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NEW APPROACH TO THE QUANTIZATION OF THE YANG–MILLS FIELD

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We review papers on a new method for quantizing the Yang–Mills field applicable both in perturbation theory and beyond it. We show that in the modified formulation of the Yang–Mills theory leading to a formal perturbation theory that coincides with the standard one, there exist soliton solutions of the classical equations of motion.

Keywords: non-Abelian gauge invariance, quantization, quantization nonuniqueness, soliton

1. Introduction

Progress in physics is as a rule related to introducing a new symmetry. Very few problems of high energy physics can be solved exactly. But based on symmetry, we can make some predictions, which can be checked experimentally. Gauge field theory illustrates this thesis well. The classical Faraday–Maxwell electrodynamics is completely described by the electromagnetic stress tensor $F_{\mu\nu}$. But if we want to describe the interaction of the electromagnetic field with matter fields, with an electron for example, we discover that it is impossible to construct a local Hermitian Hamiltonian describing this interaction using only the stress tensor $F_{\mu\nu}$. Therefore, the electromagnetic field is described by the four-vector A_μ, in terms of which the stress tensor can be expressed. The vector A_μ has more components than needed for describing the experiment: it is known that the electromagnetic field is three-dimensionally transverse, i.e., the electric and magnetic polarization vectors are perpendicular to the three-dimensional momentum. From the experimental standpoint, it is necessary to know only the three-dimensionally transverse components of A_μ. But A_μ, in addition to the transverse components, also has the timelike component A_0 and the component parallel to the three-dimensional momentum, which is usually denoted by A_3. Clearly, the theory based on the vector A_μ has more components than necessary. But a new symmetry—gradient (gauge) invariance—is introduced simultaneously with A_μ. This invariance provides the decoupling of A_0 and A_3 from the transverse components, and observables are independent of A_0 and A_3. The same idea can be illustrated by the models based on non-Abelian gauge groups. The simplest version of such a theory is based on the group $SU(2)$ and was proposed by Yang and Mills [1]. After the transition to the quantum theory in the Yang–Mills model, in addition to the unphysical components of the vector field, the anticommuting scalar fields $\bar{c}$ and c—Faddeev–Popov ghosts—arise [2], [3]. It can be shown [4], [5] that the components A_0 and A_3 and the Faddeev–Popov ghosts decouple from the transverse components and observables are independent of unphysical particles.

The same idea underlies the renormalizable description of the Higgs model [6]–[8], which allows introducing the mass term for the vector field without breaking the gauge invariance. In this case, the complete spectrum of the theory, in addition to A_i and Faddeev–Popov ghosts, also includes the Goldstone particles

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B_a. As before, it can be shown that the component A_0, Faddeev–Popov ghosts $\bar{c}$ and c, and Goldstone fields decouple from the physical components of the vector field A_i, $i = 1, 2, 3$, and observables depend on only these components.

In these examples, we observe the following law: for a consistent description of observables, the spectrum of the theory must be extended by introducing unphysical excitations. Of course, we should care about the new symmetry arising in the theory, leading to the decoupling of the unphysical degrees of freedom from the physical ones.

We use this observation to construct a scheme for quantizing non-Abelian gauge fields applicable beyond perturbation theory. We also show that the Yang–Mills theory in the modified formulation, leading to the same formal perturbation theory in the quantum case as the standard formulation, has soliton solutions of the classical equations of motion. This contradicts the generally accepted view that the classical Yang–Mills theory does not have soliton solutions and agrees with the view that the confinement of color objects is related to the existence of quasiparticle solutions of classical equations with a localized finite energy.

Speaking about the quantization of gauge fields beyond perturbation theory, we have in mind the problem of the nonuniqueness of quantization, first noted by Gribov [9]. It is known that to quantize the gauge field, some gauge condition selecting a unique representative in the class of gauge-equivalent configurations must be imposed. In the Maxwell theory, this means that the gauge condition (e.g., the Coulomb gauge)

$$\partial_i A_i = 0 \tag{1}$$

has only a trivial solution for the function Φ, where gauge-equivalent configurations are

$$A_i + \partial_i \Phi. \tag{2}$$

Indeed, if condition (1) is satisfied, then the function by which two gauge-equivalent configurations differ must satisfy the equation

$$\Delta\Phi = 0, \tag{3}$$

i.e., must be a harmonic function. It is known that a harmonic function has an extremal value at the boundary. Because $\Phi(x)$ must vanish at the spatial infinity, we conclude that this function is identically zero. Therefore, in the case of an Abelian gauge group, the Coulomb gauge selects a unique representative in the class of gauge-equivalent configurations.

But in the case of the simplest non-Abelian gauge group $SU(2)$, the equation corresponding to (3) is

$$\Delta\Phi^a + g\,\partial_i(\varepsilon^{abc} A_i^b \Phi^c) = 0. \tag{4}$$

This equation has nontrivial solutions even if $\Phi(x) \to 0$ as $|x| \to \infty$. In the framework of the perturbation theory in the coupling constant, the solution of Eq. (4) is trivial. Indeed, if we seek the solution for Φ as a formal series

$$\Phi = \Phi_0 + g\Phi_1 + g^2\Phi_2 + \ldots, \tag{5}$$

then in the perturbation theory framework, we obtain

$$\Phi_0 = \Phi_1 = \Phi_2 = \cdots = 0. \tag{6}$$

For large g, Eq. (4) has nontrivial solutions that tend to zero at spatial infinity. Therefore, the Faddeev–Popov–De Witt quantization scheme, based on the assumption that the Coulomb gauge condition selects a unique representative in the class of gauge-equivalent configurations, is strictly speaking inapplicable beyond the perturbation theory. If we try to use it for large g, then this leads to singularities appearing in the path integral for the scattering matrix. Singer generalized this result to any differential gauge [10].

We can hope that it is possible to use so-called algebraic gauges, for example, the Hamiltonian gauge $A_0 = 0$. In these cases, the problem of the gauge choice uniqueness does not arise. But setting $A_0 = 0$ in the Lagrangian, we lose the constraint $D_i P_i^a = 0$, which must be satisfied for the observable quantities. This condition cannot be imposed on the fields, which are assumed to be independent in the quantization process. This condition can be imposed on the allowed state vectors

$$\widehat{D}_i \widehat{P}_i |\Phi\rangle = 0. \tag{7}$$

This condition can be solved in perturbation theory, but the question about the existence of solutions of (7) beyond perturbation theory is open. Analogous problems arise in other algebraic gauges. Moreover, from the practical standpoint, these gauges are unsatisfactory because they destroy the manifest Lorentz invariance, which complicates calculations considerably. We hence see that quantizing non-Abelian gauge fields in the standard formulation is possible only in the perturbation theory framework. A possible way out of this situation was proposed in papers by Zwanziger [11].

Here, we use another possibility. We show that by the mechanism described above, i.e., by expanding the spectrum of unphysical fields and introducing a new symmetry, we can quantize the non-Abelian gauge theory beyond perturbation theory. The modified theory coincides with the standard theory in the formal perturbation theory framework but has soliton solutions of the classical equation of motions of the type of the 't Hooft–Polyakov magnetic monopole [12], [13].

2. The modified Yang–Mills theory

We begin by considering the modified $SU(2)$ theory described by the Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu}^i F_{\mu\nu}^i + (D_\mu \varphi^+)^i (D_\mu \varphi^-)^i + \mathrm{i}(D_\mu b)^i (D_\mu e)^i. \tag{8}$$

Here, $F_{\mu\nu}^i$ is the usual stress tensor for the Yang–Mills field, and D_μ denotes the covariant derivative. The fields $\varphi^\pm$, b, and e are Hermitian and have zero spin. The fields $\varphi^\pm$ and b, e are respectively commuting and anticommuting elements of the Grassmann algebra. The fields $\varphi^\pm$, b, and e belong to the adjoint representation of the group $SU(2)$.

We first consider the topologically trivial sector. Shifting the fields φ^- along the third axis,

$$\varphi_i^- \to \varphi_i^- - \hat{\mu}, \qquad \hat{\mu} = \delta^{i3} m g^{-1}, \tag{9}$$

we obtain the Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu}^i F_{\mu\nu}^i + (D_\mu \varphi^+)^i (D_\mu \varphi^-)^i + \mathrm{i}(D_\mu b)^i (D_\mu e)^i - (D_\mu \varphi^+)^i (D_\mu \hat{\mu})^i. \tag{10}$$

This Lagrangian is obviously invariant under the "shifted" gauge transformations

$$\begin{aligned}
&\delta A_\mu^i = \partial_\mu \eta^i + g\varepsilon^{ijk} A_\mu^j \eta^k, \\
&\delta \varphi_i^+ = g\varepsilon^{ijk} \varphi_j^+ \eta^k, \qquad \delta \varphi_i^- = -m\varepsilon^{i3k} \eta^k + g\varepsilon^{ijk} \varphi_j^- \eta^k, \\
&\delta b^i = g\varepsilon^{ijk} b^j \eta^k, \qquad \delta e^i = g\varepsilon^{ijk} e^j \eta^k
\end{aligned} \tag{11}$$

(here $i,j,k=1,2,3$). Lagrangian (10) is also invariant under the supersymmetry transformation

$$\delta\varphi_i^- = \mathrm{i} b_i \epsilon, \qquad \delta e_i = \varphi_i^+ \epsilon, \qquad \delta b^i = \delta\varphi_i^+ = 0. \tag{12}$$

This is a new symmetry, which was absent in the standard Yang–Mills Lagrangian and which plays the main role in the proof that the unphysical fields $\varphi^\pm$, b, and e decouple from the physical components of A_μ^{tr}.

We note that the transformations shift the fields $\varphi_{1,2}^-$ by arbitrary functions. Therefore, as in the Higgs model, the new gauge field $\varphi_{1,2}^-$ arises in addition to the gauge field A_μ. We choose the gauge

$$\varphi_{1,2}^- = 0. \tag{13}$$

This gauge is obviously algebraic and does not require introducing Faddeev–Popov ghosts. At the same time, condition (13) is manifestly Lorentz invariant. In this model, we thus succeeded in introducing a manifestly Lorentz-invariant algebraic gauge and, as is seen, in preserving the renormalizability of the theory. The remaining gauge invariance, related to rotations around the third axis in the charge space, is Abelian and does not produce the Gribov ambiguity.

Nevertheless, the gauge $\varphi_{1,2}^- = 0$ is still not unique. Applying gauge transformations (11) to the fields $\varphi_{1,2}^-$, we obtain equations that must be satisfied by the gauge function to provide the uniqueness of the gauge:

$$(m+g\varphi_3^-)\eta^2 = 0, \qquad (m+g\varphi_3^-)\eta^1 = 0. \tag{14}$$

We note that in perturbation theory, with $\eta^a = \eta_0^a + g\eta_1^a + \dots$, the only solution of Eqs. (14) is $\eta^{1,2}=0$. But for the values of φ_3^- for which $m - g\varphi_3^- = 0$, the choice of the gauge is not unique, as before.

To eliminate this nonuniqueness, we change the variables in the classical Lagrangian:

$$\begin{aligned}
&\varphi_3^- = -\frac{m}{g}\left(\exp\left\{\frac{gh}{m}\right\} - 1\right), \qquad \varphi_3^+ = M^{-1}\tilde\varphi_3^+, \\
&\varphi_{1,2}^- = M\tilde\varphi_{1,2}^-, \qquad \varphi_{1,2}^+ = M^{-1}\tilde\varphi_{1,2}^+, \\
&e = M^{-1}\tilde e, \qquad b = M\tilde b,
\end{aligned} \tag{15}$$

where

$$M = \left(1 + \frac{g}{m}\varphi_3^-\right) = \exp\left\{\frac{gh}{m}\right\}. \tag{16}$$

Instead of the gauge $\varphi_{1,2}^- = 0$, we impose the condition

$$\tilde\varphi_{1,2}^- = 0. \tag{17}$$

We thus obtain

$$\delta\tilde\varphi_{1,2}^- = \pm m\eta^{1,2}. \tag{18}$$

As can be seen, the nonuniqueness of the gauge fixing is completely absent. The effective Lagrangian in gauge (17) is

$$\begin{aligned}
L_{\mathrm{eff}} = {} & -\frac{1}{4}F_{\mu\nu}^i F_{\mu\nu}^i + \partial_\mu h\,\partial_\mu\tilde\varphi_3^+ - \frac{g}{m}\partial_\mu h\,\partial_\mu h\,\tilde\varphi_3^+ - \\
& - \left[D_\mu\tilde b + \frac{g}{m}\tilde b\,\partial_\mu h\right]^i \left[D_\mu\tilde e - \frac{g}{m}\tilde e\,\partial_\mu h\right]^i + \\
& + mg(A_\mu^a)^2\tilde\varphi_3^+ + g\,\partial_\mu h\,A_\mu^a\tilde\varphi_a^+, m\varepsilon^{3ab}\tilde\varphi_a^+\,\partial_\mu A_\mu^b.
\end{aligned} \tag{19}$$

where $a, b = 1, 2$ and $i = 1, 2, 3$. The free propagators determined by Lagrangian (19) are

$$\begin{aligned} &\Delta^{\mathrm{tr}}(A^a_\mu A^b_\nu) = -\mathrm{i}\delta^{ab}\frac{T_{\mu\nu}}{p^2}, \qquad \Delta(A^a_\mu \tilde{\varphi}^+_b) = \varepsilon^{3ab}\frac{p_\mu}{mp^2}, \\ &\Delta(\tilde{b}^i \tilde{e}^j) = \mathrm{i}\delta^{ij}\frac{1}{p^2}, \qquad \Delta(h\tilde{\varphi}^+_3) = \frac{1}{p^2}, \end{aligned} \tag{20}$$

where $T_{\mu\nu}$ is the transverse projector. The remaining propagators corresponding to the Abelian subgroup of rotations around the third axis can be set equal to

$$\Delta(A^3_\mu A^3_\nu) = -\mathrm{i}\frac{T_{\mu\nu}}{p^2}. \tag{21}$$

It is easy to calculate the divergence index of an arbitrary diagram, which is equal to

$$n = 4 - 2L_{\tilde{\varphi}^+_3} - 2L_{\tilde{\varphi}^+_a} - L_A - L_e - L_b - L_h. \tag{22}$$

where L_c denotes the number of external lines of the field c. Because the interaction Lagrangian includes only trilinear vertices with one derivative and four linear vertices without derivatives, the theory is obviously renormalizable.

3. Unitarity of the theory in the physical space

The spectrum of our theory includes several unphysical particles. They are the zeroth and third components of A^i_μ, the fields $\varphi^\pm_a$, φ^+_3, and h, and the anticommuting fields b^i and e^i. We recall that the original Lagrangian was invariant under gauge transformations (11) and supersymmetry transformations (12). The asymptotic gauge transformations are unchanged in terms of the new variables, and the supersymmetry transformations are changed as

$$\delta\tilde{\varphi}^-_a = \mathrm{i}\tilde{b}^a\epsilon, \qquad \delta h = \mathrm{i}\tilde{b}^3\epsilon, \qquad \delta\tilde{e}^i = \tilde{\varphi}^+_i. \tag{23}$$

The remaining components of the asymptotic fields are not transformed. We say nothing about the Abelian gauge transformations related to rotations around the third axis. These transformations do not introduce any complications.

We fix the gauge corresponding to rotation about the first and second axes by adding the expression

$$s_1 \int d^4x\, \bar{c}^a \tilde{\varphi}^-_a = \int d^4x [\lambda^a \tilde{\varphi}^-_a + \bar{c}^a c^a] \tag{24}$$

to the effective action, where s_1 is the nilpotent operator, similar to the usual BRST operator, determined by the gauge transformation. This leaves effective Lagrangian (19) written in terms of the transformed variables invariant, and the action of s_1 on the ghost fields and the field λ is defined by the formula

$$(s_1 c)^a = 0, \qquad (s_1 \bar{c})^a = \lambda^a, \qquad (s_1 \lambda)^a = 0. \tag{25}$$

The gauge-fixed effective action is

$$A_{\mathrm{eff}} = \int d^4x \big(L(x) + \lambda^a \tilde{\varphi}^-_a + m\bar{c}^a c^a - \bar{c}^a \tilde{b}^a\big), \tag{26}$$

where L denotes Lagrangian (19), invariant under simultaneous gauge transformations and supersymmetry transformations written in terms of λ, h, $\tilde{\varphi}^{\pm}$, A_μ, $\tilde{\varphi}_3^+$, $\tilde{b}$, and $\tilde{e}$. The canonical gauge fixing does not include the term $\bar{c}^a\tilde{b}^a$, but this term can be easily generated by the change of variables $c \to c - \tilde{b}m^{-1}$. Explicitly integrating over $\bar{c}^a$ and c^a, we obtain an action that is invariant under simultaneous BRST transformations and supersymmetry transformations with the change $c \to \tilde{b}m^{-1}$.

According to the Noether theorem, an invariance of the effective action leads to the existence of a conserved charge Q, which allows separating physical excitations by requiring their annihilation by the asymptotic charge

$$Q^0|\Phi\rangle_{\mathrm{as}}^{\mathrm{ph}} = 0, \tag{27}$$

where Q^0 is the asymptotic operator acting on the asymptotic fields as

$$\begin{aligned} &Q^0 A_\mu^i = \mathrm{i}\frac{\partial_\mu \tilde{b}^i \epsilon}{m}, \qquad Q^0\tilde{b}^a = 0, \qquad Q^0\tilde{b}^3 = 0,\\ &Q^0 h = \mathrm{i}\tilde{b}^3\epsilon, \qquad Q^0\tilde{\varphi}_a^+ = 0, \qquad Q^0\tilde{\varphi}_3^+ = 0,\\ &Q^0\tilde{e}^a = \tilde{\varphi}_a^+\epsilon, \qquad Q^0\tilde{e}^3 = \tilde{\varphi}_3^+\epsilon. \end{aligned} \tag{28}$$

It can be seen from formula (28) that unphysical fields enter in the form of BRST doublets. If we identify the field $\tilde{e}^a$ with the antighost field $\bar{c}$ and the field $\tilde{b}^a m^{-1}$ with the ghost field c, then these transformations provide the decoupling of the fields $\tilde{\varphi}_a^+$, $\tilde{e}^a$, and $\tilde{b}^a$ and the unphysical components of the Yang–Mills field from the transverse components. The remaining transformations provide the decoupling of the components $\tilde{e}^3$, $\tilde{b}^3$, h, and $\tilde{\varphi}_3^+$. Also fixing the Abelian degree of freedom of the vector field, we conclude that all unphysical fields decouple from the transverse fields.

We have thus shown that the modified Lagrangian containing additional unphysical degrees of freedom leads to the same result as does the standard Yang–Mills Lagrangian. The nonuniqueness of the gauge fixing does not arise in quantizing this Lagrangian, and it can be used both in the perturbation theory framework and beyond it. The results presented in this review were first obtained in [14]–[18]. Of course, a complete study of the model includes the question whether renormalization preserves the model symmetry; this was investigated in [17].

Up to now, we considered Yang–Mills fields with a zero mass. This theory is the basis of quantum chromodynamics (QCD). Modern QCD uses one more essential postulate, the color confinement hypothesis. The confinement of color objects cannot be explained in the framework of the perturbation theory in the coupling constant. Lattice simulations are commonly used to explain quark confinement. The models used for this as a rule contain quasiparticle excitations, which we call solitons. On the other hand, it is known that the classical equation of motions in the standard formulation of the Yang–Mills theory have no soliton solutions [19]–[21]. But the arguments forbidding the existence of solitons in the Yang–Mills theory are inapplicable to the modified formulation presented above. In the following sections, we show that classical solutions of the soliton type indeed exist in the QCD based on the modified formulation of the Yang–Mills theory.

4. Expansion in the coupling constant

We choose the original Lagrangian in the form

$$L = -\frac{1}{4}F_{\mu\nu}^i F_{\mu\nu}^i + \frac{1}{2}(D_\mu\varphi)^i(D_\mu\varphi)^i - \frac{1}{2}(D_\mu\chi)^i(D_\mu\chi)^i + \mathrm{i}(D_\mu b)^i(D_\mu e)^i. \tag{29}$$

As before, for simplicity, we consider the group $SU(2)$. Instead of the fields $\varphi^{\pm}$ in this formula, we explicitly introduce the fields χ, which have negative energy. The fields φ and χ commute, and the fields b and e anticommute. All these fields belong to the adjoint representation of the group $SU(2)$.

We assume that the fields φ and χ have a nontrivial asymptotic behavior:

$$|\varphi| \underset{r\to\infty}{\to} \left|\frac{m}{g}\right|, \qquad |\chi| \underset{r\to\infty}{\to} \left|\frac{m\alpha}{g}\right|, \qquad r = |\mathbf{x}|. \tag{30}$$

The parameter $|\alpha| \le 1$, and $\alpha \to 1$ as $g \to 0$. For example,

$$\alpha = \frac{g^{-n} - g^n}{g^{-n} + g^n} = 1 - g^{2n} + \ldots, \tag{31}$$

and hence $\alpha = 1 - O(g^{2n})$. Choosing a sufficiently large n, we obtain results in the formal perturbation theory coinciding with the standard Yang–Mills theory to an arbitrary order in g. In Eq. (30), m is a constant with the dimension of mass.

We call the power series in the coupling constant independently of whether it converges the formal perturbation theory. Separate terms in this series might not even exist in the limit when some intermediate regularization is removed. In QCD, the limit for separate terms of the series may not exist because of infrared divergences. Numerous attempts to construct the Yang–Mills theory analogously to quantum electrodynamics failed because of the nonlinear interaction of the Yang–Mills quanta.

If the coupling constant is small, as in electrodynamics, then the usual relations of the type of unitarity or causality conditions are satisfied in the formal perturbation theory at any order in the coupling constant. But in QCD, the coupling constant is not small, and separate terms in the formal perturbation theory might not even exist because of infrared singularities. Nevertheless, in the formal perturbation theory for the Yang–Mills field, fulfillment of the unitarity and causality conditions is usually required. This standpoint is supported by the observation of jets, which is evidence that QCD is based on some non-Abelian gauge theory. But no rigorous statements can be made, because the hadronization process, which plays an important role in the formation of jets, is essentially nonperturbative.

The only sensible object in the perturbative quantum theory of the Yang–Mills field can be the generating functional for gauge-invariant operators. Below, we show that this functional in our formulation coincides with the standard expression.

We first consider the topologically trivial sector corresponding to the perturbation theory. In this case, we can choose the direction in which the fields do not vanish as the third axis in the charge space. Shifting the variables φ and χ as

$$\varphi^i = \tilde{\varphi}^i + \delta^{i3} m g^{-1}, \qquad \chi^i = \tilde{\chi}^i - \delta^{i3} m\alpha g^{-1}, \tag{32}$$

we obtain the Lagrangian in which $\tilde{\varphi} = 0$ and $\tilde{\chi} = 0$ at infinity, which is necessary for constructing the perturbation theory. We want to prove that the scattering matrix obtained after the shift in the perturbation theory framework (in the limit $\alpha \to 1$) coincides with the standard scattering matrix in the Yang–Mills theory. If $\alpha \ne 1$, then we can speak about the coincidence of the scattering matrices up to an arbitrary order of the formal perturbation theory. Of course, the on-shell scattering matrix does not exist strictly speaking, because of infrared singularities, but we can speak about the vanishing of the matrix elements corresponding to transitions between the states containing only physical excitations and the states containing some unphysical excitations. In both cases, the physical excitations correspond to the transverse components of the Yang–Mills field.

In the topologically trivial sectors, our theory differs from the standard theory: the Yang–Mills theory in the standard formulation has no soliton excitations, but the modified formulation describes classical solitons.

For now, we are interested in only the perturbation theory results and can therefore set the parameter $\alpha = 1$ because $\alpha = 1 - O(g^{2n})$, where n is an arbitrary number. Clearly, no mass term arises for the

Yang–Mills field in this case: the contributions of the terms depending on the fields φ and χ mutually cancel because they have different signs.

The Lagrangian describing the modified theory after shift (32) is

$$\begin{aligned} L = & -\frac{1}{4}F^i_{\mu\nu}F^i_{\mu\nu} + D_\mu\tilde{\varphi}^i_+\, D_\mu\tilde{\varphi}^i_- + \mathrm{i}D_\mu\tilde{b}^i\, D_\mu\tilde{e}^i + \\ & + m\frac{1+\alpha}{\sqrt{2}}D_\mu\tilde{\varphi}^i_+\,\varepsilon^{ij3}A^j_\mu + m\frac{1-\alpha}{\sqrt{2}}D_\mu\tilde{\varphi}^i_-\,\varepsilon^{ij3}A^j_\mu + \frac{m^2(1-\alpha^2)}{2}A^a_\mu A^a_\mu. \end{aligned} \tag{33}$$

Here, we use the obvious notation

$$\tilde{\varphi}^i_\pm = \frac{\tilde{\varphi}^i \pm \tilde{\chi}^i}{\sqrt{2}}, \quad i = 1, 2, 3. \tag{34}$$

This Lagrangian for any α is invariant under the “shifted” gauge transformations

$$\begin{aligned} &\delta A^i_\mu = \partial_\mu\eta^i + g\varepsilon^{ijk}A^j_\mu\eta^k, \\ &\delta\tilde{\varphi}^1_- = -\frac{1+\alpha}{\sqrt{2}}m\eta^2 + g\varepsilon^{1jk}\tilde{\varphi}^j_-\eta^k, \qquad \delta\tilde{\varphi}^1_+ = -\frac{1-\alpha}{\sqrt{2}}m\eta^2 + g\varepsilon^{1jk}\tilde{\varphi}^j_+\eta^k, \\ &\delta\tilde{\varphi}^2_- = \frac{1+\alpha}{\sqrt{2}}m\eta^1 + g\varepsilon^{2jk}\tilde{\varphi}^j_-\eta^k, \qquad \delta\tilde{\varphi}^2_+ = \frac{1-\alpha}{\sqrt{2}}m\eta^1 + g\varepsilon^{2jk}\tilde{\varphi}^j_+\eta^k, \\ &\delta\tilde{\varphi}^3_- = g\varepsilon^{3jk}\tilde{\varphi}^j_-\eta^k, \qquad \delta\tilde{\varphi}^3_+ = g\varepsilon^{3jk}\tilde{\varphi}^j_+\eta^k, \\ &\delta\tilde{b}^i = g\varepsilon^{ijk}\tilde{b}^j\eta^k, \qquad \delta\tilde{e}^i = g\varepsilon^{ijk}\tilde{e}^j\eta^k. \end{aligned} \tag{35}$$

The Gribov ambiguity is absent from the perturbation theory, and we can therefore choose the gauge $\partial_\mu A_\mu = 0$, simultaneously introducing the Faddeev–Popov ghosts $\bar{c}$ and c.

We can write the scattering matrix at $\alpha = 1$ as a path integral

$$\begin{aligned} S = \int d\mu\, \exp\biggl\{\mathrm{i}\biggl[\int d^4x\biggl(&-\frac{1}{4}F^i_{\mu\nu}F^i_{\mu\nu} + D_\mu\tilde{\varphi}^i_+\, D_\mu\tilde{\varphi}^i_- + \\ &+ \lambda^i\partial_\mu A^i_\mu + \mathrm{i}\,\partial_\mu\bar{c}^i\, D_\mu c^i + \mathrm{i}\, D_\mu\tilde{b}^i\, D_\mu\tilde{e}^i + m\sqrt{2}\, D_\mu\tilde{\varphi}^i_+\,\varepsilon^{ij3}A^j_\mu\biggr)\biggr]\biggr\}, \end{aligned} \tag{36}$$

where the integration measure $d\mu$ is the product of differentials of all the fields in the Lagrangian.

For $\alpha = 1$, Lagrangian (33) is invariant under the supersymmetry transformation

$$\delta\tilde{\varphi}^i_- = \tilde{b}^i\epsilon, \qquad \delta\tilde{e}^i = \tilde{\varphi}^i_+\epsilon, \qquad \delta\tilde{b}^i = \delta\tilde{\varphi}^i_+ = 0. \tag{37}$$

It is easy to see that these transformations are nilpotent:

$$\delta^2\tilde{\varphi}^i_- = 0, \qquad \delta^2\tilde{e}^i = 0. \tag{38}$$

This invariance provides the decoupling of excitations corresponding to the fields $\tilde{\varphi}_\pm$, $\tilde{b}$, and $\tilde{e}$. As in the preceding section, the invariance of the effective action under the BRST transformations and the supersymmetry transformations according to the Noether theorem generates the conserved charges Q_B and Q_S, and we can choose the asymptotic states such that they satisfy the equations

$$Q^0_\mathrm{B}|\psi\rangle_\mathrm{ph} = 0, \qquad Q^0_\mathrm{S}|\psi\rangle_\mathrm{ph} = 0, \qquad [Q^0_\mathrm{B}, Q^0_\mathrm{S}]_+ = 0, \tag{39}$$

where Q^0_{B} and Q^0_{S} are asymptotic charges. Any vector satisfying Eqs. (39) has the form

$$|\psi\rangle_{\mathrm{ph}} = |\psi\rangle_{\mathrm{tr}} + |N\rangle, \tag{40}$$

where $|\psi\rangle_{\mathrm{tr}}$ is a vector containing only transverse quanta of the Yang–Mills field and $|N\rangle$ is a vector with a zero norm. It hence follows that the scattering matrix in our formulation coincides with the scattering matrix in the Yang–Mills theory.

But this proof is formal: because of infrared singularities, the scattering matrix does not exist in the Yang–Mills theory. Nevertheless, we can speak about the vanishing of the matrix elements corresponding to the transitions between physical and unphysical states.

The only sensible nontrivial objects in the perturbative Yang–Mills theory are the correlation functions of gauge-invariant operators. It is easy to see that these correlation functions coincide in the standard and modified formulation up to an arbitrary order of the perturbation theory. Indeed, we can repeat the considerations presented above and show that these correlation functions are given by the path integral

$$\begin{aligned} Z = \int d\mu \biggl\{ \exp\biggl[\mathrm{i} \int dx \biggl(-\frac{1}{4} F^i_{\mu\nu} F^i_{\mu\nu} + D_\mu \tilde{\varphi}^i_+ \, D_\mu \tilde{\varphi}^i_- + m\sqrt{2}\, D_\mu \tilde{\varphi}^i_+ \, \varepsilon^{ij3} A^j_\mu + {} \\ + \lambda^i \, \partial_\mu A^i_\mu + \mathrm{i}\, \partial_\mu \bar{c}^i \, D_\mu c^i + \mathrm{i}\, D_\mu b^i \, D_\mu e^i + J(x) O(x) \biggr) \biggr] \biggr\}, \end{aligned} \tag{41}$$

where $J(x)$ is a source and $O(x)$ is a gauge-invariant operator depending only on $A_\mu(x)$. The boundary conditions for all the fields in (41) correspond to vacuum states.

In generating functional (41) for $m = 0$, we can integrate explicitly over the fields $\tilde{\varphi}_\pm$, e, and b. The determinants arising after such an integration compensate each other, because the fields $\tilde{\varphi}_\pm$ and b, e obey different statistics, and for the generating functional of the correlation functions of gauge-invariant operators in the Yang–Mills theory, we obtain the expression

$$Z = \int d\tilde{\mu} \biggl\{ \exp\biggl[\mathrm{i} \int dx \biggl(-\frac{1}{4} F^i_{\mu\nu} F^i_{\mu\nu} + \lambda^i \partial_\mu A^i_\mu + \mathrm{i}\, \partial_\mu \bar{c}^i \, D_\mu c^i + J(x) O(x) \biggr) \biggr] \biggr\}, \tag{42}$$

where the integration measure $d\tilde{\mu}$ is the product of differentials

$$d\tilde{\mu} = dA^i_\mu \, d\lambda^j \, d\bar{c}^k \, dc^l. \tag{43}$$

We would draw the same conclusion if we worked in a gauge applicable beyond perturbation theory, for example, $\tilde{\varphi}^a_- = 0$, $a = 1, 2$, $\partial_\mu A^3_\mu = 0$. Starting from this gauge, we can pass to any admissible gauge. In the next section, we consider the classical theory in the Hamiltonian gauge $A_0 = 0$.

5. Soliton excitations in the modified Yang–Mills theory

In this section, we show that the model constructed above has nontrivial soliton excitations of the 't Hooft–Polyakov magnetic monopole type. In our presentation, we follow [22]. All indices now range $1, 2, 3$.

We consider classical solutions for action (29) and classical solitons with the asymptotic forms

$$\varphi^i \underset{r\to\infty}{\to} \frac{x^i m}{rg}, \qquad \chi^i \underset{r\to\infty}{\to} -\frac{x^i m\alpha}{rg}. \tag{44}$$

We treat stationary solutions in the gauge $A_0 = 0$.

We work in a topologically nontrivial sector and seek the nonperturbative soliton solutions of the classical equations of motion:

$$
\begin{aligned}
&D_i F^a_{ij} + g\varepsilon^{alm}(D_j\varphi)^l\varphi^m - g\varepsilon^{akn}(D_j\chi)^k\chi^n = 0, && A^a_i \underset{r\to\infty}{\to} \varepsilon^{aij}\frac{x^j}{gr^2},\\
&D_i(D_i\varphi)^n = 0, && \varphi^n(x) \underset{r\to\infty}{\to} \frac{x^n m}{gr},\\
&D_i(D_i\chi)^n = 0, && \chi^n(x) \underset{r\to\infty}{\to} -\frac{\alpha x^n m}{gr}.
\end{aligned}
\tag{45}
$$

The chosen asymptotic conditions yield rapidly decreasing covariant derivatives of the fields φ and χ, which is important for a finite soliton energy. With the same goal in mind, we consider only solutions that are nonsingular as $r \to 0$. We cannot now neglect the terms that are small in the formal perturbation theory, because we seek soliton solutions that are unobtainable in the perturbation theory. We also note that the coupling constant is not small in QCD, and the conclusions drawn in the formal perturbation theory might therefore be wrong.

We use the 't Hooft–Polyakov ansatz

$$
\begin{aligned}
&A^i_a(x) = \varepsilon^{aij}\frac{x^j}{r}W(r), \qquad A_0(x) = 0,\\
&\varphi^i(x) = \delta^{ai}\frac{x_a}{r}F(r), \qquad \chi^i(x) = \delta^{ai}\frac{x_a}{r}G(r),\\
&W(r) \underset{r\to\infty}{\to} (gr)^{-1}, \qquad F(r) \underset{r\to\infty}{\to} F\cosh\gamma, \qquad G(r) \underset{r\to\infty}{\to} F\sinh\gamma,\\
&F\cosh\gamma = \frac{m}{g}, \qquad F\sinh\gamma = -\alpha\frac{m}{g}.
\end{aligned}
\tag{46}
$$

If g is small and $\alpha \to 1$, as happens in the electroweak models based on the Brout–Englert–Higgs mechanism, then $\varphi(x) \simeq \chi(x)$, and the equation for the Yang–Mills field has the same form as in the standard Yang–Mills theory. This equation has no soliton solutions. But we can also consider the theories in which the constant g is not small (e.g., QCD).

We can rewrite Eqs. (45) in terms of the functions

$$
K(r) = 1 - grW(r), \qquad J(r) = F(r)rg, \qquad Y(r) = G(r)rg,
\tag{47}
$$

and we then have

$$
\begin{aligned}
&r^2\frac{d^2K}{dr^2} = (K^2 + J^2 - Y^2 - 1)K(r), && K(r) \underset{r\to\infty}{\to} 0,\\
&r^2\frac{d^2J}{dr^2} = 2K^2J, && J(r) \underset{r\to\infty}{\to} Frg\cosh\gamma,\\
&r^2\frac{d^2Y}{dr^2} = 2K^2Y, && Y(r) \underset{r\to\infty}{\to} Frg\sinh\gamma = -\alpha Frg\cosh\gamma.
\end{aligned}
\tag{48}
$$

Following [23], we choose the ansatz

$$
\begin{aligned}
&J(r) = \Lambda(r)\cosh\gamma, && Y(r) = \Lambda(r)\sinh\gamma,\\
&\Lambda(r)\cosh\gamma \underset{r\to\infty}{\to} Frg\cosh\gamma, && \Lambda(r)\sinh\gamma \underset{r\to\infty}{\to} Frg\sinh\gamma.
\end{aligned}
\tag{49}
$$

Equations (48) then become

$$
\begin{aligned}
&r^2\frac{d^2K}{dr^2} = (K^2+\Lambda^2-1)K, \qquad && K \underset{r\to\infty}{\to} 0,\\
&r^2\frac{d^2\Lambda}{dr^2} = 2K^2\Lambda, && \Lambda(r) \underset{r\to\infty}{\to} Frg.
\end{aligned}
\tag{50}
$$

The solutions of this system are well known [24], [25]:

$$
K(r) = \frac{rgF}{\sinh rgF}, \qquad \Lambda(r) = \frac{rgF}{\tanh grF} - 1. \tag{51}
$$

It is easy to calculate the energy corresponding to this solution. This energy is obviously positive and bounded. It is equal to the energy of the magnetic monopole

$$
\begin{aligned}
E &= \int d^3x\left[\frac{1}{4}F^i_{lm}F^i_{lm} + \frac{1}{2}(D_l\varphi)^i(D_l\varphi)^i - \frac{1}{2}(D_l\chi)^i(D_l\chi)^i\right] =\\
&= \int d^3x\left[\frac{1}{4}F^i_{lm}F^i_{lm} + \frac{1}{2}(D_l\Lambda)^a(D_l\Lambda^a)\right].
\end{aligned}
\tag{52}
$$

We can also calculate the magnetic field created by this solution. Using the gauge-invariant definition of the electromagnetic stress tensor, we obtain

$$
F_{\mu\nu} = \hat{\Lambda}^a F^a_{\mu\nu} - g^{-1}\varepsilon^{abc}\hat{\Lambda}^a(D_\mu\hat{\Lambda})^b(D_\nu\hat{\Lambda})^c, \qquad \hat{\Lambda}^a = \frac{\Lambda^a}{|\Lambda|}, \qquad |\Lambda|^2 = \sum_a \Lambda^a\Lambda^a. \tag{53}
$$

We find that the considered excitation is a magnetic monopole, creating the magnetic field

$$
B^i(x) = \frac{x^i}{gr^3}. \tag{54}
$$

It can be seen that even for large g, the mass and magnetic field of the monopole are independent of γ and are determined by the constants F and g.

6. Results

In this review, we have demonstrated that many facts regarded as firmly established in the Yang–Mills theory (the impossibility of quantizing the theory beyond perturbation theory, the unavoidable breaking of the manifest Lorentz invariance of the theory by an algebraic renormalizable gauge, the absence of classical solutions with a finite bounded energy) are in fact related to a specific formulation of the theory. An alternative formulation of the theory is possible that gives the same results in the formal perturbation theory as the standard formulation and allows overcoming these difficulties. This formulation follows the general tendency of the development of gauge fields, introducing new unphysical degrees of freedom and enlarging the symmetry group of the theory. In particular, it is thus possible to quantize the Yang–Mills theory beyond perturbation theory. Naturally, this does not solve the problem of calculations beyond perturbation theory, but it does show that the absence of soliton excitations, which are generally thought necessary to yield the color confinement, is not an unavoidable feature of the theory. As shown, this theory allows an alternative formulation that leads to the existence of soliton solutions of the classical equations. This formulation also allows constructing an infrared regularization that preserves the manifest gauge and Lorentz invariance [18].

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