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A COUPLE OF METHODOLOGICAL COMMENTS ON THE QUANTUM YANG–MILLS THEORY

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We present methodological proposals regarding the definition of the notion of the effective action, the coupling constant renormalization, and the interpretation of dimensional transmutation. We show that the divergences that arise when quantizing a Yang–Mills field can be eliminated and lead to violation of the scaling invariance of the classical theory.

Keywords: effective action, dimensional transmutation

1. Introduction

It is already 35 years since the book written by Andrei Slavnov and myself was published [1]. That was the first book on quantum field theory where the presentation was based on Feynman's functional integral. But in hindsight, I think that we were insufficiently radical: a number of points could have been improved. In Feynman's spirit, we should not use Green–Schwinger functions but rely only on the S-matrix. Incidentally, the same ideology was also shared by N. N. Bogoliubov. The background field method must be formulated having its application to the definition of the S-matrix in mind.

In the effective action method, it is especially simple to describe the renormalization of a single parameter of the classical theory, the coupling constant g^2, which enters the formalism as a coefficient of the classical action. Divergent coefficients in the effective action are collected in a renormalized coupling constant g_{r}^2.

I have already made methodological remarks at various conferences, whose proceedings have been published [2], [3], but they have not yet appeared in scientific journals except [4].

2. Effective action

We use the following notation. The Yang–Mills field $A_\mu(x)$ takes values in the Lie algebra of the gauge group. The curvature $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_{mu} + [A_\mu, A_\nu],$$

and the classical action has the form

$$\mathcal{A}(x) = \frac{1}{4g^2} \operatorname{tr} F_{\mu\nu}^2,$$

where g plays the role of the classical coupling constant.

In the definition of the S-matrix in terms of the functional integral proposed in [1], the functional

$$\exp\left\{ i \int d^4x\, \mathcal{A}(x) \right\}$$

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is integrated over the fields $A_\mu(x)$ with a fixed asymptotic behavior at large times,

$$A_\mu \to A_\mu^{\text{in,out}}, \quad t \to \mp\infty,$$

where the free fields A_μ^{in} and A_μ^{out} have the respective prescribed incoming wave A_μ^- and outgoing wave A_μ^+. The S-matrix becomes a functional of A_μ^- and A_μ^+, i.e., a generating function for the S-matrix elements. In the functional integral, we must integrate over classes of gauge-equivalent fields, which is realized by introducing the appropriate ghosts.

To calculate the S-matrix, it is convenient to use the background field formalism. We set

$$A_\mu = B_\mu + g a_\mu,$$

where the gauge field B_μ satisfies the same asymptotic conditions as A_μ and the vector field a_μ has no incoming waves as $t \to -\infty$ and no outgoing waves as $t \to \infty$. These conditions uniquely define the quadratic form operator $\Box = \nabla_\mu^2$.

Gauge fixing is achieved by imposing the condition $\nabla_\mu a_\mu = 0$ or by adding the term $(1/2)\,\mathrm{tr}(\nabla_\mu a_\mu)^2$ to the action. The corresponding ghost term has the form

$$\mathrm{tr}\big(\nabla_\mu \bar{c}(\nabla_\mu + g a_\mu)c\big).$$

As a result, we must integrate a functional containing quadratic forms $\big(M_1(B)a, a\big)$ and $\big(M_0(B)\bar{c}, c\big)$, where

$$M_0(B)c = \nabla_\mu^2 c, \qquad M_1(B)a_\nu = \nabla_\mu^2 a_\nu + 2[F_{\mu\nu}, a_\mu],$$

and also vertices of the first, third, and fourth orders:

$$\Gamma_1(B) = \frac{1}{g}\int d^4x \ \mathrm{tr}(\nabla_\mu F_{\mu\nu} a_\nu),$$

$$\Gamma_3(B) = g\int d^4x \ \mathrm{tr}\,\nabla_\mu a_\nu [a_\mu, a_\nu],$$

$$\Gamma_4(B) = g^2\int d^4x \ \mathrm{tr}[a_\mu, a_\nu]^2,$$

$$\Omega(B) = g\int d^4x \ \mathrm{tr}\,\nabla_\mu \bar{c}[a_\mu, c].$$

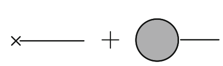
The S-matrix is represented as a functional of the effective action:

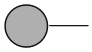
$$S = \exp\{iW(B)\}.$$

In calculating $W(B)$, it is customarily assumed that the vertex $\Gamma_1(B)$ vanishes. In other words, the background field is subjected to the condition

$$\nabla_\mu F_{\mu\nu}(B) = 0,$$

which is the classical equation of motion. In the case of finite-dimensional integrals, this is a natural setting of the stationary phase method. But we suggest imposing another condition on B_μ, which has the graphic representation


$= 0,$ (1)

where the block  is the sum of strongly connected diagrams with one external leg. The asymptotic condition can be imposed on the solution of this equation.

The term "strongly connected" is synonymous with "1-particle-irreducible." At the level of playing with words, we can say that Eq. (1) determines the self-action of asymptotic particles.

Under that condition, we can write the functional $W(B)$ in the form

$$W(B) = \frac{1}{4g^2} \int F_{\mu\nu}^2 d^4x - \frac{1}{2} \log\det M_1 + \log\det M_0 + \sum_{n=1}^{\infty} g^{2n} W_n(B), \tag{2}$$

where $W_n(B)$ is the sum of strongly connected diagrams with $n+1$ loops.

We make a brief remark. The notion of the effective action is the subject of a vast literature. Within the commonly accepted method, we must start with the functional for Green's functions, and the external field is introduced via the Legendre transformation. The external current is a nuisance in the Yang–Mills theory because it violates the manifest gauge invariance. The method proposed here is more direct and better suits the Yang–Mills theory.

3. Renormalization of the coupling constant

In the effective action formalism, renormalization amounts to redefining the coupling constant g^2. Following Landau, we regard it as a function of the regularization parameter ϵ and verify that the contribution of divergent integrals in expansion (2) contains expressions of the type $C_k(g,\epsilon) \int d^4x\, F_{\mu\nu}^2$, whose combination with the classical contribution to the action gives the renormalized charge:

$$\frac{1}{g_{\mathrm{r}}^2} = \frac{1}{g^2(\epsilon)} + \sum C_k(g,\epsilon).$$

In the terminology of the preceding section, we already described the entire procedure in [4]. Here, we only describe the one-loop contribution in greater detail than in [4], showing all the characteristic features of the renormalization procedure. After the (infinite) constant $W_0(0)$ is subtracted, this contribution becomes

$$W_0(B) - W_0(0) = -\frac{1}{2} \log\det \frac{M_1(B)}{M_1(0)} + \log\det \frac{M_0(B)}{M_0(0)}.$$

To calculate these determinants, it is natural to use the proper time method, which can be traced back to Fock [5]:

$$\log\det \frac{A}{B} = -\int_0^{\infty} \frac{ds}{s} \operatorname{Tr}(e^{-sA} - e^{-sB}).$$

The operators $e^{-M_1(B)s}$ and $e^{-M_0(B)s}$ are integral operators with the kernels $D_{\mu\nu}(x,y;s)$ and $D(x,y;s)$ and with values in the gauge algebra. The behavior of these kernels as $s \to 0$ is given by the known expansion

$$D(x,y;s) = a_{-1}(x,y)s^{-2} + a_0(x,y)s^{-1} + a_1(x,y) + a_2(x,y)s + \dots, \tag{3}$$

where the coefficients $a_i(x,y)$, $i = -1, 0, 1, 2, \dots$, can be evaluated explicitly (a procedure that returns us to the early 20th century). In our case, the first term in (3) cancels under subtraction of the contribution from $e^{-M(0)s}$. Next, we have

$$\operatorname{tr} a_0(x,x) = 0,$$

and the expression for $\operatorname{tr} D(x,x;s)$ hence starts with the term $\operatorname{tr} a_1(x,x)$, which is proportional to $\operatorname{tr} F_{\mu\nu}^2$ (which in fact follows from locality and the dimension count). Hence, we have

$$W_0 = \int_0^{\infty} \frac{ds}{s} D(B,s), \tag{4}$$

where

$$D(B,s) = \beta \int d^4x \ \mathrm{tr}\, F_{\mu\nu}^2 + O(s)$$

as $s \to 0$. We regularize the integral in (4):

$$\int_0^\infty \frac{ds}{s}(\,\cdot\,) = \int_\epsilon^\mu \frac{ds}{s}(\,\cdot\,) + \int_\mu^\infty \frac{ds}{s}(\,\cdot\,),$$

which yields

$$W_{-1}(B) + W_0(B) = \left(\frac{1}{4g^2} + \beta \log\frac{\mu}{\epsilon}\right) \int d^4x \ \mathrm{tr}\, F_{\mu\nu}^2 + \\ + \int_0^\mu \frac{ds}{s}\left(D(B,s) - \beta \int d^4x \ \mathrm{tr}\, F_{\mu\nu}^2\right) + \int_\mu^\infty \frac{ds}{s} D(B,s). \quad (5)$$

All the divergences are contained in the first term in the right-hand side. In the process of regularizing, we introduced an auxiliary value μ, and the renormalized charge acquires a dependence on it:

$$\frac{1}{g_{\mathrm{r}}^2(\mu)} = \frac{1}{g^2(\epsilon)} + 4\beta \log\frac{\mu}{\epsilon}. \quad (6)$$

The renormalized charge is customarily called the running coupling constant.

Expression (5) is obviously independent of μ. Therefore, the notion of a running coupling constant loses meaning, and we replace it with a dimensional constant m that plays the role of a separator constant in (6):

$$\frac{1}{g^2(\epsilon)} = 4\beta \log \epsilon m^2, \qquad \frac{1}{g_{\mathrm{r}}^2(\mu)} = 4\beta \log \mu m^2. \quad (7)$$

We see that the renormalized and nonrenormalized charges are the values of the same function $g^2(s)$ at the respective renormalization point $s = \mu$ and regularization point $s = \epsilon$. The constant β is easy to evaluate (see, e.g., [6]); it is negative and, apart from twos and factors of π, contains the celebrated value 11/3. This remarkable property means that formula (7) does make sense: the quantity $g(\epsilon)$ must tend to zero as $\epsilon \to 0$, and $g(\mu)$ has a meaning if $\mu m^2 < 1$. The role of the only parameter in the theory is now taken by m, which has the dimension of mass. The replacement $g^2 \to m$ is called dimensional transmutation. But it would be incorrect to assume that the quantum Yang–Mills theory is nontrivially parameterized by m. In fact, m is merely a scaling parameter.

Renormalization in higher loops does not change our argument. The independence of physical answers from the normalization μ is guaranteed by the Gell-Mann–Low equation. Along the lines of the presentation in this paper, we described this in [4].

4. Conclusions

In conclusion, if Andrei and I had attempted yet another re-edition of our book, I would have insisted on introducing the notion of effective action as a generating functional for the S-matrix and describing renormalization within that formalism. It would also be important to emphasize that divergences are not a deficiency of quantum theory. They only violate the scaling invariance of the classical theory and result in the quantum theory becoming a dynamical model free of arbitrary parameters.

The main unsolved problem in Yang–Mills theory is to describe excitations. We must confess that in defining the scattering matrix, we explicitly used asymptotic conditions based on free motion. Just this allows uniquely defining the operators inverse to M_1 and M_0. Alternatively, we could resort to the Euclidean formulation and the definition of asymptotic states under the reverse transition to the formulation with a chosen time direction. I believe that it would then be important to understand the role of the quantum equations of motion introduced in Sec. 2. In particular, as a result of the appearance of the dimensional parameter m, the existence of soliton solutions becomes possible, which must then be the true one-particle excitations. Speculations on this subject can be found in [7].

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REFERENCES

1. A. A. Slavnov and L. D. Faddeev, *Introduction to the Quantum Theory of Gauge Fields* [in Russian], Nauka, Moscow (1988); English transl.: *Gauge Fields: An Introduction to Quantum Theory* (Frontiers Phys., Vol. 83), Westview, Boulder, Colo. (1993).
2. L. D. Faddeev, *Bull. Braz. Math. Soc.*, n.s., **33**, 201–212 (2002); arXiv:0911.1013v1 [math-ph] (2009).
3. L. D. Faddeev, "Separation of scattering and selfaction revisited," arXiv:1003.4854v1 [hep-th] (2010).
4. L. D. Faddeev, *Theor. Math. Phys.*, **148**, 986–994 (2006).
5. V. Fock, *Phys. Z. Sowjetunion*, **12**, 404–425 (1937).
6. I. Jack and H. Osborn, *Nucl. Phys. B*, **207**, 474–504 (1982).
7. L. D. Faddeev, "Knots as possible excitations of the quantum Yang–Mills fields," in: *Quantum Field Theory and Beyond: Essays in Honor of Wolfhart Zimmermann* (E. Seiler, K. Sibold, eds.), World Scientific, Hackensack, N. J. (2008), pp. 156–166; arXiv:0805.1624v1 [hep-th] (2008).