructed by L-operator (4.1) has the property mentioned above. Indeed, we can define local L-operators $L_n^{(a)}(u)$, $n = 1, \dots, N$, by Eqs. (4.1) and (4.2), where the operators a_k and $a_k^\dagger$ are respectively replaced with $a_k(n)$ and $a_k^\dagger(n)$ with the commutation relations $[a_i(n), a_k^\dagger(m)] = \delta_{nm}\delta_{ik}$. We can then set

$$T^{(a)}(u) = u^{-N} L_N^{(a)}(u) \cdots L_1^{(a)}(u), \tag{8.1}$$

and this matrix obviously has an asymptotic expansion

$$T^{(a)}(u) = \mathbf{1} + \frac{c}{u} T^{(a)}[0] + O(u^{-2}), \quad u \to \infty. \tag{8.2}$$

Therefore, we can standardly define the zero modes of this monodromy matrix:

$$T^{(a)}[0] = \lim_{u\to\infty} \frac{u}{c}\big(T^{(a)}(u) - \mathbf{1}\big). \tag{8.3}$$

But in passing from L-operator (4.1) to L-operator (4.9), we multiplied $L^{(a)}(u)$ by the matrix $J = \operatorname{diag}(1, 1, -1)$ (see (4.5)). As a result, the monodromy matrix $T(u)$ given by (4.8) has an essential singularity at infinity in the continuum limit. The definition of zero modes therefore needs clarification in the case of the TCBG model. We clarify it in this section and consider an asymptotic expansion of the monodromy matrix elements $T_{ij}(u)$ at a large value of the argument. For this, we use the integral representations for $T_{ij}(u)$ obtained in Sec. 6.

If $u \to \infty$, then the expansion for the monodromy matrix contains multiple integrals of rapidly oscillating exponents. Methods for calculating rapidly oscillating integrals are well known (see, e.g., [36], [37]). In our case, the integration domain of every integration variable is a finite interval $[0, L]$, and one of the simplest ways to obtain the asymptotic expansion of $T_{ij}(u)$ is therefore integration by parts. Using this method, we can easily show that single and double integrals yield a $1/u$ behavior while all the terms with $\ell > 1$ give contributions of the order $o(u^{-1})$. Therefore, to find zero modes, it suffices to take only the first nontrivial terms of the expansion for $T(u)$. We then have

$$\begin{aligned}
T_{ij}(u) &= \delta_{ij} + \varkappa \int_0^L e^{iu(z-y)}\theta(z-y)\Psi_i^\dagger(z)\Psi_j(y)\,dz\,dy + O(\varkappa^2), \quad i, j = 1, 2,\\
T_{33}(u) &= e^{iuL} + \varkappa\, e^{iuL} \int_0^L e^{iu(y-z)}\theta(z-y)\big(\Psi_1^\dagger(y)\Psi_1(z) + \Psi_2^\dagger(y)\Psi_2(z)\big)dz\,dy + O(\varkappa^2).\\
T_{i3}(u) &= -i\sqrt{\varkappa} \int_0^L e^{iuy}\Psi_i^\dagger(y)\,dy + O(\varkappa^{3/2}), \quad i = 1, 2,\\
T_{3j}(u) &= i\sqrt{\varkappa} e^{iuL} \int_0^L e^{-iuy}\Psi_j(y)\,dy + O(\varkappa^{3/2}), \quad j = 1, 2.
\end{aligned}$$

All the terms denoted by $O(\varkappa^2)$ or $O(\varkappa^{3/2})$ contribute $O(u^{-2})$ as $u \to \infty$ and are therefore unimportant.

Integrating by parts, we obtain

$$T_{ij}(u) = \delta_{ij} + \frac{i\varkappa}{u}\int_0^L \Psi_i^\dagger(y)\Psi_j(y)\,dy + O(u^{-2}), \quad i,j=1,2, \tag{8.4}$$

$$T_{33}(u) = e^{iuL} - \frac{i\varkappa}{u}\,e^{iuL}\int_0^L \big(\Psi_1^\dagger(y)\Psi_1(y) + \Psi_2^\dagger(y)\Psi_2(y)\big)dy + O(u^{-2}), \tag{8.5}$$

$$T_{i3}(u) = -\frac{\sqrt{\varkappa}}{u}\big(e^{iuL}\Psi_i^\dagger(L) - \Psi_i^\dagger(0)\big) + O(u^{-2}), \quad i=1,2, \tag{8.6}$$

$$T_{3j}(u) = -\frac{\sqrt{\varkappa}}{u}\big(\Psi_j(L) - e^{iuL}\Psi_j(0)\big) + O(u^{-2}), \quad j=1,2. \tag{8.7}$$

We now define the zero modes as

$$T_{ij}[0] = \lim_{u\to\infty}\frac{u}{c}\big(T_{ij}(u)-\delta_{ij}\big) = -\int_0^L \Psi_i^\dagger(y)\Psi_j(y)\,dy, \quad i,j=1,2 \tag{8.8}$$

(we recall that $c=-i\varkappa$). This is the same definition as for the models considered in [12]. The zero mode $T_{33}[0]$ is defined slightly differently:

$$T_{33}[0] = \lim_{u\to\infty}\frac{u}{c}\big(e^{-iuL}T_{33}(u)-1\big) = \int_0^L \big(\Psi_1^\dagger(y)\Psi_1(y) + \Psi_2^\dagger(y)\Psi_2(y)\big)dy, \tag{8.9}$$

and therefore $T_{11}[0]+T_{22}[0] = -T_{33}[0]$.

Looking at (8.6) and (8.7), we see that we in fact have two types of zero modes for these operators. We call them left and right zero modes, respectively denoted by $T_{ij}^{(\mathrm{L})}[0]$ and $T_{ij}^{(\mathrm{R})}[0]$. Then

$$\begin{aligned} T_{k3}^{(\mathrm{R})}[0] &= \lim_{u\to -i\infty} e^{-iuL}\frac{u}{c}T_{k3}(u) = \frac{1}{i\sqrt{\varkappa}}\Psi_k^\dagger(L),\\ T_{k3}^{(\mathrm{L})}[0] &= \lim_{u\to +i\infty} \frac{u}{c}T_{k3}(u) = -\frac{1}{i\sqrt{\varkappa}}\Psi_k^\dagger(0), \end{aligned} \qquad k=1,2, \tag{8.10}$$

and

$$\begin{aligned} T_{3j}^{(\mathrm{R})}[0] &= \lim_{u\to +i\infty} \frac{u}{c}T_{3j}(u) = \frac{1}{i\sqrt{\varkappa}}\Psi_j(L),\\ T_{3j}^{(\mathrm{L})}[0] &= \lim_{u\to -i\infty} e^{-iuL}\frac{u}{c}T_{3j}(u) = -\frac{1}{i\sqrt{\varkappa}}\Psi_j(0), \end{aligned} \qquad j=1,2. \tag{8.11}$$

The sums $T_{ij}^{(\mathrm{L})}[0]+T_{ij}^{(\mathrm{R})}[0]$ play the same role as the zero modes of the monodromy matrix of type (8.1), (8.2). It is known, in particular [38], [13], that some of the zero modes $T_{ij}[0]$ annihilate on-shell Bethe vectors:

$$T_{ij}^{(a)}[0]\,\mathbb{B}_{a,b}(\bar u,\bar v) = 0, \quad i>j. \tag{8.12}$$

Similarly, it can be verified that

$$(T_{3j}^{(\mathrm{L})}[0] + T_{3j}^{(\mathrm{R})}[0])\mathbb{B}_{a,b}(\bar u,\bar v) = 0, \quad j\neq 3, \tag{8.13}$$

if $\mathbb{B}_{a,b}(\bar u,\bar v)$ is an on-shell vector. To prove (8.13), it suffices to use the formulas of the action of $T_{ij}(u)$ on Bethe vectors [25] and then to consider the limits $u\to\pm i\infty$ as in (8.11).

Finally, the obtained formulas for the zero modes allow studying the form factors of local operators in the framework of composite model (2.12). Indeed, let the partial monodromy matrix $T^{(1)}(u)$ in (2.12) correspond to an interval $[0,x]$, where x is a fixed point of the interval $[0,L]$. The partial zero modes $T^{(1)}_{ij}[0]$ and $T^{(1;\mathrm{R})}_{ij}[0]$ are given by (8.8)–(8.11), where L should be replaced with x everywhere. In particular, we obtain

$$\begin{aligned}
&\Psi_i^\dagger(x)\Psi_j(x) = -\frac{d}{dx}T^{(1)}_{ij}[0] = \frac{1}{i\varkappa}\frac{d}{dx}\lim_{u\to\infty} u\bigl(T^{(1)}_{ij}(u)-\delta_{ij}\bigr), \quad i,j=1,2,\\
&\Psi_j(x) = i\sqrt{\varkappa}\,T^{(1;\mathrm{R})}_{3j}[0] = \frac{1}{\sqrt{\varkappa}}\lim_{u\to+i\infty} uT^{(1)}_{3j}(u), \qquad j=1,2,\\
&\Psi_k^\dagger(x) = i\sqrt{\varkappa}\,T^{(1;\mathrm{R})}_{k3}[0] = \frac{1}{\sqrt{\varkappa}}\lim_{u\to-i\infty} e^{-iux}uT^{(1)}_{k3}(u), \qquad k=1,2.
\end{aligned}\tag{8.14}$$

The problem of calculating the form factors of local operators in the TCBG model thus reduces to evaluating the form factors of the partial zero modes $T^{(1)}_{ij}[0]$ and $T^{(1;\mathrm{R})}_{ij}[0]$.

9. Conclusion

We have described the TCBG model in the framework of the algebraic Bethe ansatz. Our main goal was to prove that lattice L-operator (4.9) correctly describes the TCBG model in the continuum limit and allows finding zero modes of the monodromy matrix $T(u)$. This goal was successfully achieved.

To calculate the form factors of the fields $\Psi_i(x)$ and $\Psi_i^\dagger(x)$ and their combinations $\Psi_i^\dagger(x)\Psi_j(x)$, we can now use the method in [12]. In fact, part of the results can already be predicted. Indeed, definition (8.8) of the zero modes $T_{ij}[0]$ for $i,j=1,2$ coincides with the definition used in [12]. Therefore, the form factors of the operators $\Psi_i^\dagger(x)\Psi_j(x)$ are in fact already computed.

The calculation of the form factors of the fields $\Psi_i(x)$ and $\Psi_i^\dagger(x)$ should be slightly modified. But the modification in this case affects only the limit $u\to\infty$ and does not affect the determinant representations for partial zero modes. We will consider this question in detail in our subsequent publication.

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