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A MATRIX MODEL FOR HYPERGEOMETRIC HURWITZ NUMBERS

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We present multimatrix models that are generating functions for the numbers of branched covers of the complex projective line ramified over n fixed points z_i, $i = 1, \dots, n$ (generalized Grothendieck's dessins d'enfants) *of fixed genus, degree, and ramification profiles at two points z_1 and z_n. We sum over all possible ramifications at the other $n-2$ points with a fixed length of the profile at z_2 and with a fixed total length of profiles at the remaining $n-3$ points. All these models belong to a class of hypergeometric Hurwitz models and are therefore tau functions of the Kadomtsev–Petviashvili hierarchy. In this case, we can represent the obtained model as a chain of matrices with a (nonstandard) nearest-neighbor interaction of the type $\operatorname{tr} M_i M_{i+1}^{-1}$. We describe the technique for evaluating spectral curves of such models, which opens the way for obtaining $1/N^2$-expansions of these models using the topological recursion method. These spectral curves turn out to be algebraic.*

Keywords: Hurwitz number, random complex matrix, Kadomtsev–Petviashvili hierarchy, matrix chain, bipartite graph, spectral curve

1. Introduction

It is generally considered that Hurwitz numbers pertain to combinatorial classes of ramified maps $f\colon \mathbb{CP}^1 \to \Sigma_g$ of the complex projective line onto a genus-g Riemann surface. The notion of single or double Hurwitz numbers correspond to the cases whose ramification profiles (defined by the corresponding Young tableau λ or λ and μ) are respectively determined at one (∞) or two (∞ and 1) distinct points, while we assume that m other distinct ramification points exist, which admit only simple ramifications.

Generating functions for Hurwitz numbers have long been considered in mathematical physics. Issues of the integrability of character expansions were addressed in [1] (see a more recent paper for their classification [2]).

We note that Okounkov and Pandharipande showed that the exponential of the generating function for double Hurwitz numbers is a tau function of the Kadomtsev–Petviashvili (KP) hierarchy [3]. Orlov and Shcherbin obtained the same result using the Schur function technique [4], [5]. The general conditions on KP tau functions were formulated by Takasaki [6] and Goulden and Jackson [7] using Plucker relations.

Orlov and Shcherbin also addressed the case of the generating function in the case of Grothendieck *dessins d'enfants*, where we have only three ramification points with multiple ramifications and the ramification profile is fixed at one or two of these points [4]. In this case, they also concluded that the exponentials of the corresponding generating functions must be tau functions of the KP hierarchy. In fact, the results in [4] describe a wider class of generating functions for *hypergeometric Hurwitz numbers* (this term was

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coined there) in which we have a fixed number n of ramification points in $\mathbb{CP}^1$ and we fix profiles at two of these points and a sum over the profiles at all the other points with weights proportional to the *lengths* of the remaining $n-2$ profiles. Harnad and Orlov recently demonstrated that all such generating functions are in turn tau functions of the KP hierarchy [8].

The interest in Hurwitz numbers corresponding to Belyi pairs was revived by Zograf [9] (also see [10]), who obtained recurrence relations for the generating function of the Grothendieck *dessins d'enfants* enumerating the Belyi pairs (C, f), where C is a smooth algebraic curve and f a meromorphic function $f: C \to \mathbb{CP}^1$ ramified only over the points $0, 1, \infty \in \mathbb{CP}^1$. In [11], we proposed a matrix-model description of Belyi pairs, clean Belyi morphisms, and two-profile Belyi pairs and thus again showed that all these cases are in the category of KP tau functions. The corresponding matrix models are the standard Hermitian one-matrix model or the Kontsevich–Penner matrix model [12] in the case of a single fixed profile and the generalized Kontsevich model [13] in the two-profile case. Almost immediately, a multimatrix-model representation for hypergeometric Hurwitz numbers was constructed in [14] but with a complicated interaction between the matrices in the chain. Here, we propose a more standard description of hypergeometric Hurwitz numbers in the case where we fix profiles at two ramification points, fix the length of the profile at a third point, and fix the total length of profiles at the other $n-3$ points. In this case, we can obtain the spectral curve equation in the framework of the $1/N^2$-expansion.

We recall some mathematical facts relating Belyi pairs to Galois groups.

Theorem 1 [15]**.** *A smooth complex algebraic curve C is defined over the field of algebraic numbers $\overline{\mathbb{Q}}$ if and only if there is a nonconstant meromorphic function f defined on C ($f: C \to \mathbb{CP}^1$) and ramified only over the points $0, 1, \infty \in \mathbb{CP}^1$.*

For a Belyi pair (C, f), let g be the genus of C and d be the degree of f. If we take the preimage $f^{-1}([0,1]) \subset C$ of the real line segment $[0,1] \in \mathbb{CP}^1$, then we obtain a connected bipartite fat graph with d edges with vertices being preimages of 0 and 1 and such that the cyclic ordering of edges entering a vertex comes from the orientation of the curve C. This led Grothendieck to formulate the following lemma.

Lemma 1 [16]**.** *There is a one-to-one correspondence between the isomorphism classes of Belyi pairs and connected bipartite fat graphs.*

A Grothendieck *dessin d'enfant* is therefore a connected bipartite fat graph representing a Belyi pair. It is well known that we can naturally extend the dessin $f^{-1}([0,1]) \subset C$ corresponding to a Belyi pair (C, f) to a bipartite triangulation of the curve C. For this, we cut the complex plane along the (real) line containing the points $0, 1, \infty$ and color the upper half-plane white and the lower half-plane gray. This defines a partition of C into white and gray triangles such that white triangles have common edges only with gray triangles. We then consider a dual graph with three types of edges.

In this paper, we consider *generalized Belyi pairs*, which are maps ($f: C \to \mathbb{CP}^1$) with allowed ramifications over n fixed points $z_i \in \mathbb{CP}^1$, $i = 1, \ldots, n$. We then have the splitting of the curve C into bipartite n-gons with edges of n colors (the corresponding fat graphs are then coverings of the basic graph shown in Fig. 1 for $n = 5$): the color of an edge depends on which of n segments of $\mathbb{RP}^1$ ($f^{-1}([\infty_-, z_2]) \subset C$, $f^{-1}([z_2, z_3]) \subset C$, $\ldots$, $f^{-1}([z_{n-1}, z_n]) \subset C$, $f^{-1}([z_n, \infty_+]) \subset C$) its image intersects (we identify z_1 with the point at infinity and let $\infty_\pm$ indicate the directions of approaching this point along the real axis in $\mathbb{CP}^1$). Each face of the dual partition then contains a preimage of exactly one of the points $z_1, \ldots, z_n$, and these faces are hence of n sorts (bordered by solid, dash-dotted, or dashed lines in the figure). We call such a graph a *generalized Belyi fat graph*.

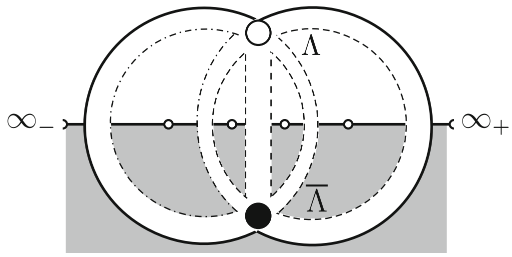


Fig. 1. The generalized Belyi fat graph Γ_1 corresponding to possible ramifications at $n = 5$ points (commonly taken to be ∞, $-(1+\sqrt{5})/2$, 0, 1, and $(3+\sqrt{5})/2$, denoted here by small circles): this graph describes the generalized Belyi pair $(\mathbb{CP}^1, \mathrm{id})$; $\infty_\pm$ indicate directions of approaching the point at infinity in $\mathbb{CP}^1$. The symbols Λ and $\overline{\Lambda}$ indicate the insertions of the external field in the matrix-model formalism in Sec. 2. For example, this graph contributes the term $N^2\gamma_1\gamma_2\gamma_3^2 t_1\,\mathrm{tr}(\Lambda\overline{\Lambda})$.

The type of ramification at infinity is determined by the set of faces bounded by solid lines in a generalized Belyi fat graph: the order of branching is r for a $2r$-gon. We can therefore introduce a generating function that distinguishes between different types of branching at infinity or at z_1. Moreover, we also distinguish between different types of ramifications at the nth point (the point $(3+\sqrt{5})/2$ in Fig. 1). This situation is customarily called a *two-profile* generating function for Hurwitz numbers because we fix two ramification patterns at two distinct branching points; each such pattern can be represented by its Young tableau. We let k_i denote the numbers of respective cycles (preimages of the points z_i on the Riemann surface C) and let $k_1^{(r)}$ and $k_n^{(r)}$ denote the numbers of cycles of length $2r$ centered at preimages of the respective points z_1 and z_n in a generalized Belyi fat graph.

As shown in [14] and [8], the exponential of the generating function

$$\mathcal{F}[\{t_m\}, \{\mathfrak{t}_r\}, \gamma_2, \ldots, \gamma_{n-1}; N] = \sum_\Gamma \frac{1}{|\operatorname{Aut}\Gamma|} N^{2-2g} \prod_{r=1}^{\infty} t_r^{k_1^{(r)}} \prod_{s=1}^{\infty} \mathfrak{t}_s^{k_n^{(s)}} \prod_{j=2}^{n-1} \gamma_j^{k_j} \tag{1.1}$$

is a tau function of the KP hierarchy in times t_r or $\mathfrak{t}_r$. Although a matrix-model description of this generating function was proposed in those papers, the possibility of solving it in terms of a *topological recursion* (see [17]–[19]) remained obscure. We construct a matrix model describing a subclass of generating functions (1.1) with $\gamma_3 = \gamma_4 = \cdots = \gamma_{n-1}$ and arbitrary $\gamma_2 > \gamma_3$.

Our goal here is therefore to construct and solve a matrix model whose free energy is the generating function

$$\mathcal{F}[\{t_m\}, \{\mathfrak{t}_r\}, \gamma_2, \gamma_3; N] = \sum_\Gamma \frac{1}{|\operatorname{Aut}\Gamma|} N^{2-2g} \prod_{r=1}^{\infty} t_r^{k_1^{(r)}} \prod_{s=1}^{\infty} \mathfrak{t}_s^{k_n^{(s)}} \gamma_2^{k_2} \gamma_3^{k_3+\cdots+k_{n-1}}, \tag{1.2}$$

where N, γ_2, γ_3, t_r, and $\mathfrak{t}_r$ are formal independent parameters and the sum ranges all (connected) generalized Belyi fat graphs. Below, we consider a matrix model with an external matrix field $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{\gamma_3 N})$; the corresponding times are

$$\mathfrak{t}_r = \mathrm{tr}[(\Lambda\overline{\Lambda})^r]. \tag{1.3}$$

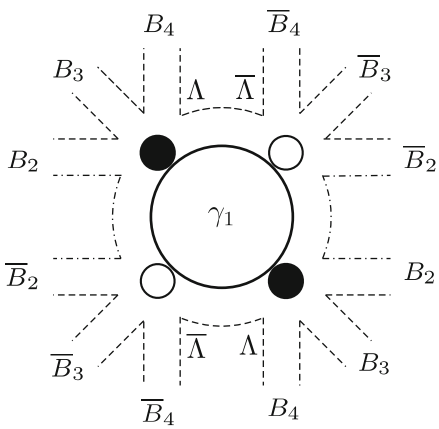
Factors $\gamma_1^{k_1}$ and $\gamma_n^{k_n}$ are sometimes added, but they can always be absorbed into the times t_r and $\mathfrak{t}_r$ by scaling $t_r \to \gamma_1 t_r$ and $\mathfrak{t}_r \to \gamma_n \mathfrak{t}_r$ for all r.

The paper has the following structure. In Sec. 2, we show that generating function (1.2) is the free energy of a special multimatrix model represented as a chain of matrices with somewhat nonstandard interaction terms $\operatorname{tr} M_i M_{i+1}^{-1}$. We express this model as an integral over eigenvalues of these matrices in a form similar to that of the standard generalized Kontsevich model (GKM) [13]. We adapt the technique of Eynard and Prats Ferrer [20] for evaluating spectral curves for chains of matrices with these nonstandard interaction terms in Sec. 4. Although we derive the spectral curve only in the first nontrivial case $n = 4$ (i.e., in the case of one intermediate field), our technique can be straightforwardly generalized to all higher n, which we will do in a separate publication. We conclude in Sec. 5 with the discussion of our results.

Throughout the entire text, we disregard all factors that are independent of the external fields and the times t_r; all equalities in the paper must therefore be understood modulo such irrelevant factors.

2. The model

To take the profile at the point at infinity into account, we first contract all solid cycles (centered at preimages of ∞) assigning the time t_r to each contracted cycle of length $2r$. New interaction vertices arise from the thus contracted solid cycles. For example, for a cycle of length four, we obtain the correspondence



$$\sim \frac{1}{2} N t_2 \operatorname{tr}[(B_2 B_3 B_4 \Lambda \overline{\Lambda}\, \overline{B}_4 \overline{B}_3 \overline{B}_2)^2],$$

where the factor 1/2 takes the cyclic symmetry of the four-cycle into account.

The matrix-valued fields B_i, $i = 2, \ldots, n-1$, are general complex-valued matrices such that B_2 is a rectangular matrix of the size $\gamma_2 N \times \gamma_3 N$ and we always assume that $\gamma_2 > \gamma_3$, and all other matrices $B_3, \ldots, B_{n-1}$ are square matrices of the size $\gamma_3 N \times \gamma_3 N$.

The matrix-model integral whose free energy is generating function (1.2) is

$$\int DB_2 \cdots DB_{n-1} \exp\left\{ N \sum_{r=1}^{\infty} \frac{t_r}{r} \operatorname{tr}[(B_2 \cdots B_{n-1} \Lambda \overline{\Lambda}\, \overline{B}_{n-1} \cdots \overline{B}_2)^r] - \sum_{j=2}^{n-1} N \operatorname{tr}(B_j \overline{B}_j) \right\}. \tag{2.1}$$

We next change the variable as

$$\begin{aligned} \mathfrak{B}_2 &= B_2 B_3 \cdots B_{n-1}, \\ \mathfrak{B}_3 &= B_3 \cdots B_{n-1}, \\ &\vdots \\ \mathfrak{B}_{n-1} &= B_{n-1} \end{aligned} \tag{2.2}$$

and assume that all matrices $\mathfrak{B}_3, \ldots, \mathfrak{B}_{n-1}$ *are invertible* (the matrix $\mathfrak{B}_2$ remains rectangular). With the

Jacobian of transformation (2.2) taken into account, integral (2.1) becomes

$$\int D\mathfrak{B}_2\cdots D\mathfrak{B}_{n-1}\exp\biggl\{-\gamma_2 N\,\mathrm{tr}\log(\mathfrak{B}_3\overline{\mathfrak{B}}_3)-\sum_{j=4}^{n-1}\gamma_3 N\,\mathrm{tr}\log(\mathfrak{B}_j\overline{\mathfrak{B}}_j)+ \\ +\sum_{r=1}^{\infty}N\frac{t_r}{r}\,\mathrm{tr}[(\mathfrak{B}_2|\Lambda|^2\overline{\mathfrak{B}}_2)^r]-N\,\mathrm{tr}[\mathfrak{B}_2\mathfrak{B}_3^{-1}\overline{\mathfrak{B}}_3^{-1}\overline{\mathfrak{B}}_2]-N\,\mathrm{tr}[\mathfrak{B}_3\mathfrak{B}_4^{-1}\overline{\mathfrak{B}}_4^{-1}\overline{\mathfrak{B}}_3]-\cdots \\ \cdots-N\,\mathrm{tr}[\mathfrak{B}_{n-2}\mathfrak{B}_{n-1}^{-1}\overline{\mathfrak{B}}_{n-1}^{-1}\overline{\mathfrak{B}}_{n-2}]-N\,\mathrm{tr}[\mathfrak{B}_{n-1}\overline{\mathfrak{B}}_{n-1}]\biggr\}. \tag{2.3}$$

Here it becomes clear why we require that all matrices except $\mathfrak{B}_2$ be quadratic: we must be able to invert them in order to write the corresponding generating function as a free energy of a chain of Hermitian matrices, as we demonstrate below.

We now recall [21] that we can write an integral over general complex matrices $\mathfrak{B}_i$ in terms of positive-definite Hermitian matrices X_i after the change of variables

$$X_i:=\overline{\mathfrak{B}}_i\mathfrak{B}_i,\quad i=2,\ldots,n-1. \tag{2.4}$$

All the matrices X_i, $i=2,\ldots,n-1$, are of the same size $\gamma_3 N\times\gamma_3 N$. Changing the integration measure for rectangular complex matrices introduces only a simple logarithmic term (see, e.g., [11], [22]), and the resulting integral becomes

$$\int DX_{2\geq 0}\cdots DX_{n-1\geq 0}\exp\biggl\{N\sum_{r=1}^{\infty}\frac{t_r}{r}\,\mathrm{tr}[(X_2|\Lambda|^2)^r]-N\,\mathrm{tr}(X_2X_3^{-1})-\cdots \\ \cdots-N\,\mathrm{tr}(X_{n-2}X_{n-1}^{-1})-N\,\mathrm{tr}\,X_{n-1}+ \\ +(\gamma_2-\gamma_3)N\,\mathrm{tr}\log X_2-\gamma_2 N\,\mathrm{tr}\log X_3-\gamma_3 N\,\mathrm{tr}\log(X_4\cdots X_{n-1})\biggr\}. \tag{2.5}$$

The logarithmic term in X_2 stabilizes the equilibrium distribution of eigenvalues of this matrix in the domain of positive real numbers; in the case where $\gamma_2=\gamma_3$, we lose this term and must use the technique of matrix models with hard walls (see, e.g., [23] for a review).

Scaling $X_i\to X_i|\Lambda|^{-2}$ for all the integration variables, we reduce (2.5) to a more familiar form of an integral over a chain of matrices,

$$\int DX_{2\geq 0}\cdots DX_{n-1\geq 0}\exp\biggl\{N\sum_{r=1}^{\infty}\frac{t_r}{r}\,\mathrm{tr}(X_2^r)-N\,\mathrm{tr}(X_2X_3^{-1})-\cdots \\ \cdots-N\,\mathrm{tr}(X_{n-2}X_{n-1}^{-1})-N\,\mathrm{tr}(X_{n-1}|\Lambda|^{-2})- \\ +(\gamma_2-\gamma_3)N\,\mathrm{tr}\log X_2-\gamma_2 N\,\mathrm{tr}\log X_3-\gamma_3 N\,\mathrm{tr}\log(X_4\cdots X_{n-1})\biggr\}. \tag{2.6}$$

We use this expression when deriving the spectral curve equation in the next section. We now proceed further and express integral (2.6) in terms of the eigenvalues $x_i^{(k)}$ of the X_k, $k=2,\ldots,n-1$.

We apply the Mehta–Itzykson–Zuber integration formula to every term in the chain of matrices in (2.6). Taking into account that the integral over the unitary group for the term $e^{-N\,\mathrm{tr}\,X_kX_{k+1}^{-1}}$, for instance, gives

$$\int DU\exp\biggl[-N\sum_{i,j=1}^{\gamma_3 N}U_{ij}x_i^{(k)}U_{ij}^*[x_j^{(k+1)}]^{-1}\biggr]=\frac{\det_{i,j}[e^{-Nx_i^{(k)}/x_j^{(k+1)}}]}{\Delta(x^{(k)})\Delta(1/x^{(k+1)})}$$

and that

$$\frac{1}{\Delta(1/x^{(k+1)})}=\prod_{i=1}^{\gamma_3 N}[x_i^{(k+1)}]^{\gamma_3 N-1}\frac{1}{\Delta(x^{(k+1)})},$$

we finally write expression (2.6) in terms of eigenvalues of the X_k:

$$\int_0^\infty \prod_{i=1}^{\gamma_3 N} dx_i^{(2)}\frac{\Delta(x^{(2)})}{\Delta(|\Lambda|^{-2})}\prod_{k=3}^{n-1}\biggl(\prod_{i=1}^{\gamma_3 N}\frac{dx_i^{(k)}}{x_i^{(k)}}\biggr)\prod_{i=1}^{\gamma_3 N}\biggl\{\biggl(\frac{x_i^{(2)}}{x_i^{(3)}}\biggr)^{(\gamma_2-\gamma_3)N}\times$$

$$\times\exp\biggl[N\sum_{r=1}^{\infty}\frac{t_r}{r}(x_i^{(2)})^r-\frac{Nx_i^{(2)}}{x_i^{(3)}}-\dots-\frac{Nx_i^{(n-2)}}{x_i^{(n-1)}}-Nx_i^{(n-1)}|\Lambda|_i^{-2}\biggr]\biggr\}. \tag{2.7}$$

Finally, if we introduce the logarithmic quantities

$$\varphi_i^{(r)}=\log x_i^{(r)},\quad r=3,\dots,n-1,$$

then we can rewrite integral (2.7) in a more transparent form resembling the Toda chain Lagrangian:

$$\int_0^\infty \prod_{i=1}^{\gamma_3 N} dx_i^{(2)}\frac{\Delta(x^{(2)})}{\Delta(|\Lambda|^{-2})}\prod_{i=1}^{\gamma_3 N}\biggl\{\int_{-\infty}^{\infty}\prod_{k=3}^{n-1}d\varphi_i^{(k)}\times$$

$$\times\exp\biggl[N\sum_{r=1}^{\infty}\frac{t_r}{r}(x_i^{(2)})^r+(\gamma_2-\gamma_3)N\log x_i^{(2)}-(\gamma_2-\gamma_3)N\varphi_i^{(3)}-$$

$$-Nx_i^{(2)}e^{-\varphi_i^{(3)}}-Ne^{\varphi_i^{(3)}-\varphi_i^{(4)}}-\dots-Ne^{\varphi_i^{(n-2)}-\varphi_i^{(n-1)}}-Ne^{\varphi_i^{(n-1)}}|\Lambda|_i^{-2}\biggr]\biggr\}. \tag{2.8}$$

In this form, it is clear that all integrals over $\varphi_i^{(k)}$ converge.

3. The case of the two-profile generating function for Belyi pairs for $n = 3$

We now recall the results in [11], where the case $n=3$ was considered. In this case, we do not have "intermediate" integrations over φ_i in (2.8), and the partition function is described by the following lemma.

Lemma 2. *In the case where only three ramification points* 0, 1, *and* ∞ *are allowed, the generating function*

$$\mathcal{F}[\{t_1,t_2,\dots\},\{\mathfrak{t}_1,\mathfrak{t}_2,\dots\},\beta;N]=\sum_{\Gamma}\frac{1}{|\operatorname{Aut}\Gamma|}N^{2-2g}\beta^{n_2}\prod_{i=1}^{n_1}t_{r_i}\prod_{k=1}^{n_3}\mathfrak{t}_{s_k} \tag{3.1}$$

of Belyi pairs, where we fix two sets of ramification profiles $\{t_{r_1},\dots,t_{r_{n_1}}\}$ *at infinity and* $\{\mathfrak{t}_{s_1},\dots,\mathfrak{t}_{s_{n_3}}\}$ *at* 1 *and we sum over profiles at zero, is given by the integral over positive-definite Hermitian matrices* X *of size* $\gamma N\times\gamma N$ *with the external matrix field* $\widetilde{\Lambda}:=|\Lambda|^{-2}$:

$$\mathcal{Z}[t,\mathfrak{t}]=\prod_{k=1}^{\gamma N}|\lambda_k|^{-2\beta N}\int_{\gamma N\times\gamma N}DX_{\geq 0}\times$$

$$\times\exp\biggl\{N\operatorname{tr}[-X|\Lambda|^{-2}+\sum_{m=1}^{\infty}\frac{t_m}{m}X^m+(\beta-\gamma)\log X]\biggr\}. \tag{3.2}$$

Here, $\mathfrak{t}_s=\operatorname{tr}[(\Lambda\overline{\Lambda})^s]$.

Integral (3.2) is a GKM integral [13]; after integration over the eigenvalues x_k of X, it becomes the ratio of two determinants,

$$\mathcal{Z}[t,\mathfrak{t}] = \prod_{k=1}^{\gamma N} |\lambda_k|^{-2\beta N} \frac{1}{\Delta(\tilde\lambda)} \left\| \frac{\partial^{k_1-1}}{\partial \tilde\lambda_{k_2}^{k_1-1}} f(\tilde\lambda k_2) \right\|_{k_1,k_2=1}^{\gamma N}, \tag{3.3}$$

where

$$f(\tilde\lambda) = \int_0^\infty x^{N(\beta-\gamma)} \exp\biggl\{-Nx\tilde\lambda + N\sum_{m=1}^\infty \frac{t_m}{m}x^m\biggr\}. \tag{3.4}$$

Because any GKM integral (in the appropriate normalization) is a tau function of the KP hierarchy (this was shown for a model with a logarithmic term in the potential in [24]), we immediately conclude that the exponential $e^{\mathcal{F}[\{t\},\{\mathfrak{t}\},\gamma;N]}$ of generating function (3.1) modulo the normalization factor $\prod_{k=1}^{\gamma N}|\lambda_k|^{-2\beta N}$ is a tau function of the KP hierarchy (i.e., it satisfies the bilinear Hirota relations) in the times $\mathfrak{t}_s$ described in Lemma 2.

4. Spectral curve and topological recursion

In this section, we propose a method for deriving the spectral curve of model (2.6), adapting the technique in [20] to our case of a nonstandard interaction between matrices in the matrix chain. We restrict ourself here to a technically more transparent case of the three-matrix model given by the integral

$$\int DM_1\,DM_2\,DM_3\,\exp\bigl\{N\,\mathrm{tr}[V(M_1)+M_1M_2^{-1}-\gamma_2\log M_2+M_2M_3+U(M_3)]\bigr\}, \tag{4.1}$$

where the integrations are over positive-definite Hermitian matrices of size $\gamma_3 N\times\gamma_3 N$ and the potentials $V(x)$ and $U(x)$ are two Laurent polynomials of the respective positive degrees n and r (this consideration can be easily generalized to the case where $V'(x)$ and $U'(x)$ are two rational functions).

Model (4.1) satisfies equations of the two-dimensional Toda chain hierarchy (see [8], [25]), and these two classes of models are in fact closely related. Hence, solving the problem of finding the spectral curve in one model can be standardly translated to solving the corresponding problem in the other model. Because finding spectral curves for multimatrix models is more transparent technically than finding spectral curves for models with external matrix fields, we stay with the first choice.

We consider the variations of the matrix fields M_i:

$$\begin{aligned}\delta M_1 &= \frac{1}{x-M_1}\xi(M_2,M_3),\\ \delta M_2 &= M_2\frac{1}{x-M_1}\eta(M_1,M_3),\\ \delta M_3 &= \frac{1}{x-M_1}\rho(M_1,M_2),\end{aligned} \tag{4.2}$$

where ξ, η, and ρ are Laurent polynomials in their arguments. We introduce the standard notation for the leading term of the $1/N^2$-expansion of the one-loop mean of the matrix field M_1:

$$\omega_1(x) := \frac{1}{N}\biggl\langle \mathrm{tr}\,\frac{1}{x-M_1}\biggr\rangle_0. \tag{4.3}$$

Here and hereafter, the subscript 0 of a correlation function indicates the contribution of the leading order of the $1/N^2$-expansion. A single trace symbol in the angle brackets pertains to the whole expression inside the corresponding brackets.

The exact loop equations obtained using variations (4.2) are

$$\frac{1}{N^2}\left\langle \operatorname{tr}\frac{1}{x-M_1}\operatorname{tr}\frac{1}{x-M_1}\xi(M_2,M_3)\right\rangle^{\mathrm{c}} + [\omega_1(x)+V'(x)]\left\langle \operatorname{tr}\frac{1}{x-M_1}\xi(M_2,M_3)\right\rangle + \\ + \left\langle \operatorname{tr}\frac{V'(M_1)-V'(x)}{x-M_1}\xi(M_2,M_3)\right\rangle + \left\langle \operatorname{tr} M_2^{-1}\frac{1}{x-M_1}\xi(M_2,M_3)\right\rangle = 0, \tag{4.4}$$

$$\left\langle \operatorname{tr}\frac{-M_1}{x-M_1}\eta(M_1,M_3)M_2^{-1}\right\rangle + \left\langle \operatorname{tr} M_3M_2\frac{1}{x-M_1}\eta(M_1,M_3)\right\rangle + \\ + (\gamma_2-\gamma_3)\left\langle \operatorname{tr}\frac{1}{x-M_1}\eta(M_1,M_3)\right\rangle = 0, \tag{4.5}$$

$$\left\langle \operatorname{tr} M_2\frac{1}{x-M_1}\rho(M_1,M_2)\right\rangle + \left\langle \operatorname{tr} U'(M_3)\frac{1}{x-M_1}\rho(M_1,M_2)\right\rangle = 0. \tag{4.6}$$

Complete information about the model is encoded in these loop equations; solving them, a topological recursion procedure for evaluating all terms in the $1/N^2$-expansion can be developed. But our goal here is more modest: we only derive the spectral curve (this nevertheless ensures all the necessary ingredients of the topological recursion [17]–[19], also see [26], which are the spectral curve itself and two meromorphic differentials defined on this curve).

Because we obtain the spectral curve in the large-N limit, we disregard the first term in (4.4), which is of the next order in $1/N^2$. All other terms in all three equations contribute to the leading order.

We next make several substitutions that allow producing the required identities; in all identities below, we keep only leading terms in the large-N limit.

The first substitution $\xi(M_2,M_3) = (U'(M_3)-U'(z))/(M_3-z)$ yields

$$[\omega_1(x)+V'(x)]\left\langle \operatorname{tr}\frac{1}{x-M_1}\frac{U'(M_3)-U'(z)}{M_3-z}\right\rangle_0 + \\ + \left\langle \operatorname{tr}\frac{U'(M_3)-U'(z)}{M_3-z}\frac{V'(M_1)-V'(x)}{x-M_1}\right\rangle_0 + \left\langle \operatorname{tr} M_2^{-1}\frac{1}{x-M_1}\frac{U'(M_3)-U'(z)}{M_3-z}\right\rangle_0 = 0. \tag{4.7}$$

In the last term in (4.7), we use Eq. (4.5):

$$\left\langle \operatorname{tr} M_2^{-1}\frac{1}{x-M_1}\frac{U'(M_3)-U'(z)}{M_3-z}\right\rangle_0 = \left\langle \operatorname{tr} M_2M_1^{-1}\frac{1}{x-M_1}\frac{U'(M_3)-U'(z)}{M_3-z}M_3\right\rangle_0 + \\ + (\gamma_3-\gamma_2)\left\langle \operatorname{tr} M_1^{-1}\frac{1}{x-M_1}\frac{U'(M_3)-U'(z)}{M_3-z}\right\rangle_0 = \\ = \frac{1}{x}\left\langle \operatorname{tr} M_2\frac{1}{x-M_1}\frac{U'(M_3)-U'(z)}{M_3-z}(M_3-z+z)\right\rangle_0 + \\ + \frac{1}{x}\left\langle \operatorname{tr} M_2M_1^{-1}\frac{U'(M_3)-U'(z)}{M_3-z}M_3\right\rangle_0 + \\ + (\gamma_3-\gamma_2)\frac{1}{x}\left\langle \operatorname{tr}\frac{1}{x-M_1}\frac{U'(M_3)-U'(z)}{M_3-z}\right\rangle_0 + \\ + (\gamma_3-\gamma_2)\frac{1}{x}\left\langle \operatorname{tr} M_1^{-1}\frac{U'(M_3)-U'(z)}{M_3-z}\right\rangle_0 =$$

$$
\begin{aligned}
&= \frac{z}{x}\left\langle \operatorname{tr} M_2 \frac{1}{x-M_1}\frac{U'(M_3)-U'(z)}{M_3-z}\right\rangle_0 + \\
&+ \frac{1}{x}\left\langle \operatorname{tr} M_2 \frac{1}{x-M_1}(U'(M_3)-U'(z))\right\rangle_0 + \\
&+ (\gamma_3-\gamma_2)\frac{1}{x}\left\langle \operatorname{tr} \frac{1}{x-M_1}\frac{U'(M_3)-U'(z)}{M_3-z}\right\rangle_0 + \\
&+ \frac{1}{x}\left\langle \operatorname{tr} M_2^{-1} \frac{U'(M_3)-U'(z)}{M_3-z}\right\rangle_0,
\end{aligned} \tag{4.8}
$$

where we again use substitution (4.5) in the last term (in the opposite direction). We introduce the polynomials

$$
\begin{aligned}
P_{n-1,r-1}(x,z) &:= \left\langle \operatorname{tr} \frac{U'(M_3)-U'(z)}{M_3-z}\frac{V'(M_1)-V'(x)}{x-M_1}\right\rangle_0, \\
Q_{r-1}(z) &:= \left\langle \operatorname{tr} M_2^{-1}\frac{U'(M_3)-U'(z)}{M_3-z}\right\rangle_0.
\end{aligned} \tag{4.9}
$$

Equation (4.7) then becomes

$$
\begin{aligned}
&\left[\omega_1(x)+V'(x)+\frac{\gamma_3-\gamma_2}{x}\right]\left\langle \operatorname{tr} \frac{1}{x-M_1}\frac{U'(M_3)-U'(z)}{M_3-z}\right\rangle_0 + \\
&+ \frac{z}{x}\left\langle \operatorname{tr} M_2 \frac{1}{x-M_1}\frac{U'(M_3)-U'(z)}{M_3-z}\right\rangle_0 + \frac{1}{x}\left\langle \operatorname{tr} M_2 \frac{1}{x-M_1}(U'(M_3)-U'(z))\right\rangle_0 + \\
&+ P_{n-1,r-1}(x,z) + \frac{1}{x}Q_{r-1}(z) = 0,
\end{aligned} \tag{4.10}
$$

and it remains only to evaluate the term $\langle \operatorname{tr} M_2(1/(x-M_1))(U'(M_3)-U'(z))\rangle_0$. We first note that from (4.6), we have

$$
\left\langle \operatorname{tr} M_2 \frac{1}{x-M_1}U'(M_3)\right\rangle_0 = \left\langle \operatorname{tr} M_2^2 \frac{1}{x-M_1}\right\rangle_0,
$$

and we can evaluate the correlation functions $\langle \operatorname{tr} M_2 \frac{1}{x-M_1}\rangle_0$ and $\langle \operatorname{tr} M_2^2 \frac{1}{x-M_1}\rangle_0$ by consecutively substituting $\xi(M_2,M_3)=M_2$ and $\xi(M_2,M_3)=M_2^2$ in (4.4). We introduce two more polynomials

$$
\begin{aligned}
\widehat{P}_{n-1}(x) &:= \left\langle \operatorname{tr} \frac{V'(M_1)-V'(x)}{x-M_1}M_2\right\rangle_0, \\
\widehat{\widehat{P}}_{n-1}(x) &:= \left\langle \operatorname{tr} \frac{V'(M_1)-V'(x)}{x-M_1}M_2^2\right\rangle_0.
\end{aligned} \tag{4.11}
$$

Substituting $\xi(M_2,M_3)=M_2$ yields the equation

$$
[\omega_1(x)+V'(x)]\left\langle \operatorname{tr} M_2 \frac{1}{x-M_1}\right\rangle_0 + \widehat{P}_{n-1}(x) + \omega_1(x) = 0,
$$

substituting $\xi(M_2,M_3)=M_2^2$ yields

$$
[\omega_1(x)+V'(x)]\left\langle \operatorname{tr} M_2^2 \frac{1}{x-M_1}\right\rangle_0 + \widehat{\widehat{P}}_{n-1}(x) + \left\langle \operatorname{tr} M_2 \frac{1}{x-M_1}\right\rangle_0 = 0,
$$

and we obtain

$$\left\langle \operatorname{tr} M_2 \frac{1}{x-M_1} \right\rangle_0 = -\frac{\omega_1(x) + \widehat{P}_{n-1}(x)}{\omega_1(x) + V'(x)}, \tag{4.12}$$

$$\left\langle \operatorname{tr} M_2^2 \frac{1}{x-M_1} \right\rangle_0 = \frac{1}{\omega_1(x) + V'(x)} \left[-\widehat{\widehat{P}}_{n-1}(x) + \frac{\omega_1(x) + \widehat{P}_{n-1}(x)}{\omega_1(x) + V'(x)} \right]. \tag{4.13}$$

Equation (4.10) therefore becomes

$$\left[\omega_1(x) + V'(x) + \frac{\gamma_3 - \gamma_2}{x} \right] \left\langle \operatorname{tr} \frac{1}{x-M_1} \frac{U'(M_3) - U'(z)}{M_3 - z} \right\rangle_0 + \\ + \frac{z}{x} \left\langle \operatorname{tr} M_2 \frac{1}{x-M_1} \frac{U'(M_3) - U'(z)}{M_3 - z} \right\rangle_0 + s(x,z) = 0, \tag{4.14}$$

where $s(x,z)$ is a rational function

$$s(x,z) = P_{n-1,r-1}(x,z) + \frac{1}{x} Q_{r-1}(z) + \\ + \frac{1}{x} \left[\frac{1}{\omega_1 + V'(x)} \left(-\widehat{\widehat{P}}_{n-1}(x) + \frac{\omega_1(x) + \widehat{P}_{n-1}(x)}{\omega_1(x) + V'(x)} \right) + U'(z) \frac{\omega_1(x) + \widehat{P}_{n-1}(x)}{\omega_1(x) + V'(x)} \right]. \tag{4.15}$$

Finally, substituting $\xi(M_2, M_3) = \frac{U'(M_3) - U'(z)}{M_3 - z} M_2$ in (4.4), we obtain

$$[\omega_1(x) + V'(x)] \left\langle \operatorname{tr} M_2 \frac{1}{x-M_1} \frac{U'(M_3) - U'(z)}{M_3 - z} \right\rangle_0 + \\ + \left\langle \operatorname{tr} \frac{1}{x-M_1} \frac{U'(M_3) - U'(z)}{M_3 - z} \right\rangle_0 + t(x,z) = 0, \tag{4.16}$$

where

$$t(x,z) := \widehat{P}_{n-1,r-1}(x,z) := \left\langle \operatorname{tr} M_2 \frac{U'(M_3) - U'(z)}{M_3 - z} \frac{V'(M_1) - V'(x)}{x - M_1} \right\rangle_0 \tag{4.17}$$

is again a polynomial function. We now treat Eqs. (4.14) and (4.16) as a system of two linear equations for two unknowns

$$\left\langle \operatorname{tr} \frac{1}{x-M_1} \frac{U'(M_3) - U'(z)}{M_3 - z} \right\rangle_0 \qquad \text{and} \qquad \left\langle \operatorname{tr} M_2 \frac{1}{x-M_1} \frac{U'(M_3) - U'(z)}{M_3 - z} \right\rangle_0 .$$

We are interested in the case where this system is degenerate, which imposes the constraint on the variable z

$$\det \begin{bmatrix} \omega_1(x) + V'(x) + \frac{\gamma_3 - \gamma_2}{x} & \frac{z}{x} \\ 1 & \omega_1(x) + V'(x) \end{bmatrix} = 0 \tag{4.18}$$

and gives

$$z = x(\omega_1(x) + V'(x)) \left(\omega_1(x) + V'(x) + \frac{\gamma_3 - \gamma_2}{x} \right). \tag{4.19}$$

Introducing the new variable

$$y := \omega_1(x) + V'(x) \tag{4.20}$$

is a standard trick in multimatrix models. The solvability condition for the system of linear equations (4.14) and (4.16) in the degenerate case is then exactly the *spectral curve equation*

$$s(x,z) - \left(y + \frac{\gamma_3 - \gamma_2}{x}\right) t(x,z) = 0, \qquad z = xy^2 + (\gamma_3 - \gamma_2)y. \tag{4.21}$$

Despite its complexity even in the simplest cases (for example, we obtain a hyperelliptic curve of maximum genus three for the Gaussian potentials $V(x)$ and $U(z)$ in Example 1 below), we still have algebraic curves in all these cases in contrast to the case of Hurwitz numbers for branching points with only simple ramifications for which it was conjectured in [27] and shown in [28], [29] that the corresponding spectral curve (in the case of one-profile Hurwitz numbers) is the Lambert curve given by a nonpolynomial equation $x = ye^{-y}$.

Example 1. We consider the case of Gaussian potentials $V(x) = x^2/2$ and $U(z) = z^2/2$. All the polynomials $P_{n-1,r-1}$, $\widehat{P}_{n-1,r-1}$, $\widehat{P}_{n-1}$, $\widehat{\widehat{P}}_{n-1}$, and Q_{r-1} are then constants; moreover, $P_{n-1,r-1} = 1$ and $\widehat{P}_{n-1,r-1} = \widehat{P}_{n-1}$. After all cancelations, we then obtain the spectral curve equation

$$y - x + \widehat{P} - \widehat{\widehat{P}}y + xy^2 + Qy^2 + y^2(y-x)(xy + \gamma_2 - \gamma_3) = 0, \tag{4.22}$$

which describes a hyperelliptic curve of genus three for general values of the constants.

5. Conclusion

We have constructed a representation in the form of a matrix chain for the generating functions for the numbers of generalized Belyi fat graphs for hypergeometric Hurwitz numbers with ramifications at n distinct points and with ramification profiles fixed at two of these n points. We also distinguished between fat graphs with different numbers of preimages of other ramification points. The corresponding partition functions are in the class of generalized Kontsevich matrix models and are hence tau functions of the KP hierarchy, as was previously shown in [14], [8] from the standpoint of the character expansion. We constructed the matrix-chain representation with a nonstandard interaction $\sum_{i=3}^{n} \operatorname{tr}(M_{i-1}M_i^{-1})$ between adjacent positive-definite Hermitian matrices in the chain in the case where the variables of $n-3$ cycles are all equal. We successfully proposed a method for solving models with such interactions. For simplicity, we here considered only the simplest nontrivial case of the two-dimensional Toda chain with one intermediate matrix (the case $n = 4$), but our method is straightforwardly generalizable to the case of $n-3$ intermediate matrices with the last (nth) matrix being an external matrix field $|\Lambda|^{-2}$.

Having such a generalization in prospect, it would nevertheless be interesting to establish other relations. For instance, generating function (1.2) in the case of so-called clean Belyi morphisms is related [11] to the free energy of the Kontsevich–Penner matrix model [30], which is known (see [31]–[33]) to be the generating function of the numbers of integer points in moduli spaces $\mathcal{M}_{g,s}$ of curves of genus g with s holes with fixed (integer) perimeters; the very same model is also related [31] by a canonical transformation to two copies of the Kontsevich matrix model expressed in times related to the discretization of the moduli spaces $\mathcal{M}_{g,s}$. It is tempting to generalize these discretization patterns to the cut-and-join operators in [9] and [14] in the case of hypergeometric Hurwitz numbers and to the Hodge integrals in [34].

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