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MAJORIZATION AND ADDITIVITY FOR MULTIMODE BOSONIC GAUSSIAN CHANNELS

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We obtain a multimode extension of the majorization theorem for bosonic Gaussian channels, in particular, giving sufficient conditions under which the Glauber coherent states are the only minimizers for concave functionals of the output state of such a channel. We discuss direct implications of this multimode majorization for the positive solution of the famous additivity problem in the case of Gaussian channels. In particular, we prove the additivity of the output Rényi entropies of arbitrary order $p > 1$. Finally, we present an alternative, more direct derivation of a majorization property of the Husimi function established by Lieb and Solovej.

Keywords: quantum information theory, bosonic Gaussian communication channel, classical capacity, gauge invariance, minimal output entropy, Gaussian optimizer, additivity

1. Introduction

The longstanding *Gaussian optimizer conjecture* in quantum information theory was recently proved for the class of bosonic Gaussian gauge-covariant or contravariant channels [1]. The conjecture states that the minimum output entropy of a bosonic Gaussian channel is attained on the vacuum state (and also on any coherent state). This result was strengthened in [2] for one-mode channels by establishing that the output for the vacuum or coherent input *majorizes* the output for any other input, in that it minimizes a broad class of concave functionals of the output states. A detailed discussion of the motivation and of applications of these advances to quantum optics and communications can be found in [1], [2].

Here, we obtain further results in this direction. In Sec. 2, we give the multimode extension of the result in [2] and, in particular, a precise formulation of sufficient conditions under which the coherent states are the *only* minimizers. We also discuss direct implications of this multimode majorization for the positive solution of one more famous conjecture, namely, the *additivity problem* for Gaussian channels. In particular, we demonstrate the additivity of the output Rényi entropies of arbitrary order $p > 1$, which generalizes a result of Giovannetti and Lloyd [3] for integer p and special channels.

In Sec. 4, based on the method in [1], we generalize the majorization result of Lieb and Solovej [4]. Wehrl [5] introduced the *classical entropy* of a quantum state ρ by the formula

$$S_{\mathrm{cl}}(\rho) = -\int_{\mathbb{C}^s} \langle z|\rho|z\rangle \log\langle z|\rho|z\rangle \,\frac{d^{2s}z}{\pi^s},$$

where $\langle z|\rho|z\rangle$ is the Husimi function, $|z\rangle$ are the Glauber coherent vectors, and s is the number of modes. Lieb [6] used exact constants in the Hausdorff–Young inequality (Fourier transform) and Young inequality (convolution) to prove the Wehrl conjecture [5]: $S_{\mathrm{cl}}(\rho)$ *is minimized by any coherent state* $\rho = |\zeta\rangle\langle\zeta|$. Lieb

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and Solovej [4] recently gave another derivation based on the limit version of a similar result for Bloch spin coherent states. Moreover, they could thus establish the majorization property of the Glauber coherent states. In Sec. 4, we suggest yet a different (and perhaps most natural) approach to the proof of this property and its generalization motivated by the recent solution of the Gaussian optimizers problem [1].

2. Majorization for gauge-covariant channels

We start by presenting some definitions and notation from [1], restricting to the case of channels with identical input and output spaces. We consider an s-dimensional complex Hilbert space $\mathbf{Z}$ that can be regarded as a $2s$-dimensional real space equipped with the symplectic form $z, z' \to 2\,\mathrm{Im}\, z^* z'$. We regard vectors in $\mathbf{Z}$ as s-dimensional complex column vectors, in which case (complex-linear) operators in $\mathbf{Z}$ are represented by complex $s\times s$ matrices, and the superscript asterisk denotes Hermitian conjugation. The gauge group acts in $\mathbf{Z}$ as multiplication by $e^{i\phi}$, where ϕ is a real number called the phase. The Weyl quantization is described by the unitary displacement operators $D(z)$ acting irreducibly in the representation space $\mathcal{H}$ and satisfying the canonical commutation relation

$$D(z)\, D(z') = e^{-i\,\mathrm{Im}\, z^* z'}\, D(z+z'). \tag{1}$$

Introducing the annihilation–creation operators of the system a_j, $a_j^\dagger$, $j = 1, \ldots, s$, which satisfy the commutation relations $[a_j a_k^\dagger] = \delta_{jk} I$, we can express the operator $D(z)$ as

$$D(z) = \exp\left[\sum_{j=1}^{s} (z_j a_j^\dagger - \bar{z}_j a_j)\right]. \tag{2}$$

The gauge group has the unitary representation $\phi \to U_\phi = e^{i\phi N}$ in $\mathcal{H}$, where $N = \sum_{j=1}^{s} a_j^\dagger a_j$ is the total number operator. The representation of the gauge group in $\mathcal{H}$ acts according to the relation $U_\phi^*\, D(z)\, U_\phi = D(e^{i\phi} z)$, $\phi \in [0, 2\pi]$. A state ρ is then said to be gauge invariant if it commutes with all U_ϕ or, equivalently, if its characteristic function $\phi(z) = \mathrm{Tr}\, \rho D(z)$ is invariant under the action of the gauge group. In particular, Gaussian gauge-invariant states are given by a characteristic function of the form

$$\phi(z) = e^{-z^* \alpha z}, \tag{3}$$

where α is a complex correlation matrix satisfying $\alpha \geq I/2$, where I is the unit $s\times s$ matrix. The vacuum state $|0\rangle\langle 0|$ corresponds to $\alpha = I/2$.

A channel Φ in $\mathcal{H}$ is a completely positive trace-preserving map of the Banach space of trace-class operators in $\mathcal{H}$ (see, e.g., [7] for details). The channel is said to be *gauge covariant* if

$$\Phi[U_\phi \rho U_\phi^*] = U_\phi \Phi[\rho] U_\phi^*. \tag{4}$$

In the Heisenberg picture, a bosonic Gaussian gauge-covariant channel Φ [1] is described by the action of its adjoint Φ^* onto the displacement operators as

$$\Phi^*[D(z)] = D(K^* z) e^{-z^* \mu z}, \tag{5}$$

where K is a complex matrix and μ is a Hermitian matrix satisfying the inequality

$$\mu \geq \pm\frac{1}{2}(I - KK^*). \tag{6}$$

A gauge-covariant channel is *quantum-limited* if μ is a minimal solution of inequality (6). Special cases of maps (5) are provided by the attenuator and amplifier channels characterized by a matrix K satisfying the respective inequalities $KK^* \le I$ and $KK^* \ge I$. We are particularly interested in the *quantum-limited attenuator*, which corresponds to

$$KK^* \le I, \qquad \mu = \frac{1}{2}(I - KK^*), \tag{7}$$

and the *quantum-limited amplifier*,

$$KK^* \ge I, \qquad \mu = \frac{1}{2}(KK^* - I). \tag{8}$$

These channels are diagonalizable: using the singular value decomposition $K = V_B K_{\mathrm{d}} V_A$ where V_A and V_B are unitaries and K_{d} is a diagonal matrix with nonnegative values on the diagonal, we have $KK^* = V_B K_{\mathrm{d}} K_{\mathrm{d}}^* V_B^*$ and

$$\Phi[\rho] = U_B \Phi_{\mathrm{d}}[U_A \rho U_A^*] U_B^*, \tag{9}$$

where

$$\Phi_{\mathrm{d}} = \bigotimes_{j=1}^{s} \Phi_j \tag{10}$$

is a tensor product of one-mode quantum-limited channels defined by the matrix K_{d} and U_A and U_B are the canonical unitary transformations acting on $\mathcal{H}$ such that

$$U_B^*\, D(z)\, U_B = D(V_B^* z), \qquad U_A^*\, D(z)\, U_A = D(V_A^* z)$$

(we note that $U_A|0\rangle = |0\rangle$ and $U_B|0\rangle = |0\rangle$).

Theorem 1. 1. *Let Φ be a Gaussian gauge-covariant channel and f be a concave function on* $[0, 1]$ *such that $f(0) = 0$. Then*

$$\operatorname{Tr} f(\Phi[\rho]) \ge \operatorname{Tr} f(\Phi[|\zeta\rangle\langle\zeta|]) = \operatorname{Tr} f(\Phi[|0\rangle\langle 0|]) \tag{11}$$

for all states ρ and any coherent state $|\zeta\rangle\langle\zeta|$ (the value on the right is the same for all coherent states by the displacement covariance property of a Gaussian channel [7]).

2. *If f is strictly concave and the channel Φ satisfies one of the two conditions*

a. *K is invertible and*[1]

$$\mu > \frac{1}{2}(KK^* - I), \tag{12}$$

b. *$KK^* > I$ and $\mu = (KK^* - I)/2$ (hence Φ is a quantum-limited amplifier),*

then the equality in (11) *is attained only when ρ is a coherent state.*

Such a result was obtained in [2] in the case of one mode. Our goal here is to generalize it to the case of many modes, in particular by making the conditions in statement 2 in Theorem 1 precise.

Proof. 1. By the concavity of f, it suffices to prove (11) for pure states $\rho = |\psi\rangle\langle\psi|$. As shown in [1] (also see Proposition 2 in Appendix A), any gauge-covariant channel can be represented as a concatenation $\Phi = \Phi_2 \circ \Phi_1$ of a quantum-limited attenuator Φ_1 with an operator K_1 and a quantum-limited amplifier

[1]For Hermitian matrices M and N, the strict inequality $M > N$ means that $M - N$ is positive definite.

Φ_2 with an operator K_2. An argument similar to [2] then shows that it suffices to prove (11) only for the amplifier Φ_2. Indeed, if

$$\operatorname{Tr} f(\Phi_2[|\psi\rangle\langle\psi|]) \geq \operatorname{Tr} f(\Phi_2[|0\rangle\langle 0|]) \tag{13}$$

for any state vector $|\psi\rangle$, then we can consider the spectral decomposition

$$\Phi_1[|\psi\rangle\langle\psi|] = \sum_j p_j |\phi_j\rangle\langle\phi_j|,$$

where $p_j > 0$. Then

$$\begin{aligned}\operatorname{Tr} f(\Phi[|\psi\rangle\langle\psi|]) &= \operatorname{Tr} f(\Phi_2[\Phi_1[|\psi\rangle\langle\psi|]]) \geq \\ &\geq \sum_j p_j \operatorname{Tr} f(\Phi_2[|\phi_j\rangle\langle\phi_j|]) \geq \\ &\geq \operatorname{Tr} f(\Phi_2[|0\rangle\langle 0|]) = \\ &= \operatorname{Tr} f(\Phi_2[\Phi_1[|0\rangle\langle 0|]]) = \operatorname{Tr} f(\Phi[|0\rangle\langle 0|]) \end{aligned} \tag{14}$$

because the vacuum is an invariant state of a quantum-limited attenuator.

We now prove (13). Because

$$\min_\rho \operatorname{Tr} f(\Phi_2[\rho]) = \min_\rho \operatorname{Tr} f(U_B \Phi_{\mathrm{d}}[U_A^* \rho U_A] U_B^*) = \min_\rho \operatorname{Tr} f(\Phi_{\mathrm{d}}[\rho]),$$

it suffices to consider the diagonal amplifier. The proof for a one-mode quantum-limited amplifier is based on the fact that the complementary channel has the representation (also based on Proposition 2 in Appendix A)

$$\tilde{\Phi}_2 = \mathrm{T} \circ \Phi_2 \circ \Phi_1', \tag{15}$$

where T is transposition defined by the relation $\mathrm{T}[D(z)] = D(-\bar{z})$, $\bar{z}$ is the complex conjugate vector, and Φ_1' is another quantum-limited attenuator defined by the operator $K_1' = \sqrt{I - K_2^{-2}}$. But for a diagonal multimode amplifier, the expression for the complementary channel and also representation (15) (with a diagonal Φ_1') follows from the results for each mode.

Representation (15) implies that nonzero spectra of the density operators $\Phi_2[\rho]$ and $\Phi_2 \circ \Phi_1'[\rho]$ coincide for pure inputs $\rho = |\psi\rangle\langle\psi|$ [1]. Then similarly to (14),

$$\begin{aligned}\operatorname{Tr} f(\Phi_2[|\psi\rangle\langle\psi|]) &= \operatorname{Tr} f(\Phi_2[\Phi_1'[|\psi\rangle\langle\psi|]]) \geq \\ &\geq \sum_j p_j' \operatorname{Tr} f(\Phi_2[|\phi_j'\rangle\langle\phi_j'|]), \end{aligned} \tag{16}$$

where

$$\Phi_1'[|\psi\rangle\langle\psi|] = \sum_j p_j' |\phi_j'\rangle\langle\phi_j'|, \quad p_j' > 0, \tag{17}$$

is the spectral decomposition of the output of the quantum-limited attenuator Φ_1'. For the moment, we assume that f is strictly concave. We then conclude that for any pure minimizer $\rho = |\psi\rangle\langle\psi|$ of $\operatorname{Tr} f(\Phi_2[|\psi\rangle\langle\psi|])$, sum (17) necessarily contains only one term, i.e.,

$$\Phi_1'[|\psi\rangle\langle\psi|] = |\phi'\rangle\langle\phi'|. \tag{18}$$

Indeed, otherwise the inequality in (16) by the strict concavity of f is strict, contradicting the assumption that $|\psi\rangle\langle\psi|$ is a minimizer of $\operatorname{Tr} f(\Phi_2[|\psi\rangle\langle\psi|])$ (strict concavity of f also excludes nonpure minimizers). Next, we first consider the amplifier with $K_2 > I$. The associated attenuator Φ_1' is then defined by the operator $K_1' = \sqrt{I - K_2^{-2}}$ such that $0 < K_1' < I$. We then apply the following lemma.

Lemma 1. *Let Φ_1' be the diagonal quantum-limited attenuator defined by an operator K_1' such that $0 < K_1' < I$. Then* (18) *implies that $|\psi\rangle\langle\psi|$ is a coherent state.*

For one mode, this is Lemma 2 in [2], which implies that any pure input ρ such that $\Phi_1'[\rho]$ is also a pure state is a coherent state. The proof is based on the explicit expression for the complementary channel $\tilde{\Phi}_1'$. By using this expression for each mode, we can generalize the proof to the case of a diagonal multimode channel Φ_1'.

This proves (13) for a strictly concave f and for the amplifiers Φ_2 with $K_2 > I$. An arbitrary concave f can then be monotonically approximated by strictly concave functions by setting $f_\varepsilon(x) = f(x) - \varepsilon x^2$ and passing to the limit $\varepsilon \downarrow 0$ in (13).

In the case of a diagonal amplifier Φ_2 with $K_2 \geq I$, we take any sequence of diagonal operators $K^{(n)} > I$, $K^{(n)} \to K_2$, and consider the corresponding diagonal amplifiers $\Phi_2^{(n)}$. Then $\|\Phi_2^{(n)}[\rho] - \Phi_2[\rho]\|_1 \to 0$ and $\operatorname{Tr} f(\Phi_2^{(n)}[\rho]) \to \operatorname{Tr} f(\Phi_2[\rho])$ for any concave polygonal function f on $[0,1]$ such that $f(0) = 0$. This follows because any such function is Lipschitz, $|f(x) - f(y)| \leq \varkappa|x - y|$, and hence

$$|\operatorname{Tr} f(\Phi_2^{(n)}[\rho]) - \operatorname{Tr} f(\Phi_2[\rho])| \leq \varkappa\|\Phi_2^{(n)}[\rho] - \Phi_2[\rho]\|_1.$$

This implies that (13) holds for polygonal concave functions f and all quantum-limited amplifiers. Hence, by (14), the inequality (11) with such f holds for for all Gaussian gauge-covariant channels. For an arbitrary concave f on $[0,1]$, there is a monotonically nondecreasing sequence of concave polygonal functions f_m converging to f pointwise. Passing to the limit $m \to \infty$ gives the first statement.

2. Case *a*: We note that the conditions on the channel Φ imply that the attenuator Φ_1 in the decomposition $\Phi = \Phi_2 \circ \Phi_1$ is defined by an operator K_1 such that $0 < K_1^* K_1 < I$ (see Remark 1 in Appendix A). Applying the argument involving relations (16) with a strictly concave f to relations (14), we find that for any pure minimizer $\rho = |\psi\rangle\langle\psi|$ of $\operatorname{Tr} f(\Phi[|\psi\rangle\langle\psi|])$, the output of the quantum-limited attenuator $\Phi_1[|\psi\rangle\langle\psi|]$ is necessarily a pure state. Applying Lemma 1 to the attenuator Φ_1, we conclude that $|\psi\rangle\langle\psi|$ is necessarily a coherent state.

Case *b*: In case *b*, we just apply the argument involving relations (16) with strictly concave f to the quantum-limited amplifier $\Phi = \Phi_2$. ■

Theorem 1 can be extended to a Gaussian gauge-contravariant channel satisfying $\Phi[U_\phi \rho U_\phi^*] = U_\phi^* \Phi[\rho] U_\phi$ instead of (4). The proof follows because the complement $\tilde{\Phi}_2$ of the diagonal quantum-limited amplifier Φ_2 is just a diagonal quantum-limited gauge-contravariant channel (see [1] for details).

3. Implications for the additivity

For any $p > 1$, the output purity of a channel Φ is defined as

$$\nu_p(\Phi) = \sup_{\rho \in \mathfrak{S}(\mathcal{H})} \operatorname{Tr} \Phi[\rho]^p.$$

Corollary 1. *For any Gaussian gauge-covariant channel Φ, the output purity is equal to $\nu_p(\Phi) = \operatorname{Tr} \Phi[|0\rangle\langle 0|]^p$. The multiplicativity property*

$$\nu_p(\Phi \otimes \Psi) = \nu_p(\Phi)\nu_p(\Psi) \tag{19}$$

holds for any two Gaussian gauge-covariant channels Φ and Ψ.

Proof. The first statement follows from Theorem 1 by taking $f(x) = -x^p$ such that

$$\nu_p(\Phi) = -\min_{\rho} \operatorname{Tr} f(\Phi[\rho]).$$

The second statement then follows because the channel $\Phi \otimes \Psi$ is also gauge-covariant and from the multiplicativity of the vacuum state. ■

The output purity for channel (5) can be computed explicitly as

$$\nu_p(\Phi) = \det\left[\left(\mu + \frac{KK^*}{2} + \frac{I}{2}\right)^p - \left(\mu + \frac{KK^*}{2} - \frac{I}{2}\right)^p\right].$$

The formula follows because the state $\Phi[|0\rangle\langle 0|]$ is Gaussian with the covariance matrix $\mu + KK^*/2$ and from the expression for the spectrum of a Gaussian density operator [8].

The minimal output Rényi entropy of a channel Φ is expressed via its output purity as

$$\check{R}_p(\Phi) = \frac{1}{1-p} \log \nu_p(\Phi),$$

and multiplicativity property (19) can be rewritten as the additivity of the minimal output Rényi entropy:

$$\check{R}_p(\Phi \otimes \Psi) = \check{R}_p(\Phi) + \check{R}_p(\Psi). \tag{20}$$

In the limit $p \downarrow 1$ (or taking $f(x) = -x \log x$), we recover the additivity of the minimal output von Neumann entropy established in [1]:

$$\min_{\rho_{12}} H((\Phi \otimes \Psi)[\rho_{12}]) = \min_{\rho_1} H(\Phi[\rho_1]) + \min_{\rho_2} H(\Phi[\rho_2]).$$

The additivity result in [1] is more general in that it allows the case where one of the channels is gauge-covariant, while the other is contravariant. On the other hand, the proof in [1] is restricted to states with finite second moments, while the present proof does not require this.

4. Majorization for quantum–classical Gaussian channel

It is helpful to regard the map $\rho \to \langle z|\rho|z\rangle$ as a "quantum–classical Gaussian channel" transforming Gaussian density operators into Gaussian probability densities. We consider a more general transformation

$$\rho \to p_\rho(z) = \operatorname{Tr} \rho\, D(z)\, \rho_0\, D(z)^*,$$

where $D(z)$ are the displacement operators and ρ_0 is the Gaussian gauge-invariant state with the quantum characteristic function $\phi_0(z) = e^{-z^* \alpha_0 z}$, where $\alpha_0 \geq I/2$. We note that $p_\rho(z) = \langle z|\rho|z\rangle$ if ρ_0 is the vacuum state corresponding to $\alpha_0 = I/2$.

The function $p_\rho(z)$ is bounded by 1 and is a continuous probability density, and the normalization follows from the resolution of the identity operator in $\mathcal{H}$,

$$\int_{\mathbb{C}^s} D(z)\, \rho_0\, D(z)^* \, \frac{d^{2s}z}{\pi^s} = I_{\mathcal{H}}.$$

Proposition 1. *Let f be a concave function on $[0,1]$ such that $f(0) = 0$. Then for an arbitrary state ρ,*

$$\int_{\mathbb{C}^s} f(p_\rho(z))\, \frac{d^{2s}z}{\pi^s} \geq \int_{\mathbb{C}^s} f(p_{|\zeta\rangle\langle\zeta|}(z))\, \frac{d^{2s}z}{\pi^s}. \tag{21}$$

Proof. For any $c > 0$, we consider the "measure-reprepare" channel Φ_c defined by the relation

$$\Phi_c[\rho] = \int \frac{d^{2s}z}{\pi^s c^{2s}} \,\mathrm{Tr}[\rho\, D(c^{-1}z)\, \rho_0\, D^*(c^{-1}z)]\, D(z)\, \rho_0'\, D^*(z), \tag{22}$$

where ρ_0' is another gauge-invariant Gaussian state with the characteristic function $\phi_0'(z) = e^{-z^*\alpha_0' z}$. Map (22) is a gauge-covariant bosonic Gaussian channel that acts on $D(z)$ in the Heisenberg representation as

$$\Phi_c^*[D(z)] = D(cz)\, e^{-z^*(\alpha_0' + c^2\alpha_0)z}$$

(cf. [1]). Therefore, by Theorem 1,

$$\mathrm{Tr}\, f(\Phi_c[\rho]) \geq \mathrm{Tr}\, f(\Phi_c[|\zeta\rangle\langle\zeta|]) \tag{23}$$

for all states ρ and any coherent state $|\zeta\rangle\langle\zeta|$. We prove the proposition by taking the limit $c \to \infty$.

In the proof, we also use a simple generalization of the Berezin–Lieb inequalities [9],

$$\int_{\mathbb{C}^s} f(\underline{p}(z))\, \frac{d^{2s}z}{\pi^s} \leq \mathrm{Tr}\, f(\sigma) \leq \int_{\mathbb{C}^s} f(\bar{p}(z))\, \frac{d^{2s}z}{\pi^s}, \tag{24}$$

which holds for any quantum state admitting the representation

$$\sigma = \int_{\mathbb{C}^s} \underline{p}(z)\, D(z)\, \rho_0'\, D(z)^*\, \frac{d^{2s}z}{\pi^s}$$

with a probability density $\underline{p}(z)$. In the right-hand side of (24),

$$\bar{p}(z) = \mathrm{Tr}\, \sigma D(z)\, \rho_0'\, D^*(z).$$

The original inequalities refer to the case where ρ_0 is a pure state, but the proof applies to the more general case (see Appendix B). In inequalities (24), we must assume that f is defined on $[0, \infty)$ (in fact, $\underline{p}(z)$ can be unbounded). We assume this for now.

Taking $\sigma = \Phi_c[\rho]$, we obtain

$$\underline{p}(z) = \frac{1}{c^{2s}}\, \mathrm{Tr}\, \rho\, D(c^{-1}z)\, \rho_0\, D^*(c^{-1}z) = \frac{1}{c^{2s}}\, p_\rho(c^{-1}z)$$

from (22), while

$$\bar{p}(z) = \mathrm{Tr}\, \Phi_c[\rho]\, D(z)\, \rho_0'\, D(z)^* = \int_{\mathbb{C}^s} \underline{p}(w)\, \mathrm{Tr}\, \rho_0'\, D(z-w)\, \rho_0'\, D(z-w)^*\, \frac{d^{2s}w}{\pi^s}. \tag{25}$$

Using the quantum Parceval formula [10], we obtain

$$\pi^{-s}\, \mathrm{Tr}\, \rho_0'\, D(z)\, \rho_0'\, D(z)^* = \int_{\mathbb{C}^s} \phi_0'(w)^2 e^{2i\,\mathrm{Im}\, z^* w}\, \frac{d^{2s}w}{\pi^{2s}} =$$

$$= \pi^{-s} \det(2\alpha_0')^{-1} e^{-z^*[\alpha_0']^{-1}z/2} \equiv q_{\alpha_0'}(z),$$

which is the probability density of a Gaussian distribution. Substituting this in (25), we obtain

$$\bar{p}(z) = \int d^{2s}w\, \underline{p}(w)\, q_{\alpha_0'}(z-w) = $$
$$= \int d^{2s}w'\, p_\rho(w')\, q_{\alpha_0'}(z-cw') = $$
$$= \frac{1}{c^{2s}}\, p_\rho * q_{\alpha_0'/c^2}(c^{-1}z). \tag{26}$$

Here, $q_{\alpha_0'/c^2}(z) = c^{2s} q_{\alpha_0'}(cz)$ is the probability density of a Gaussian distribution tending to the δ-function as $c \to \infty$.

With the change of the integration variable $c^{-1}z \to z$, inequalities (24) become

$$\int_{\mathbb{C}^s} f(c^{-2s}p_\rho(z))\, \frac{d^{2s}z}{\pi^s} \le c^{-2s}\, \mathrm{Tr}\, f(\Phi_c[\rho]) \le \int_{\mathbb{C}^s} f\big(c^{-2s}p_\rho * q_{\alpha_0'/c^2}(z)\big)\, \frac{d^{2s}z}{\pi^s}.$$

Substituting $\rho = |\zeta\rangle\langle\zeta|$, we obtain

$$\int_{\mathbb{C}^s} f\big(c^{-2s}p_{|\zeta\rangle\langle\zeta|}(z)\big)\frac{d^{2s}z}{\pi^s} \le c^{-2s}\, \mathrm{Tr}\, f(\Phi_c[|\zeta\rangle\langle\zeta|]) \le \int_{\mathbb{C}^s} f\big(c^{-2s}p_{|\zeta\rangle\langle\zeta|} * q_{\alpha_0'/c^2}(z)\big)\, \frac{d^{2s}z}{\pi^s}.$$

Combining the last two displayed formulas with (23), we obtain

$$\int_{\mathbb{C}^s} g(p_\rho(z))\, \frac{d^{2s}z}{\pi^s} - \int_{\mathbb{C}^s} g(p_{|\zeta\rangle\langle\zeta|}(z))\, \frac{d^{2s}z}{\pi^s} \ge$$
$$\ge \int_{\mathbb{C}^s} g(p_\rho(z))\, \frac{d^{2s}z}{\pi^s} - \int_{\mathbb{C}^s} g\big(p_\rho * q_{\alpha_0'/c^2}(z)\big)\, \frac{d^{2s}z}{\pi^s}, \tag{27}$$

where we set $g(x) = f(c^{-2s}x)$, which is again a concave function. Moreover, an arbitrary concave polygonal function g on $[0,1]$ satisfying $g(0) = 0$ can be thus obtained by defining

$$f(x) = \begin{cases} g(c^{2s}x), & x \in [0, c^{-2s}], \\ g(1) + g'(1)(x - c^{-2s}), & x \in [c^{-2s}, \infty), \end{cases}$$

and (27) hence holds for any such function. The right-hand side of inequality (27) then tends to zero as $c \to \infty$. Indeed, for a polygonal function, we have $|g(x) - g(y)| \le \varkappa|x-y|$, and the asserted convergence follows from the convergence $p_\rho * q_{\alpha_0'/c^2} \longrightarrow p_\rho$ in L_1: if $p(z)$ is a bounded continuous probability density, then

$$\lim_{c\to\infty} \int_{\mathbb{C}^s} |p * q_{\alpha_0'/c^2}(z) - p(z)|\, d^{2s}z = 0.$$

We thus obtain (21) for concave polygonal functions f. But for an arbitrary concave f on $[0,1]$, there is a monotonically nondecreasing sequence of concave polygonal functions f_n converging to f. Applying the Beppo–Levy theorem, we obtain the statement. ■

Appendix A

The concatenation $\Phi = \Phi_2 \circ \Phi_1$ of two Gaussian gauge-covariant channels Φ_1 and Φ_2 obeys the rule

$$K = K_2 K_1, \tag{28}$$

$$\mu = K_2 \mu_1 K_2^* + \mu_2. \tag{29}$$

Proposition 2 [1]. *Any bosonic Gaussian gauge-covariant channel Φ is a concatenation of a quantum-limited attenuator Φ_1 and a quantum-limited amplifier Φ_2.*

Proof. Substituting

$$\mu_1 = \frac{1}{2}(I - K_1 K_1^*) = \frac{1}{2}(I - |K_1^*|^2), \qquad \mu_2 = \frac{1}{2}(K_2 K_2^* - I) = \frac{1}{2}(|K_2^*|^2 - I)$$

in (29) and using (28), we obtain

$$|K_2^*|^2 = K_2 K_2^* = \mu + \frac{1}{2}(KK^* + I) \geq \begin{cases} I, \\ KK^* \end{cases} \tag{30}$$

from inequality (6). Using the operator monotonicity of the square root, we obtain

$$|K_2^*| \geq I, \qquad |K_2^*| \geq |K^*|.$$

The first inequality in (30) implies that choosing

$$K_2 = |K_2^*| = \sqrt{\mu + \frac{1}{2}(KK^* + I)} \tag{31}$$

and the corresponding $\mu_2 = (|K_2^*|^2 - I)/2$, we obtain a (diagonalizable) quantum-limited amplifier.

With

$$K_1 = |K_2^*|^{-1} K, \tag{32}$$

taking the second inequality in (30) into account, we then obtain

$$K_1^* K_1 = K^* |K_2^*|^{-2} K = K^* \left[\mu + \frac{1}{2}(KK^* + I)\right]^{-1} K \leq I, \tag{33}$$

which implies $K_1^* K_1 \leq I$. Hence, K_1 with the corresponding $\mu_1 = (I - K_1 K_1^*)/2$ gives a quantum-limited attenuator.

Remark 1. Inequality (12) via (33) implies $K_1^* K_1 < I$. The invertibility of K implies $K_1^* K_1 > 0$.

Appendix B

For completeness, we sketch the proof of the required generalization of the Berezin–Lieb inequalities. Let $\mathcal{X}$ be a measurable space with a σ-finite measure μ, and let $P(x)$ be a weakly measurable function on $\mathcal{X}$ whose values are density operators in a separable Hilbert space $\mathcal{H}$ such that

$$\int_{\mathcal{X}} P(x)\, \mu(dx) = I_{\mathcal{H}},$$

where the integral converges in the sense of weak operator topology. Let ρ be a density operator in $\mathcal{H}$ admitting the representation

$$\rho = \int_{\mathcal{X}} \underline{p}(x) P(x)\, \mu(dx),$$

where $\underline{p}(x)$ is a bounded probability density. We set $\bar{p}(x) = \operatorname{Tr} \rho P(x)$, which is a probability density uniformly bounded by 1. For a concave function f defined on $[0, \infty)$ and satisfying $f(0) = 0$, we then have

$$\int_{\mathcal{X}} f(\underline{p}(x))\, \mu(dx) \leq \operatorname{Tr} f(\rho) \leq \int_{\mathcal{X}} f(\bar{p}(x))\, \mu(dx). \tag{34}$$

We set $k = \max\{1, \sup_x \underline{p}(x)\}$ and consider the restriction of f to $[0, k]$. Then there is a monotonically nondecreasing sequence of concave polygonal functions f_n converging to f pointwise on $[0, k]$ and satisfying $f_n(0) = 0$. Because $|f_n(x)| \leq \varkappa_n |x|$, the integrals and the trace in (34) with f replaced with f_n are finite for all n. We prove (34) for concave polygonal functions f_n and then take the limit $n \to \infty$. This also shows that the integrals and trace in (34) are well defined although they may take the value $+\infty$.

The second inequality follows from $\operatorname{Tr} f(\rho) P(x) \leq f(\operatorname{Tr} \rho P(x))$, which is a consequence of the Jensen inequality applied together with the spectral decomposition of ρ. To prove the first inequality, we consider the positive operator-valued measure

$$M(B) = \int_B P(x)\, \mu(dx), \quad B \subseteq \mathcal{X},$$

and its Naimark dilation to a projection-valued measure $\{E(B)\}$ in a larger Hilbert space $\widetilde{\mathcal{H}} \supseteq \mathcal{H}$. We consider the bounded operator $R = \int_{\mathcal{X}} \underline{p}(x)\, E(dx)$ in $\widetilde{\mathcal{H}}$. Then

$$f(R) = \int_{\mathcal{X}} f(\underline{p}(x))\, E(dx)$$

and

$$\rho = PRP, \qquad Pf(R)P = \int_{\mathcal{X}} f(\underline{p}(x))\, \mu(dx),$$

where P is the projection from $\widetilde{\mathcal{H}}$ onto $\mathcal{H}$. The required inequality then follows from the more general fact $\operatorname{Tr} Pf(R)P \leq \operatorname{Tr} f(PRP)$ [11].

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