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DETERMINANT REPRESENTATIONS FOR FORM FACTORS IN QUANTUM INTEGRABLE MODELS WITH THE $GL(3)$-INVARIANT R-MATRIX

© **S. Z. Pakuliak,* E. Ragoucy,† and N. A. Slavnov‡**

We obtain determinant representations for the form factors of the monodromy matrix elements in quantum integrable models solvable by the nested algebraic Bethe ansatz and having the $GL(3)$-invariant R-matrix. These representations can be used to calculate correlation functions in physically interesting models.

Keywords: nested algebraic Bethe ansatz, scalar product, form factor

1. Introduction

The form-factor approach is one of the most effective methods for calculating correlation functions of quantum integrable models. Therefore, finding explicit and compact representations for the form factors is an important task. There are currently several methods for studying form factors of integrable systems. One of the first methods developed was the so-called form factor bootstrap approach, which has been successfully applied to integrable quantum field theory [1]–[7]. This method is closely related to the method based on conformal field theory and its perturbation [8]–[11]. We also mention the approach developed in [12]–[14], where the form factors were studied via the representation theory of quantum affine algebras. All the methods listed above deal with quantum integrable models in an infinite volume. Form factors of models in a finite volume were studied in [15], [16] using the algebraic Bethe ansatz [17]–[20]. In particular, this method was found to be very effective for quantum spin chain models, for which the solution of the quantum inverse scattering problem is known [16], [21]. Determinant representations for form factors obtained in this framework were successfully used to calculate correlation functions [22]–[25].

The results listed above mostly concern models based on the $GL(2)$ symmetry or its q-deformation. Models with a higher-rank symmetry have been studied much less. At the same time, such models play an important role in various applications. For instance, integrability was proved to be a very effective tool for calculating scattering amplitudes in super-Yang–Mills theories [26]–[28]. Calculating these amplitudes can be related to calculating scalar products of Bethe vectors. In particular, in the $SU(3)$ subsector of the theory, we need the $SU(3)$-invariant Bethe vectors. Hence, knowing the form factors in this context is very essential.

Form factors of integrable models with higher-rank symmetries also appear in condensed matter physics, in particular, in a two-component Bose (or Fermi) gas and in studying models of cold atoms (e.g., ferro-

*Joint Institute for Nuclear Research, Dubna, Moscow Oblast, Russia; Moscow Institute of Physics and Technology, Dolgoprudny, Moscow Oblast, Russia; Institute of Theoretical and Experimental Physics, Moscow, Russia, e-mail: pakuliak@theor.jinr.ru.

†Laboratoire d'Annecy-le-Vieux de Physique Théorique, CNRS – Université de Savoie, Annecy-le-Vieux, France, e-mail: ragoucy@lapth.cnrs.fr.

‡Steklov Mathematical Institute, RAS, Moscow, Russia, e-mail: nslavnov@mi.ras.ru.

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magnetism or phase separation). We can also mention two-band Hubbard models (mostly in the half-filled regime) in the context of strongly correlated electron systems. In this case, the symmetry increases when spin and orbital degrees of freedom are assumed to play a symmetric role, leading to an $SU(4)$ or even an $SO(8)$ symmetry (see, e.g., [29], [30]). All these studies require seeking integrable models with an $SU(N)$ symmetry, the first step being the $SU(3)$ case. In this context, we mention [31], where the form factors in the model of a two-component Bose gas were studied.

Here, we give determinant representations for form factors in $GL(3)$-invariant quantum integrable models solvable by the nested algebraic Bethe ansatz [32]–[34]. More precisely, we calculate matrix elements of the monodromy matrix operators $T_{ij}(z)$ between on-shell Bethe vectors (i.e., between eigenstates of the transfer matrix). The determinant representations that we give here are based on formulas obtained in [35]–[37]. But there we had slightly different representations for the form factors of the diagonal elements $T_{ii}(z)$ and of the elements $T_{ij}(z)$ with $|i-j|=1$. Furthermore, in the case of the operators $T_{ii}(z)$, we had to consider two different cases depending on whether two Bethe vectors coincided or differed. Here, we give more uniform determinant representations for all form factors. We also announce determinant formulas for the form factors of the operators $T_{13}(z)$ and $T_{31}(z)$. To derive these formulas, we used a new approach, which requires a detailed description to be given in a separate publication.

This paper is organized as follows. In Sec. 2, we introduce the model under consideration. In Sec. 3, we recall the results for form factors in the models with $GL(2)$ symmetries. In Sec. 4, we present our main results. In Sec. 5, we briefly describe the methods for deriving them and, in particular, introduce the notion of a twisted transfer matrix, which seems very effective for calculating form factors of diagonal elements. In Sec. 6, we present a proof of some determinant representations given in Sec. 4. In Sec. 7, we discuss some prospects. The appendix contains several summation identities needed for proving the determinant representations.

2. Bethe vectors and form factors

In this section, we describe the model under consideration, introduce the necessary notation, and define the object of our study.

2.1. Generalized $GL(3)$-invariant model. The models considered below are described by the $GL(3)$-invariant R-matrix acting in the tensor product of two auxiliary spaces $V_1 \otimes V_2$, where $V_k \sim \mathbb{C}^3$, $k=1,2$:

$$R(x,y) = \mathbf{I} + g(x,y)\mathbf{P}, \qquad g(x,y) = \frac{c}{x-y}. \tag{2.1}$$

In this definition, $\mathbf{I}$ is the identity matrix in $V_1 \otimes V_2$, $\mathbf{P}$ is the permutation matrix that exchanges V_1 and V_2, and c is an arbitrary nonzero constant.

The monodromy matrix $T(w)$ satisfies the algebra

$$R_{12}(w_1,w_2)T_1(w_1)T_2(w_2) = T_2(w_2)T_1(w_1)R_{12}(w_1,w_2). \tag{2.2}$$

Equation (2.2) holds in the tensor product $V_1 \otimes V_2 \otimes \mathcal{H}$, where $V_k \sim \mathbb{C}^3$, $k=1,2$, are the auxiliary linear spaces and $\mathcal{H}$ is the Hilbert space of the Hamiltonian of the model. The matrices $T_k(w)$ act nontrivially in $V_k \otimes \mathcal{H}$.

The trace in the auxiliary space $V \sim \mathbb{C}^3$ of the monodromy matrix, $\operatorname{tr} T(w)$, is called the transfer matrix. It is a generating functional of integrals of motion of the model. The eigenvectors of the transfer matrix are called on-shell Bethe vectors (or simply on-shell vectors). They can be parameterized by sets of complex parameters satisfying Bethe equations (see Sec. 2.3).

2.2. Notation. We use the same notation and conventions as in [36]. In addition to the function $g(x,y)$, we also introduce the function

$$f(x,y) = 1 + g(x,y) = \frac{x-y+c}{x-y}. \tag{2.3}$$

We also use two other auxiliary functions

$$h(x,y) = \frac{f(x,y)}{g(x,y)} = \frac{x-y+c}{c}, \qquad t(x,y) = \frac{g(x,y)}{h(x,y)} = \frac{c^2}{(x-y)(x-y+c)}. \tag{2.4}$$

Because of the obvious property $g(-x,-y) = g(y,x)$, all the functions introduced above have similar properties:

$$f(-x,-y) = f(y,x), \qquad h(-x,-y) = h(y,x), \qquad t(-x,-y) = t(y,x). \tag{2.5}$$

Before describing the Bethe vectors, we formulate a convention for the notation. We designate sets of variables by a bar: $\bar{w}$, $\bar{u}$, $\bar{v}$, etc. Individual elements of the sets are denoted by subscripts, w_j, u_k, etc., and $\bar{u}_i$ and $\bar{v}_i$, for example, respectively mean $\bar{u} \setminus u_i$ and $\bar{v} \setminus v_i$.

To avoid too cumbersome formulas, we use a shorthand notation for products of functions g, f, and h. Namely, if these functions depend on sets of variables, then the product is taken over the corresponding set. For example,

$$h(z,\bar{w}) = \prod_{w_j \in \bar{w}} h(z,w_j), \qquad g(u_i,\bar{u}_i) = \prod_{\substack{u_j \in \bar{u}, \\ u_j \neq u_i}} g(u_i,u_j), \qquad f(\bar{u},\bar{v}) = \prod_{u_j \in \bar{u}} \prod_{v_k \in \bar{v}} f(u_j,v_k).$$

We also use a special notation $\Delta'_n(\bar{x})$ and $\Delta_n(\bar{x})$ for the products

$$\Delta'_n(\bar{x}) = \prod_{j<k}^{n} g(x_j,x_k), \qquad \Delta_n(\bar{x}) = \prod_{j>k}^{n} g(x_j,x_k). \tag{2.6}$$

2.3. Bethe vectors. We now describe Bethe vectors. Generic Bethe vectors are denoted by $\mathbb{B}^{a,b}(\bar{u};\bar{v})$. They are parameterized by two sets of complex parameters $\bar{u} = u_1,\dots,u_a$ and $\bar{v} = v_1,\dots,v_b$ with $a,b = 0,1,\dots$. Dual Bethe vectors are denoted by $\mathbb{C}^{a,b}(\bar{u};\bar{v})$. They also depend on two sets of complex parameters $\bar{u} = u_1,\dots,u_a$ and $\bar{v} = v_1,\dots,v_b$. The state with $\bar{u} = \bar{v} = \varnothing$ is called a pseudovacuum vector $|0\rangle$. Similarly, the dual state with $\bar{u} = \bar{v} = \varnothing$ is called a dual pseudovacuum vector $\langle 0|$. These vectors are annihilated by the operators $T_{ij}(w)$, where $i > j$ for $|0\rangle$ and $i < j$ for $\langle 0|$. At the same time, both vectors are eigenvectors for the diagonal elements of the monodromy matrix,

$$T_{ii}(w)|0\rangle = \lambda_i(w)|0\rangle, \qquad \langle 0|T_{ii}(w) = \lambda_i(w)\langle 0|, \tag{2.7}$$

where $\lambda_i(w)$ are some scalar functions. In the framework of the generalized model, $\lambda_i(w)$ remain free functional parameters. In fact, we can always normalize the monodromy matrix $T(w) \to \lambda_2^{-1}(w)T(w)$ such that we deal with only the ratios

$$r_1(w) = \frac{\lambda_1(w)}{\lambda_2(w)}, \qquad r_3(w) = \frac{\lambda_3(w)}{\lambda_2(w)}. \tag{2.8}$$

If the parameters $\bar{u}$ and $\bar{v}$ of a Bethe vector[1] satisfy a special system of equations (Bethe equations), then the vector becomes an eigenvector of the transfer matrix (on-shell Bethe vector). The system of Bethe equations can be written in the form

$$\begin{aligned} r_1(u_i) &= \frac{f(u_i,\bar{u}_i)}{f(\bar{u}_i,u_i)} f(\bar{v},u_i), \quad i=1,\dots,a, \\ r_3(v_j) &= \frac{f(\bar{v}_j,v_j)}{f(v_j,\bar{v}_j)} f(v_j,\bar{u}), \quad j=1,\dots,b, \end{aligned} \tag{2.9}$$

and we recall that $\bar{u}_i = \bar{u}\setminus u_i$ and $\bar{v}_j = \bar{v}\setminus v_j$.

If $\bar{u}$ and $\bar{v}$ satisfy system (2.9), then

$$\begin{aligned} \operatorname{tr} T(w)\, \mathbb{B}^{a,b}(\bar{u};\bar{v}) &= \tau(w|\bar{u},\bar{v})\, \mathbb{B}^{a,b}(\bar{u};\bar{v}), \\ \mathbb{C}^{a,b}(\bar{u};\bar{v}) \operatorname{tr} T(w) &= \tau(w|\bar{u},\bar{v})\, \mathbb{C}^{a,b}(\bar{u};\bar{v}), \end{aligned} \tag{2.10}$$

where

$$\tau(w) \equiv \tau(w|\bar{u},\bar{v}) = r_1(w) f(\bar{u},w) + f(w,\bar{u}) f(\bar{v},w) + r_3(w) f(w,\bar{v}). \tag{2.11}$$

Remark 1. We note that system of Bethe equations (2.9) is equivalent to the statement that the function $\tau(w|\bar{u},\bar{v})$ in (2.11) has no poles at the points $w=u_i$ and $w=v_j$.

Form factors of the monodromy matrix elements are defined as

$$\mathcal{F}^{(i,j)}_{a,b}(z) \equiv \mathcal{F}^{(i,j)}_{a,b}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B) = \mathbb{C}^{a',b'}(\bar{u}^C;\bar{v}^C) T_{ij}(z) \mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B), \tag{2.12}$$

where both $\mathbb{C}^{a',b'}(\bar{u}^C;\bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)$ are on-shell Bethe vectors and

$$a' = a + \delta_{i1} - \delta_{j1}, \qquad b' = b + \delta_{j3} - \delta_{i3}. \tag{2.13}$$

We use the superscripts B and C here to distinguish the sets of parameters in these two vectors. In other words, unless explicitly specified, the variables $\{\bar{u}^B;\bar{v}^B\}$ in $\mathbb{B}^{a,b}$ and $\{\bar{u}^C;\bar{v}^C\}$ in $\mathbb{C}^{a,b}$ are not assumed to be related. The parameter z is an arbitrary complex number. Acting with the operator $T_{ij}(z)$ on $\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)$ via formulas obtained in [38], we reduce the form factor to a linear combination of scalar products in which $\mathbb{C}^{a',b'}(\bar{u}^C;\bar{v}^C)$ is an on-shell vector.

2.4. Relations between form factors. Obviously, there exist nine form factors of $T_{ij}(z)$ in models with the $GL(3)$-invariant R-matrix; they are not all independent. In particular, because the R-matrix is invariant under transposition with respect to both spaces, the map[2]

$$\psi\colon T_{ij}(u) \;\mapsto\; T_{ji}(u) \tag{2.14}$$

defines an antimorphism of algebra (2.2). Acting on the Bethe vectors, this antimorphism sends them into the dual ones, and vice versa,

$$\psi(\mathbb{B}^{a,b}(\bar{u};\bar{v})) = \mathbb{C}^{a,b}(\bar{u};\bar{v}), \qquad \psi(\mathbb{C}^{a,b}(\bar{u};\bar{v})) = \mathbb{B}^{a,b}(\bar{u};\bar{v}). \tag{2.15}$$

[1]For simplicity here and hereafter, we do not distinguish between vectors and dual vectors.

[2]For simplicity, maps (2.14), (2.15), and (2.16) acting on the operators, vectors, and form factors are denoted by the same letter ψ. The same holds for maps (2.17), (2.18), and (2.19) below.

Therefore, we have

$$\psi(\mathcal{F}_{a,b}^{(i,j)}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B))=\mathcal{F}_{a',b'}^{(j,i)}(z|\bar{u}^B,\bar{v}^B;\bar{u}^C,\bar{v}^C), \tag{2.16}$$

and the form factor $\mathcal{F}_{a,b}^{(i,j)}(z)$ can hence be obtained from $\mathcal{F}_{a,b}^{(j,i)}(z)$ using the replacements $\{\bar{u}^C,\bar{v}^C\}\leftrightarrow\{\bar{u}^B,\bar{v}^B\}$ and $\{a,b\}\leftrightarrow\{a',b'\}$.

One more relation between form factors arises because of the map φ,

$$\varphi\colon T_{ij}(u)\;\mapsto\;T_{4-j,4-i}(-u), \tag{2.17}$$

which defines an isomorphism of algebra (2.2) [38]. This isomorphism implies the Bethe vector transformation

$$\varphi(\mathbb{B}^{a,b}(\bar{u};\bar{v}))=\mathbb{B}^{b,a}(-\bar{v};-\bar{u}),\qquad\varphi(\mathbb{C}^{a,b}(\bar{u};\bar{v}))=\mathbb{C}^{b,a}(-\bar{v};-\bar{u}). \tag{2.18}$$

Because the map φ relates the operators T_{11} and T_{33}, it also leads to the replacement of functions $r_1\leftrightarrow r_3$. Hence,

$$\varphi(\mathcal{F}_{a,b}^{(i,j)}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B))=\mathcal{F}_{b,a}^{(4-j,4-i)}(-z|-\bar{v}^C,-\bar{u}^C;-\bar{v}^B,-\bar{u}^B)\big|_{r_1\leftrightarrow r_3}. \tag{2.19}$$

In total, we are left with not more than four independent form factors, for example, the form factors of the operators $T_{11}(z)$, $T_{22}(z)$, $T_{12}(z)$, and $T_{13}(z)$.

3. Form factors in $GL(2)$-based models

Before giving our main results, we recall the determinant representations for form factors previously obtained in the integrable models with the $GL(2)$-invariant R-matrix [15], [16]. In fact, these results can be treated as particular cases of form factors in the models with the $GL(3)$-invariant R-matrix, which correspond to special Bethe vectors with $a=0$ or $b=0$. Below, we set $b=0$ for definiteness. Let

$$\mathbb{C}^a(\bar{u})=\mathbb{C}^{a,0}(\bar{u};\varnothing),\qquad\mathbb{B}^a(\bar{u})=\mathbb{B}^{a,0}(\bar{u};\varnothing). \tag{3.1}$$

Bethe vectors (3.1) become on-shell if the parameters $\bar{u}$ satisfy the system of Bethe equations

$$r_1(u_i)=\frac{f(u_i,\bar{u}_i)}{f(\bar{u}_i,u_i)}=(-1)^{a-1}\frac{h(u_i,\bar{u})}{h(\bar{u},u_i)},\quad i=1,\dots,a. \tag{3.2}$$

Then

$$\begin{aligned}\big(T_{11}(w)+T_{22}(w)\big)\mathbb{B}^a(\bar{u})&=\tau_2(w|\bar{u})\,\mathbb{B}^a(\bar{u}),\\ \mathbb{C}^a(\bar{u})\big(T_{11}(w)+T_{22}(w)\big)&=\tau_2(w|\bar{u})\,\mathbb{C}^a(\bar{u}),\end{aligned} \tag{3.3}$$

where

$$\tau_2(w)\equiv\tau_2(w|\bar{u})=r_1(w)f(\bar{u},w)+f(w,\bar{u}). \tag{3.4}$$

The form factors of the monodromy matrix elements in the $GL(2)$-based models are defined as

$$\mathcal{F}_a^{(i,j)}(z)\equiv\mathcal{F}_a^{(i,j)}(z|\bar{u}^C;\bar{u}^B)=\mathbb{C}^{a'}(\bar{u}^C)T_{ij}(z)\mathbb{B}^a(\bar{u}^B), \tag{3.5}$$

where both vectors are on-shell. For brevity, we use the notation $a'=a+j-i$.

All the representations for the form factors of the operators $T_{ij}(z)$, $i,j=1,2$, are based on the determinant formula for the scalar product of an on-shell Bethe vector and a generic Bethe vector [39]. This formula immediately implies such representations for $\mathcal{F}_a^{(1,2)}(z)$ and $\mathcal{F}_a^{(2,1)}(z)$. Namely, let $\bar{x}=\{\bar{u}^B,z\}$. Then

$$\mathcal{F}_a^{(1,2)}(z)=\Delta'_{a'}(\bar{u}^C)\Delta_{a'}(\bar{x})\,\det_{a'}n_{jk}, \tag{3.6}$$

where

$$n_{jk} = \frac{c}{g(x_k, \bar{u}^C)} \frac{\partial \tau_2(x_k|\bar{u}^C)}{\partial u_j^C}. \tag{3.7}$$

The result for $\mathcal{F}_a^{(2,1)}(z)$ can be obtained from (3.6) and (3.7) by replacing $\bar{u}^C \leftrightarrow \bar{u}^B$ and $a' \leftrightarrow a$:

$$\mathcal{F}_a^{(2,1)}(z) = \Delta'_a(\bar{u}^B)\Delta_a(\bar{y}) \det_a \left(\frac{c}{g(y_k, \bar{u}^B)} \frac{\partial \tau_2(y_k|\bar{u}^B)}{\partial u_j^B} \right), \tag{3.8}$$

where $\bar{y} = \{\bar{u}^C, z\}$.

There are several equivalent formulas for form factors of the diagonal elements $T_{ss}(z)$, $s = 1, 2$. Here, we give representations in the form of determinants of matrices of the size $(a+1) \times (a+1)$. We have

$$\mathcal{F}_a^{(s,s)}(z) = \Delta'_a(\bar{u}^C)\Delta_{a+1}(\bar{x}) \det_{a+1} n_{jk}^{(s)}, \quad s = 1, 2, \tag{3.9}$$

where $\bar{x} = \{\bar{u}^B, z\}$. The elements $n_{jk}^{(s)}$ of the matrices $n^{(s)}$ in the first a rows ($j = 1, \dots, a$) coincide with the elements of matrix (3.7). But we note that the number of elements in the set $\bar{u}^C$ in (3.7) is equal to $a+1$ while we have $\#\bar{u}^C = a$ for the form factor $\mathcal{F}_a^{(s,s)}(z)$. We can say that $\#\bar{u}^C = a'$ in both cases. In the last row, we have

$$n_{a+1,k}^{(1)} = (-1)^a r_1(x_k) h(\bar{u}^B, x_k), \qquad n_{a+1,k}^{(2)} = h(x_k, \bar{u}^B). \tag{3.10}$$

Remark 2. We note that by virtue of Bethe equations (3.2), we have

$$n_{a+1,k}^{(1)} + n_{a+1,k}^{(2)} = 0 \quad \text{for } k = 1, \dots, a$$

(i.e., if $x_k \in \bar{u}^B$). Therefore, the form factor of the transfer matrix $T_{11}(z) + T_{22}(z)$ reduces to the eigenvalue $\tau(z|\bar{u}^B)$ multiplied by the scalar product of the vectors $\mathbb{C}^a(\bar{u}^C)$ and $\mathbb{B}^a(\bar{u}^B)$. This result, of course, follows immediately from the definition of on-shell Bethe vectors.

Replacing $\bar{u}^C \leftrightarrow \bar{u}^B$ in (3.9) and (3.10), we obtain alternative determinant representations for the form factors of the operators $T_{ss}(z)$. Despite the very different appearance of these two types of representations, their equivalence can be proved (see, e.g., [40]).

Therefore, we see that the form factors of the monodromy matrix elements in the $GL(2)$-based models are proportional to the Jacobians of the eigenvalues $\tau_2(w)$ on the left or right Bethe vector (up to a possible modification of one row).

4. Main results

The results given in Sec. 3 suggest their possible generalization to models with the $GL(3)$-invariant R-matrix. Indeed, it seems quite reasonable to expect that the form factors of the monodromy matrix elements in such models are also proportional to the Jacobians of the transfer matrix eigenvalue. But this conjecture is only partly confirmed. In this section, we show that determinant representations of form factors of the operators $T_{ij}(z)$ in the $GL(3)$-based models are more sophisticated.

4.1. Form factors of off-diagonal elements. The determinant representations of form factors of the operators $T_{ij}(z)$ with $|i-j| = 1$ have the simplest structure. They were calculated in [37]. We start our exposition with the form factor $\mathcal{F}_{a,b}^{(1,2)}(z)$:

$$\mathcal{F}_{a,b}^{(1,2)}(z) \equiv \mathcal{F}_{a,b}^{(1,2)}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}^{a',b'}(\bar{u}^C; \bar{v}^C) T_{12}(z) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \tag{4.1}$$

where both $\mathbb{C}^{a',b'}(\bar{u}^C;\bar{v}^C)$ and $\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)$ are on-shell Bethe vectors. As in the $GL(2)$ case, we use the a' and b' notation, whose definition depends on the form factor we consider. For $\mathcal{F}_{a,b}^{(1,2)}(z)$, we have $a'=a+1$ and $b'=b$.

To describe the determinant representation for this form factor, we introduce a set of variables $\bar{x}=\{x_1,\dots,x_{a'+b}\}$ as the union of three sets $\bar{x}=\{\bar{u}^B,\bar{v}^C,z\}$ and define a scalar function $\mathcal{H}_{a',b}$ as

$$\mathcal{H}_{a',b}=\frac{h(\bar{x},\bar{u}^B)h(\bar{v}^C,\bar{x})}{h(\bar{v}^C,\bar{u}^B)}\,\Delta'_{a'}(\bar{u}^C)\Delta'_b(\bar{v}^B)\Delta_{a+b+1}(\bar{x}). \tag{4.2}$$

The following proposition was proved in [37].

Proposition 4.1. *The form factor* $\mathcal{F}_{a,b}^{(1,2)}(z)$ *admits the determinant representation*

$$\mathcal{F}_{a,b}^{(1,2)}(z)=\mathcal{H}_{a',b}\,\det_{a'+b}\mathcal{N}, \tag{4.3}$$

where the $(a'+b)\times(a'+b)$ *matrix* $\mathcal{N}$ *has the elements*

$$\mathcal{N}_{j,k}=\frac{c}{f(x_k,\bar{u}^B)f(\bar{v}^C,x_k)}\frac{g(x_k,\bar{u}^B)}{g(x_k,\bar{u}^C)}\frac{\partial\tau(x_k|\bar{u}^C,\bar{v}^C)}{\partial u_j^C},\qquad j=1,\dots,a', \tag{4.4}$$

$$\mathcal{N}_{a'+j,k}=\frac{-c}{f(x_k,\bar{u}^B)f(\bar{v}^C,x_k)}\frac{g(\bar{v}^C,x_k)}{g(\bar{v}^B,x_k)}\frac{\partial\tau(x_k|\bar{u}^B,\bar{v}^B)}{\partial v_j^B},\quad j=1,\dots,b. \tag{4.5}$$

We see that this representation involves two eigenvalues of the transfer matrix. Namely, the elements in the first $a+1$ rows of $\mathcal{N}$ are proportional to the derivatives of the eigenvalue $\tau(x_k|\bar{u}^C,\bar{v}^C)$ on the left vector, while the elements in the last b rows of $\mathcal{N}$ are proportional to the derivatives of the eigenvalue $\tau(x_k|\bar{u}^B,\bar{v}^B)$ on the right vector. Hence, as mentioned at the beginning of this section, this determinant representation is not a straightforward generalization of formula (3.6). Nevertheless, it can be easily seen that Eq. (4.3) at $b=0$ reproduces result (3.6).

Determinant representations for other form factors $\mathcal{F}_{a,b}^{(i,j)}(z)$ with $|i-j|=1$ can be derived from (4.3) by maps (2.16) and (2.19). First, we give explicit formulas for the form factor of the operator T_{23}:

$$\mathcal{F}_{a,b}^{(2,3)}(z)\equiv\mathcal{F}_{a,b}^{(2,3)}(z|\bar{u}^C,\bar{v}^C;\bar{u}^B,\bar{v}^B)=\mathbb{C}^{a',b'}(\bar{u}^C;\bar{v}^C)T_{23}(z)\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B), \tag{4.6}$$

where we now have $a'=a$ and $b'=b+1$.

We introduce a set of variables $\bar{y}=\{y_1,\dots,y_{a+b'}\}$ as the union of three sets, $\bar{y}=\{\bar{u}^C,\bar{v}^B,z\}$, and a function

$$\widetilde{\mathcal{H}}_{a,b'}=\frac{h(\bar{y},\bar{u}^C)h(\bar{v}^B,\bar{y})}{h(\bar{v}^B,\bar{u}^C)}\,\Delta'_a(\bar{u}^B)\Delta'_{b'}(\bar{v}^C)\Delta_{a+b+1}(\bar{y}). \tag{4.7}$$

The following proposition was also proved in [37].

Proposition 4.2. *The form factor* $\mathcal{F}_{a,b}^{(2,3)}(z)$ *admits the determinant representation*

$$\mathcal{F}_{a,b}^{(2,3)}(z)=\widetilde{\mathcal{H}}_{a,b'}\,\det_{a+b'}\widetilde{\mathcal{N}}, \tag{4.8}$$

where the $(a+b')\times(a+b')$ *matrix* $\widetilde{\mathcal{N}}$ *has the elements*

$$\widetilde{\mathcal{N}}_{j,k}=\frac{c}{f(y_k,\bar{u}^C)f(\bar{v}^B,y_k)}\frac{g(y_k,\bar{u}^C)}{g(y_k,\bar{u}^B)}\frac{\partial\tau(y_k|\bar{u}^B,\bar{v}^B)}{\partial u_j^B},\qquad j=1,\dots,a, \tag{4.9}$$

$$\widetilde{\mathcal{N}}_{a+j,k}=\frac{-c}{f(y_k,\bar{u}^C)f(\bar{v}^B,y_k)}\frac{g(\bar{v}^B,y_k)}{g(\bar{v}^C,y_k)}\frac{\partial\tau(y_k|\bar{u}^C,\bar{v}^C)}{\partial v_j^C},\quad j=1,\dots,b'. \tag{4.10}$$

Using (2.5), we can easy verify that the representation of $\mathcal{F}^{(2,3)}_{a,b}(z)$ can be obtained from the one for $\mathcal{F}^{(1,2)}_{a,b}(z)$ by replacing

$$\bar{u}^C \leftrightarrow -\bar{v}^C, \qquad \bar{u}^B \leftrightarrow -\bar{v}^B, \qquad r_1 \leftrightarrow r_3, \qquad a \leftrightarrow b, \tag{4.11}$$

as prescribed by isomorphism (2.19).

At the same time, we can see that the formulas for these two form factors are also related by the replacements

$$\{\bar{u}^C, \bar{v}^C\} \leftrightarrow \{\bar{u}^B, \bar{v}^B\}, \qquad \{a, b\} \leftrightarrow \{a', b'\}. \tag{4.12}$$

But care is necessary in doing these transformations: the definition of a' and b' changes when going from $\mathcal{F}^{(1,2)}_{a,b}(z)$ to $\mathcal{F}^{(2,3)}_{a,b}(z)$ (and vice versa).

Applying map (2.16) to representations (4.3) and (4.8), we obtain the following proposition [37].

Proposition 4.3. *The form factor $\mathcal{F}^{(3,2)}_{a,b}(z)$ admits the determinant representation*

$$\mathcal{F}^{(3,2)}_{a,b}(z) = \mathcal{H}_{a',b} \det_{a'+b} \mathcal{N}, \tag{4.13}$$

where $\mathcal{H}_{a',b}$ and $\mathcal{N}$ are respectively given by (4.2) *and* (4.3).

The form factor $\mathcal{F}^{(2,1)}_{a,b}(z)$ admits the determinant representation

$$\mathcal{F}^{(2,1)}_{a,b}(z) = \widetilde{\mathcal{H}}_{a,b'} \det_{a+b'} \widetilde{\mathcal{N}}, \tag{4.14}$$

where $\widetilde{\mathcal{H}}_{a,b'}$ and $\widetilde{\mathcal{N}}$ are respectively given by (4.7) *and* (4.8).

Remark 3. We again emphasize that although representations (4.13) and (4.14) formally coincide with (4.3) and (4.8), the values of a' and b' differ in these formulas. Indeed, we have $a' = a+1$ and $b' = b$ in (4.3), while $a' = a$ and $b' = b-1$ in (4.13). Similarly, $a' = a$ and $b' = b+1$ in (4.8), while $a' = a-1$ and $b' = b$ in (4.14). Therefore, in particular, the matrices $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ have the size $(a+b+1)\times(a+b+1)$ in (4.3) and (4.8) and $(a+b)\times(a+b)$ in (4.13) and (4.14).

4.2. Form factors of diagonal elements. The form factors of diagonal elements of the monodromy matrix,

$$\mathcal{F}^{(s,s)}_{a,b}(z) \equiv \mathcal{F}^{(s,s)}_{a,b}(z|\bar{u}^C, \bar{v}^C; \bar{u}^B, \bar{v}^B) = \mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) T_{ss}(z) \mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B), \tag{4.15}$$

were calculated in [36]. Here, we give different representations for them. In a sense, they are analogous to the determinant formulas for form factors in the $GL(2)$-based models (see Sec. 3). Namely, they are based on the determinant of the matrix $\mathcal{N}$ given by (4.4) and (4.5), but one row of this matrix should be modified.

As before, we combine the sets $\bar{u}^B$ and $\bar{v}^C$ and the parameter z into the set $\bar{x} = \{\bar{u}^B, \bar{v}^C, z\}$. We also introduce three $(a+b+1)$-component vectors $Y^{(s)}$, $s = 1, 2, 3$, as

$$\begin{aligned}
Y^{(s)}_k &= \delta_{s2} - \delta_{s1} + \frac{u^B_k}{c}(\delta_{s1} - \delta_{s3})\left(\frac{f(\bar{v}^B, u^B_k)}{f(\bar{v}^C, u^B_k)} - 1\right), \qquad k = 1, \dots, a, \\
Y^{(s)}_{a+k} &= \delta_{s2} - \delta_{s3} + \frac{v^C_k + c}{c}(\delta_{s1} - \delta_{s3})\left(\frac{f(v^C_k, \bar{u}^C)}{f(v^C_k, \bar{u}^B)} - 1\right), \quad k = 1, \dots, b.
\end{aligned} \tag{4.16}$$

The values of $Y^{(s)}_{a+b+1}$ are essential only in the case where $\mathbb{C}^{a,b}(\bar{u}^C; \bar{v}^C) = (\mathbb{B}^{a,b}(\bar{u}^B; \bar{v}^B))^\dagger$, i.e., $\bar{u}^B = \bar{u}^C = \bar{u}$ and $\bar{v}^B = \bar{v}^C = \bar{v}$. We define them as

$$Y^{(1)}_{a+b+1} = \frac{r_1(z) f(\bar{u}, z)}{f(\bar{v}, z) f(z, \bar{u})}, \qquad Y^{(2)}_{a+b+1} = 1, \qquad Y^{(3)}_{a+b+1} = \frac{r_3(z) f(z, \bar{v})}{f(\bar{v}, z) f(z, \bar{u})}. \tag{4.17}$$

We can here set $\bar{v} = \bar{v}^C$ or $\bar{v} = \bar{v}^B$ and also $\bar{u} = \bar{u}^C$ or $\bar{u} = \bar{u}^B$.

Proposition 4.4. *We define an* $(a+b+1)\times(a+b+1)$ *matrix* $\mathcal{N}^{(s)}$ *as*

$$
\begin{aligned}
&\mathcal{N}^{(s)}_{j,k} = \mathcal{N}_{j,k}, \quad j=1,\dots,a+b,\\
&\mathcal{N}^{(s)}_{a+b+1,k} = Y^{(s)}_k.
\end{aligned}
\tag{4.18}
$$

Here, the matrix $\mathcal{N}$ *is given by* (4.4) *and* (4.5). *Then*

$$
\mathcal{F}^{(s,s)}_{a,b}(z) = (-1)^b \mathcal{H}_{a',b} \det_{a+b+1} \mathcal{N}^{(s)}, \tag{4.19}
$$

where $\mathcal{H}_{a',b}$ *is given by* (4.2).

Remark 4. It should be remembered that $a'=a$ in the case of $\mathcal{F}^{(s,s)}_{a,b}(z)$ and $a'=a+1$ in the case of $\mathcal{F}^{(1,2)}_{a,b}(z)$. Therefore, the function $\mathcal{H}_{a',b}$ in (4.19) is given by (4.2), where $a'=a$ should be set. The same remark concerns the elements of the matrix $\mathcal{N}^{(s)}$.

We prove Proposition 4.4 in Sec. 6, reducing representation (4.19) to the formulas obtained in [36]. But before we do this, we mention that similarly to the $GL(2)$-case representation, (4.19) implies several alternative determinant formulas for the form factors of the diagonal elements of the monodromy matrix. They can be obtained from (4.19) via morphisms (2.16) and (2.19).

We also mention that

$$
\begin{aligned}
&\sum_{s=1}^{3} Y^{(s)}_k = 0, \quad k=1,\dots,a+b,\\
&\sum_{s=1}^{3} Y^{(s)}_{a+b+1} = \frac{\tau(z|\bar u,\bar v)}{f(z,\bar u)f(\bar v,z)}.
\end{aligned}
\tag{4.20}
$$

Therefore, the form factor of the transfer matrix reduces to its eigenvalue $\tau(z|\bar u,\bar v)$ multiplied by the minor of the matrix $\mathcal{N}^{(s)}$ built on the first $a+b$ rows and columns. This minor vanishes if $\mathbb{C}^{a,b}(\bar u^{C};\bar v^{C}) \neq (\mathbb{B}^{a,b}(\bar u^{B};\bar v^{B}))^{\dagger}$ (see [35] and Sec. 6.1), and the form factor of the transfer matrix between different states is therefore equal to zero, as it should be. Otherwise, if $\mathbb{C}^{a,b}(\bar u^{C};\bar v^{C}) = (\mathbb{B}^{a,b}(\bar u^{B};\bar v^{B}))^{\dagger}$, then the form factor of the transfer matrix is equal to the eigenvalue $\tau(z|\bar u,\bar v)$ multiplied by square of the norm of Bethe vector (see Sec. 6.2).

4.3. Form factor of $T_{13}(z)$. The form factor of the matrix element $T_{13}(z)$ is defined as

$$
\mathcal{F}^{(1,3)}_{a,b}(z) \equiv \mathcal{F}^{(1,3)}_{a,b}(z|\bar u^{C},\bar v^{C};\bar u^{B},\bar v^{B}) = \mathbb{C}^{a',b'}(\bar u^{C};\bar v^{C})T_{13}(z)\mathbb{B}^{a,b}(\bar u^{B};\bar v^{B}), \tag{4.21}
$$

where $a'=a+1$ and $b'=b+1$. As already mentioned, calculating this form factor relies on a new method, which will be presented elsewhere. But to have a complete overview here of the form factors in the $GL(3)$ case, we preview the result. The determinant representation of the $T_{13}(z)$ form factor is similar to the ones for the form factors of the diagonal elements $T_{ss}(z)$. We again combine the sets $\bar u^{B}$ and $\bar v^{C}$ and the parameter z into the set $\bar x = \{\bar u^{B},\bar v^{C},z\}$. But this set now contains $a'+b'$ (i.e., $a+b+2$) elements. We also introduce the $(a'+b')$-component vector $Y^{(1,3)}$ as

$$
Y^{(1,3)}_k = (-1)^{b'} \frac{r_3(x_k)h(x_k,\bar v^{B})}{f(x_k,\bar u^{B})h(\bar v^{C},x_k)} + \frac{h(\bar v^{B},x_k)}{h(\bar v^{C},x_k)}. \tag{4.22}
$$

Proposition 4.5. *We define an* $(a'+b')\times(a'+b')$ *matrix* $\mathcal{N}^{(1,3)}$ *as*

$$\begin{aligned} &\mathcal{N}^{(1,3)}_{j,k} = \mathcal{N}_{j,k}, \quad j = 1,\dots,a'+b, \\ &\mathcal{N}^{(1,3)}_{a'+b',k} = Y^{(1,3)}_k. \end{aligned} \tag{4.23}$$

Here, the matrix $\mathcal{N}$ *is given by* (4.4) *and* (4.5). *Then*

$$\mathcal{F}^{(1,3)}_{a,b}(z) = (-1)^{b'}\mathcal{H}_{a',b} \det_{a'+b+1} \mathcal{N}^{(1,3)}, \tag{4.24}$$

where $\mathcal{H}_{a',b}$ *is given by* (4.2).

We note that an alternative determinant representation for the form factor $\mathcal{F}^{(1,3)}_{a,b}(z)$ can be obtained by applying map (2.19) to result (4.24). In turn, applying antimorphism (2.16) to (4.24) leads to a determinant representation for the form factor $\mathcal{F}^{(3,1)}_{a,b}(z)$.

5. Calculation of form factors

As already mentioned, the determinant representation for the scalar product of an on-shell Bethe vector and a generic Bethe vector plays a key role in calculating form factors in $GL(2)$-based models. An analogue of such a determinant representation is unknown in the case of the $GL(3)$ group. Therefore, calculating the form factor becomes much more involved. Details of these calculations can be found in [35]–[37]. Here, we only give a general description of the method used in those papers.

Studying form factors is based on an explicit representation for the scalar products of Bethe vectors obtained in [41]–[43]. The scalar product is defined as

$$S_{a,b} \equiv S_{a,b}(\bar u^C, \bar v^C; \bar u^B, \bar v^B) = \mathbb{C}^{a,b}(\bar u^C; \bar v^C)\mathbb{B}^{a,b}(\bar u^B; \bar v^B). \tag{5.1}$$

Here, the Bethe parameters $\{\bar u^C, \bar v^C\}$ and $\{\bar u^B, \bar v^B\}$ are assumed to be generic complex numbers. The representation obtained in [41] describes the scalar product as a sum over partitions of Bethe parameters into subsets (the so-called sum formula). This representation is not generally reducible to a more compact form. But when calculating the form factors, we deal with very particular scalar products in which most of the parameters satisfy Bethe equations (2.9). In such cases, this sum over partitions is reducible to a single determinant.

As an example, we consider the form factor of the operator $T_{12}(z)$. The action of $T_{12}(z)$ on $\mathbb{B}^{a,b}(\bar u^B; \bar v^B)$ is (see [38])

$$\begin{aligned} T_{12}(z)\mathbb{B}^{a,b}(\bar u^B; \bar v^B) &= f(\bar v^B, z)\,\mathbb{B}^{a+1,b}(\{\bar u^B, z\}; \bar v^B) + \\ &+ \sum_{i=1}^{b} g(z, v^B_i) f(\bar v^B_i, v^B_i)\,\mathbb{B}^{a+1,b}(\{\bar u^B, z\}; \{\bar v^B_i, z\}). \end{aligned} \tag{5.2}$$

Hence, the form factor of $T_{12}(z)$ is equal to

$$\begin{aligned} \mathcal{F}^{(1,2)}_{a,b}(z) &= f(\bar v^B, z)\, S_{a+1,b}(\bar u^C, \bar v^C; \{\bar u^B, z\}, \bar v^B) + \\ &+ \sum_{i=1}^{b} g(z, v^B_i) f(\bar v^B_i, v^B_i)\, S_{a+1,b}(\bar u^C, \bar v^C; \{\bar u^B, z\}, \{\bar v^B_i, z\}), \end{aligned} \tag{5.3}$$

and we reduce the original problem to calculating the scalar products, where only z is an arbitrary complex number and the other variables satisfy Bethe equations (2.9).

Formally, other form factors can be calculated similarly. It was proved in [38] that the action of the monodromy matrix elements on Bethe vectors reduces to a linear combination of the last ones. Therefore, the form factors of $T_{ij}(z)$ can always be expressed in terms of a linear combination of scalar products. But each specific case has its own peculiarities. In particular, as explained above, there is no need to especially consider the other form factors $\mathcal{F}_{a,b}^{(i,j)}(z)$ with $|i-j|=1$, because they can all be obtained from $\mathcal{F}_{a,b}^{(1,2)}(z)$ via maps (2.16) and (2.19).

The form factors of the diagonal operators $T_{ss}(z)$ can also be calculated in the framework of the scheme described above. But the action of $T_{ss}(z)$ on $\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)$ is much more involved than (5.2). In particular, it contains a double sum over the Bethe parameters. This fact makes a straightforward calculation of $\mathcal{F}_{a,b}^{(s,s)}(z)$ technically very complicated. Therefore, in the case of form factors of the diagonal elements of the monodromy matrix, it is more convenient to apply a special trick based on using the *twisted transfer matrix*. We describe this method in the next subsection.

Finally, calculating the form factors $\mathcal{F}_{a,b}^{(i,j)}(z)$ with $|i-j|=2$ should also be included in the general scheme. But we could not reduce the summation over partitions to a single determinant in this case, because of technical problems. This seems rather strange because the action of the operator $T_{13}(z)$ on the Bethe vectors is the simplest,

$$T_{13}(z)\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)=\mathbb{B}^{a+1,b+1}(\{\bar{u}^B,z\};\{\bar{v}^B,z\}), \tag{5.4}$$

and the form factor of $T_{13}(z)$ is therefore given by a single scalar product:

$$\mathcal{F}_{a,b}^{(1,3)}(z)=S_{a+1,b+1}(\bar{u}^C,\bar{v}^C;\{\bar{u}^B,z\},\{\bar{v}^B,z\}). \tag{5.5}$$

Nevertheless, despite this simplicity, the method for calculating the sums over partitions of the Bethe parameters arising in (5.5) is not yet developed. We therefore use another approach for studying the form factors $\mathcal{F}_{a,b}^{(i,j)}(z)$ with $|i-j|=2$, which will be described in a separate publication. Here we only mention that the form factors $\mathcal{F}_{a,b}^{(1,3)}(z)$ and $\mathcal{F}_{a,b}^{(3,1)}(z)$ are related by map (2.16).

5.1. Twisted transfer matrix. The $GL(3)$-invariance of R-matrix (2.1) means that $[\hat{\kappa}_1\hat{\kappa}_2,R_{12}]=0$ for arbitrary $\hat{\kappa}\in GL(3)$. It is easy to see [41], [44]–[46] that because of this property, a *twisted monodromy matrix* $\hat{\kappa}T(w)$ satisfies algebra (2.2). If the matrix $\hat{\kappa}T(w)$ has the same pseudovacuum and dual pseudovacuum vectors as the original matrix $T(w)$, then we can apply all the tools of the nested algebraic Bethe ansatz to the twisted monodromy matrix. In particular, we can find the spectrum of the twisted transfer matrix $\operatorname{tr}\hat{\kappa}T(w)$. Its eigenvectors are called twisted on-shell Bethe vectors (or simply twisted on-shell vectors).

We consider a matrix $\hat{\kappa}=\operatorname{diag}(\kappa_1,\kappa_2,\kappa_3)$, where κ_i are arbitrary complex numbers. Obviously, the corresponding twisted monodromy matrix has the same pseudovacuum and dual pseudovacuum vectors. Actually, multiplying $T(w)$ by $\hat{\kappa}$ reduces to replacing the original eigenvalues $\lambda_i(w)$ given by (2.7) with $\kappa_i\lambda_i(w)$. Therefore, like the standard on-shell vectors, the twisted on-shell vectors can be parameterized by a set of complex parameters satisfying the twisted Bethe equations, which have form (2.9) with $r_k(z)$ replaced with $r_k(z)\,\kappa_k/\kappa_2$. Below, we need these equations in a logarithmic form. Namely, let

$$\Phi_j=\log r_1(u_j)-\log\frac{f(u_j,\bar{u}_j)}{f(\bar{u}_j,u_j)}-\log f(\bar{v},u_j), \qquad j=1,\dots,a, \tag{5.6}$$

$$\Phi_{a+j}=\log r_3(v_j)-\log\frac{f(\bar{v}_j,v_j)}{f(v_j,\bar{v}_j)})-\log f(v_j,\bar{u}), \quad j=1,\dots,b. \tag{5.7}$$

The system of twisted Bethe equations then has the form

$$\begin{aligned}\Phi_j &= \log\kappa_2 - \log\kappa_1 + 2\pi i\ell_j, \qquad j=1,\dots,a,\\ \Phi_{a+j} &= \log\kappa_2 - \log\kappa_3 + 2\pi i m_j, \quad j=1,\dots,b,\end{aligned} \tag{5.8}$$

where ℓ_j and m_j are some integers. The Jacobian of (5.6) and (5.7) is closely related to the norm of the on-shell Bethe vector and the expectation values of the operators $T_{ss}(z)$ [36].

Using the notion of the twisted transfer matrix, we can calculate the form factors of the diagonal elements of the monodromy matrix. We consider the expectation value

$$Q_{\bar\kappa}(z) = \mathbb{C}^{a,b}_{\bar\kappa}(\bar u^C;\bar v^C)\big(\operatorname{tr}\hat\kappa T(z) - \operatorname{tr} T(z)\big)\mathbb{B}^{a,b}(\bar u^B;\bar v^B), \tag{5.9}$$

where $\mathbb{C}^{a,b}_{\bar\kappa}(\bar u^C;\bar v^C)$ and $\mathbb{B}^{a,b}(\bar u^B;\bar v^B)$ are respectively twisted and standard on-shell vectors. Here and hereafter, we set $\bar\kappa = \{\kappa_1,\kappa_2,\kappa_3\}$. Obviously,

$$Q_{\bar\kappa}(z) = \mathbb{C}^{a,b}_{\bar\kappa}(\bar u^C;\bar v^C)\sum_{j=1}^{3}(\kappa_j-1)T_{jj}(z)\mathbb{B}^{a,b}(\bar u^B;\bar v^B), \tag{5.10}$$

and therefore

$$\frac{dQ_{\bar\kappa}(z)}{d\kappa_s}\Big|_{\bar\kappa=1} = \mathbb{C}^{a,b}_{\bar\kappa}(\bar u^C;\bar v^C)\Big|_{\bar\kappa=1} T_{ss}(z)\mathbb{B}^{a,b}(\bar u^B;\bar v^B). \tag{5.11}$$

Here, $\bar\kappa=1$ means that $\kappa_i=1$ for $i=1,2,3$. We see that after we set $\bar\kappa=1$, the vector $\mathbb{C}^{a,b}_{\bar\kappa}(\bar u^C;\bar v^C)$ becomes the standard on-shell vector $\mathbb{C}^{a,b}(\bar u^C;\bar v^C)$. Hence, we obtain the form factor of $T_{ss}(z)$ in the right-hand side of (5.11),

$$\frac{dQ_{\bar\kappa}(z)}{d\kappa_s}\Big|_{\bar\kappa=1} = \mathcal{F}^{(s,s)}_{a,b}(z|\bar u^C,\bar v^C;\bar u^B,\bar v^B). \tag{5.12}$$

On the other hand,

$$Q_{\bar\kappa}(z) = \big(\tau_{\bar\kappa}(z|\bar u^C;\bar v^C) - \tau(z|\bar u^B;\bar v^B)\big)\mathbb{C}^{a,b}_{\bar\kappa}(\bar u^C;\bar v^C)\mathbb{B}^{a,b}(\bar u^B;\bar v^B), \tag{5.13}$$

where $\tau(z|\bar u^B;\bar v^B)$ is the eigenvalue of $\operatorname{tr}T(z)$ in (2.11) and $\tau_{\bar\kappa}(z|\bar u^C;\bar v^C)$ is the eigenvalue of the twisted transfer matrix $\operatorname{tr}\hat\kappa T(z)$:

$$\tau_{\bar\kappa}(z) \equiv \tau_{\bar\kappa}(z|\bar u,\bar v) = \kappa_1 r_1(z) f(\bar u,z) + \kappa_2 f(z,\bar u) f(\bar v,z) + \kappa_3 r_3(z) f(z,\bar v). \tag{5.14}$$

We thus obtain

$$\mathcal{F}^{(s,s)}_{a,b}(z) = \frac{d}{d\kappa_s}\big[\big(\tau_{\bar\kappa}(z|\bar u^C;\bar v^C) - \tau(z|\bar u^B;\bar v^B)\big)\mathbb{C}^{a,b}_{\bar\kappa}(\bar u^C;\bar v^C)\mathbb{B}^{a,b}(\bar u^B;\bar v^B)\big]_{\bar\kappa=1}, \tag{5.15}$$

and we see that the form factors $\mathcal{F}^{(s,s)}_{a,b}(z)$ can be calculated as κ_s-derivatives of the scalar product of twisted on-shell and standard on-shell vectors.

6. Proof of Proposition 4.4

In this section, we prove Proposition 4.4. More precisely, we show that the determinant representations given by Proposition 4.4 are equivalent to those obtained in [36].

Dealing with the form factors of diagonal elements $T_{ss}(z)$, we should distinguish two cases:

- $\mathbb{C}^{a,b}(\bar u^C;\bar v^C) \neq (\mathbb{B}^{a,b}(\bar u^B;\bar v^B))^\dagger$,
- $\mathbb{C}^{a,b}(\bar u^C;\bar v^C) = (\mathbb{B}^{a,b}(\bar u^B;\bar v^B))^\dagger$.

We consider these two cases separately.

6.1. Proof for different states. In this section, $\mathbb{C}^{a,b}(\bar{u}^C;\bar{v}^C) \neq (\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B))^\dagger$. This means that there exists at least one $w \in \{\bar{u}^C, \bar{v}^C\}$ such that $w \notin \{\bar{u}^B, \bar{v}^B\}$. Then

$$\mathbb{C}^{a,b}_{\bar{\kappa}}(\bar{u}^C;\bar{v}^C)\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)\big|_{\bar{\kappa}=1} = 0 \tag{6.1}$$

as a product of two eigenstates corresponding to the different eigenvalues of the transfer matrix. Hence, the κ_s-derivative in (5.15) should be applied only to this scalar product. We obtain

$$\mathcal{F}^{(s,s)}_{a,b}(z) = \big(\tau(z|\bar{u}^C;\bar{v}^C) - \tau(z|\bar{u}^B;\bar{v}^B)\big)\frac{d}{d\kappa_s}\mathbb{C}^{a,b}_{\bar{\kappa}}(\bar{u}^C;\bar{v}^C)\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)\Big|_{\bar{\kappa}=1}. \tag{6.2}$$

The κ_s-derivatives of the scalar product of twisted on-shell and standard on-shell vectors were calculated in [36]. We describe this result.

First, we introduce an $(a+b)$-component vector Ω as

$$\begin{aligned} \Omega_j &= \frac{g(u^C_j,\bar{u}^C_j)}{g(u^C_j,\bar{u}^B)}, \qquad j=1,\dots,a, \\ \Omega_{a+j} &= \frac{g(v^B_j,\bar{v}^B_j)}{g(v^B_j,\bar{v}^C)}, \quad j=1,\dots,b. \end{aligned} \tag{6.3}$$

It is easy to see that because $\mathbb{C}^{a,b}(\bar{u}^C;\bar{v}^C) \neq \big(\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)\big)^\dagger$, this vector has at least one nonzero component. Without loss of generality, we assume that $\Omega_{a+b} \neq 0$. The result for the κ_s-derivative of the scalar product is then

$$\frac{d}{d\kappa_s}\,\mathbb{C}^{a,b}_{\bar{\kappa}}(\bar{u}^C;\bar{v}^C)\mathbb{B}^{a,b}(\bar{u}^B;\bar{v}^B)\Big|_{\bar{\kappa}=1} = \Omega^{-1}_{a+b}H_{a,b}\widehat{\mathcal{N}}_{a+b,a+b+1}, \tag{6.4}$$

where

$$H_{a,b} = \frac{(-1)^b\mathcal{H}_{a,b}}{f(z,\bar{u}^B)f(\bar{v}^C,z)} = \frac{h(\bar{w},\bar{u}^B)h(\bar{v}^C,\bar{w})}{h(\bar{v}^C,\bar{u}^B)}\,\Delta'_a(\bar{u}^C)\Delta'_b(\bar{v}^B)\Delta_{a+b}(\bar{w}), \tag{6.5}$$

$\mathcal{H}_{a,b}$ is given by (4.2), and $\bar{w} = \{\bar{u}^B, \bar{v}^C\}$. The factor $\widehat{\mathcal{N}}_{a+b,a+b+1}$ in (6.4) is the cofactor to the element $\mathcal{N}^{(s)}_{a+b,a+b+1}$ of the matrix $\mathcal{N}^{(s)}$ given by (4.18),

$$\widehat{\mathcal{N}}_{a+b,a+b+1} = -\det_{\substack{j\neq a+b,\\ k\neq a+b+1}} \mathcal{N}^{(s)}_{j,k}. \tag{6.6}$$

We reproduce this result starting from determinant representation (4.19). First, we give the elements of the matrix $\mathcal{N}$ more explicitly:

$$\mathcal{N}_{j,k} = (-1)^{a'-1}t(u^C_j,x_k)\frac{r_1(x_k)h(\bar{u}^C,x_k)}{f(\bar{v}^C,x_k)h(x_k,\bar{u}^B)} + t(x_k,u^C_j)\frac{h(x_k,\bar{u}^C)}{h(x_k,\bar{u}^B)}, \qquad j=1,\dots,a', \tag{6.7}$$

$$\mathcal{N}_{a'+j,k} = (-1)^{b-1}t(x_k,v^B_j)\frac{r_3(x_k)h(x_k,\bar{v}^B)}{f(x_k,\bar{u}^B)h(\bar{v}^C,x_k)} + t(v^B_j,x_k)\frac{h(\bar{v}^B,x_k)}{h(\bar{v}^C,x_k)}, \quad j=1,\dots,b. \tag{6.8}$$

We note that $a'=a$ and $b'=b$ in the case under consideration. But we use the symbol a' in (6.7) and (6.8) because these equations in this form still hold for the form factor $\mathcal{F}^{(1,2)}_{a,b}(z)$, where $a'=a+1$.

Let

$$S(x_k) = \sum_{j=1}^{a'}\Omega_j\,\mathcal{N}_{j,k} + \sum_{j=1}^{b}\Omega_{a'+j}\,\mathcal{N}_{a'+j,k}. \tag{6.9}$$

Using (A.1), we can then easily obtain

$$S(x_k)=\frac{\tau(x_k|\bar u^C,\bar v^C)-\tau(x_k|\bar u^B,\bar v^B)}{f(\bar v^C,x_k)f(x_k,\bar u^B)}. \tag{6.10}$$

It is straightforward to verify that $S(u^B_k)=S(v^C_k)=0$ because of the Bethe equations. In fact, this can be seen without any calculations. Indeed, the Bethe equations are equivalent to the statement that the function $\tau(x_k|\bar u,\bar v)$ has no poles at the points $x_k=u_j$ and $x_k=v_j$ (see Remark 1). The factor $f^{-1}(\bar v^C,x_k)f^{-1}(x_k,\bar u^B)$ then immediately yields the equalities $S(u^B_k)=S(v^C_k)=0$.

We now multiply the first $a+b-1$ rows of the matrix $\mathcal{N}^{(s)}$ by the factors Ω_j/Ω_{a+b} and add them to the $(a+b)$th row. We then obtain a modified $(a+b)$th row with the components

$$\begin{aligned}
&\mathcal{N}^{(s),\mathrm{mod}}_{a+b,k}=0,\quad k=1,\dots,a+b,\\
&\mathcal{N}^{(s),\mathrm{mod}}_{a+b,a+b+1}=\Omega^{-1}_{a+b}\frac{\tau(z|\bar u^C,\bar v^C)-\tau(z|\bar u^B,\bar v^B)}{f(\bar v^C,z)f(z,\bar u^B)}.
\end{aligned} \tag{6.11}$$

The determinant $\det\mathcal{N}^{(s)}$ reduces to the product of the element $\mathcal{N}^{(s),\mathrm{mod}}_{a+b,a+b+1}$ times the corresponding cofactor, and we obtain

$$\det_{a+b+1}\mathcal{N}^{(s)}=\Omega^{-1}_{a+b}\frac{\tau(z|\bar u^C,\bar v^C)-\tau(z|\bar u^B,\bar v^B)}{f(\bar v^C,z)f(z,\bar u^B)}\widehat{\mathcal{N}}_{a+b,a+b+1}. \tag{6.12}$$

We draw attention to the fact that the matrix element $Y^{(s)}_{a+b+1}$ has disappeared from the game. Substituting this result in (4.19), we immediately reproduce (6.4).

6.2. Proof for the same states. In this section, $\mathbb{C}^{a,b}(\bar u^C;\bar v^C)=\big(\mathbb{B}^{a,b}(\bar u^B;\bar v^B)\big)^\dagger$, and we set $\bar u^C=\bar u^B=\bar u$ and $\bar v^C=\bar v^B=\bar v$. In this case,

$$\tau_{\bar\kappa}(z|\bar u^C;\bar v^C)-\tau(z|\bar u^B;\bar v^B)=0\quad\text{for }\bar\kappa=1,\quad \bar u^C=\bar u^B=\bar u,\quad \bar v^C=\bar v^B=\bar v, \tag{6.13}$$

and the κ_s-derivative in (5.15) should hence act only on the difference of the eigenvalues $\tau_{\bar\kappa}$ and τ. We then find

$$\mathcal{F}^{(s,s)}(z|\bar u,\bar v;\bar u,\bar v)=\|\mathbb{B}^{a,b}(\bar u;\bar v)\|^2\frac{d\tau_{\bar\kappa}(z|\bar u^C;\bar v^C)}{d\kappa_s}\bigg|_{\bar\kappa=1}, \tag{6.14}$$

and we should set $\bar u^C=\bar u$ and $\bar v^C=\bar v$ after taking the derivative of $\tau_{\bar\kappa}(z|\bar u^C;\bar v^C)$ with respect to κ_s. Below in this section, we always assume that the condition $\bar\kappa=1$ automatically yields $\bar u^C=\bar u^B=\bar u$ and $\bar v^C=\bar v^B=\bar v$.

The squared norm of the on-shell Bethe vector $\|\mathbb{B}^{a,b}(\bar u;\bar v)\|^2$ was calculated in [41], [35]. It is proportional to the minor of the matrix $\mathcal{N}^{(s)}$ built on the first $a+b$ rows and columns:[3]

$$\|\mathbb{B}^{a,b}(\bar u;\bar v)\|^2=H_{a,b}\det_{a+b}\mathcal{N}, \tag{6.15}$$

where $H_{a,b}$ is given by (6.5) for $\bar u^C=\bar u^B=\bar u$ and $\bar v^C=\bar v^B=\bar v$.

We present the elements of $\mathcal{N}$ explicitly in the case $\bar u^C=\bar u^B=\bar u$ and $\bar v^C=\bar v^B=\bar v$ (see [41], [35]). For $j,k=1,\dots,a$, we have

$$\mathcal{N}_{j,k}=\delta_{jk}\bigg(-c\log' r_1(u_k)-\sum_{\ell=1}^{a}\frac{2c^2}{u^2_{k\ell}-c^2}+\sum_{m=1}^{b}t(v_m,u_k)\bigg)+\frac{2c^2}{u^2_{jk}-c^2}, \tag{6.16}$$

[3] It is important that this minor is independent of s.

where $u_{k\ell} = u_k - u_\ell$. The elements of the second diagonal block are

$$\mathcal{N}_{a+j,a+k} = \delta_{jk}\left(c\log' r_3(v_k) - \sum_{m=1}^{b}\frac{2c^2}{v_{km}^2 - c^2} + \sum_{\ell=1}^{a} t(v_k, u_\ell)\right) + \frac{2c^2}{v_{jk}^2 - c^2}, \tag{6.17}$$

where $v_{km} = v_k - v_m$ and $j, k = 1, \dots, b$. The antidiagonal blocks have a simpler structure,

$$\mathcal{N}_{j,a+k} = t(v_k, u_j), \quad j = 1, \dots, a, \quad k = 1, \dots, b, \tag{6.18}$$

$$\mathcal{N}_{a+j,k} = t(v_j, u_k), \quad j = 1, \dots, b, \quad k = 1, \dots, a. \tag{6.19}$$

We note that the matrix $\mathcal{N}$ is symmetric: $\mathcal{N}_{jk} = \mathcal{N}_{kj}$. It is also easy to verify (see [35]) that

$$\begin{aligned} \mathcal{N}_{j,k} &= -c\frac{\partial\Phi_j}{\partial u_k}, \quad j = 1, \dots, a+b, \quad k = 1, \dots, a, \\ \mathcal{N}_{j,a+k} &= c\frac{\partial\Phi_j}{\partial v_k}, \quad j = 1, \dots, a+b, \quad k = 1, \dots, b, \end{aligned} \tag{6.20}$$

where Φ_j is given by (5.6) and (5.7).

We reproduce result (6.14) starting from representation (4.19). The elements of the matrix $\mathcal{N}^{(s)}_{j,k}$ with $j, k = 1, \dots, a+b$ coincide with those defined in (6.16)–(6.19). In the last row, we have

$$\begin{aligned} \mathcal{N}^{(s)}_{a+b+1,k} &= Y^{(s)}_k = \delta_{s2} - \delta_{s1}, \quad k = 1, \dots, a, \\ \mathcal{N}^{(s)}_{a+b+1,k} &= Y^{(s)}_k = \delta_{s2} - \delta_{s3}, \quad k = a+1, \dots, a+b. \end{aligned} \tag{6.21}$$

Finally, the last column has the components

$$\begin{aligned} \mathcal{N}^{(s)}_{j,a+b+1} &= \frac{c}{f(z,\bar u)f(\bar v,z)}\frac{\partial\tau(z|\bar u,\bar v)}{\partial u_j}, \qquad j = 1, \dots, a, \\ \mathcal{N}^{(s)}_{a+j,a+b+1} &= -\frac{c}{f(z,\bar u)f(\bar v,z)}\frac{\partial\tau(z|\bar u,\bar v)}{\partial v_j}, \quad j = 1, \dots, b, \\ \mathcal{N}^{(s)}_{a+b+1,a+b+1} &= \frac{1}{f(z,\bar u)f(\bar v,z)}\frac{\partial\tau_{\bar\kappa}(z|\bar u^{C},\bar v^{C})}{\partial\kappa_s}\bigg|_{\bar\kappa=1}. \end{aligned} \tag{6.22}$$

We have thus described the $(a+b+1)\times(a+b+1)$ matrix $\mathcal{N}^{(s)}$ in the limit $\bar u^{C} = \bar u^{B} = \bar u$ and $\bar v^{C} = \bar v^{B} = \bar v$. We show that $\det\mathcal{N}^{(s)}$ is reducible to the determinant of the $(a+b)\times(a+b)$ block of this matrix given by (6.16)–(6.19). For this, we introduce three $(a+b)$-component vectors $\widetilde\Omega^{(s)}_j$ as

$$\begin{aligned} \widetilde\Omega^{(s)}_j &= \frac{1}{c}\frac{du^{C}_j}{d\kappa_s}\bigg|_{\bar\kappa=1}, \qquad j = 1, \dots, a, \\ \widetilde\Omega^{(s)}_{a+j} &= -\frac{1}{c}\frac{dv^{C}_j}{d\kappa_s}\bigg|_{\bar\kappa=1}, \quad j = 1, \dots, b. \end{aligned} \tag{6.23}$$

It is easy to show that

$$\sum_{j=1}^{a+b+1}\widetilde\Omega^{(s)}_j\mathcal{N}^{(s)}_{j,k} = 0, \quad k = 1, \dots, a+b. \tag{6.24}$$

Indeed, differentiating system of twisted Bethe equations (5.8) with respect to κ_s at $\bar{\kappa}=1$, we obtain

$$\sum_{\ell=1}^{a} \frac{\partial \Phi_j}{\partial u_\ell} \frac{du_\ell^C}{d\kappa_s}\Big|_{\bar{\kappa}=1} + \sum_{m=1}^{b} \frac{\partial \Phi_j}{\partial v_m} \frac{dv_m^C}{d\kappa_s}\Big|_{\bar{\kappa}=1} = Y_k^{(s)}. \tag{6.25}$$

Taking (6.20) and the symmetry of $\mathcal{N}_{j,k}^{(s)}$ for $j,k=1,\dots,a+b$ into account, we immediately obtain (6.24). Adding all other rows times the coefficients $\widetilde{\Omega}_j^{(s)}$ to the last row of $\mathcal{N}_{j,k}^{(s)}$, we thus obtain zeros everywhere except the element $j,k=a+b+1$, where we have

$$\begin{aligned}\sum_{j=1}^{a+b+1} \widetilde{\Omega}_j^{(s)} \mathcal{N}_{j,a+b+1}^{(s)} &= \frac{1}{f(z,\bar{u})f(\bar{v},z)} \Big\{ \frac{\partial \tau(z|\bar{u}^C,\bar{v}^C)}{\partial \kappa_s}\Big|_{\bar{\kappa}=1} + \\ &+ \sum_{\ell=1}^{a} \frac{\partial \tau(z|\bar{u},\bar{v})}{\partial u_\ell} \frac{du_\ell^C}{d\kappa_s}\Big|_{\bar{\kappa}=1} + \sum_{m=1}^{b} \frac{\partial \tau(z|\bar{u},\bar{v})}{\partial v_m} \frac{dv_m^C}{d\kappa_s}\Big|_{\bar{\kappa}=1} \Big\} = \\ &= \frac{1}{f(z,\bar{u})f(\bar{v},z)} \frac{d\tau(z|\bar{u}^C,\bar{v}^C)}{d\kappa_s}\Big|_{\bar{\kappa}=1}.\end{aligned} \tag{6.26}$$

We thus obtain

$$\mathcal{F}_{a,b}^{(s,s)}(z|\bar{u},\bar{v};\bar{u},\bar{v}) = \frac{d\tau(z|\bar{u},\bar{v})}{d\kappa_s}\Big|_{\bar{\kappa}=1} \cdot H_{a,b} \det_{a+b} \mathcal{N}. \tag{6.27}$$

Comparing (6.27) and (6.15), we obtain representation (6.14).

7. Discussion

We have considered the form factors of the monodromy matrix elements in models with the $GL(3)$-invariant R-matrix and obtained determinant representations for them. The question arises of generalizing the obtained results to models with a higher-rank symmetry group. For this, it is useful to compare the structure of the determinant formulas for the models with $GL(2)$ and $GL(3)$ symmetries.

For $GL(3)$-based models, all the representations have a similar structure and are based on the determinants of the matrices $\mathcal{N}$ or $\widetilde{\mathcal{N}}$ (the latter can be obtained from $\mathcal{N}$ by replacing $\{\bar{u}^C,\bar{v}^C\} \leftrightarrow \{\bar{u}^B,\bar{v}^B\}$). In these matrices, all rows and columns are associated with one of the Bethe parameters or with the external variable z. For example, in $\mathcal{N}$, the first a columns correspond to the set $\bar{u}^B$, the next b columns correspond to the set $\bar{v}^C$, and the last column is associated with the variable z. The rows of this matrix are associated with the parameters $\bar{u}^C$ and $\bar{v}^B$. For the form factor of the diagonal elements and also for the operator $T_{13}(z)$, the matrix $\mathcal{N}$ has an additional row.

It is hardly possible to predict such a structure based on the results obtained for the models with the $GL(2)$ symmetry. For example, we could expect that the columns of the matrices should correspond to the parameters of one Bethe vector (e.g., $\{\bar{u}^B,\bar{v}^B\}$) while the rows should correspond to the parameters of another Bethe vector (in this case, $\{\bar{u}^C,\bar{v}^C\}$). But we see that this is not the case, and we should “mix” the parameters from different Bethe vectors to label the rows and the columns. Such mixing of the Bethe parameters makes it very problematic to generalize our results straightforwardly to models with a $GL(N)$ symmetry with $N>3$. There is also one more argument ruling out a simple generalization of these results to of higher-rank symmetry groups. We see that the matrix whose determinant describes form factors has a block structure

$$\mathcal{N} = \begin{pmatrix} \mathcal{N}_\ell \\ \\ \mathcal{N}_{\mathrm{r}} \end{pmatrix}, \quad \text{where} \quad \begin{aligned} (\mathcal{N}_\ell)_{j,k} &\sim \frac{\partial \tau(x_k|\bar{u}^C,\bar{v}^C)}{\partial u_j^C}, \\ (\mathcal{N}_{\mathrm{r}})_{j,k} &\sim \frac{\partial \tau(x_k|\bar{u}^B,\bar{v}^B)}{\partial v_j^B}. \end{aligned} \tag{7.1}$$

The upper and lower blocks are proportional to the Jacobians of the transfer matrix eigenvalues on the respective left and right Bethe vectors. On the other hand, the block structure is also related to the fact that Bethe vectors depend on two sets of parameters. But the Bethe vectors depend on $N-1$ sets of variables in the case of the $GL(N)$ group [32]. Hence, it is natural to expect that if there are determinant representations for form factors in the models with the symmetry group $GL(4)$, for example, then the corresponding matrices should have a 3×3 block structure. At the same time, we still have only two vectors and hence only two eigenvalues.

Of course, the arguments above do not mean that determinant representations for form factors do not exist in models with a $GL(N)$-invariant R-matrix. These arguments can only tell that a determinant representations based on the Jacobians of the transfer matrix eigenvalues are hardly possible for models with a higher symmetry group. Nevertheless, on the other hand, we cannot exclude the existence of determinant representations with a different structure.

Concluding this paper, we say a few words about possible applications. One application immediately arises for the quantum models admitting an explicit solution of the quantum inverse scattering problem [16], [21]. In particular, we have the representation

$$E_m^{\alpha,\beta} = (\operatorname{tr} T(0))^{m-1} T_{\beta\alpha}(0) (\operatorname{tr} T(0))^{-m} \tag{7.2}$$

for the local operators in the $SU(3)$-invariant XXX Heisenberg chain. Here, $E_m^{\alpha,\beta}$, $\alpha,\beta = 1,2,3$, is an elementary unit $((E^{\alpha,\beta})_{jk} = \delta_{j\alpha}\delta_{k\beta})$ associated with the mth site of the chain. Because the action of the transfer matrix $\operatorname{tr} T(0)$ on on-shell Bethe vectors is trivial, we see that the form factors of $E_m^{\alpha,\beta}$ are proportional to those of $T_{\beta\alpha}$,

$$\mathbb{C}^{a',b'}(\bar u^C;\bar v^C) E_m^{\alpha,\beta} \mathbb{B}^{a,b}(\bar u^B;\bar v^B) = \frac{\tau^{m-1}(0|\bar u^C,\bar v^C)}{\tau^m(0|\bar u^B,\bar v^B)} \mathcal{F}_{a,b}^{(\beta,\alpha)}(0|\bar u^C,\bar v^C;\bar u^B,\bar v^B). \tag{7.3}$$

Therefore, if we have explicit, compact representations for the form factors of $T_{\beta,\alpha}$, then we can study the problem of two-point and multipoint correlation functions, expanding them in series in the form factors.

Appendix: Summation formulas

In this section, we prove several identities for the vector Ω introduced in (6.3).

Proposition 7.1. *Let Ω be defined as in* (6.3). *Then*

$$\begin{aligned}
&\sum_{j=1}^{a} t(u_j^C, z)\Omega_j = \frac{h(\bar u^B, z)}{h(\bar u^C, z)}\left(1 - \frac{f(\bar u^C, z)}{f(\bar u^B, z)}\right),\\
&\sum_{j=1}^{a} t(z, u_j^C)\Omega_j = \frac{h(z, \bar u^B)}{h(z, \bar u^C)}\left(\frac{f(z, \bar u^C)}{f(z, \bar u^B)} - 1\right),\\
&\sum_{j=1}^{b} t(v_j^B, z)\Omega_{j+a} = \frac{h(\bar v^C, z)}{h(\bar v^B, z)}\left(1 - \frac{f(\bar v^B, z)}{f(\bar v^C, z)}\right),\\
&\sum_{j=1}^{b} t(z, v_j^B)\Omega_{j+a} = \frac{h(z, \bar v^C)}{h(z, \bar v^B)}\left(\frac{f(z, \bar v^B)}{f(z, \bar v^C)} - 1\right).
\end{aligned} \tag{A.1}$$

All the identities above can be proved similarly. As an example, we consider the first identity.

Proof. Let

$$\sum_{j=1}^{a} t(u_j^C, z)\Omega_j = W(z). \tag{A.2}$$

The sum in the left-hand side of (A.2) can be computed using an auxiliary integral

$$I = \frac{1}{2\pi i}\oint_{|\omega|=R\to\infty} \frac{c\,d\omega}{(\omega-z)(\omega-z+c)} \prod_{\ell=1}^{a} \frac{\omega-u_\ell^B}{\omega-u_\ell^C}. \tag{A.3}$$

The integral is taken over the counterclockwise-oriented contour $|\omega| = R$, and we consider the limit $R \to \infty$. Then $I = 0$ because the integrand behaves as $1/\omega^2$ for $\omega \to \infty$. On the other hand, the same integral is equal to the sum of residues inside the integration contour. Obviously, the sum of the residues at $\omega = u_\ell^C$ gives $W(z)$. There are also two additional poles at $\omega = z$ and $\omega = z - c$. We then have

$$I = 0 = W(z) - \prod_{\ell=1}^{a} \frac{z-u_\ell^B-c}{z-u_\ell^C-c} + \prod_{\ell=1}^{a} \frac{z-u_\ell^B}{z-u_\ell^C}, \tag{A.4}$$

whence we obtain the first identity in (A.1).

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