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BÄCKLUND TRANSFORMATIONS RELATING DIFFERENT HAMILTON–JACOBI EQUATIONS

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We discuss one of the possible finite-dimensional analogues of the general Bäcklund transformation relating different partial differential equations. We show that different Hamilton–Jacobi equations can be obtained from the same Lax matrix. We consider Hénon–Heiles systems on the plane, Neumann and Chaplygin systems on the sphere, and two integrable systems with velocity-dependent potentials as examples.

Keywords: general Bäcklund transformation, Hamilton–Jacobi equation, separation of variables, Lax matrix

1. Introduction

Bäcklund transformations, invented in 1875 in studying negative-curvature surfaces, continue to be a modern and rather popular tool for seeking solutions (in particular, soliton solutions) of nonlinear differential equations. It is believed that the Bäcklund transformations (BTs) yield

a. relations between solutions of a given equation (auto-BT),

b. relations between solutions of different equations (hetero-BT), and

c. a method for constructing discrete systems.

The existence of a nontrivial symmetry of an equation (auto-BTs) in essence is a property providing its integrability (all necessary references can be found in recently published monographs [1], [2]). Using autotransformations for discretization is discussed in [3]–[5]. On the other hand, the general BT (hetero-BT) is a somewhat more complicated object, describing more the relations between equations than a bijective map between their solutions [6].

The analogue of auto-BTs in the finite-dimensional case is represented by the canonical coordinate transformations

$$(u, p_u) \to (v, p_v), \qquad \{u_i, p_{u_j}\} = \{v_i, p_{v_j}\} = \delta_{ij}, \tag{1}$$

preserving not only the Poisson bracket but also the form of the Hamilton–Jacobi equations

$$H_i\left(u, \frac{\partial S}{\partial u}\right) = \alpha_i, \qquad H_i\left(v, \frac{\partial S}{\partial v}\right) = 0 \tag{2}$$

corresponding to a family of integrals of motion $H_1, \ldots, H_n$ in involution. This analogue of auto-BTs, i.e., a canonical map preserving the algebraic form of the integrals of motion for a finite-dimensional integrable system, was first introduced by Wojciechowski [7].

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To construct discrete or difference integrable equations using auto-BT (1), it suffices to regard the variables v as the same variables u taken at the next instant of a discrete time variable. Dozens of papers are devoted to this interpretation of BTs in the finite-dimensional case; appropriate references can be found in [5], [8], [9].

Up to now, there is no widely recognized analogue of the general BT relating different partial differential equations. This is likely to be some canonical transformation of variables (1) that allows constructing different Hamilton–Jacobi equations

$$H_i\left(u, \frac{\partial S}{\partial u}\right) = 0, \qquad \widetilde{H}_i\left(v, \frac{\partial \widetilde{S}}{\partial v}\right) = 0 \tag{3}$$

and satisfies some additional conditions. Imposing additional conditions is necessary to construct a nontrivial theory.

One example of such additional conditions can be found in the theory of superintegrable or degenerate systems. For instance, we consider the two-dimensional harmonic oscillator with the integrals of motion

$$H_1 = p_x^2 + p_y^2 + a(x^2 + y^2), \qquad H_2 = p_x^2 - p_y^2 + a(x^2 - y^2).$$

The related Hamilton–Jacobi equations admit a separation of variables in the Cartesian coordinates $u = (x, y)$.

Because this is a superintegrable system, there is another set of integrals of motion

$$\widetilde{H}_1 = p_x^2 + p_y^2 + a(x^2 + y^2), \qquad \widetilde{H}_2 = xp_y - yp_x,$$

for which the corresponding Hamilton–Jacobi equations admit a separation of variables in the polar coordinates $v = (r, \varphi)$ on the plane. This allows stating that the canonic transformation of variables

$$(u, p_u) = (x, y, p_x, p_y) \to (v, p_v) = (r, \varphi, p_r, p_\varphi) \tag{4}$$

specifies the relation between two different systems of Hamilton–Jacobi equations

$$H_{1,2}\left(x, y, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}\right) = \alpha_{1,2}, \qquad \widetilde{H}_{1,2}\left(r, \varphi, \frac{\partial \widetilde{S}}{\partial r}, \frac{\partial \widetilde{S}}{\partial \varphi}\right) = \tilde{\alpha}_{1,2}.$$

In this case, the additional condition is the requirement that the Hamilton–Jacobi equation for the Hamilton function $H_1 = \widetilde{H}_1$ (the mechanical energy) admits a simultaneous separation of variables in terms of the u and v variables. This additional condition is often (and rather successfully) used in the theory of classical and quantum superintegrable systems [10].

Because this expression relates two different systems of Hamilton–Jacobi equations corresponding to the same superintegrable system, we can speak about the semiauto-BT (semiauto-BT) or about the semigeneral BT (semihetero-BT). The problem of constructing relations between completely different Hamilton–Jacobi equations remains open.

Here, we consider several relations of this type that can be treated as BTs of general form [11]. The basic problem is to prove the usefulness of imposing the additional conditions that

1. the required canonical transformation of variables (1) is an auto-BT for the original dynamical system with the integrals of motion $H_1, \dots, H_n$, which therefore admits a simultaneous separation of variables in both the u and the v variables, and
2. the Hamilton–Jacobi equations corresponding to the different integrals of motion H_k and $\widetilde{H}_k$ admit a simultaneous separation of variables in terms of the v variables obtained after the autotransformation.

Below, we show that there are examples of such transformations and that these transformations can be used to construct and classify integrable systems.

2. Integrable systems on the plane

We recall that the Hamilton–Jacobi equation

$$H(q,p)=E$$

admits an additive separation of variables if the complete integral of this equation has the form

$$S(u_1,\dots,u_n;\alpha_1,\dots,\alpha_n)=\sum_{i=1}^{n}S_i(u_i;\alpha_1,\dots,\alpha_n),$$

where u_k are the separation variables in involution with respect to the canonical Poisson brackets. In this case, the second Jacobi equations

$$p_{u_i}=\frac{\partial S_i(u_i;\alpha_1,\dots,\alpha_n)}{\partial u_i},\quad i=1,2,\dots,n, \tag{5}$$

define their conjugate momenta, which allows proving that the solutions of separated equations (5) for $\alpha_i=H_i(u,p_u)$ are in involution with each other.

There are four systems of orthogonal curvilinear coordinates on the plane: Cartesian, polar, parabolic, and elliptic coordinates. The Hamilton–Jacobi equation for the Hamilton function of natural form

$$H_1=p_1^2+p_2^2+V(q_1,q_2) \tag{6}$$

admits an additive separation of variables in one of these coordinate systems if the potential takes one of the forms

$$\begin{aligned}
&1.\quad V(q_1,q_2)=U_1(x)+U_2(y), &&2.\quad V(q_1,q_2)=U_1(r)+\frac{U_2(\phi)}{r^2},\\
&3.\quad V(q_1,q_2)=\frac{U_1(u_1)+U_2(u_2)}{u_1+u_2}, &&4.\quad V(q_1,q_2)=\frac{U_1(\zeta_1)+U_2(\zeta_2)}{\zeta_1^2-\zeta_2^2}.
\end{aligned} \tag{7}$$

Here, x and y are the Cartesian coordinates, r and ϕ are the polar coordinates, u_1 and u_2 are the parabolic coordinates, and ζ_1 and ζ_2 are the elliptic coordinates on the plane (see [12]).

Hence, we have a complete description and complete classification of all integrable systems on the plane that admit a separation of variables in one of the orthogonal curvilinear coordinate systems. A Stäckel-type generalization of this classification of integrable systems to Riemann spaces of constant curvature can be found in [13]. Applying the auto-BT to these integrable systems with integrals of motion quadratic in momenta, we obtain a family of new canonical coordinates (v,p_v) from the classical curvilinear coordinates (u,p_u), which can be regarded as separation variables for the integrable systems with higher-degree integrals of motion. To classify the integrable systems thus obtained, we can use an elaborated classification of orthogonal systems of coordinates and the Stäckel-type integrable systems corresponding to them.

Hereafter, we restrict ourself to considering dynamical systems on the plane that admit a separation of variables in the parabolic coordinates because, on one hand, these systems are rather interesting and, on the other hand, the corresponding formulas are not so cumbersome as those for the elliptic coordinates, which can be found in [11]. We therefore consider the parabolic coordinates on the plane

$$\lambda-2q_2-\frac{q_1^2}{\lambda}=\frac{(\lambda-u_1)(\lambda-u_2)}{\lambda}, \tag{8}$$

for which separated equations (5) are

$$p_{u_i}^2 + U_i(u_i) = H_1 + \frac{H_2}{u_i}, \quad i = 1, 2. \tag{9}$$

Adding Eqs. (9), we obtain another integrable system with the Hamiltonian

$$\widetilde{H}_1 = \frac{1}{2}(p_{u_1}^2 + U_1(u_1) + p_{u_2}^2 + U_2(u_2)) = H_1 + \frac{H_2}{2}\left(\frac{1}{u_1} + \frac{1}{u_2}\right), \tag{10}$$

which can be regarded as an integrable perturbation of the original Hamiltonian H_1 given by (6). Subtracting the second equation from the first, we obtain the second integral of motion

$$\widetilde{H}_2 = (p_{u_1}^2 + U_1(u_1) - p_{u_2}^2 - U_2(u_2)) = H_2\left(\frac{1}{u_1} - \frac{1}{u_2}\right).$$

The integrals of motion thus obtained have no physical meaning.

To give a physical meaning to these integrals, we can use the auto-BTs of the original dynamical system

$$(u_i, p_{u_i}) \to (v_i, p_{v_i}),$$

which preserve the original integrals of motion H and H_2 and change the second integrals of motion:

$$\widetilde{H}_1 = H_1 + \frac{H_2}{2}\left(\frac{1}{v_1} + \frac{1}{v_2}\right), \qquad \widetilde{H}_2 = H_2\left(\frac{1}{v_1} - \frac{1}{v_2}\right). \tag{11}$$

If there is a particular transformation in the variety of auto-BTs such that the Hamiltonian $\widetilde{H}$ after its application becomes a meaningful function (from the physical standpoint) of the original physical variables (q, p), then this transformation can be called an analogue of the general BT relating different Hamilton–Jacobi equations.

2.1. Hénon–Heiles systems. There are three integrable Hénon–Heiles systems on the plane with a third-degree potential. These systems can be obtained using finite-dimensional reductions of the following partial differential equations: the fifth-order Korteweg–de Vries (KdV) equation, the Kaup–Kupershmidt equation, and the Sawada–Kotera equation [14]. The explicit integration of the equations of motion in all three cases was discussed in [15].

The first Hamilton function, related to the KdV equation,

$$H_1 = p_1^2 + p_2^2 - 16aq_2(q_1^2 + 2q_2^2), \quad a \in \mathbb{R}, \tag{12}$$

generates a Hamilton–Jacobi equation admitting a separation of variables in parabolic coordinates on the plane.

The second Hamilton function, related to the Kaup–Kupershmidt equation, is

$$\widetilde{H}_1 = p_1^2 + p_2^2 - 2aq_2(3q_1^2 + 16q_2^2). \tag{13}$$

Because the Kaup–Kupershmidt equation is gauge equivalent to the Sawada–Kotera equation, there is a canonical transformation

$$(q_{1,2}, p_{1,2}) \to (P_{1,2}, Q_{1,2})$$

that identifies $\widetilde{H}_1$ given by (13) with the Hamiltonian

$$\widehat{H}_1 = P_1^2 + P_2^2 - 2aQ_2(3Q_1^2 + Q_2^2), \tag{14}$$

which admits a separation of variables in the Cartesian coordinates [16]. Different properties of this transformation and its generalizations were discussed in [17], [18].

Below, we present a previously unknown relation between Hénon–Heiles systems (12) and (13), which can be considered an analogue of the general BT. We could have constructed the sought analogue of the general BT using the finite-dimensional reduction method [14] if we had known the BT (hetero-BT) relating the fifth-order KdV equation to either the Kaup–Kupershmidt or Sawada–Kotera equation, but we do not know such a transformation.

Because we believe that all information about the integrable system, including its relation to other systems, must be contained in the "correct" Lax representation, we need only learn how to obtain this information. Therefore, we below obtain all relations among the Hénon–Heiles systems directly from the 2×2 Lax matrix for Hénon–Heiles system (12) without using any additional information.

2.2. Bäcklund transformations. Using the orthogonal coordinate systems on the plane, we can construct a continuum of 2×2 Lax matrices [19], [20] corresponding to integrable systems with the potentials $V(q_1, q_2)$ given by (7). The appropriate Lax matrix for the Hénon–Heiles system has the form

$$L(\lambda) = \begin{pmatrix} \dfrac{p_2}{2} + \dfrac{p_1 q_1}{2\lambda} & \lambda - 2q_2 - \dfrac{q_1^2}{\lambda} \\ a\lambda^2 + 2aq_2\lambda + a(q_1^2 + 4q_2^2) + \dfrac{p_1^2}{4\lambda} & -\dfrac{p_2}{2} - \dfrac{p_1 q_1}{2\lambda} \end{pmatrix}, \quad a \in \mathbb{R}. \tag{15}$$

The spectral curve of this Lax matrix is given by the equation

$$C\colon \mu^2 - a\lambda^3 - \frac{H_1}{4} + \frac{H_2}{\lambda} = 0, \tag{16}$$

including the integrals of motion H_1 given by (12) and

$$H_2 = aq_1^2(q_1^2 + 4q_2^2) + \frac{p_1(q_2 p_1 - q_1 p_2)}{2}.$$

Auto-BTs preserve the form of integrals of motion [7] defined by the characteristic polynomial of the Lax matrix, and each auto-BT can therefore be associated with the similarity transformation

$$\widehat{L} = VLV^{-1}$$

for the Lax matrix [8]. Because there are infinitely many auto-BTs for any dynamical system, there are also infinitely many similarity transformations associated with them [8], [5].

We consider the special similarity transformations given by the matrix

$$V = \begin{pmatrix} L_{12} & 0 \\ 4(L_{11} - \widehat{L}_{11}(\lambda)) & 4L_{12} \end{pmatrix}, \tag{17}$$

where L_{ij} are the elements of the original Lax matrix (15) and

$$\widehat{L}_{11}(\lambda) = \frac{p_2}{2} + \frac{q_1 p_1(\lambda - 2q_2)}{2q_1^2}.$$

After the transformation, we obtain a unique matrix $\widehat{L}(\lambda) = VLV^{-1}$ satisfying the following three conditions:

1. Zeros of the upper off-diagonal element

$$\widehat{L}_{12}(\lambda) = \frac{(\lambda - u_1)(\lambda - u_2)}{4\lambda}$$

are the original parabolic coordinates (8) on the plane.

2. The lower off-diagonal element

$$\widehat{L}_{21} = 4a(\lambda - v_1)(\lambda - v_2) = \\ = 4a\lambda^2 + \frac{(8aq_1^2q_2 - p_1^2)\lambda}{q_1^2} + 4a(q_1^2 + 4q_2^2) + \frac{2p_1(p_1q_2 - p_2q_1)}{q_1^2} \tag{18}$$

also has only two zeros, which are the functionally independent functions $v_{1,2}(q,p)$ in involution with respect to the canonical Poisson brackets.

3. The conjugate momenta for the coordinates $u_{1,2}$ and for the coordinates $v_{1,2}$ are the values of the diagonal element of the Lax matrix:

$$p_{u_i} = \widehat{L}_{11}(\lambda)\big|_{\lambda=u_i}, \qquad p_{v_i} = \widehat{L}_{11}(\lambda)\big|_{\lambda=v_i}, \quad i = 1,2.$$

Because

$$\det(L(\lambda) - \mu) = \det(\widehat{L}(\lambda) - \mu) = \mu^2 - \widehat{L}_{11}^2(\lambda) - \widehat{L}_{12}(\lambda)\widehat{L}_{21}(\lambda),$$

we readily obtain separated equations of form (9) by substituting $\lambda = u_{1,2}$ and $\lambda = v_{1,2}$ in these equations.

The two families of the canonical variables (u, p_u) and (v, p_v) are related by the canonical transformation that is a finite-dimensional analogue of the auto-BT for the nonlinear evolution equations.

Proposition 1. *The auto-BT for Hénon–Heiles system* (12) *consists of the canonical transformation of parabolic coordinates on the plane and their conjugate momenta*

$$(u_1, u_2, p_{u_1}, p_{u_2}) \to (v_1, v_2, p_{v_1}, p_{v_2})$$

and the relations

$$\Phi(\lambda,\mu) = \mu^2 - a\lambda^3 = \frac{H_1}{4} - \frac{H_2}{\lambda}, \qquad \lambda = u_{1,2}, v_{1,2}, \qquad \mu = p_{u_{1,2}}, p_{v_{1,2}}, \tag{19}$$

which allow constructing the Hamilton–Jacobi equations

$$H_{1,2}\left(\lambda, \frac{\partial S}{\partial \lambda}\right) = \alpha_{1,2}, \quad \lambda = u, v,$$

which have the same form in both the (u, p_u) and the (v, p_v) variables.

From the geometric standpoint, auto-BTs describe a change of coordinates (shift) on the Jacobian of the hyperelliptic curve (spectral curve of Lax matrix [8]).

Supplementing this autotransformation with the additional rule to seek other Hamilton–Jacobi equations, we can obtain an analogue of the general BT. For instance, substituting the v variables in definition (11), we obtain

$$\widetilde{H}_1 = H_1 - \frac{d\log(\lambda - v_1)(\lambda - v_2)}{d\lambda}\bigg|_{\lambda=0} = p_1^2 + \frac{p_2^2}{2} - 2aq_2(6q_1^2 + 16q_2^2). \tag{20}$$

This Hamiltonian coincides with Hamiltonian (13) for the second Hénon–Heiles system up to the canonical transformation

$$p_1 \to \sqrt{2}p_1, \qquad q_1 \to \frac{q_1}{\sqrt{2}}.$$

On the other hand, the canonical transformation

$$(q,p) \to (Q,P), \qquad P_{1,2} = \frac{p_{v_1} \pm p_{v_2}}{\sqrt{2}}, \qquad Q_{1,2} = \frac{v_1 \pm v_2}{\sqrt{2}}, \tag{21}$$

transforms this Hamiltonian into the Hamiltonian of the third Hénon–Heiles system

$$\widehat{H}_1 = P_1^2 + P_2^2 - 2aQ_2(3Q_1^2 + Q_2^2).$$

According to [14], [16], canonical transformation (21) is a finite-dimensional analogue of the gauge equivalence of the Kaup–Kupershmidt and Sawada–Kotera equations.

Hence, the Hamilton–Jacobi equations for all three integrable Hénon–Heiles systems (12)–(14) on the plane admit a simultaneous separation of variables in terms of the v variables. We can regard this as the existence of a finite-dimensional analogue of the general BT relating different systems of Hamilton–Jacobi equations.

Proposition 2. *An analogue of the general BT for Hénon–Heiles systems* (12)–(14) *consists of the same canonical transformation of the parabolic coordinates on the plane and their conjugate momenta*

$$(u_1, u_2, p_{u_1}, p_{u_2}) \to (v_1, v_2, p_{v_1}, p_{v_2})$$

relations $\Phi(\lambda,\mu)$ *given by* (19) *and the two additional relations*

$$\widetilde{H}_{1,2} = \Phi(v_1, p_{v_1}) \pm \Phi(v_2, p_{v_2}),$$

which allow constructing two different systems of Hamilton–Jacobi equations

$$H_{1,2}\left(u, \frac{\partial S}{\partial u}\right) = \alpha_{1,2}, \qquad \widetilde{H}_{1,2}\left(v, \frac{\partial \widetilde{S}}{\partial v}\right) = \widetilde{\alpha}_{1,2},$$

whose complete integrals can be additively separated in the v *variables.*

This analogue of the general BT relates two different common level surfaces of the integrals of motion $H_{1,2}$ and $\widetilde{H}_{1,2}$. Similar relations between different Abelian manifolds also appear in the theory of superintegrable systems.

2.3. The integrable system with the potential depending linearly on the velocity. If we assume that the functions $U_{1,2}(u)$ in definition (7) are the same Nth-degree homogeneous polynomials, then we can write explicit expressions for the Hamiltonian admitting a separation in parabolic coordinates on the plane [21]:

$$H_1 = p_1^2 + p_2^2 + V_N(q_1, q_2), \qquad V_N = 4a \sum_{k=0}^{[N/2]} 2^{1-2k} \begin{pmatrix} N-k \\ k \end{pmatrix} q_1^{2k} q_2^{N-2k}. \tag{22}$$

If $N = 3$, then we obtain the Hamiltonian for Hénon–Heiles system (12), and if $N = 4$, then we obtain the Hamiltonian for the so-called system (1:12:16),

$$H_1 = p_1^2 + p_2^2 - 4a(q_1^4 + 12q_1^2q_2^2 + 16q_2^4). \tag{23}$$

The corresponding Lax matrix

$$L(\lambda)=\begin{pmatrix} \dfrac{p_2}{2}+\dfrac{p_1q_1}{2\lambda} & \lambda-2q_2-\dfrac{q_1^2}{\lambda} \\ a\lambda^3+2aq_2\lambda^2+a(q_1^2+4q_2^2)\lambda+4aq_2(q_1^2+2q_2^2)+\dfrac{p_1^2}{4\lambda} & -\dfrac{p_2}{2}-\dfrac{p_1q_1}{2\lambda} \end{pmatrix} \tag{24}$$

after the similarity transformation with the matrix V given by (17), where

$$\widehat{L}_{11}(\lambda)=\sqrt{a}\,\lambda^2-\frac{4\sqrt{a}q_2q_1-p_1}{2q_1}\lambda-\frac{2\sqrt{a}q_1^3+2p_1q_2-p_2q_1}{2q_1},$$

transforms into the matrix $\widehat{L}$ whose two off-diagonal elements $\widehat{L}_{12}(\lambda)$ and $\widehat{L}_{21}(\lambda)$ generate two families of separation variables. Again, the momenta associated with them are generated by the values of the diagonal element of $\widehat{L}$.

Similar to the Hénon–Heiles system, the first family is represented by the parabolic coordinates on the plane $u_{1,2}$, and the second family consists of the coordinates $v_{1,2}$ that are zeros of the polynomial

$$\widehat{L}_{21}=\frac{4(4aq_1q_2-\sqrt{a}p_1)}{q_1}\lambda^2+\frac{8aq_1^2(q_1^2+2q_2^2)+4\sqrt{a}q_1(2p_1q_2-p_2q_1)-p_1^2}{q_1^2}\lambda+$$
$$+\frac{16aq_1^2q_2(q_1^2+2q_2^2)+2p_1(p_1q_2-p_2q_1)}{q_1^2}=\frac{4(4aq_1q_2-\sqrt{a}p_1)}{q_1}(\lambda-v_1)(\lambda-v_2).$$

The canonical transformation between these two families of separation variables is the auto-BT for the system (1:12:16). As before, we can regard this transformation as a general BT if we add rule (11) to define the second system of Hamilton–Jacobi equations.

In our case, the second dynamical system is given by integrable Hamiltonian (11) with the potential linearly depending on the velocities:

$$\widetilde{H}_1=\frac{p_1^2}{2}+p_2^2+4\sqrt{a}p_1q_1q_2-2\sqrt{a}p_2q_1^2-8aq_2^2(5q_1^2+8q_2^2).$$

Using the canonical transformations, we can rewrite this Hamiltonian in the more symmetric form

$$\widetilde{H}_1=p_1^2+p_2^2-3\sqrt{a}p_2q_1^2+2a(q_1^4-12q_1^2q_2^2-32q_2^4). \tag{25}$$

The corresponding second integral

$$\widetilde{H}_2^2=p_1^4+4q_1^4(q_1^4-8q_1^2q_2^2-112q_2^4)a^2+4q_1^3(64p_1q_2^3-p_2q_1^3-12p_2q_1q_2^2)a^{3/2}+$$
$$+q_1^2(4p_1^2q_1^2-48p_1^2q_2^2+32p_1p_2q_1q_2+p_2^2q_1^2)a-6a^{1/2}p_1^2p_2q_1^2$$

is a fourth-degree polynomial in momenta. The physical meaning of these integrals was discussed, for example, in [22], where a detailed list of necessary references can also be found.

We note that using an analogue of the general BT, we have not only found the expressions for new integrals of motion but also constructed the separation variables and separated equations. Moreover, using the quite nontrivial canonical transformations (21), we can prove that the integrable system with potential (25) linearly depending on velocities is equivalent to the system with the Hamiltonian

$$\widehat{H}_1=P_1^2+P_2^2-a(Q_1^4+6Q_1^2Q_2^2+Q_2^4),$$

similar to the relation between the second and the third Hénon–Heiles systems.

If $N\geq 5$ in relations (22), i.e., for fifth- and higher-order potentials, the genus of the spectral curve of the corresponding Lax matrix is larger than the number of degrees of freedom, and the analogous similarity transformations for the Lax matrices have not been found.

3. Integrable systems on the sphere

There are two systems of orthogonal coordinates on the sphere: the spherical and the elliptic (spheroconical) coordinates. We restrict ourself to considering dynamical systems admitting separation in elliptic coordinates and to their BTs.

If q_1, q_2, and q_3 are the Cartesian coordinates of the three-dimensional Euclidean space $\mathbb{R}^3$, where the two-dimensional sphere $\mathbb{S}^2$ is placed, then the elliptic coordinates u_1 and u_2 on the sphere can be determined standardly:

$$\frac{(\lambda-u_1)(\lambda-u_2)}{(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)}=\frac{q_1^2}{\lambda-a_1}+\frac{q_2^2}{\lambda-a_2}+\frac{q_3^2}{\lambda-a_3},\quad a_i\in\mathbb{R}, \tag{26}$$

where a_i are the parameters defining the coordinate domain,

$$a_1<u_1<a_2<u_2<a_3.$$

The conjugate momenta p_u are the values of the function

$$h(\lambda)=\frac{1}{2}\left(\frac{p_1q_1}{\lambda-a_1}+\frac{p_2q_2}{\lambda-a_2}+\frac{p_3q_3}{\lambda-a_3}\right) \tag{27}$$

at $\lambda=u_{1,2}$. As usual, constraints are imposed on the redundant coordinates $q_k\in\mathbb{R}^3$ and momenta $p_k\in T^*\mathbb{R}^3$ not included in these definitions,

$$q_1^2+q_2^2+q_3^2=1,\qquad p_1q_1+p_2q_2+p_3q_3=0,$$

defining the embedding of the sphere and its cotangent bundle into the respective Euclidean space and its cotangent bundle.

If the Hamilton–Jacobi equation defined by the Hamiltonian of the natural form

$$H=\sum g_{ij}(q)p_ip_j+V(q)$$

admits a separation of variables in the elliptic coordinates on the sphere, then relative separated equations (5) have the form

$$p_{u_i}^2+U_i(u_i)=\frac{K_1}{u_i-a_1}+\frac{K_2}{u_i-a_1}+\frac{K_3}{u_i-a_1},\quad i=1,2,$$

where K_j are three functionally dependent integrals of motion in involution. It is convenient to write these equations in the form

$$(u_i+a_1)(u_i+a_2)p_{u_i}^2+U_i(u_i)=H+\frac{H_2}{u_i-a_3},\quad i=1,2, \tag{28}$$

where H and H_2 are two functionally independent integrals of motion in involution.

Similar to the systems on the plane, we can add and subtract these separated equations to obtain a new integrable system. We then seek an auto-BT $(u,p_u)\to(v,p_v)$ that gives the Hamiltonians

$$\widetilde{H}=H+\frac{H_2}{2}\left(\frac{1}{v_1-a_3}+\frac{1}{v_2-a_3}\right),\qquad \widetilde{H}_2=H_2\left(\frac{1}{v_1-a_3}-\frac{1}{v_2-a_3}\right) \tag{29}$$

a physical meaning in the original variables (q,p). If such an auto-transformation exists, then we call it the analogue of the general BT relating different Hamilton–Jacobi equations.

3.1. The Neumann and Chaplygin systems. We take the Lax matrix for the Neumann system on the sphere

$$L(\lambda) = \begin{pmatrix} \dfrac{1}{2}\displaystyle\sum_{k=1}^{3}\frac{q_k p_k}{\lambda - a_k} & \displaystyle\sum_{k=1}^{3}\frac{q_k^2}{\lambda - a_k} \\ -\dfrac{1}{4}\left(1 + \displaystyle\sum_{i=1}^{3}\frac{p_k^2}{\lambda - a_k}\right) & -\dfrac{1}{2}\displaystyle\sum_{k=1}^{3}\frac{q_k p_k}{\lambda - a_k} \end{pmatrix}. \tag{30}$$

In the equation

$$C\colon\ 4(\lambda - a_1)(\lambda - a_2)\mu^2 + \lambda - H_1 - \frac{H_2}{\lambda - a_3} = 0$$

defining the spectral curve of the given matrix, we have a Hamiltonian function

$$-H_1 = J_1^2 + J_2^2 + J_3^2 + a_1 q_1^2 + a_2 q_2^2 + a_3 q_3^2, \tag{31}$$

where J_k are the components of the angular momentum vector $J = q \times p$.

According to [11], there exists a unique similarity transformation $\widehat{L} = VLV^{-1}$ with the matrix V given by (17) that allows simultaneously obtaining two families of separation variables from the off-diagonal elements of the matrix

$$\widehat{L} = \begin{pmatrix} \dfrac{(p_1q_3 - p_3q_1)q_1}{2q_3(\lambda - a_1)} + \dfrac{(p_2q_3 - p_3q_2)q_2}{2q_3(\lambda - a_2)} & \dfrac{1}{4}\left(\displaystyle\sum_{i=1}^{n}\frac{q_i^2}{\lambda - a_i}\right) \\ -1 - \dfrac{(p_1q_3 - p_3q_1)^2}{q_3^2(\lambda - a_1)} - \dfrac{(p_2q_3 - p_3q_2)^2}{q_3^2(\lambda - a_2)} & -\dfrac{(p_1q_3 - p_3q_1)q_1}{2q_3(\lambda - a_1)} - \dfrac{(p_2q_3 - p_3q_2)q_2}{2q_3(\lambda - a_2)} \end{pmatrix}.$$

The first family is represented by the elliptic coordinates on the sphere $u_{1,2}$, and the second family consists of the coordinates $v_{1,2}$ that are zeros of the second off-diagonal element $\widehat{L}_{21}$.

Proposition 3. *The auto-BT for Neumann system* (31) *consists of the canonical transformation of the elliptic coordinates on the sphere and their conjugate momenta*

$$(u_1, u_2, p_{u_1}, p_{u_2}) \to (v_1, v_2, p_{v_1}, p_{v_2})$$

and the relations

$$\Phi(\lambda, \mu) = 4(\lambda - a_1)(\lambda - a_2)\mu^2 + \lambda = H_1 + \frac{H_2}{\lambda - a_3}, \tag{32}$$

which allow constructing the Hamilton–Jacobi equations

$$H_{1,2}\left(\lambda, \frac{\partial S}{\partial \lambda}\right) = \alpha_{1,2}, \quad \lambda = u, v,$$

which have the same form in both the (u, p_u) *and the* (v, p_v) *variables.*

As before, supplementing this auto-BT with an additional rule to seek other Hamilton–Jacobi equations, we can obtain an analogue of the general BT. In fact, substituting the variables $v_{1,2}$ in the definition of $\widetilde{H}_{1,2}$ given by (29), we obtain the Hamilton function

$$\widetilde{H}_1 = J_1^2 + J_2^2 + 2J_3^2 - 2a_1 q_2^2 - 2a_2 q_1^2 - (a_1 + a_2) q_3^2, \tag{33}$$

which defines the known Chaplygin system [23], which describes the motion of a solid in an ideal incompressible liquid.

Proposition 4. *Neumann system (31) and Chaplygin system (33) are related by an analogue of the general BT consisting of the canonical transformations of the elliptic coordinates and their conjugate momenta*

$$(u_1, u_2, p_{u_1}, p_{u_2}) \to (v_1, v_2, p_{v_1}, p_{v_2}),$$

relations $\Phi(\lambda, \mu)$ given by (32), and two additional relations

$$\widetilde{H}_{1,2} = \Phi(v_1, p_{v_1}) \pm \Phi(v_2, p_{v_2}),$$

which allow constructing two different systems of Hamilton–Jacobi equations

$$H_{1,2}\left(u, \frac{\partial S}{\partial u}\right) = \alpha_{1,2}, \qquad \widetilde{H}_{1,2}\left(v, \frac{\partial \widetilde{S}}{\partial v}\right) = \tilde{\alpha}_{1,2},$$

whose complete integrals are additively separable in the v variables.

As with the Hénon–Heiles systems, this analogue of the general BT relates two different consistent level surfaces of the integrals of motion $H_{1,2}$ and $\widetilde{H}_{1,2}$.

3.2. Integrable system with a potential depending linearly on the velocity. For systems on the sphere, similarly to systems (22) on the plane, there is a general expression for the homogeneous polynomial potentials

$$V_N = \frac{P_N(u_1) - P_N(u_2)}{u_1 - u_2},$$

which can be rewritten in terms of the redundant variables q_i [24]. For $N = 2$, we obtain Neumann system (31), and for $N = 3$, we obtain a system with a fourth-degree potential. The homogeneous potentials can be added to each other and singular terms can be added to them.

We consider the Lax matrix for one of these systems:

$$L(\lambda) = \begin{pmatrix} \dfrac{1}{2}\displaystyle\sum_{k=1}^{3} \frac{q_k p_k}{\lambda - a_k} & \displaystyle\sum_{k=1}^{3} \frac{q_k^2}{\lambda - a_k} \\ \dfrac{1}{4}\left(a\lambda - a(a_1 q_1^2 + a_2 q_2^2 + a_3 q_3^2) - b - \displaystyle\sum_{i=1}^{3} \frac{p_k^2}{\lambda - a_k}\right) & -\dfrac{1}{2}\displaystyle\sum_{k=1}^{3} \frac{q_k p_k}{\lambda - a_k} \end{pmatrix},$$

constructed using standard rules [19], [20]. The equation for the spectral curve for this Lax matrix,

$$4(\lambda - a_1)(\lambda - a_2)\mu^2 - \lambda(a\lambda - a(a_1 + a_2) - b) = H_1 + \frac{H_2}{\lambda - a_3},$$

contains the Hamilton function

$$H_1 = J_1^2 + J_2^2 + J_3^2 - a(a_1 q_1^2 + a_2 q_2^2 + a_3 q_3^2)^2 + a(a_1^2 q_1^2 + a_2^2 q_2^2 + a_3^2 q_3^2) - b(a_1 q_1^2 + a_2 q_2^2 + a_3 q_3^2)$$

and the second integral of motion H_2, which we do not write explicitly for brevity.

As before, in this case, there is a unique similarity transformation with the matrix V given by (17), where

$$\widehat{L}_{11} = \frac{\sqrt{a}}{2} + \frac{q_1^2(a_1 - a_3)\sqrt{a} + (q_3 p_1 - q_1 p_3) q_1/q_3}{2(\lambda - a_1)} + \frac{q_2^2(a_2 - a_3)\sqrt{a} + (q_3 p_2 - q_2 p_3) q_2/q_3}{2(\lambda - a_2)},$$

that allows obtaining a Lax matrix whose off-diagonal elements generate two families of separation variables.

The first family is represented by the elliptic coordinates on the sphere $u_{1,2}$, and the second family consists of the coordinates $v_{1,2}$ that are zeros of the second off-diagonal element,

$$\widehat{L}_{21} = \frac{2\sqrt{a}p_3}{q_3} - 2a(a_1q_1^2 + a_2q_2^2 - a_3(q_1^2 + q_2^2)) - \frac{\delta_1}{\lambda - a_1} - \frac{\delta_2}{\lambda - a_2},$$

where

$$\delta_i = a(a_i - a_3)^2 q_i^2 + \frac{2\sqrt{a}q_i(q_3p_i - q_ip_3)(a_i - a_3)}{q_3} + \frac{(q_3p_i - q_ip_3)^2}{q_3^2}, \quad i = 1, 2.$$

Substituting the variables $v_{1,2}$ in the definition of $\widetilde{H}_{1,2}$ given by (29) and applying the canonical transformations

$$J_1 = J_1 + \sqrt{a}(a_2 - a_3)q_2q_3, \qquad J_2 = J_2 - \sqrt{a}(a_1 - a_3)q_1q_3, \qquad J_3 = J_3 + \sqrt{a}(a_1 - a_2)q_1q_2,$$

and

$$q_k = \frac{q_k}{\sqrt{a_1 - a_2}}, \quad k = 1, 2, 3,$$

we obtain the expressions for the Hamilton function and the second integral of motion:

$$\begin{aligned} \widetilde{H} &= J_1^2 + J_2^2 + 2J_3^2 - 2\sqrt{a}(q_2q_3J_1 + q_1q_3J_2 - 2q_1q_2J_3) + b(q_1^2 - q_2^2), \\ \widetilde{H}_2^2 &= (J_1^2 - J_2^2 - 2\sqrt{a}q_3(q_2J_1 + q_1J_2) + bq_3^2)^2 + 4J_1^2J_2^2. \end{aligned} \tag{34}$$

We again emphasize that we have simultaneously obtained integrals of motion, separation variables, and separated equations. Similarly to [11], we can construct different additional integrable perturbations of this system by properly varying the separated equations.

We recall that Sokolov [25], [26] found an analogous integrable perturbation for the Kovalevskaya Hamiltonian

$$\widetilde{H} = J_1^2 + J_2^2 + 2J_3^2 + a(q_1J_3 - q_3J_1) + bq_2 \tag{35}$$

with a potential depending on velocities. A similar integrable perturbation for the Goryachev–Chaplygin system

$$\widetilde{H} = J_1^2 + J_2^2 + 4J_3^2 + a(2q_1J_3 - q_3J_1) + bq_2 \tag{36}$$

was discussed in [27], [28]. Both a discussion of the physical meaning of such integrable systems with potentials depending on the velocities and appropriate references can be found in those papers.

4. Conclusion

We have discussed possible analogues of the general BT that relate different Hamilton–Jacobi equations. In essence, the idea of such transformations was formulated by Jacobi [29]: "*The main trouble when integrating given differential equations is to find convenient variables. No general rule exists for finding these variables. We must therefore go in the opposite direction, and finding a remarkable substitution, we must look for the problems in which it can be successfully exploited.*" That is, substituting the separation variables (however found) in arbitrary separable equations, we obtain a family of integrable Hamiltonians that simultaneously admit an additive separation in a complete integral of the appropriate Hamilton–Jacobi equations. But the theory of simultaneously separable systems thus obtained is too general because the integrable Hamiltonians obtained in most cases do not have any physical meaning.

We suggest restricting the general Jacobi theory a little and obtaining the new separated equations from old ones (9) by simultaneously applying the coordinate shift on the Jacobian of the appropriate algebraic curve. The geometric meaning of such addition and subtraction of separated equations defining the Jacobian of the hyperelliptic genus-g=n=2 curve is not yet clear. But this construction of the new separated equations obviously allows generalizing to the n-dimensional case.

Another part of the imposed constraints is using the auto-BTs (the coordinate shift on the Jacobian) to give a physical meaning to the perturbed Hamiltonian $\widetilde{H}$. If we regard the v variables as the original u variables at the next instant of some new time variable $\tilde{t}$ when discretizing, then we propose regarding the v variables as the separation variables for some new integrable system with the Hamiltonian $\widetilde{H}$ in our case. That is, we propose using the BTs as a source of new coordinate systems constructed from the classical orthogonal coordinate systems. This part of the construction, much more complicated for the n-dimensional generalization, also requires additional study.

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REFERENCES

1. Y. C. Li and A. Yurov, *Lie–Bäcklund–Darboux Transformations* (Surv. Mod. Math., Vol. 8), International Press, Somerville, Mass. (2014).
2. C. Rogers and W. K. Schief, *Bäcklund and Darboux Transformations*: *Geometry and Modern Applications in Soliton Theory* (Cambridge Texts Appl. Math., Vol. 30), Cambridge Univ. Press, Cambridge (2002).
3. A. I. Bobenko and Yu. B. Suris, *Discrete Differential Geometry*: *Integrable Structure* [in Russian], RKhD, Izhevsk (2010); English transl. prev. ed. (Grad. Stud. Math., Vol. 98), Amer. Math. Soc., Providence, R. I. (2008).
4. F. W. Nijhoff, *Discrete Systems and Integrability*, Univ. of Leeds, Leeds, UK (2006).
5. Yu. B. Suris, *The Problem of Integrable Discretization*: *Hamiltonian Approach* (Progr. Math., Vol. 219), Springer, Berlin (2003).
6. M. J. Ablowitz and H. Segur, *Solitons and the Inverse Scattering Transform* (SIAM Stud. Appl. Math., Vol. 4), SIAM, Philadelphia (1981).
7. S. Wojciechowski, *J. Phys. A*: *Math. Gen.*, **15**, L653–L657 (1982).
8. V. Kuznetsov and P. Vanhaecke, *J. Geom. Phys.*, **44**, 1–40 (2002).
9. F. Zullo, *J. Phys. A*: *Math. Theor.*, **46**, 145203 (2013); arXiv:1207.0387v2 [nlin.SI] (2012).
10. W. Miller, S. Post, and P. Winternitz, *J. Phys. A*: *Math. Theor.*, **46**, 423001 (2013).
11. A. V. Tsiganov, *Reg. Chaotic Dyn.*, **20**, 74–93 (2015).
12. L. D.Landau and E. M. Lifshitz, *Course of Theoretical Physics* [in Russian], Vol. 1, *Mechanics*, Fizmatlit, Moscow (2004); English transl. prev. ed., Pergamon, London (1980).
13. E. G. Kalnins, *Separation of Variables for Riemann Spaces of Constant Curvature* (Pitman Monogr. Surv. Pure Appl. Math., Vol. 28), Wiley, New York (1986).
14. A. P. Fordy, *Phys. D*, **52**, 204–210 (1991).
15. R. Conte, M. Musette, and C. Verhoeven, *J. Nonlinear Math. Phys.*, **12**, Suppl. 1, 212–227 (2005); arXiv: nlin/0412057v1 (2004).
16. V. Ravoson, L. Gavrilov, and R. Caboz, *J. Math. Phys.*, **34**, 2385–2393 (1993).
17. M. Salerno, V. Z. Enol'skii, and D. V. Leykin, *Phys. Rev. E*, **49**, 5897–5899 (1994).
18. S. Rauch-Wojciechowski, and A. V. Tsiganov, *J. Phys. A*: *Math. Gen.*, **29**, 7769–7778 (1996).
19. A. V. Tsiganov, *J. Math. Phys.*, **40**, 279–298 (1999).
20. A. V. Tsiganov, *Theor. Math. Phys.*, **139**, 636–653 (2004).
21. J. Hietarinta, *Phys. Rep.*, **147**, 87–154 (1987).
22. G. Pucacco, *Celestial Mech. Dynam. Astronomy*, **90**, 109–123 (2004).

23. S. A. Chaplygin, “New special solution of the problem of the motion of a solid body in liquid,” in: *Trudy otdeleniya fizich. nauk Obshch. lyubit. estestvozhaniya*, Vol. 11, N. N. Sharapov, Moscow (1902), pp. 10–19.
24. O. I. Bogoyavlenskii, *Math. USSR-Izv.*, **27**, 203–218 (1986).
25. V. V. Sokolov, *Theor. Math. Phys.*, **129**, 1335–1340 (2001).
26. V. V. Sokolov, “Generalized Kowalevski top: New integrable cases on $e(3)$ and $so(4)$,” in: *The Kowalevski Property* (CRM Proc. Lect. Notes, Vol. 32, V. B. Kuznetsov, ed.), Amer. Math. Soc., Providence, R. I. (2002), pp. 307–313.
27. V. V. Sokolov and A. V. Tsiganov, *Theor. Math. Phys.*, **131**, 543–549 (2002).
28. V. V. Sokolov and A. V. Tsiganov, *Theor. Math. Phys.*, **133**, 1730–1743 (2002).
29. C. G. J. Jacobi, *Vorlesungen über Dynamik*, G. Reimer, Berlin (1866).