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LOCAL ALGEBRAIC ANALYSIS OF DIFFERENTIAL SYSTEMS

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We propose a new approach for studying the compatibility of partial differential equations. This approach is a synthesis of the Riquier method, Gröbner basis theory, and elements of algebraic geometry. As applications, we consider systems including the wave equation and the sine-Gordon equation.

Keywords: compatibility of differential equations, reduction, Gröbner basis

1. Introduction

The problem of the compatibility of partial differential equations began to be extensively studied on the cusp of the twentieth century in papers by König, Riquier, Janet, Cartan, and others. The papers from that time were reviewed in [1], [2]. After a quiet period, interest in this problem was revived by arising applications and the appearance of new problems. At this stage, the language of fibration, homology, and commutative algebra came into use in the theory of differential equations [3]–[6]. The central focus of studies recently shifted to the algorithmic and computation problems [7]–[9]. This is explained by successes in developing computer algebra. Most progress was achieved in studying the compatibility of systems of algebraic equations thanks to the Gröbner basis theory [10], [11]. Overdetermined systems arise in applications from studying the symmetries and conservation laws of equations in mathematical physics [12], [13]. Differential relations have been successfully used to construct solutions of equations of gas dynamics [14] and quantum mechanics [15]. The inverse scattering method [16] uses the compatibility conditions of linear systems of partial differential equations.

Here, we propose an algebraic-analytic approach for studying some local properties of partial differential equations. In this approach, we use ideas from algebraic geometry and the Gröbner basis. The algebraic structure in which the construction is done consists of a local differential ring, relations in the ring, and a semigroup acting on the ring.

In Sec. 2, we introduce an infinite-dimensional space, an analogue of the space of infinite jets. Each point a in this space is associated with a local differential algebra $\mathcal{F}(a)$ of convergent series. Each subset $S \subset \mathcal{F}(a)$ generates a differential ideal, and the passive sets of finitely generated differential ideals of $\mathcal{F}(a)$ become the basic object of study. A passive set is similar to the Gröbner basis of ideals of a polynomial ring, but our definition of passivity does not use the order on the ring. At the end of Sec. 2, we obtain a necessary condition for a set to be passive.

In Sec. 3, we prove statements analogous to those known in the theory of Gröbner bases. In contrast to the standard approach applied to the polynomial algebra, we use a partition of the algebra $\mathcal{F}(a)$, the semigroup action compatible with this partition, and the Weierstrass division theorem for power series. We define the reduction of a series with respect to some subsets of $\mathcal{F}(a)$, the normal form of a series, the τ-series (analogue of the S-polynomial), and the canonical set of a differential ideal. We describe a way

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to reduce a convergent differential series to the normal form. We obtain a convenient passivity criterion in terms of the τ-series. We prove that if a differential ideal has a passive set, then there exists a unique canonical set generating this ideal.

Finally, in Sec. 4, we describe the scheme for constructing a passive set for a given differential ideal. This scheme relies on a certain refinement, proved here, of the implicit function theorem. Although this scheme is not universal, it can be useful in studying the compatibility of concrete systems of partial differential equations. We consider two examples as applications of this scheme. In the first example, we study the compatibility of a system consisting of the sine-Gordon equation and a third-order equation defined by a higher symmetry. In the second example, we construct passive sets associated with the three-dimensional sound wave equation in a nonhomogeneous medium. We note how the passive sets can be used to construct solutions of the equations.

2. Passive sets

Let $\mathbb{N}$ denote the set of nonnegative integers, and $\mathbb{N}_m = \{1, \dots, m\}$. Then $\mathbb{N}^n$ is a semigroup generated by the generators

$$e_1 = (1, 0, \dots, 0), \quad \dots, \quad e_n = (0, \dots, 0, 1).$$

Let $\mathbb{K}$ be a field complete with respect to some norm. We let $\mathbb{K}^\infty$ denote the set of families $\{a^i_\alpha\}^{i\in\mathbb{N}_m}_{\alpha\in\mathbb{N}^n}$, $a^i_\alpha \in \mathbb{K}$. The operation of adding families

$$\{a^i_\alpha\} + \{b^i_\alpha\} = \{a^i_\alpha + b^i_\alpha\}$$

and of multiplying by the field elements

$$c\{a^i_\alpha\} = \{ca^i_\alpha\}$$

define the structure of a vector space on $\mathbb{K}^\infty$. The space $\mathbb{K}^n \times \mathbb{K}^\infty$ is denoted by $\mathbb{K}^{n+\infty}$. Such spaces are often called jet spaces or prolongation spaces.

We associate each point $a \in \mathbb{K}^{n+\infty}$ with some $\mathbb{K}$-algebra of convergent series as follows. If L is a finite-dimensional coordinate subspace in $\mathbb{K}^{n+\infty}$, then $\mathbb{K}^{n+\infty}$ is representable as a direct sum $L \oplus M$, where M is the direct complement of L. We let π_L denote the projector from $\mathbb{K}^{n+\infty}$ to L along M. The space L is mapped into $\mathbb{K}^s$, $s = \dim L$, by the coordinate isomorphism ϕ. We associate a point $a \in \mathbb{K}^{n+\infty}$ with a point $a_L = \phi(\pi_L(a)) \in \mathbb{K}^s$ and let $\mathcal{F}(a_L)$ denote the set of convergent power series depending on a finite number of variables with the center at a_L.

If the subspace L is contained in a subspace L', then we consider that the ring $\mathcal{F}(a_L)$ is canonically embedded in $\mathcal{F}(a_{L'})$. The union

$$\mathcal{F}(a) = \bigcup_{L\in\mathbb{K}^{n+\infty}} \mathcal{F}(a_L) \tag{2.1}$$

is then a $\mathbb{K}$-algebra corresponding to the point a. The Cartesian coordinates on $\mathbb{K}^{n+\infty}$ are denoted by $x_1, \dots, x_n, \dots, u^i_\alpha, \dots$, and the whole set of Cartesian coordinates is denoted by Y. It is split into two subsets

$$X = \{x_1, \dots, x_n\}, \qquad U = \{u^i_\alpha\colon\ i \in \mathbb{N}_m,\ \alpha \in \mathbb{N}^n\}. \tag{2.2}$$

According to [17], the algebra $\mathcal{F}(a)$ is local with the maximal ideal

$$\mathfrak{M}(a) = \{f \in \mathcal{F}(a)\colon\ f(a) = 0\}. \tag{2.3}$$

The n differentiation operators

$$D_i f = \frac{\partial f}{\partial x_i} + \sum_{\substack{j\in\mathbb{N}_m,\\ \alpha\in\mathbb{N}^n}} \frac{\partial f}{\partial u^j_\alpha} u^j_{\alpha+e_i}$$

act on $\mathcal{F}(a)$. Consequently, $\mathcal{F}(a)$ is a differential algebra. The elements of $\mathcal{F}(a)$ are called *convergent differential series*.

It is useful to consider the disjoint union of algebras (the "bunch" of local algebras over $\mathbb{K}^{n+\infty}$)

$$\mathcal{F} = \bigcup_{a \in \mathbb{K}^{n+\infty}} \mathcal{F}(a). \tag{2.4}$$

Definition 1. An ordered triple $(\mathbb{K}, Y, S)$ of sets, where S is a subset in $\mathcal{F}$, is called a *local analytic differential system*.

Local analytic differential systems can be regarded as an algebraic formalization of partial differential equations.

We let $\langle\langle S \rangle\rangle$ denote the differential ideal in $\mathcal{F}(a)$ generated by the set S, and D^α denote the product $D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ of operators. Let $\mathrm{iv}(f)$ be the set of variables on which the series $f \in \mathcal{F}$ depends. If Y' is a nonempty subset of Y, then

$$\mathbb{K}[\langle Y' \rangle]_a = \{f \in \mathcal{F}(a)\colon\ \mathrm{iv}(f) \subseteq Y'\} \tag{2.5}$$

is a subalgebra of $\mathcal{F}(a)$.

Definition 2. We call the set

$$O(S) = \{D^\alpha s\colon\ \alpha \in \mathbb{N}^n,\ s \in S\}$$

the *orbit of the set* $S \subset \mathcal{F}$.

In analogy with both algebraic geometry and the theory of local analytic algebras [18], we introduce the following concepts.

Definition 3. Let $S \subset \mathcal{F}(a)$, where a is a point in $\mathbb{K}^{n+\infty}$ such that the orbit of the set S is contained in the maximal ideal $\mathfrak{M}(a)$ given by (2.3). Then the quotient algebra

$$\mathcal{O}_S(a) = \mathcal{F}(a)/\langle\langle S \rangle\rangle$$

is called the *local algebra* of a. If $\mathcal{O}_S(a)$ is isomorphic to some algebra $\mathbb{K}[\langle Y' \rangle]_a$, then a is called a *nonsingular point* for S.

Further, we obtain conditions for a point to be nonsingular. In the case of local analytic algebras, such conditions follow from the Jacobi implicit function theorem [18].

Definition 4. The set $\mathcal{B} \subset \mathcal{F}(a)$ is said to be *normalized* if the following conditions are satisfied:

1. Any element $f \in \mathcal{B}$ has the form
$$f = u^i_\alpha + g, \tag{2.6}$$
in which case the variables u^i_α form a set $\mathcal{L}$ in U and $\mathrm{iv}(g) \in Y \setminus \mathcal{L}$.

2. If $f_1 = u^i_\alpha + g_1$ and $f_2 = u^i_\alpha + g_2 \in \mathcal{B}$, then $g_1 = g_2$.

In this case, $\mathcal{L}$ is called the set of *pivotal variables*, and $\mathcal{P} = Y \setminus \mathcal{L}$ is called the set of *parametric variables* of the system $\mathcal{B}$.

Remark 1. The notions of pivotal and parametric derivatives of orthonormal systems are used in the Janet–Riquier theory [19].

Definition 5. The generator system $\mathcal{B}$ of the ideal $\mathcal{F}(a)$ is called the *normalized system* if $\mathcal{B}$ is a normalized set in $\mathcal{F}(a)$.

Everywhere below, we use the notion of a system of ideal generators in the standard sense [17], although the considered ideals are often differentiable.

Remark 2. The normalized generator system for a given differential ideal can be nonunique. Sufficient conditions for the existence of a normalized generator system were obtained in [20].

Statement 1. *A point $a \in \mathbb{K}^{n+\infty}$ is nonsingular for $S \subset \mathcal{F}(a)$ if the ideal $\langle\langle S \rangle\rangle$ has a normalized generator system $\mathcal{B}$ and $\langle\langle S \rangle\rangle \neq \mathcal{F}(a)$. In this case, the local algebra $\mathcal{O}_S(a)$ of a is isomorphic to the algebra $\mathbb{K}[\langle \mathcal{P} \rangle]_a$, where $\mathcal{P}$ is the set of parametric variables of $\mathcal{B}$.*

Proof. We take an arbitrary series $f \in \mathcal{F}(a)$. If f is independent of the pivotal variables of the normalized generator system $\mathcal{B}$ of the ideal $\langle\langle S \rangle\rangle$, then $f \in \mathbb{K}[\langle \mathcal{P} \rangle]_a$.

We suppose that f depends on the pivotal variables $y_1, \ldots, y_p$ of the generator system $\mathcal{B}$. Then there are series $f_1, \ldots, f_p \in \mathcal{B}$ such that

$$f_i = y_i + h_i, \quad i = 1, \ldots, p, \quad h_i \in \mathbb{K}[\langle \mathcal{P} \rangle]_a.$$

Using the Weierstrass division theorem [18], [20], [21], we can represent the series f as

$$f = \sum_{i=1}^{p} q_i f_i + r, \quad r, q_i \in \mathcal{F}(a),$$

where r is independent of the pivotal variables in $\mathcal{L}$ and is uniquely defined. Consequently, r belongs to $\mathbb{K}[\langle \mathcal{P} \rangle]_a$.

Definition 6. A convergent differential series $f \in \mathcal{F}$ of form (2.6) is said to be *solvable* for u^i_α if g is independent of the elements of the orbit $O(u^i_\alpha)$.

Definition 7. A set $S \subset \mathcal{F}(a)$ is said to be *passive* (at the point a) if

1. there exists a normalized generator system $\mathcal{B}$ of the differential ideal $\langle\langle S \rangle\rangle$ and
2. each series $f \in S$ is solvable for some element u^i_α, and the union of orbits of u^i_α moreover coincides with the set of pivotal variables of $\mathcal{B}$.

In this case, we let $O(\operatorname{st} S)$ denote the union of orbits of u^i_α and $\operatorname{st} f$ denote the element for which the series $f \in S$ is solvable.

Below, we give a necessary condition for a set to be passive.

Lemma 1. *If S is a passive set, then any element of the ideal $\langle\langle S \rangle\rangle$ depends on at least one element of the orbit $O(\operatorname{st} S)$.*

Proof. We suppose that there is a nonzero set $h \in \langle\langle S \rangle\rangle$ independent of the elements of the orbit $O(\operatorname{st} S)$. Because S is a passive set and the set h belongs to the ideal generated by the normalized generator system $\mathcal{B}$, we have

$$h = \sum_{j=1}^{p} a_j b_j, \tag{2.7}$$

where $a_j \in \mathcal{F}(a)$ and $b_j \in \mathcal{B}$. Because $b_1, \ldots, b_p$ belongs to the normalized set $\mathcal{B}$, the Jacobi matrix of these sets in the variables $\operatorname{st} b_1, \ldots, \operatorname{st} b_p$ is the unit matrix.

We now let $y_1, \ldots, y_r$ denote the variables on which the series $a_1, \ldots, a_p, b_1, \ldots, b_p$ depend, assuming that $y_1 = \operatorname{st} b_1, \ldots, y_p = \operatorname{st} b_p$ in this case. We perform the change of variables

$$z_1 = b_1, \quad \ldots, \quad z_p = b_p.$$

According to the implicit function theorem, the last relations can be solved for $y_1, \ldots, y_p$. Substituting the found $y_1, \ldots, y_p$ in (2.7), we obtain the representation

$$h = \sum_{j=1}^{p} \tilde{a}_j z_j,$$

where h can depend on only $y_{p+1}, \ldots, y_r$. Because the right-hand side of the last relation depends on at least one of the variables $z_1, \ldots, z_p$, we obtain a contradiction.

We would like to direct attention to the following question: Is it true that if a is a nonsingular point of $S \subset \mathcal{F}(a)$, then the ideal $\langle\langle S \rangle\rangle$ has a passive set? An affirmative answer to this question would mean that passivity is equivalent to regularity.

Remark 3. A passive set is an analogue of the Gröbner basis of an ideal of a polynomial ring. But we do not use the order in our definition. On the other hand, the order and other algebraic structures are useful for obtaining the criterion for a set to be passive.

3. Stratified sets and reductions

We first introduce the definitions that we soon need. Hereafter, $(\Gamma, \leq)$ denotes a well-ordered set. Let a semigroup G act on the set A, i.e., the map $(g, a) \to ga$ of the set $G \times A$ into A is defined and satisfies the condition

$$g_1(g_2 a) = (g_1 g_2)a \quad \text{for all } a \in A, \quad g_1, g_2 \in G.$$

Definition 8. Let $\{A_\gamma\}_{\gamma \in \Gamma}$ be a partition of the set A. A semigroup G acting on A *preserves the partition* if for any $g \in G$, any $\gamma \in \Gamma$, and all $a, b \in A_\gamma$, there exists a γ' such that $ga, gb \in A_{\gamma'}$.

Definition 9. Let a partition $\{A_\gamma\}_{\gamma \in \Gamma}$ of the set A and a semigroup G acting on A and conserving this partition be given. The set A is called a *stratified G-set* if a reflexive (or irreflexive) transitive relation $\prec$ is given on it satisfying the conditions that for all $g \in G$,

1. $a \prec b$ implies $ga \prec gb$ and

2. $a \prec ga$ for any $a \in A$.

We introduce the sets

$$\widehat{\mathcal{F}}(a) = \mathcal{F}(a) \setminus \mathbb{K}[\langle X\rangle]_a, \qquad \widehat{\mathcal{F}} = \bigcup_{a\in\mathbb{K}^{n+\infty}} \widehat{\mathcal{F}}(a),$$

where $\mathbb{K}[\langle X\rangle]_a$ is given by formula (2.5). Any partition $\{U_\gamma\}_{\gamma\in\Gamma}$ of the set U generates partitions of the sets $\widehat{\mathcal{F}}(a)$ and $\widehat{\mathcal{F}}$ as follows. We consider the family of the sets

$$Y_\gamma = X \cup \Big(\bigcup_{\gamma'\le\gamma} U_{\gamma'}\Big).$$

We take a point $a \in \mathbb{K}^{n+\infty}$ and the subalgebra

$$\mathcal{F}_\gamma(a) = \{f \in \mathcal{F}(a)\colon\ \mathrm{iv}(f) \in Y_\gamma\}.$$

Then the families $\{\mathcal{F}_\gamma(a)\}_{\gamma\in\Gamma}$ and $\{\mathcal{F}_\gamma\}_{\gamma\in\Gamma}$, where

$$\mathcal{F}_\gamma = \bigcup_{a\in\mathbb{K}^{n+\infty}} \mathcal{F}_\gamma(a),$$

generate the partitions $\{\Phi_\gamma(a)\}_{\gamma\in\Gamma}$ and $\{\Phi_\gamma\}_{\gamma\in\Gamma}$ of the sets $\widehat{\mathcal{F}}(a)$ and $\widehat{\mathcal{F}}$ into the blocks

$$\Phi_\gamma(a) = \mathcal{F}_\gamma(a) \setminus \Big(\bigcup_{\gamma'<\gamma} \mathcal{F}_{\gamma'}(a)\Big), \qquad \Phi_\gamma = \mathcal{F}_\gamma \setminus \Big(\bigcup_{\gamma'<\gamma} \mathcal{F}_{\gamma'}\Big). \tag{3.1}$$

The partition $\{\Phi_\gamma\}_{\gamma\in\Gamma}$ induces the irreflexive transitive relation on $\widehat{\mathcal{F}}$

$$f \prec g \iff \exists\gamma,\gamma'\colon\ f \in \Phi_\gamma,\ g \in \Phi_{\gamma'},\ \gamma < \gamma' \tag{3.2}$$

and the reflexive transitive relation

$$f \preceq g \iff \exists\gamma,\gamma'\colon\ f \in \Phi_\gamma,\ g \in \Phi_{\gamma'},\ \gamma \le \gamma'. \tag{3.3}$$

The relations $\prec$ and $\preceq$ on $\widehat{\mathcal{F}}(a)$ can be introduced similarly.

The action of the semigroup $\mathbb{N}^n_{-0} = \mathbb{N}^n \setminus \vec{0}$ (where $\vec{0}$ is a block of zeros) on the sets U, $\widehat{\mathcal{F}}(a)$, and $\widehat{\mathcal{F}}$ is given by the formulas

$$\alpha u^i_\beta = u^i_{\alpha+\beta}, \qquad \alpha f = D^\alpha(f). \tag{3.4}$$

Everywhere below, we assume that $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{F}}(a)$ are stratified $\mathbb{N}^n_{-0}$-sets with relation (3.2). It was shown in [20] that if U is a stratified $\mathbb{N}^n_{-0}$-set, then $\widehat{\mathcal{F}}$ is also a stratified $\mathbb{N}^n_{-0}$-set.

Definition 10. A subset S in $\mathcal{F}$ is said to be *weakly solvable* if each series $f \in S$ has the form $f = u^i_\alpha + h$ with some $u^i_\alpha \in U$ and $h \in \mathcal{F}$ such that $h \prec u^i_\alpha$. In this case, the element u^i_α is called the *highest term* of f and is denoted by $\mathrm{lt}\, f$.

The next statement allows introducing "partial division" in the algebra $\mathcal{F}(a)$.

Statement 2. *Let f be an arbitrary series in the algebra $\mathcal{F}(a)$ and $u^i_\alpha \in \mathrm{iv}(f)$. We assume that the set $g \in \mathcal{F}(a)$ has the form $g = u^i_\beta + h$, where $u^i_\beta \in U$ and $h \prec u^i_\beta$, and that there is an element $\delta \in \mathbb{N}^n$ such that $u^i_\alpha = u^i_{\beta+\delta}$. Then there are unique series $q, r \in \mathcal{F}(a)$ satisfying the relations*

$$f = qD^\delta g + r, \tag{3.5}$$

$$q \preceq f, \qquad r \preceq f, \qquad u^i_\alpha \notin \mathrm{iv}(r). \tag{3.6}$$

Proof. The existence and uniqueness of the sets $q, r \in \mathcal{F}(a)$ satisfying both (3.5) and the relations

$$\mathrm{iv}(q) \subseteq \big(\mathrm{iv}(f) \cup \mathrm{iv}(D^\delta g)\big), \qquad \mathrm{iv}(r) \subset \big(\mathrm{iv}(f) \cup \mathrm{iv}(D^\delta g)\big), \quad u^i_\alpha \notin \mathrm{iv}(r), \tag{3.7}$$

follow from the Weierstrass division theorem [18], [21]. Because $\mathrm{lt}(D^\delta g) = u^i_\alpha \in \mathrm{iv}(f)$, we have

$$D^\delta g \preceq f. \tag{3.8}$$

Then (3.6) follows from formulas (3.7) and (3.8).

We say that $f \in \mathcal{F}(a)$ is *one-step reducible* to the set $r \in \mathcal{F}(a)$ with respect to g if the conditions of Statement 2 are satisfied. The one-step reduction is denoted by $f \underset{g}{\rightarrow} r$.

Definition 11. Let $f \in \mathcal{F}(a)$ and $S = \{f_1, \ldots, f_k\} \subset \mathcal{F}(a)$ be a weakly solvable subset. The series f *reduces to a series* $r \in \mathcal{F}(a)$ *with respect to* S if there is a finite set of one-step reductions of the form

$$f \underset{f_{i_1}}{\longrightarrow} r_1 \underset{f_{i_2}}{\longrightarrow} r_2 \underset{f_{i_3}}{\longrightarrow} \cdots \underset{f_{i_p}}{\longrightarrow} r, \tag{3.9}$$

where $f_{i_j} \in S$. This sequence of reductions is briefly denoted by $f \underset{S}{\rightarrow} r$.

Definition 12. We say that a series $f \in \mathcal{F}(a)$ is *irreducible* with respect to a weakly solvable set S if $\mathrm{iv}(f) \cap O(\mathrm{lt}\, S) = \varnothing$, where

$$\mathrm{lt}\, S = \{\mathrm{lt}\, f\colon\ f \in S\}.$$

Definition 13. A series $r \in \mathcal{F}(a)$ is called the *normal form of a series* $f \in \mathcal{F}(a)$ *with respect to a weakly solvable subset* $S \subset \mathcal{F}(a)$ if $f \underset{S}{\rightarrow} r$ and r is a series irreducible with respect to S.

The normal form of a series f with respect to S is denoted by $\mathrm{NF}(f \downarrow S)$.

Lemma 2. *Let S be a weakly solvable subset in $\mathcal{F}(a)$. Then the series $f \in \mathcal{F}(a)$ is either reducible to the normal form* $\mathrm{NF}(f \downarrow S)$ *with respect to S or irreducible with respect to S.*

Proof. If $\mathrm{iv}(f) \cap O(\mathrm{lt}\, S) = \varnothing$, then f cannot be reduced with respect to S. We now suppose that $\mathrm{iv}(f) \cap O(\mathrm{lt}\, S) \neq \varnothing$ and show that f reduces to the normal form $\mathrm{NF}(f \downarrow S)$ with respect to S.

For any $\gamma \in \Gamma$ and any $\phi \in \mathcal{F}(a)$, we introduce the set

$$V_\gamma(\phi) = O(\mathrm{lt}\, S) \cap \mathrm{iv}(\phi) \cap U_\gamma.$$

This set is either finite or empty because $\mathrm{iv}(\phi)$ is finite. We define an element

$$\gamma_1 = \max\{\gamma \in \Gamma\colon\ V_\gamma(f) \neq \varnothing\}.$$

Let the set $V_{\gamma_1}(f)$ consist of m_{γ_1} elements. We consider an arbitrary element $y \in V_{\gamma_1}(f)$. According to our supposition, there exists $g \in S$ such that $y \in O(\mathrm{lt}\, g)$. Consequently, f is one-step reducible to some $r \in \mathcal{F}_{\gamma_1}(a)$ with respect to g, and r is therefore independent of y, and $V_{\gamma_1}(r) \subset V_{\gamma_1}(g)$,because S is a weakly solvable set.

If $V_{\gamma_1}(r) \neq \varnothing$, then we chose an arbitrary element $\tilde{y} \in V_{\gamma_1}(r)$ and a series $\tilde{g} \in S$ such that $\tilde{y} \in O(\mathrm{lt}\, \tilde{g})$ and apply the one-step reduction of r with respect to $\tilde{g}$ to some $\tilde{r}$ such that $\tilde{y} \notin \mathrm{iv}(\tilde{r})$ and $V_{\gamma_1}(\tilde{r}) \subset V_{\gamma_1}(r)$. In m_{γ_1} steps, we reduce f to some $r_{m_{\gamma_1}}$ with respect to S, and the set $V_{\gamma_1}(r_{m_{\gamma_1}})$ is therefore empty.

If there is an element γ such that $V_\gamma(r_{m_{\gamma_1}}) \neq \varnothing$, then we set

$$\gamma_2 = \max\{\gamma \in \Gamma\colon\ V_\gamma(r_{m_{\gamma_1}}) \neq \varnothing\}.$$

Following the above steps, we reduce $r_{m_{\gamma_1}}$ to some $r_{m_{\gamma_2}}$ with respect to S, and therefore $V_{\gamma_2}(r_{m_{\gamma_2}}) = \varnothing$. Continuing this process, we obtain a strictly decreasing sequence of elements $\gamma_1 > \gamma_2 > \dots$. This sequence cannot be infinite, because any subset in the well-ordered set Γ has a minimum element. Consequently, f in this case reduces to the normal form $\mathrm{NF}(f \downarrow S)$.

The proof of Lemma 2 includes a method for transforming a convergent differential series to the normal form. In the general case, the normal form of a series f is not unique; nevertheless, the following statement holds for passive sets.

Proposition 1. *Let S be a weakly solvable passive set in $\mathcal{F}(a)$ such that $\operatorname{lt} S = \operatorname{st} S$. If $f \in \mathcal{F}(a)$ and $\operatorname{iv}(f) \cap O(\operatorname{lt} S) \neq \varnothing$, then f reduces to a unique normal form with respect to S.*

Proof. Because $\operatorname{iv}(f) \cap O(\operatorname{lt} S) \neq \varnothing$, the existence of the reduction to the normal form follows from the Lemma 2. We show the uniqueness of the normal form. Let r_1 and r_2 be two different normal forms of the series f with respect to S. Then the differences $f - r_1$, $f - r_2$, and $r_1 - r_2$ belong to the ideal $\langle\langle S \rangle\rangle$. According to the Lemma 1, the difference $r_1 - r_2$ depends on at least one element of the orbit $O(\operatorname{lt} S)$. On the other hand, because r_1 and r_2 are normal forms, they are independent of the elements of the orbit $O(\operatorname{lt} S)$. Consequently, $\operatorname{iv}(r_1 - r_2) \cap O(\operatorname{lt} S) = \varnothing$, and we obtain a contradiction.

We notice that a similar uniqueness property of the normal form in the case of the Buchberger polynomial ring [10] is used in the definition of the Gröbner basis.

We introduce the binary operation

$$\alpha \diamond \beta = (\mu_1, \dots, \mu_n), \quad \text{where } \mu_i = \max(\alpha_i, \beta_i) - \alpha_i,$$

on the semigroup $\mathbb{N}^n$.

Definition 14. Let two convergent differential series in $\mathcal{F}(a)$ be given:

$$f_1 = u^i_\alpha + h_1, \qquad f_2 = u^i_\beta + h_2, \tag{3.10}$$

where $u^i_\alpha = \operatorname{lt} f_1$ and $u^i_\beta = \operatorname{lt} f_2$. We then call the difference

$$D^{\alpha\diamond\beta} f_1 - D^{\beta\diamond\alpha} f_2 \tag{3.11}$$

the τ-*series* of f_1 and f_2 and let $\tau(f_1, f_2)$ denote it.

Using the τ-series, we obtain the passivity criterion.

Lemma 3. *Let S be a weakly solvable subset in $\mathcal{F}(a)$. The set S is passive and $\operatorname{st} S = \operatorname{lt} S$ if and only if the series $\tau(f_1, f_2)$ reduces to zero with respect to S for all pairs $f_1, f_2 \in S$.*

Proof. We suppose that S is a passive set. Because $f_1, f_2 \in S$, $\tau(f_1, f_2) \in \langle\langle S\rangle\rangle$. If the series $\tau(f_1, f_2)$ is nonzero, then it depends on the elements of the orbit $O(\operatorname{lt} S)$ according to Lemma 1. According to Proposition 1, this τ-series reduces to the normal form with respect to S. This normal form is independent of the elements of the orbit $O(\operatorname{lt} S)$ and belongs to the ideal $\langle\langle S\rangle\rangle$. Consequently, it is zero.

We now suppose that the series $\tau(f_1, f_2)$ reduces to zero with respect to S for all pairs $f_1, f_2 \in S$. According to relations (3.5), (3.6), and (3.9), this means that there exists a finite sequence of the series $\{r_i\}_{i=1}^{p+1} \in \mathcal{F}(a)$ related by the equalities

$$r_i = q_i D^{\delta_i} g_i + r_{i+1}, \tag{3.12}$$

and the series $q_i, g_i, r_i \in \mathcal{F}(a)$ moreover satisfy the conditions

$$r_0 = \tau(f_1, f_2), \quad r_{p+1} = 0, \qquad q_i \preceq r_i, \qquad D^{\delta_i} g_i \preceq r_i, \qquad r_{i+1} \preceq r_i. \tag{3.13}$$

Therefore, the series $\tau(f_1, f_2)$ can be represented in the form

$$\tau(f_1, f_2) = \sum_{i=1}^{p} q_i D^{\delta_i} g_i. \tag{3.14}$$

By (3.3), it is easily seen that the relation $\preceq$ is such that $h_1 \preceq h$ and $h_2 \preceq h$ imply $h_1 h_2 \preceq h$, where $h_1, h_2, h \in \mathcal{F}(a)$. It then follows from formulas (3.13), (3.14), and (3.3) that

$$q_i D^{\delta_i} g_i \preceq r_i, \qquad q_i D^{\delta_i} g_i \preceq \tau(f_1, f_2).$$

Consequently, S is a passive set according to Theorem 1 in [20].

Definition 15. A passive set $S \subset \mathcal{F}(a)$ is called the *canonical* set if each series $f \in S$ is irreducible with respect to $S \setminus f$.

Lemma 4. *Let S be a weakly solvable passive set in $\mathcal{F}(a)$ and $\operatorname{lt} S = \operatorname{st} S$. Then there is a unique canonical set R in $\mathcal{F}(a)$ such that $\langle\langle S\rangle\rangle = \langle\langle R\rangle\rangle$.*

Proof. The existence of the canonical set can be proved by direct construction. Because S is a finite set, there exist elements $\gamma_1 < \gamma_2 < \cdots < \gamma_p$ such that

$$S = S_{\gamma_1} \cup S_{\gamma_2} \cup \cdots \cup S_{\gamma_p},$$

where $S_{\gamma_j} = S \cap \Phi_{\gamma_j}$ and the blocks Φ_{γ_j} are given by formula (3.1). We consider the set S_{γ_1}. Because S is a passive set and $\gamma_1 = \min\{\gamma \in \Gamma\colon\ S \cap \Phi_\gamma \neq \varnothing\}$, any $f \in S_{\gamma_1}$ is independent of elements of the orbit $O(\operatorname{lt}(S \setminus f))$.

For each j (where $2 \le j \le p$), we introduce the two sets

$$Q_j = R_{\gamma_1} \cup \cdots \cup R_{\gamma_j},$$

$$R_{\gamma_j} = \{\operatorname{NF}(f_i \downarrow Q_{j-1}) \neq \varnothing\colon\ f_i \in S_{\gamma_j}\} \cup \{f_i \in S_{\gamma_j}\colon\ f_i \nrightarrow_{Q_{j-1}}\},$$

where the notation $f_i \nrightarrow_{Q_{j-1}}$ means that f_i is irreducible with respect to Q_{j-1}. Moreover, we assume that $R_{\gamma_1} = S_{\gamma_1}$. We show that Q_p is a canonical set in $\mathcal{F}(a)$. We must show that any $g \in Q_p$ is irreducible with respect to $Q_p \setminus g$.

Let a series g belong to R_{γ_j}. Obviously, this series is irreducible with respect to $R \setminus R_{\gamma_j}$. We note that if the series $r_i = \mathrm{NF}(f_i \downarrow Q_{j-1})$ (where $f_i \in S_j$) is nonzero, then it belongs to Φ_{γ_j}. In fact, if $r_i \in \Phi_{\gamma'}$ (where $\gamma' < \gamma_j$), then $\mathrm{iv}(r_i) \cap O(\mathrm{lt}\, S) = \varnothing$. On the other hand, any element in $\langle\langle S \rangle\rangle$ depends on at least one element in $O(\mathrm{lt}\, S)$ according to Lemma 1. We obtain a contradiction. Consequently, $R_{\gamma_j} \subset \Phi_{\gamma_j}$.

We now show that if $g, h \in R_{\gamma_j}$ and $\mathrm{lt}\, g = \mathrm{lt}\, h$, then $g = h$. We suppose that this is not the case. Then $g - h \neq 0$ and $g - h \in \Phi_{\gamma'}$, where $\gamma' < \gamma_j$. Because the series g and h are irreducible with respect to Q_{j-1}, their difference is also irreducible with respect to Q_{j-1}. According to Lemma 1, any element in $\langle\langle S \rangle\rangle$ depends on at least one element in $O(\mathrm{lt}\, S)$. Therefore, $g - h = 0$, and the existence is proved.

It remains to prove the uniqueness. We suppose that there is a canonical set $\widetilde{R}$ distinct from R. By the condition, S is a passive set and $\langle\langle R \rangle\rangle = \langle\langle \widetilde{R} \rangle\rangle = \langle\langle S \rangle\rangle$. Consequently, the equalities

$$O(\mathrm{lt}\, S) = O(\mathrm{lt}\, R) = O(\mathrm{lt}\, \widetilde{R}), \qquad \mathrm{lt}\, R = \mathrm{lt}\, \widetilde{R}$$

hold. We take two elements $g \in R$ and $\tilde{g} \in \widetilde{R}$ such that $\mathrm{lt}\, g = \mathrm{lt}\, \tilde{g}$ and show that $g = \tilde{g}$. We first suppose that this is false. Then $g - \tilde{g} \neq 0$ and $\mathrm{iv}(g - \tilde{g}) \cap O(\mathrm{lt}\, S) \neq \varnothing$ according to Lemma 1. Because the sets R and $\widetilde{R}$ are canonical, the series g and $\tilde{g}$ are independent of the elements of the orbits $O(\mathrm{lt}(R \setminus g)) = O(\mathrm{lt}(\widetilde{R} \setminus \tilde{g}))$. Consequently, the difference $g - \tilde{g}$ is independent of the elements of the orbit $O(\mathrm{lt}(R \setminus g))$. Obviously, the formulas

$$O(\mathrm{lt}\, S) = O(\mathrm{lt}\, R) = O(\mathrm{lt}(R \setminus g)) \cup O(\mathrm{lt}\, g), \qquad \mathrm{iv}(g - \tilde{g}) \cap O(\mathrm{lt}\, g) = \varnothing$$

hold. Therefore, $\mathrm{iv}(g - \tilde{g}) \cap O(\mathrm{lt}\, S) = \varnothing$.

4. Construction of passive sets and examples of differential systems

If a set $S \subset \mathcal{F}(a)$ is not passive, then the problem of constructing a passive set $P \subset \mathcal{F}(a)$ such that $\langle\langle S \rangle\rangle = \langle\langle P \rangle\rangle$ arises. We need a convenient existence criterion for a passive set of a given differential ideal. Such problems have been considered in both classical and modern papers but apparently were not ultimately resolved. We describe a scheme for constructing a passive set, note problems in realizing it, and give some examples.

We first prove a statement that can be used to study passivity. We let $\mathbb{K}_n(c)$ denote the algebra of convergent power series (with the center at the point $c \in \mathbb{K}^n$) depending on n variables $x_1, \ldots, x_n$. The set

$$\mathfrak{m}_n(c) = \{f \in \mathbb{K}_n(c) \colon f(c) = 0\}$$

forms the maximal ideal of the algebra $\mathbb{K}_n(c)$. We need the following refinement of the implicit function theorem.

Proposition 2. *Let $f_1, \ldots, f_k \in \mathfrak{m}_n(c)$ and the Jacobian $\frac{\partial(f_1,\ldots,f_k)}{\partial(x_1,\ldots,x_k)}(c)$ be nonzero. Then there exist series $g_1, \ldots, g_k \in \mathfrak{m}_n(c)$ of the form*

$$g_i = x_i + h_i \tag{4.1}$$

and a kth-order square matrix A with elements in $\mathbb{K}_n(c)$ such that

$$(f_1, \ldots, f_k) = (g_1, \ldots, g_k)A, \qquad A(c) = \left(\frac{\partial f_j}{\partial x_l}\right)(c), \tag{4.2}$$

where $j, l \in \mathbb{N}_k$, and the series h_i are independent of $x_1, \ldots, x_k$.

Proof. It follows from the conditions in the proposition and the implicit function theorem that there exist series $g_1, \dots, g_k \in \mathfrak{m}_n(c)$ of form (4.1) such that

$$f_i(-h_1, \dots, -h_k, x_{k+1}, \dots, x_n) = 0, \quad i \in \mathbb{N}_k. \tag{4.3}$$

The Weierstrass division theorem [18] implies the representation [20]

$$f_j = \sum_{i=1}^{k} g_i a_{ji} + r_j, \quad j \in \mathbb{N}_k, \tag{4.4}$$

where $r_j, a_{i,j} \in \mathbb{K}_n(c)$ and $\mathrm{iv}(r_j) \cap \{x_1, \dots, x_k\} = \varnothing$. We substitute the series $-h_1, \dots, -h_k$ for $x_1, \dots, x_k$ in (4.4). It then follows from relations (4.1) and (4.3) that the series r_j are zero. Consequently, the first formula in (4.2) holds.

We prove the validity of the second formula in (4.2). Differentiating equality (4.4) with respect to x_l (at $r = 0$, $l \in \mathbb{N}_k$), we obtain

$$\frac{\partial f_j}{\partial x_l} = \sum_{i=1}^{k} \frac{\partial g_i}{\partial x_l} a_{ji} + \sum_{i=1}^{k} g_i \frac{\partial a_{ji}}{\partial x_l}. \tag{4.5}$$

Because the equalities

$$g_i(c) = 0, \qquad \frac{\partial g_i}{\partial x_l}(c) = \delta_l^i$$

hold for all $i, l \in \mathbb{N}_k$ (where δ_l^i is the Kronecker symbol), it follows from (4.5) that $\frac{\partial f_j}{\partial x_l}(c) = a_{jl}(c)$. This proves the second formula in (4.2).

We return to studying the properties of the algebra $\mathcal{F}(a)$.

Definition 16. We say that the blocks $\vec{f}^{\,1}, \vec{f}^{\,2} \in \mathcal{F}^k(a)$ are *similar* if there is a kth-order square matrix A nondegenerate at the point a with elements in $\mathcal{F}(a)$ such that $\vec{f}^{\,1} = \vec{f}^{\,2} A$. Two sets $R_1, R_2 \subset \mathcal{F}(a)$ of k elements are said to be *equivalent* if the blocks $\vec{f}^{\,1}, \vec{f}^{\,2} \in \mathcal{F}^k(a)$ composed of the elements of the respective sets R_1 and R_2 are similar.

Obviously, equivalent sets generate coinciding differential ideals. In certain cases, we can use equivalent sets to solve the problem of constructing a passive set for a given differential ideal generated by the finite subsets in $\mathcal{F}(a)$. We present the scheme for constructing a passive set.

Let $\widehat{\mathcal{F}}$ be a stratified $\mathbb{N}^n_{-0}$-set. We assume that $S = \{f_1, \dots, f_k\}$ is not a weakly solvable set in $\mathcal{F}$ and there is a point c where all series in S vanish and the rank of the Jacobi matrix $(\partial f_j/\partial z_i)(c)$ is equal to $p > 0$ (here z_i are the variables on which the series $f_j \in S$ depend). Then there is a nonzero minor

$$\frac{\partial(f_{i_1}, \dots, f_{i_p})}{\partial(z_{i_1}, \dots, z_{i_p})}(c).$$

According to Proposition 2, there exist series of the form

$$g_1 = z_{i_1} + h_1, \quad \dots, \quad g_p = z_{i_p} + h_p$$

such that the blocks $(f_{i_1}, \dots, f_{i_p})$ and $g_1 \dots, g_p$ are similar. If $h_1 \prec z_{i_1}, \dots, h_p \prec z_{i_p}$, then the set $\widetilde{S} = \{g_1, \dots, g_p\}$ is weakly solvable. We verify that any series $f \in S$ reduces to zero with respect to $\widetilde{S}$. We then reduce all τ-series of form (3.11) composed of all pairs $f_1, f_2 \in \widetilde{S}$ to the normal forms with respect to $\widetilde{S}$. We let S_1 denote the set of normal nonzero forms. If the set S_1 is not weakly solvable, then we apply the procedure described above to it and obtain the weakly solvable set $\widetilde{S}_1$. We next check whether the set $\widetilde{S} \cup \widetilde{S}_1$ is passive by reducing the τ-series to the normal forms with respect to $\widetilde{S} \cup \widetilde{S}_1$. If $\widetilde{S} \cup \widetilde{S}_1$ is not passive, then using the described method, we obtain the set $\widetilde{S}_2$ and the set $\widetilde{S} \cup \widetilde{S}_1 \cup \widetilde{S}_2$. We then check whether the last set is passive, and so on. As we show below, we obtain a passive set after a finite number of steps.

To prove that the number of steps in the scheme described above is finite, we consider the ring $\mathcal{M}$ of polynomials $\mathbb{K}[x_1, \ldots, x_n]$ and the vector space $\mathbb{K}U$ over $\mathbb{K}$ generated by the set U given by (2.2). It is simple to prove [20] that $\mathbb{K}U$ can be equipped with the structure of a left $\mathcal{M}$-module and this module is isomorphic to the Noether module $\mathcal{M}^n$. We let Q_i denote the union $\widetilde{S} \cup \widetilde{S}_1 \cup \cdots \cup \widetilde{S}_i$ and Q_0 denote $\widetilde{S}$. We assume that the set Q_i for any i is strictly contained in Q_{i+1}. This strictly increasing chain of sets corresponds to the strictly increasing chain of highest terms

$$\operatorname{lt} Q_0 \subset \operatorname{lt} Q_1 \subset \cdots \subset \operatorname{lt} Q_i \subset \ldots .$$

The inclusions in this chain are strict because $\widetilde{S}_i$ consists of normal forms with respect to Q_{i-1}. In turn, the last chain corresponds to the chain of submodules

$$[\operatorname{lt} Q_0] \subset [\operatorname{lt} Q_1] \subset \cdots \subset [\operatorname{lt} Q_i] \subset \ldots,$$

where $[\operatorname{lt} Q_i]$ is the submodule of the $\mathcal{M}$-module $\mathbb{K}U$ generated by the set Q_i. Because the module $\mathbb{K}U$ is Noether, it cannot have a strictly increasing infinite chain of submodules. We obtain a contradiction.

In the general case, we cannot guarantee that the scheme described above works, because the conditions of Proposition 2 might not be be satisfied.

We give examples of constructing passive sets.

Example 1. Let $\mathbb{K} = \mathbb{R}$ be the field of real numbers $m = 1$, $n = 2$. We consider the set $S = \{f_1, f_2\} \subset \mathcal{F}$ consisting of two convergent differential series

$$f_1 = u_{(1,1)} - \sin u_{(0,0)}, \qquad f_2 = u_{(0,3)} + \frac{1}{2} u^3_{(0,1)}.$$

The set U can be separated into the subsets $U_i = \{u_\alpha \colon |\alpha| = i\}$ and becomes a stratified $\mathbb{N}^2_{=0}$-set. We want to find a passive set $Q \subset \mathcal{F}(a)$ such that $\langle\!\langle Q \rangle\!\rangle = \langle\!\langle S \rangle\!\rangle$. Restrictions on the point $a \in \mathbb{K}^{n+\infty}$ appear in the course of the calculations.

Below, we use the more familiar notation

$$D_t = D_1, \quad D_x = D_2, \qquad u = u_{(0,0)}, \quad u_t = u_{(1,0)}, \quad u_x = u_{(0,1)}, \quad u_{tx} = u_{(1,1)}.$$

The set S corresponds to the system of partial differential equations

$$u_{tx} - \sin u = 0, \qquad u_{xxx} + \frac{1}{2} u_x^3 = 0. \tag{4.6}$$

We calculate the τ-set of f_1 and f_2:

$$\tau(f_1, f_2) = D_x^2 f_1 - D_t f_2 = u_x^2 \sin u - u_{xx} \cos u - \frac{3}{2} u_{tx} u_x^2.$$

Obviously, the set $\tau(f_1, f_2)$ reduces to the normal form

$$f_3 = -u_{xx} \cos u - \frac{1}{2} u_x^2 \sin u$$

with respect to S. Consequently, the set S is not passive according to Lemma 1. If $\cos u \neq 0$, then f_3 is equivalent to the series $\tilde{f}_3 = u_{xx} + u_x^2 \tan(u)/2$.

We must now check the set $S_1 = \{f_1, f_2, \tilde{f}_3\}$ for passivity. Computing the τ-series

$$\tau(f_1, \tilde{f}_3) = D_x(u_{tx} - \sin u) - D_t\left(u_{xx} + \frac{u_x^2}{2}\tan u\right) = -u_x \cos u - u_{tx}u_x \tan u - \frac{u_t u_x^2}{2\cos^2 u}$$

and reducing it to its normal form with respect to S_1, we obtain

$$f_4 = -\frac{u_x}{2\cos^2 u}(u_t u_x + 2\cos u).$$

If $u_x \neq 0$, then f_4 is equivalent to the series

$$\tilde{f}_4 = u_t + \frac{2\cos u}{u_x}.$$

It can be easily calculated that the series f_1 and f_2 belong to the ideal $\langle\langle f_3, f_4\rangle\rangle$ and the series $\tau(\tilde{f}_3, \tilde{f}_4)$ reduces to zero with respect to the set $\{\tilde{f}_3, \tilde{f}_4\}$.

Therefore, $\{\tilde{f}_3, \tilde{f}_4\}$ is the required passive set. This passive set generates the same differential ideal as the original set $\{f_1, f_2\}$. It can be used to find solutions of system (4.6). For this, it suffices to integrate the system

$$\tilde{f}_3 = 0, \qquad \tilde{f}_4 = 0. \tag{4.7}$$

We first consider the first equation

$$u_{xx} + \frac{u_x^2}{2}\tan u = 0$$

of system (4.7). We divide this equation by u_x and find the first integral

$$\frac{u_x}{\sqrt{\cos u}} = c,$$

where c is still an arbitrary function of t. Substituting the first integral in the second equation of the system, Eq. (4.7), we obtain the first-order system

$$u_x - c\sqrt{\cos u} = 0, \qquad u_t + \frac{2\sqrt{\cos u}}{c} = 0. \tag{4.8}$$

Composing the τ-series of the left parts of the system (4.8) and reducing it to the normal form, we see that the normal form is equal to zero only if $c = \text{const}$. In this case, the implicit solution of system (4.8) is

$$\int \frac{du}{\sqrt{\cos u}} = cx - \frac{2t}{c} + c_1,$$

where c_1 is an arbitrary constant. The obtained solution of the sine-Gordon equation is invariant under the dilatation and translation transformations.

Example 2. We consider the acoustic equation in a nonhomogeneous medium [22] as the second example:

$$\frac{p_{tt}}{\rho c^2} = \left(\frac{p_x}{\rho}\right)_x + \left(\frac{p_y}{\rho}\right)_y + \left(\frac{p_z}{\rho}\right)_z, \tag{4.9}$$

where p is the pressure, ρ is the density, and c is the speed of sound. The pressure is the sought function, while the density and the speed of sound are given functions of x, y, and z. We first assume that ρ and c are arbitrary positive functions. If we introduce a new function $P = p/\sqrt{\rho}$, then Eq. (4.9) becomes

$$P_{tt} = c^2(\Delta P + qP). \tag{4.10}$$

Here, Δ is the three-dimensional Laplacian, and the function q is expressed in terms of ρ:

$$q = -\frac{\Delta(\sqrt{\rho})}{\sqrt{\rho}}.$$

We want to find some solutions of Eq. (4.10). We assume that P satisfies the additional equation

$$P_t = aP_x + bP_y, \tag{4.11}$$

regarding a, b, and c as functions of only x.

Equations (4.10) and (4.11) correspond to the two differential series

$$e_0 = P_{tt} - c^2(\Delta P + qP), \qquad e_1 = P_t - aP_x - bP_y.$$

The set $S = \{e_0, e_1\}$ is not passive for arbitrary functions a, b, and c. We obtain conditions such that the set S becomes passive if they are satisfied. For this, we reduce e_0 to the normal form with respect to e_1, i.e., eliminate the term P_{tt} in e_0. It can be easily seen that the normal form is

$$\begin{aligned} e_2 &= e_0 - D_t e_1 - aD_x e_1 - be_1 = \\ &= (a^2 - c^2)P_{xx} + a(a' + 2b)P_x - c^2(P_{yy} + P_{zz}) + (ab' + b^2 - qc^2)P. \end{aligned}$$

We now assume that $c = a$ and $b = -a'/2$. Then e_2 is considerably simplified:

$$e_2 = -a^2(P_{yy} + P_{zz}) + \left(-\frac{aa''}{2} + \frac{(a')^2}{4} - qa^2\right)P.$$

For $a \neq 0$, the series e_2 is equivalent to

$$\tilde{e}_3 = P_{yy} + P_{zz} + \left(\frac{aa''}{2a} + \frac{(a')^2}{4a^2} - q\right)P.$$

We now assume that

$$q = -\frac{aa''}{2a} + \frac{(a')^2}{4a^2} + h,$$

where h depends only on y and z. The series $\tilde{e}_3$ is then

$$\tilde{e}_3 = P_{yy} + P_{zz} + hP.$$

It is easy to see that the manifold $S_1 = \{e_1, \tilde{e}_3\}$ is passive and

$$e_0 = D_t e_1 + aD_x e_1 + be_1 - a^2\tilde{e}_3.$$

Consequently, some solutions of Eqs. (4.10) can be found by solving the system

$$e_1 = 0, \qquad \tilde{e}_3 = 0. \tag{4.12}$$

Solving the first equation

$$P_t - aP_x + a'\frac{P}{2} = 0 \tag{4.13}$$

in system (4.12) reduces to integrating the ordinary differential equations

$$\frac{dt}{1} = \frac{dx}{a} = -2\frac{dP}{a'P}.$$

Consequently, the general solution of Eq. (4.13) has the form

$$P = \sqrt{a(x)}\,f\left(t - \int \frac{dx}{a(x)}, y, z\right).$$

According to the second equation of system (4.12), the function f must satisfy the equation

$$f_{yy} + f_{zz} + hf = 0. \tag{4.14}$$

If $h = 0$, then we obtain the Laplace equation. For a special choice of the function h, Eq. (4.14) reduces to the Laplace equation [23]. If the functions f and h are independent of z, then (4.14) becomes the linear ordinary equation

$$f'' + hf = 0,$$

which can be explicitly integrated for a special choice of h [24].

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