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CABLING PROCEDURE FOR THE COLORED HOMFLY POLYNOMIALS

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We discuss using the cabling procedure to calculate colored HOMFLY polynomials. We describe how it can be used and how the projectors and $\mathcal{R}$-matrices needed for this procedure can be found. The constructed matrix expressions for the projectors and $\mathcal{R}$-matrices in the fundamental representation allow calculating the HOMFLY polynomial in an arbitrary representation for an arbitrary knot. The computational algorithm can be used for the knots and links with $|Q|m \leq 12$, where m is the number of strands in a braid representation of the knot and $|Q|$ is the number of boxes in the Young diagram of the representation. We also discuss the justification of the cabling procedure from the group theory standpoint, deriving expressions for the fundamental $\mathcal{R}$-matrices and clarifying some conjectures formulated in previous papers.

Keywords: Chern–Simons theory, knot theory, representation theory

1. Introduction

As is known, some quantum effects are nonperturbative phenomena. Sometimes they already arise in quantum mechanics, for example, the Aharonov–Bohm effect [1], [2], but many more arise in gauge field theory [3], [4]. The nonperturbative phenomena are currently attracting ever more attention. One of the basic reasons for this is that a huge number of exactly solvable models arise in considering such phenomena, models in which, unlike the realistic quantum field theory, the answer itself is well defined and hence available for a rigorous analysis. On the other hand, there is a hope that some crucial observable phenomena, including confinement in quantum chromodynamics [5], could thus be successfully explained. Studying nonperturbative effects has yielded several new types of theories such as Seiberg–Witten theory [6], [7] and conformal field theory [8] in which nonperturbative effects play a key role.

An important class of such nonperturbative theories comprise the so-called topological field theories [9]. They are a special class of quantum field theories in which the observables (e.g., amplitudes) are unaffected by small perturbations of the coupling constant, for example, and in this sense are topological invariants. Investigating theories of this type led to studying various topological objects both previously known and newly discovered. The most direct way to use quantum field theory to study topological objects is to consider the action that in a sense is a total derivative, and the corresponding partition function

$$\mathcal{Z} = \int_{\mathcal{M}} [D\mathcal{A}] e^{-S[\mathcal{A}]} \tag{1.1}$$

is hence unchanged under smooth deformations of the manifold $\mathcal{M}$ [10], [11]. Such a partition function can be constructed quite sophisticatedly if we consider a manifold with nontrivial topological properties (see,

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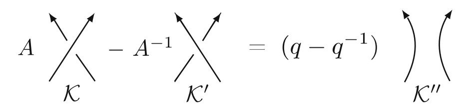


Fig. 1. Skein relation in the topological framing (see Sec. 5).

e.g., [12]). Another way is to consider a very simple manifold (e.g., a sphere) but to introduce an additional structure in integral (1.1), i.e., to try to study the averages of some observables. In gauge theories, the Wilson average [13]

$$\langle W^{\mathcal{K}}\rangle = \frac{1}{\mathcal{Z}}\int_{\mathcal{M}}[D\mathcal{A}]\,\mathrm{Tr}\,\mathrm{Pexp}\left(\oint_{\mathcal{K}}\mathcal{A}\,dx\right)e^{(i/\hbar)S[\mathcal{A}]}$$

is often considered. The Wilson average is an average of the Wilson loop, which is the path exponential of the integral of the connection $\mathcal{A}$ over a closed contour $\mathcal{K}$. A nontrivial embedding of the contour $\mathcal{K}\to\mathcal{M}$ can then be considered, even for a topologically trivial manifold. These embeddings can be characterized by the corresponding Wilson loop averages. The simplest theory in which observables of this type can be studied is three-dimensional topological gauge theory, the three-dimensional Chern–Simons theory [14]. The Wilson averages in it correspond to knots, i.e., embeddings $S_1\subset S_3$. This theory has a cubic Lagrangian,

$$S_{\mathrm{CS}} = \frac{k}{4\pi}\int\left(\mathcal{A}\wedge d\mathcal{A} - \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)d^3x.$$

The Wilson loop average $\langle W_Q^{\mathcal{K}}\rangle_{\mathrm{CS}(N,k)}$ is then defined for a knot $\mathcal{K}$ in the Chern–Simons theory with the coupling constant k in a representation Q of the gauge group $SU(N)$. In the case of a link, i.e., several intertwined knots, we can set its own representation Q_i on each link component.

In 1989, Witten suggested that the Wilson averages of the Chern–Simons theory with the $SU(2)$ gauge group are equal to the Jones polynomials known from the mathematical theory of knots [15]. This fact can be generalized to the gauge group $SU(N)$ of an arbitrary rank, and the Jones polynomials must then be replaced with HOMFLY polynomials [16]. The HOMFLY polynomials $H^{\mathcal{K}}(A,q)$ (in the case of the fundamental representation of $SU(N)$) in the mathematical theory of knots can be introduced using the skein relations [17], [18]. The skein relations are the set of equations for the HOMFLY polynomials of a given knot $\mathcal{K}$ and of the knots $\mathcal{K}'$ and $\mathcal{K}''$ respectively obtained by inverting and resolving one of the crossings (see Fig. 1).

The HOMFLY polynomial of an arbitrary knot is uniquely defined by the skein relations and the HOMFLY polynomial of the unknot. Namely, any knot can be reduced to the unknot using the skein relations, and the obtained polynomial is independent of the order in which the crossings are disentangled.[1] The proof that the HOMFLY polynomial is topologically invariant, i.e., that the answer is unchanged under smooth deformations of the knot, is provided, for example, in [17]. It is also known that the HOMFLY polynomial for a knot is a Laurent polynomial in A and q up to the factor $(q-1/q)$ [17], [18] and diverges as $q\to 1$ for an n-component link as $(q-q^{-1})^{-n}$ [20]. The skein relations thus provide a constructive, although quite complicated, definition of the HOMFLY polynomials. In any event, invariants for a huge collection of knots were obtained by this method [21].

According to Witten's ideas [15], the HOMFLY polynomial is equal to the Wilson average in the Chern–Simons theory:

$$\langle W^{\mathcal{K}}\rangle_{\mathrm{CS}(N,k)} = H^{\mathcal{K}}(A,q)\big|_{A=q^N,\;q=e^{2\pi i/(k+N)}}.$$

[1] Skein relations also exist for colored HOMFLY polynomials (see further in Sec. 1), but they are much more complicated and, generally speaking, cannot be used to reduce an arbitrary knot to the unknot [19].

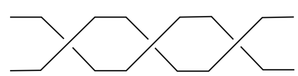
The connection between the Chern–Simons theory and knot theory has been widely studied recently [22]–[39], as have the Chern–Simons theory itself [40]–[43] and also knot theory [44]–[47], [48]–[81].

The approach relating knot invariants to the Chern–Simons theory [15], [22]–[39], like the more mathematical approaches based on Hecke algebras [44]–[47], [78]–[81], involves a construction in which a vector space V is associated with each connected component of a link. More precisely, V is a space of a Lie group representation. Defined thus, a knot invariant (for a given knot and a given group) depends on the discrete variable specifying the chosen representation. This variable is called the color of the knot invariant [26]. We can then define the so-called colored HOMFLY polynomials, generalizing the HOMFLY polynomials previously defined by the skein relations presented in Fig. 1. This in fact yields the HOMFLY polynomials in the fundamental representation. The Reshetikhin–Turaev method [82]–[84], which was used, for example, in [85]–[93], leads to a possible way to define colored HOMFLY polynomials based only on this construction.

Here, we use the method to evaluate the HOMFLY polynomials based on the Reshetikhin–Turaev formalism. According to this formalism, the HOMFLY polynomials can be described as a specially weighted trace of the product of the $\mathcal{R}$-matrices:

$$H^{\mathcal{K}}_{T_1\otimes T_2\cdots} = \mathrm{Tr}_{T_1\otimes T_2\cdots}\prod_{\alpha}\widetilde{\mathcal{R}}_{\alpha},$$

where α labels all the crossings in the braid. In general, the Reshetikhin–Turaev formalism does not require the braid representation of the knot, but we describe all the methods using this representation of the knot for simplicity. Each knot corresponds to a closure of a braid. A knot can be represented as the closure of different braids, even with different numbers of strands. The braid representation of the trefoil knot is shown as

 .

One of the basic ideas of the described approach is that all the vectors of any irreducible representation are eigenvectors of the $\mathcal{R}$ and, moreover, of the same eigenvalue:

$$\widetilde{\mathcal{R}} = \sum_{Q}\lambda_Q\, I_{d_Q}\otimes\mathcal{R},$$

where d_Q is the dimension of the representation Q. Because the $\mathcal{R}$-matrix acts on any irreducible representation as a constant, we can say that the $\mathcal{R}$-matrix acts in the space of intertwining operators. This can be understood as if the $\mathcal{R}$-matrix acts on the highest-weight vector ξ_i of the representation Q such that $\mathcal{R}_{ij} = [\mathcal{R}\xi_i]_{\xi_j}$. In this sense, we can say that the $\mathcal{R}$-matrices act in the space of irreducible representations.

The weighted trace [17], [46] is defined such that the trace over all vectors in an irreducible representation Q gives the character of the representation $S^*_Q(A,q)$ (Schur polynomial [94]): $\mathrm{Tr}_Q\, I_{d_Q} = S^*_Q(A,q)$. These properties of the $\mathcal{R}$-matrices and of the weighted trace lead to the character expansion formula for the HOMFLY polynomial [86]:

$$H^{\mathcal{K}}_{T_1\otimes T_2\otimes\cdots}(A,q) = \sum_{Q\vdash T_1\otimes T_2\otimes\cdots} h^{\mathcal{K},Q}_{T_1\otimes T_2\otimes\cdots}(q)S^*_Q(A,q). \tag{1.2}$$

The coefficients $h^{\mathcal{K},Q}_{T_1\otimes T_2\otimes\cdots}$ are Laurent polynomials in only one variable[2] q and are hence independent of N. The dependence of the HOMFLY polynomial on N is totally described by the Schur polynomials. It

[2]The coefficients $h^{\mathcal{K},Q}_{T_1\otimes T_2\otimes\cdots}$ can also depend on A with some choices of the framing (see Sec. 5). This dependence is quite simple and can be derived from the fact that these coefficients depend only on q in the vertical framing.

follows from the existence of the character expansion formula that the coefficients corresponding to different irreducible representations can be studied separately. A detailed description and analysis of this method can be found, for example, in [86], [90].

We note that we somewhat reformulate the approach presented in [86]. Namely, if the fundamental representation is considered, then there exists a basis (see (3.2) below) in which the form of all the $\mathcal{R}$-matrices acting in the braid is easily defined. We find this form explicitly in Sec. 3 and then calculate the fundamental HOMFLY polynomials directly in terms of the $\mathcal{R}$-matrices. Hence, there is no need for U-matrices (see [86]) in the case of the fundamental representation. Inter alia, this observation simplifies computer calculations of the fundamental HOMFLY polynomials.

Character expansion (1.2) also allows constructing the so-called extended HOMFLY polynomial. For this, we must replace the Schur polynomials depending on the variables A and q with Schur polynomials depending on the time variables t_k and leave the coefficients $h^{\mathcal{K},Q}_{T_1\otimes T_2\otimes\cdots}$ unchanged. The extended HOMFLY polynomials thus obtained [85] are closely related to integrable systems [95], [96].

The HOMFLY polynomials in the fundamental representation and also colored HOMFLY polynomials for several important examples were directly calculated using the Reshetikhin–Turaev formalism in [87], [89]. Here, we discuss a slightly different topic. There is an approach that allows studying colored knot invariants using the results obtained in the fundamental representation. This is the so-called cabling procedure [97]. We describe how to use the cabling procedure in the Reshetikhin–Turaev formalism and also discuss its group theory interpretation.

Results obtained using the described approach and auxiliary formulas are given in the extended electronic version of this paper [98].

This paper is organized as follows. In Secs. 2–5, we explain what the cabling procedure is, the elements needed for using it, and how to find them. In Secs. 6–8, we present some results obtained using the cabling procedure. We discuss the group theory properties of the cabling procedure in Sec. 9. Finally, in Secs. 10–12, in the language of the cabling procedure, we formulate and explain some conjectures previously suggested in [61], [89].

We present the notation used in this paper. The corresponding concepts are introduced and explicated below.

Irreducible representations of the sl_n algebra are described by Young diagrams, i.e., by sets of natural numbers $Q = [q_1, \dots, q_k]$ such that $q_1 \geq \cdots \geq q_k$ [94]. We usually let a Young diagram be denoted by a string $[q_1 \dots q_k]$, omitting the square brackets when the intended meaning is clear. A Young diagram Q can be represented graphically as k rows containing $q_1 \geq \cdots \geq q_k$ boxes. The total number $|Q| = \sum_{i=1}^{k} q_i$ of boxes in a Young diagram Q is called the size or level of the corresponding representation.

Writing $Q \vdash T_1 \otimes \cdots \otimes T_m$ means that the representation Q appears in the decomposition of the tensor product $T_1 \otimes \cdots \otimes T_m$.

The component of the product of the representations $T_1, \dots, T_m$ corresponding to the representation Q is denoted by $T_1, \dots, T_m|Q$.

The $\mathcal{R}$-matrices are always regarded as acting in the space of intertwining operators.

In a braid description, we call the strand in the ith position in a certain section through the braid the ith strand. When we say that the ith strand intersects the $(i{+}1)$th strand, we implicity mean the section through the braid that contains this crossing unless we explicitly state otherwise.

The colored $\mathcal{R}$-matrix corresponding to the crossing of the ith and $(i{+}1)$th strands in the braid is denoted by $\mathcal{R}_{Q_1\otimes\cdots(Q_i\otimes Q_{i+1})\cdots\otimes Q_k}$. Here, $Q_1, \dots, Q_i, Q_{i+1}, \dots, Q_k$ denote the representations corresponding to all the strands in the order of their position in the section through the braid in which this crossing is located. We can omit the symbol $\otimes$ if no confusion results.

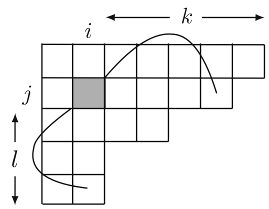


Fig. 2. The hook length $h_{i,j}$ for a box (i,j) in a Young diagram is defined as $h_{i,j} = k + l + 1$, where k is the distance to the end of the row and l is the distance to the end of the column.

The fundamental $\mathcal{R}$-matrix corresponding to the crossing of the ith and $(i{+}1)$th strands in the fundamental braid is denoted by R_i.

The projector from a representation T onto a representation Q is denoted by P_Q^T, and P_Q corresponds to the projector onto a representation Q from the representation $1^{|Q|}$. The representation Q can be reducible, for example, $Q = Q_1 \otimes Q_2$.

The Racah matrix [86], [99] describing the transition between the bases $Q_1 \otimes (Q_2 \otimes Q_3)$ and $(Q_1 \otimes Q_2) \otimes Q_3$ is denoted by $\mathcal{U}_{Q_1 \otimes Q_2 \otimes Q_3}$.

The Schur polynomials S_Q are usually defined as functions of an infinite set of time variables $\{t_k\}$ [94]. But for the HOMFLY polynomials, we need functions of A and q defined as

$$S_Q^*(A,q) = S_Q\{t\} \quad \text{for } t_k = \frac{A^k - A^{-k}}{q^k - q^{-k}}.$$

The subspace $\{t_k\}$ of such variables is called the topological locus. Passing from the usual HOMFLY polynomials to the extended ones corresponds to leaving the topological locus. For the polynomials $S_Q^*(A,q)$, there is also an explicit hook formula, which allows calculating them easily (see Fig. 2):

$$S_Q^*(A,q) = \prod_{(i,j)\in Q} \frac{Aq^{i-j} - A^{-1}q^{j-i}}{q^{h_{i,j}} - q^{-h_{i,j}}}.$$

The quantum number for n is denoted by

$$[n]_q \equiv \frac{q^n - q^{-n}}{q - q^{-1}}.$$

2. Cabling procedure

There are different ways to calculate the colored HOMFLY polynomials. But it is not yet clear how to directly calculate colored HOMFLY polynomials for representations that are neither completely symmetric nor completely antisymmetric. In such cases, another approach is often used (see, e.g., [78]) based on the so-called cabling procedure [97]. The main idea behind this approach is as follows. As previously mentioned, HOMFLY polynomials can be represented as character expansion (1.2). The size of the Young diagrams Q over which we sum in the right-hand side is equal to the sum of the sizes of the Young diagrams T_1 and T_2 in the left-hand side. It is therefore natural to reduce calculating the invariant for an initial knot or link in the representations $T_1, T_2, \ldots$ to calculating the same invariant in the fundamental representation for the knot or link with $|T_1| + |T_2| + \cdots$ strands in the braid representation. The cabling procedure is constructed in just this way.

Instead of the initial knot, we consider a satellite knot, i.e., a certain braid along the initial knot. If this braid comprises $|T_1|$ parallel strands, then the HOMFLY polynomial of the satellite knot in a representation

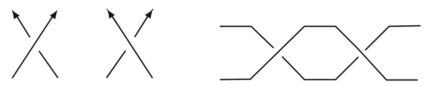


Fig. 3. Direct and inverse crossings.

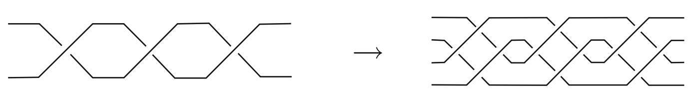
T_2 is equal to the HOMFLY polynomial of the initial knot in the representation $T_2^{|T_1|}$:

$$H^{\mathcal{K}}_{T_2^{|T_1|}} = H^{\mathcal{K}^{|T_1|}}_{T_2}.$$

The cabling method is based on just this relation. To obtain the HOMFLY polynomial in some other representation Q of size $|T_2|^{|T_1|}$, we must calculate a certain linear combination of polynomials for several different satellite knots. Constructing this linear combination corresponds to projecting from the representation $T_2^{|T_1|}$ onto the representation Q. In other words, instead of calculating the linear combination, we can multiply the product of the $\mathcal{R}$-matrices by a certain linear operator, by the projector. In the Reshetikhin–Turaev formalism, each crossing in the knot projection corresponds to an $\mathcal{R}$-matrix. Hence, the linear combination of polynomials for the satellite knots (differing by a set of several crossings) corresponds to a certain linear combination of $\mathcal{R}$-matrices. The projector can therefore be written as a polynomial in $\mathcal{R}$-matrices (see Sec. 4 for details).

In summary, to express the colored HOMFLY polynomials in terms of HOMFLY polynomials in the fundamental representation, we must perform the following procedure (the cabling procedure):

- Replace the strand on which the representation T_i is located with $|T_i|$ parallel strands. The result of this step in calculating the trefoil knot in the representation $|T_i| = 2$ is shown in the picture



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- Construct the projector from the representation $1^{|T_i|}$ onto the representation T_i.
- Calculate the polynomial for the obtained knots and links in the fundamental representation with the addition of the projection operator.

3. The $\mathcal{R}$-matrices

We use $\mathcal{R}$-matrices [82] as "building blocks" of the considered construction (see Sec. 1 above for the details). Such an $\mathcal{R}$-matrix is located at each crossing in the planar projection of the knot. Generally speaking, the $\mathcal{R}$-matrices are certain operators with four indices corresponding to the crossings of strands in the planar projection of an oriented knot or link. To obtain the matrix form of these operators, it is convenient to consider a planar knot projection of special form called a braid representation of the knot. We then introduce an analogue of the usual $\mathcal{R}$-matrix acting not only on the two crossing strands but also on all strands in this section through the braid (it acts as the unit operator on all strands in the given section except those that cross). We write the matrix form of this operator below. The $\mathcal{R}$-matrix acting in a section through a braid is still constructed using the operator with four indices. Therefore, different matrices correspond to crossings located in different places in a section through a braid. It can also be shown that the inverse $\mathcal{R}$-matrix [82] corresponds to the inverse crossing (see Fig. 3).

3.1. Diagonal $\mathcal{R}$-matrices. In the framework of the construction in which $\mathcal{R}$-matrices act on braids, the form of these matrices depends not only on the representations located at strand crossings but also on the distribution of these strands relative to the remaining strands in the braid. Each $\mathcal{R}$-matrix can be diagonalized (but not all simultaneously). Here, unless otherwise stated, we use a basis in which the first $\mathcal{R}$-matrix (i.e., the matrix corresponding to the crossing of the strands in the first and second positions in the section through the braid) is diagonal.

The simplest case is the $\mathcal{R}$-matrix acting on a braid of two strands with representations T_1 and T_2. The $\mathcal{R}$-matrix is then diagonal in the basis of irreducible representations $Q_i \vdash T_1 \otimes T_2$, its eigenvalues are [23], [86]

$$\lambda_i = \pm q^{\varkappa_{Q_i} - \varkappa_{T_1} - \varkappa_{T_2}}, \quad \text{where } \varkappa_Q = \frac{1}{2} \sum_{\{i,j\} \in Q} (j - i), \tag{3.1}$$

and i and j in the summation range all boxes in the Young diagram of Q. The eigenvalues of the $\mathcal{R}$-matrix are defined up to a common factor, which does not change the Yang–Baxter equation [23]. This factor can be chosen differently depending on the studied quantities. The eigenvalues in (3.1) correspond to the so-called vertical framing of the knot (see Sec. 5).

The situation is a bit trickier when there are more than two strands in the braid. Let the representations $T_1, \ldots, T_m$ be placed on the braid. The $\mathcal{R}$-matrix corresponding to the crossing between the strands in positions i and $i+1$ is then diagonal in the basis of irreducible representations arising in the expansion

$$T_1 \otimes \cdots \otimes (T_i \otimes T_{i+1}) \otimes \cdots \otimes T_m = T_1 \otimes \cdots \otimes \left(\sum_j Q_j \right) \otimes \cdots \otimes T_m = \sum_j \overline{Q}_j,$$

and the eigenvalues for $\overline{Q}_j$ are then the same as for the representations Q_i from which they are obtained. The inverse $\mathcal{R}$-matrix whose eigenvalues are $\lambda_i^{-1} = \pm q^{-\varkappa_{Q_i} + \varkappa_{T_1} + \varkappa_{T_2}}$ corresponds to the inverse crossing (see Fig. 3).

3.2. General $\mathcal{R}$-matrices. The form of a general (nondiagonal) $\mathcal{R}$-matrix is a much more difficult question. An answer in the general case (for an arbitrary number of strands and set of representations) is unknown. The form of nondiagonal matrices can be successfully determined in only a few concrete cases, such as in the case of the fundamental representation (the R-matrix).

The R-matrices have the simplest form in the basis of irreducible representations arising in the consecutive expansion of the tensor product with the nesting (denoted by parentheses)

$$(((((1 \otimes 1) \otimes 1) \otimes \cdots) \otimes 1) \otimes 1). \tag{3.2}$$

We call this basis the standard basis. The vectors of this basis are conveniently described in terms of strands on a definite graph [92]. For each irreducible representation, we can draw a tree depicting all ways to obtain that representation in the expansion of some tensor product. In Fig. 4, we show an example of such a tree from the representation [321] arising in the expansion of the tensor product of six fundamental representations. Each arrow in this tree corresponds to multiplication by one fundamental representation. We also introduce the $2{\times}2$ block b_j

$$b_j = \begin{pmatrix} -\dfrac{1}{q^j [j]_q} & \dfrac{\sqrt{[j+1]_q [j-1]_q}}{[j]_q} \\ \dfrac{\sqrt{[j+1]_q [j-1]_q}}{[j]_q} & \dfrac{q^j}{[j]_q} \end{pmatrix}.$$

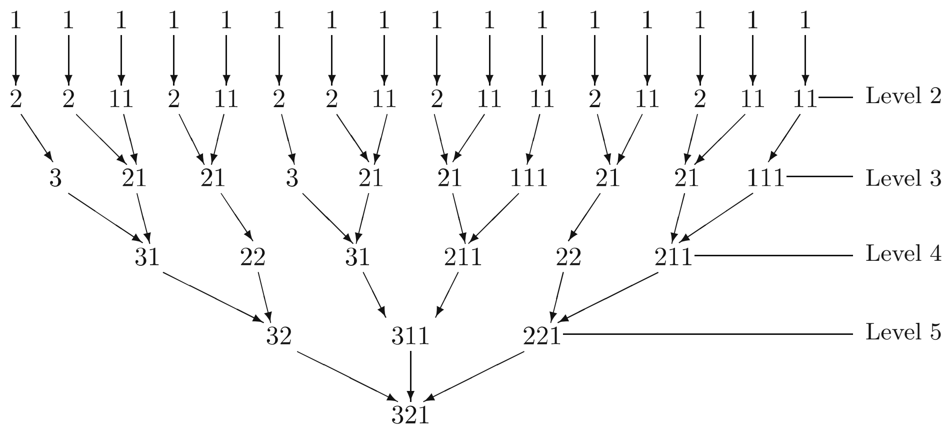


Fig. 4. Tree for the representation [321] for a six-strand braid in the fundamental representation.

$21 \to 32$ the block b_2 in the R-matrix

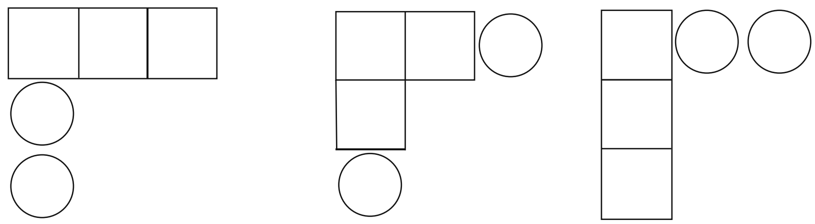
$21 \to 311$ the block b_4 in the R-matrix

Fig. 5. To choose the correct block b_j, we must draw the diagram describing the transition between paths of a doublet at the level k. If the length of the hook connecting the centers of the added boxes (shown as circles) is equal to j, then a block b_j is placed at the corresponding position in the matrix R_k.

We now describe R-matrices in basis (3.2) in explicit form. We assert that the matrix R_{k-1} corresponding to the crossing of the $(k-1)$th and kth strands in some section through the braid consists of 2×2 blocks b_k and 1×1 blocks q or $-q^{-1}$. It remains to describe the location of these blocks. For this, we use the example in Fig. 4. Each row and each column of the R-matrix corresponds to one of the leaves on the tree, i.e., to one of the points at which a path begins from a fundamental representation. Each path at a given level k enters either a doublet (in other words, is one of a pair of paths that coincide everywhere except at level k) or a singlet (otherwise). If the tree is constructed from fundamental representations, then this exhausts all possible cases, i.e., triplets, quadruplets, etc., cannot appear. If a pair of paths enter a doublet, then a block b_j, where j is the length of the hook connecting the centers of the boxes added to the Young diagram of the irreducible representation at levels k and $k+1$ (see Fig. 5), is placed at the intersection of the corresponding row and column. Sometimes, $j = k$, but this is not so in general. Strictly speaking, b_j enters the matrix R_{k-1} not as a block, but each of them (although not all simultaneously) can be transformed into a block by exchanging rows and columns.

A path is a singlet if going from the level $k-1$ to the level $k+1$ corresponds to adding two boxes either in one row or in one column of the Young diagram (Fig. 6 corresponds to $k = 4$). The diagonal element of the matrix R_k corresponding to this path is q in the first case and $-q^{-1}$ in the second case.

Any nondiagonal fundamental R-matrix in the case of fundamental representations on all strands of the braid can be constructed according to these rules. For example, the block $R_{4|321}$ of the matrix R_4 of



$3 \to 311$ the element $-q^{-1}$ in the R-matrix

$21 \to 311$ the block b_4 in the R-matrix

$111 \to 311$ the element q in the R-matrix

Fig. 6. Description of the R-matrix in terms of boxes added to the Young diagrams for the level 4 and for the final representation 311. Circles denote the boxes added to the initial diagrams.

the representation [321] (corresponding to level 4 in Fig. 4) is as follows. The diagonal elements are

$$q, \quad -\frac{1}{[2]_q q^2}, \quad -\frac{1}{[2]_q q^2}, \quad \frac{q}{[2]_q^2}, \quad \frac{q}{[2]_q^2}, \quad -\frac{1}{q}, \quad -\frac{1}{[4]_q q^4}, \quad -\frac{1}{[4]_q q^4},$$

$$\frac{q^4}{[4]_q}, \quad \frac{q^4}{[4]_q}, \quad q, \quad -\frac{1}{[2]_q q^2}, \quad -\frac{1}{[2]_q q^2}, \quad \frac{q^2}{[2]_q}, \quad \frac{q^2}{[2]_q}, \quad -\frac{1}{q}.$$

The elements of the second upper subdiagonal are

$$0, \quad \frac{\sqrt{[3]_q}}{[2]_q}, \quad \frac{\sqrt{[3]_q}}{[2]_q}, \quad 0, \quad 0, \quad 0, \quad \frac{\sqrt{[3]_q[5]_q}}{[4]_q}, \quad \frac{\sqrt{[3]_q[5]_q}}{[4]_q}, \quad 0, \quad 0, \quad 0, \quad \frac{\sqrt{[3]_q}}{[2]_q}, \quad \frac{\sqrt{[3]_q}}{[2]_q}, \quad 0, \quad 0.$$

The elements of the second lower subdiagonal are

$$0, \quad 0, \quad \frac{\sqrt{[3]_q}}{[2]_q}, \quad \frac{\sqrt{[3]_q}}{[2]_q}, \quad 0, \quad 0, \quad 0, \quad \frac{\sqrt{[3]_q[5]_q}}{[4]_q}, \quad \frac{\sqrt{[3]_q[5]_q}}{[4]_q}, \quad 0, \quad 0, \quad 0, \quad \frac{\sqrt{[3]_q}}{[2]_q}, \quad \frac{\sqrt{[3]_q}}{[2]_q}, \quad 0.$$

The remaining elements of the block are zero.

The inverse of the R-matrix described above can be constructed similarly. It suffices to replace q with q^{-1} in each instance.

We explain why the R-matrices have the form described above. First, we determine which paths can belong to the same block in the matrix R_k. Each point at a level $i \le k-1$ corresponds to a representation Q in the expansion of the tensor product $T_1 \otimes \cdots \otimes T_{i-1}$ of the representations on which R_k acts as a unit operator. Consequently, any two paths mixed by R_k coincide up to the level $k-1$. Further, the action of R_k on $T_1 \otimes \cdots \otimes T_{k+1} \otimes T_m$ is obtained from its action on $T_1 \otimes \cdots \otimes T_{k+1}$, where it does not mix the irreducible representations corresponding to different Young diagrams. Therefore, any two paths mixed by R_k must pass through the same Young diagram Q at the level $k+1$. Finally, R_k again acts as a unit operator on the representations $T_{k+1}, \dots, T_m$; hence, the corresponding parts of the paths mixed by R_k must also coincide. Therefore, all paths belonging to one block in the matrix R_k coincide (i.e., go through the same Young diagrams) everywhere except at the level k.

The size of the blocks depends on the representations in the considered product. In the fundamental case ($T_i = \Box$), each arrow in the tree corresponds to the addition of one box to the Young diagram. The paths that coincide everywhere except at level k correspond to the same Young diagram at the level $k-1$ and to the same Young diagram at the level $k+1$. Moreover, the diagram at the level $k+1$ differs from

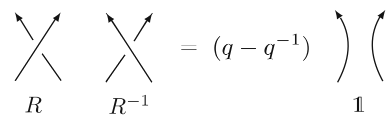


Fig. 7. Graphical description of skein relations in the vertical framing (see Sec. 5).

the diagram at the level $k-1$ by the addition of two boxes put at certain positions. Consequently, either there are two such paths (where two boxes are added at the levels $k-1$ and k in one or the other order) or only one (if the boxes are added in the same row or the same column). The first case corresponds to a 2×2 block, and the second case corresponds to 1×1 block. There are no other possibilities in the fundamental case.

The 1×1 blocks are merely the eigenvalues of the fundamental R-matrix. Because the addition of two boxes to the same row corresponds to the symmetrization of the corresponding pair of twisted representations and the addition to the same column corresponds to antisymmetrization, the first case corresponds to the eigenvalue q, and the second corresponds to $-q^{-1}$. It remains to specify the form of the 2×2 blocks, which is done in Sec. 10.

3.3. *R*-matrix properties and polynomial rings. We indicate several properties of R-matrices important for our further calculations. It is known that the R-matrices satisfy the same relations as the generators of the braid group [100]. This leads to the first two properties of the R-matrices:

$$R_iR_j - R_jR_i = 0, \quad |i-j| \neq 1, \tag{3.3}$$

$$R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1}, \tag{3.4}$$

where R_i corresponds to the crossing of the ith and $(i{+}1)$th strands. These relations hold for $\mathcal{R}$-matrices in any representation. Relation (3.3) is obvious if a braid is considered: if two crossings follow in order and have no common strand, then their order can be changed. This means that the corresponding $\mathcal{R}$-matrices must commute. Relation (3.4) in fact defines the $\mathcal{R}$-matrices. It is the famous Yang–Baxter equation.

The third property of $\mathcal{R}$-matrices, important for further considerations, substantially depends on the choice of the representation. This distinguishes the third property from the first two. This third property is essentially an equation for the eigenvalues of the $\mathcal{R}$-matrix. In the case of the fundamental representation, it has the form

$$(R_i - q)(R_i + q^{-1}) = 0. \tag{3.5}$$

This property is equivalent to the skein relations in the mathematical theory of knots; it is usually written graphically as in Fig. 7. Because an $\mathcal{R}$-matrix in the fundamental representation has only two eigenvalues, the characteristic equation is quite simple and can be used to express the polynomial of a given knot in terms of polynomials of simpler knots, as is often done in mathematics. We can also regard the characteristic equation for an $\mathcal{R}$-matrix in higher representations as a colored skein relation [17]. But such colored skein relations are not as useful as the usual skein relations: in the case of higher representations, the $\mathcal{R}$-matrix satisfies an equation of higher degree, and a way to untie knots using colored skein relations is hence unknown. For example, for the representation [2], the colored skein relations are

$$(\mathcal{R}_{22} - q^4)(\mathcal{R}_{22} + 1)(\mathcal{R}_{22} - q^{-2}) = 0.$$

Hence, the skein relations now include three terms, and it is not guaranteed that all of them are simpler than the initial knot. It is unknown if such colored skein relations can be used to calculate colored HOMFLY polynomials.

Further, the $\mathcal{R}$-matrices in the fundamental representation satisfy relations (3.3)–(3.5) and generate the Hecke algebra. This means that the polynomial ring generated by a finite number of R-matrices (e.g., on any set of the R describing a braid with m strands) is finite-dimensional. Such polynomial rings are quite important for the further calculations. Each m-strand braid can be represented as an element of a polynomial ring generated by the R-matrices $R_1, \ldots, R_{m-1}$. We note that the finite dimensionality of this polynomial ring does not generally mean that there is a finite number of prime knots: that the polynomial of some knot is equal to the sum of the polynomials of other knots does not imply that the knot itself is a combination of them. The polynomial of a composite knot is in turn equal to the product of the polynomials of its own constituent parts.

It is known from the properties of the Hecke algebra [46] that the dimension of the described polynomial ring is equal to $m!$. The basis elements can be constructed, for example, from the elementary blocks

$$\sigma_{l,0} = \mathbb{1}, \qquad \sigma_{l,k} = \prod_{i=k}^{l} R_{k+l-i}, \quad k = 1, \ldots, m-1, \quad l = k, \ldots, m-1,$$

as

$$\Xi_{k_1 \ldots k_{m-1}} = \prod_{l=1}^{m-1} \sigma_{l,k_l}, \quad k_l = 0, \ldots, l.$$

In the $q = 1$ limit, each $\sigma_{l,k}$ corresponds to a cyclic permutation of the elements labeled from k to l. In turn, each operator $\Xi_{k_1 \ldots k_{m-1}}$ corresponds to one of the $m!$ permutations. The simplest elementary blocks and the simplest basis elements of the polynomial ring for the m-strand braid are then

$$\begin{aligned}
&\sigma_{1,0} = \mathbb{1}, \quad \sigma_{1,1} = R_1, \\
&\sigma_{2,0} = \mathbb{1}, \quad \sigma_{2,1} = R_2 R_1, \quad \sigma_{2,2} = R_2, \\
&\sigma_{3,0} = \mathbb{1}, \quad \sigma_{3,1} = R_3 R_2 R_1, \quad \sigma_{3,2} = R_3 R_2, \quad \sigma_{3,3} = R_3, \\
&\quad\vdots \\
&\sigma_{m-1,0} = \mathbb{1}, \quad \sigma_{m-1,1} = R_{m-1} \cdots R_1, \quad \sigma_{m-1,2} = R_{m-1} \cdots R_2, \quad \ldots, \quad \sigma_{m-1,m-1} = R_{m-1}
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
&\Xi_{00\ldots0} = \sigma_{1,0}\sigma_{2,0}\ldots\sigma_{m-1,0} = 1, && \Xi_{10\ldots0} = \sigma_{1,1}\sigma_{2,0}\ldots\sigma_{m-1,0} = R_1, \\
&\Xi_{01\ldots0} = \sigma_{1,0}\sigma_{2,1}\ldots\sigma_{m-1,0} = R_2 R_1, && \Xi_{11\ldots0} = \sigma_{1,1}\sigma_{2,1}\ldots\sigma_{m-1,0} = R_1 R_2 R_1, \\
&\Xi_{02\ldots0} = \sigma_{1,0}\sigma_{2,2}\ldots\sigma_{m-1,0} = R_2, && \Xi_{12\ldots0} = \sigma_{1,1}\sigma_{2,2}\ldots\sigma_{m-1,0} = R_1 R_2.
\end{aligned} \tag{3.7}$$

4. Projectors

If each strand in a knot is replaced with a cable consisting, for example, of two strands and the HOMFLY polynomial is calculated in the fundamental representation, then the result is a HOMFLY polynomial in the representation $[1] \otimes [1]$. To obtain the polynomial in an irreducible representation (in this case, [2] or [11]), we must somehow construct an operator that "projects" the answer in a reducible representation onto an answer in an irreducible representation. Such operators are called projectors.

Generally speaking, each $\mathcal{R}$-matrix should be surrounded by four projectors, one for each strand that enters or leaves the corresponding crossing. From the standpoint of calculations, the key point in the considered method is that one projector suffices for the whole knot or for each link component. This is because an $\mathcal{R}$-matrix does not mix different irreducible representations [82]. This means that if a projector onto some irreducible representation is placed on one side of the $\mathcal{R}$-matrix, then the same representation must appear on the other side and hence in the whole connected component.

There are several ways to construct the projectors.

4.1. Path description of the projectors. The easiest projectors to describe are the $\mathcal{P}_Q$ that project the first n strands of a braid onto some irreducible representation Q at the level n. In standard basis (3.2), which is used here, these projectors are diagonal by definition and can be described similarly to the R-matrices above. For example, we consider the tree in Fig. 4. We can use it to construct a projector onto the level 2 for a three-strand knot or a projector onto the level 3 for a two-strand knot. For example, to describe the representation with the diagram [321] arising from the expansion of $[2]\otimes[1]^4$, we must keep only those paths in Fig. 4 that pass through the representation [2] on the level 2. By definition, each path in Fig. 4 corresponds to a representation that is an eigenvector of $\mathcal{P}_{[2]\otimes[1]^4}$. Moreover, the eigenvalue 1 corresponds to the retained paths (passing through [2]), and the eigenvalue 0 corresponds to the rejected paths (passing through [11]). Therefore, in basis (3.2), all the nondiagonal elements of the projector $\mathcal{P}_{[2]\otimes[1]^4}$ are zero, and each diagonal element is equal to the eigenvalue (1 or 0) for the corresponding path. Hence, the matrix of the projector is

$$P_{2\otimes 1^4|321} = \mathrm{diag}(1,1,0,1,0,1,1,0,1,0,0,1,0,1,0,0).$$

The described construction allows building only the projectors onto the first strand in the initial colored braid. Nevertheless, this suffices for calculating the colored polynomial for an arbitrary knot: as already discussed, one projector that can be inserted in the first strand is used for this. Moreover, Reidemeister moves can be used to deform any braid such that each strand relating to one of the colored components turns out to be first in some section through the braid. The projector for that strand can be placed in the corresponding section. To simplify the calculations, we can place several projectors in the first strand with one projector for every two crossings of the initial braid. Indeed, if each product of the $\mathcal{R}$-matrices corresponding to one crossing of a cabled knot is multiplied from two sides by a diagonal projector, then some rows and columns become zeros. We can then obtain the polynomial of the knot by multiplying matrices of a smaller size (the nonzero blocks) than the initial $\mathcal{R}$-matrices.

4.2. Projectors as polynomials in $\mathcal{R}$-matrices. The description of projectors in terms of strands allows calculating colored HOMFLY polynomials using the cabling procedure. Nevertheless, how to write the expressions for projectors in terms of $\mathcal{R}$-matrices remains an open question. The formulation of the cabling procedure itself implies that such a description should exist. Several satellite links are used in the cabling procedure, which differ from each other in the sets of crossings in the planar projection, and an $\mathcal{R}$-matrix corresponds to each crossing in the described construction. Therefore, we can describe any linear combination of polynomials for such knots as an element of the polynomial ring generated by the $\mathcal{R}$-matrices.

The $\mathcal{R}$-matrix description of projectors allows studying the relations between colored and fundamental HOMFLY polynomials and also verifying that the cabling procedure (defined as in Sec. 2) agrees with the path description of projectors. In addition, projectors onto the same representation placed in different strands in the braid and even in braids with different numbers of strands have similar expressions in terms of $\mathcal{R}$-matrices. Hence, it suffices to find the $\mathcal{R}$-matrix description, for example, for a projector placed in the first cable for which the path expression in known.

4.2.1. Projectors from the unknots. The most straightforward method (and introduced in [87]) for calculating the projectors is based on the idea that the form of the projector should depend not on the knot but only on the considered representation. Therefore, the form of these projectors can be obtained by considering the simplest knot, the unknot. Its HOMFLY polynomial in the representation Q can be represented two ways. On one hand, it is equal to $S^*_Q(A,q)$. On the other hand, it can be represented as a linear combination of several knots and links obtained in closing a braid of $|Q|$ strands in the fundamental representation. Equating these expressions gives an equation for the projector onto the representation Q. For this, we use the fact that the coefficients in the expression for projectors can depend on q but not on A.

We consider the case $|Q| = 2$. This is the simplest case where the cabling procedure is applied. If $|Q| = 2$, then two-strand knots and links should be used to describe the unknots in the representation Q. There are no more than two linearly independent HOMFLY polynomials among those for two-strand links because there are only two characters in character expansion (1.2) in this case. As the corresponding braids, we can choose two strands without any crossings ($H_0 = S^*_2 + S^*_{1,1}$) and two strands with one crossing ($H_1 = S^*_2 q - S^*_{1,1} q^{-1}$), which is the representation of the unknot. The HOMFLY polynomials of all other two-strand knots can be represented as linear combinations of these two with coefficients depending on q. The corresponding colored unknots are represented by the one-strand braid in either the representation [2] or the representation [11]. Hence, to construct the projectors, we must solve the system of equations

$$S^*_2(A,q) = p^0_2 H_0(A,q) + p^1_2 H_1(A,q), \qquad S^*_{11}(A,q) = p^0_{11} H_0(A,q) + p^1_{11} H_1(A,q). \tag{4.1}$$

This system can be solved in two ways. In the first way, we must use the exact form of the Schur polynomials in A and q. If the coefficients p are independent of A, then the system has a unique solution. The other approach is based on the idea that the cabling procedure is applicable to the extended HOMFLY polynomials [85]. This means that Eqs. (4.1) are satisfied for the coefficient before each Schur polynomial. Both approaches lead to the same solution:

$$p^0_2 = \frac{1}{q(q+q^{-1})}, \qquad p^1_2 = \frac{1}{q+q^{-1}}, \qquad p^0_{11} = \frac{q}{q+q^{-1}}, \qquad p^1_{11} = -\frac{1}{q+q^{-1}}.$$

Returning to the definition of H_0 and H_1, we can write the expressions for the projectors in terms of $\mathcal{R}$-matrices:

$$P_2 = \frac{1}{q(q+q^{-1})} + \frac{1}{q+q^{-1}} R_1, \qquad P_{11} = \frac{q}{q+q^{-1}} - \frac{1}{q+q^{-1}} R_1. \tag{4.2}$$

It is easy to verify that the projectors constructed this way are orthogonal:

$$\begin{aligned} P_2 P_{11} &= \frac{1}{(q+q^{-1})^2}\big(1 + R_1(q^{-1} - q) - R_1^2\big) = \\ &= \frac{1}{(q+q^{-1})^2}\Big(\big(1 + R_1(q - q^{-1})\big) - \big(R_1(q - q^{-1}) + 1\big)\Big) = 0. \end{aligned}$$

The properties $P_2^2 = P_2$ and $P_{11}^2 = P_{11}$ can be verified similarly.

Unfortunately, this method does not allow obtaining answers for higher representations. The reason is quite simple. For example, representations at level 3 must be obtained using cables with three strands. It can be shown that the space of HOMFLY polynomials for three-strand braids is three-dimensional with respect to linear combinations with coefficients depending only on q. At the same time, the polynomial ring generated by $\mathcal{R}$-matrices is six-dimensional, as discussed in Sec. 3.3. As a result, relations on polynomials for unknots are insufficient for recovering all projector coefficients. The same occurs with higher representations. It is therefore unclear how to apply the unknot method to higher representations.

4.2.2. $\mathcal{R}$-matrix description from the path description. An alternative method for obtaining the sought expression for projectors is to use the already known formulas in Sec. 4.1. We can try to find the combination of $\mathcal{R}$-matrices (elements of the corresponding polynomial ring) that coincides with the projector matrix in basis (3.2) obtained from the path description. We can seek such a combination as an expansion in the basis of the polynomial ring:

$$P_Q = \sum_I \alpha_I \Xi_I, \tag{4.3}$$

where Ξ_I are the basis elements, α_I are the coefficients to be determined, and I is the multi-index defined as in (3.7) and ranging $|Q|!$ different values. The left-hand side (4.3) is the projector in matrix form. This can be either the projector onto any of the isomorphic representations Q in the decomposition of $[1]^{|Q|}$ or onto a sum of several such representations. Because all elements of the polynomial ring have the block structure with each block corresponding to an irreducible representation Q from the decomposition $[1]^{|Q|} = \sum_i N_{Q_i} Q_i$, the number of elements of the Ξ_i that are not identically zero is equal to the sum of the squared multiplicities $\sum_i N_{Q_i}^2$. This is the number of equations in (4.3), and it can be shown that it is exactly $|Q|!$, i.e., the dimensionality of the polynomial ring (according to Sec. 3.3) and hence the number of variables α_I. Indeed, it is well known in representation theory that the multiplicities of irreducible representations can be obtained from the character expansion of degree $|Q|$ of the fundamental representation $S_1^{|Q|} = t_1^{|Q|}$ over the characters S_{Q_i} of the irreducible representations $Q_i \vdash 1^{|Q|}$. The characters satisfy the equations

$$t_1^{|Q|} = \sum_{Q_i \vdash |Q|} N_{1^{|Q|}}^{Q_i} S_{Q_i}(t_k), \qquad \left(\frac{\partial}{\partial t_1}\right)^{|Q|} = \sum_{Q_i \vdash 1^{|Q|}} N_{1^{|Q|}}^{Q_i} S_{Q_i}\left(\frac{\partial}{k\,\partial t_k}\right). \tag{4.4}$$

Furthermore, the scalar product on the characters can be defined as [101]

$$S_T\left(\frac{\partial}{k\,\partial t_k}\right) S_Q(t_k) = \delta_{T,Q}.$$

If we apply this relation to (4.4), then we obtain the identity

$$|Q|! = \sum \left(N_{1^{|Q|}}^{Q_i}\right)^2,$$

which shows that the number of equations in (4.3) is indeed equal to the number of defined coefficients α_I.

The case $|Q| = 2$. The polynomial ring has a single generator R_1 satisfying $(R_1 - q)(R_1 + q^{-1}) = 0$. According to (3.7), the basis of this ring comprises two elements ($2! = 2$). We can choose

$$\Xi_0 = \mathbb{1} = \mathrm{diag}(1,1), \qquad \Xi_1 = R_1 = \mathrm{diag}(q, -q^{-1}).$$

Consequently, the projectors must satisfy the equations

$$P_2 = \alpha_2^0 + \alpha_2^1 R_1 = \mathrm{diag}(1,0), \qquad P_{11} = \alpha_{11}^0 + \alpha_{11}^1 R_1 = \mathrm{diag}(0,1).$$

Because $[1]^2 = [2] + [11]$, each element of the ring splits into two $1{\times}1$ blocks, one for [2] and the other for [11]. As a result, we obtain exactly $1^2{+}1^2{=}2$ equations for $2! = 2$ variables. The solutions of these equations coincide with (4.2):

$$P_2 = \frac{1 + qR_1}{q[2]_q}, \qquad P_{11} = \frac{q - R_1}{[2]_q}.$$

The case $|Q| = 3$. According to (3.6) and (3.7), the ring of R-matrices has two generators R_1 and R_2. The polynomial ring has 3!=6 dimensions, and we can choose the basis

$$\Xi_{00} = \mathbb{1}, \qquad \Xi_{10} = R_1, \qquad \Xi_{01} = R_2R_1, \qquad \Xi_{11} = R_1R_2R_1, \qquad \Xi_{02} = R_2, \qquad \Xi_{12} = R_1R_2.$$

The projectors must satisfy the equations[3]

$$P_3 = \alpha_3^{00} + \alpha_3^{10} R_1 + \alpha_3^{02} R_2 + \alpha_3^{12} R_1 R_2 + \alpha_3^{01} R_2 R_1 + \alpha_3^{11} R_1 R_2 R_1 = \mathrm{diag}(1,0,0,0),$$

$$P_{\underline{21}} = \alpha_{\underline{21}}^{00} + \alpha_{\underline{21}}^{10} R_1 + \alpha_{\underline{21}}^{02} R_2 + \alpha_{\underline{21}}^{12} R_1 R_2 + \alpha_{\underline{21}}^{01} R_2 R_1 + \alpha_{\underline{21}}^{11} R_1 R_2 R_1 = \mathrm{diag}(0,1,0,0),$$

$$P_{\overline{21}} = \alpha_{\overline{21}}^{00} + \alpha_{\overline{21}}^{10} R_1 + \alpha_{\overline{21}}^{02} R_2 + \alpha_{\overline{21}}^{12} R_1 R_2 + \alpha_{\overline{21}}^{01} R_2 R_1 + \alpha_{\overline{21}}^{11} R_1 R_2 R_1 = \mathrm{diag}(0,0,1,0),$$

$$P_{111} = \alpha_{111}^{00} + \alpha_{111}^{10} R_1 + \alpha_{111}^{02} R_2 + \alpha_{111}^{12} R_1 R_2 + \\ + \alpha_{111}^{01} R_2 R_1 + \alpha_{111}^{11} R_1 R_2 R_1 = \mathrm{diag}(0,0,0,1).$$

Because $[1]^3 = [3] + 2\,[21] + [111]$, each element of the polynomial ring splits into three blocks: a $1{\times}1$ block for the representation [3], a $2{\times}2$ block for the representation [21], and a $1{\times}1$ block for the representation [111]. We therefore have exactly $1^2{+}2^2{+}1^2 = 6$ equations for $3!{=}6$ variables. The solution is

$$\begin{aligned}
P_3 &= \frac{1}{q^3[2]_q[3]_q}(1 + qR_1 + qR_2 + q^2 R_1R_2 + q^2 R_2R_1 + q^3 R_1R_2R_1),\\
P_{\underline{21}} &= \frac{1}{[3]_q}\left(1 + qR_1 - \frac{q^{-2}}{[2]_q}R_2 - \frac{q^{-1}}{[2]_q}(R_1R_2 + R_2R_1) - \frac{1}{[2]_q}R_1R_2R_1\right),\\
P_{\overline{21}} &= \frac{1}{[3]_q}\left(1 - q^{-1}R_1 + \frac{q^2}{[2]_q}R_2 - \frac{q}{[2]_q}(R_1R_2 + R_2R_1) + \frac{1}{[2]_q}R_1R_2R_1\right),\\
P_{111} &= \frac{q^3}{[2]_q[3]_q}(1 - q^{-1}R_1 - q^{-1}R_2 + q^{-2}R_1R_2 + q^{-2}R_2R_1 - q^{-3}R_1R_2R_1).
\end{aligned} \tag{4.5}$$

The rank-two projector onto the sum of the two isomorphic representations [21] can be obtained as the sum of the projectors

$$P_{21} = P_{\underline{21}} + P_{\overline{21}} = \frac{1}{[3]_q}\big(2 + (q - q^{-1})(R_1 + R_2) - R_1R_2 - R_2R_1\big). \tag{4.6}$$

Another way to obtain this projector is to write the following equation instead of (4.5):

$$P_{21} = \alpha_{21}^{00} + \alpha_{21}^{10} R_1 + \alpha_{21}^{02} R_2 + \alpha_{21}^{12} R_1 R_2 + \alpha_{21}^{01} R_2 R_1 + \alpha_{21}^{11} R_1 R_2 R_1 = \mathrm{diag}(0,1,1,0),$$

whose solution yields the same result.

Using properties (3.3)–(3.5) of the polynomial ring, we can bring expression (4.6) to the simpler form $P_{21} = (R_1 - R_2)^2/[3]_q$.

The case $|Q| = 4$. The polynomial ring has three generators R_1, R_2, and R_3. According to (3.7), there are 24 basis elements, for example,

$$\begin{array}{llll}
\Xi_{000} = \mathbb{1}, & \Xi_{001} = R_3R_2R_1, & \Xi_{002} = R_3R_2, & \Xi_{003} = R_3,\\
\Xi_{100} = R_1, & \Xi_{101} = R_1R_3R_2R_1, & \Xi_{102} = R_1R_3R_2, & \Xi_{103} = R_1R_3,\\
\Xi_{010} = R_2R_1, & \Xi_{011} = R_2R_1R_3R_2R_1, & \Xi_{012} = R_2R_1R_3R_2, & \Xi_{013} = R_2R_1R_3,\\
\Xi_{110} = R_1R_2R_1, & \Xi_{111} = R_1R_2R_1R_3R_2R_1, & \Xi_{112} = R_1R_2R_1R_3R_2, & \Xi_{113} = R_1R_2R_1R_3,\\
\Xi_{020} = R_2, & \Xi_{021} = R_2R_3R_2R_1, & \Xi_{022} = R_2R_3R_2, & \Xi_{023} = R_2R_3,\\
\Xi_{120} = R_1R_2, & \Xi_{121} = R_1R_2R_3R_2R_1, & \Xi_{122} = R_1R_2R_3R_2, & \Xi_{123} = R_1R_2R_3.
\end{array}$$

[3]There are two representations 21 in the cube of the fundamental representation: they arise from representation 2 and representation 11. There are hence two projectors onto 21, denoted by $P_{\underline{21}}$ and $P_{\overline{21}}$ here.

Because $[1]^4 = [4] + 3\,[31] + 2\,[22] + 3\,[211] + [1111]$, each element of the ring splits into five blocks: a 1×1 block for [4], a 3×3 block for [31], a 2×2 block for [22], a 3×3 block for [211], and a 1×1 block for [1111]. We therefore have exactly $1^2+3^2+2^2+3^2+1^2=24$ equations for $4!=24$ variables. The expressions for the projectors onto each representation arising in the decomposition of $[1]^4$ are quite lengthy. We therefore present them in Appendix A in [98] and here only give the formulas for the projectors onto the representations [4] and [1111] and onto the spaces of all representations [31], [22], and [211]:

$$
\begin{aligned}
P_4 = {} & \frac{1}{q^6[4]_q!}\big(1 + q(R_1+R_2+R_3) + \\
& + q^2(R_1R_2+R_2R_1+R_2R_3+R_3R_2+R_1R_3) + \\
& + q^3(R_1R_2R_1+R_2R_3R_2+R_1R_2R_3+R_2R_1R_3+R_3R_1R_2+R_3R_2R_1) + \\
& + q^4(R_1R_2R_1R_3+R_1R_2R_3R_2+R_1R_3R_2R_1+R_2R_1R_3R_2+R_2R_3R_2R_1) + \\
& + q^5(R_1R_2R_1R_3R_2+R_1R_2R_3R_2R_1+R_2R_1R_3R_2R_1) + q^6R_1R_2R_1R_3R_2R_1\big), \\
P_{31} = {} & \frac{1}{q^2[2][4]}\big(3 + (2q-q^{-1})(R_1+R_2+R_3) + \\
& + (q^2-1)(R_1R_2+R_2R_1+R_2R_3+R_3R_2) + (q^2-2)R_1R_3 + \\
& + q^3(R_1R_2R_1+R_2R_3R_2) - q(R_1R_2R_3+R_2R_1R_3+R_3R_1R_2+R_3R_2R_1) - \\
& - R_2R_1R_3R_2 - q(R_1R_2R_1R_3R_2+R_2R_1R_3R_2R_1) + \\
& + q^3R_1R_2R_3R_2R_1 - q^2R_1R_2R_1R_3R_2R_1\big), \\
P_{22} = {} & \frac{1}{[3][2]^2}\big(2 + (q-q^{-1})(R_1+R_2+R_3) - (R_1R_2+R_2R_1+R_2R_3+R_3R_2) - \\
& - (q^2+q^{-2})R_1R_3 + (R_1R_2R_1R_3+R_1R_2R_3R_2+R_1R_3R_2R_1+R_2R_3R_2R_1) + \\
& + (q^2+q^{-2})R_2R_1R_3R_2 - \\
& - (q-q^{-1})(R_1R_2R_1R_3R_2+R_1R_2R_3R_2R_1+R_2R_1R_3R_2R_1) + \\
& + 2R_1R_2R_1R_3R_2R_1\big), \\
P_{211}(q) = {} & P_{31}(-q^{-1}), \\
P_{1111}(q) = {} & P_4(-q^{-1}).
\end{aligned}
$$

4.2.3. Projectors from characteristic equations. We consider one more way to construct projectors as polynomials in R-matrices. This way (in contrast to the way in the preceding subsection) does not allow obtaining polynomials of the minimum degree or of the shortest length, but it leads to an answer with a more understandable structure, which is important for a theoretical analysis.

The method is based on the fact that if a characteristic equation for a linear operator is known,

$$\prod_{i=1}^{n}(A-\lambda_i) = 0, \tag{4.7}$$

then it is easy to construct a projector onto the subspace corresponding to each eigenvalue,

$$P_{\lambda_j} = \prod_{i\neq j} \frac{A-\lambda_i}{\lambda_j-\lambda_i} \tag{4.8}$$

(the order of the factors is inessential in (4.7) and (4.8) because they all commute), or on a sum of such subspaces,

$$P_{\lambda_{j_1},\ldots,\lambda_{j_k}} = \sum_{l=1}^{k} P_{\lambda_{j_l}}. \tag{4.9}$$

The property $P_{\lambda_j}^2 = P_{\lambda_j}$ then follows directly from (4.7). A drawback of using this method applied to R-matrices is that many of their eigenvalues coincide. For example, there are only two distinct eigenvalues in the fundamental case: q and $-q^{-1}$. Therefore, the basic task is to find combinations of R-matrices with sufficiently many distinct eigenvalues. Then we can "distinguish" all irreducible representations in the decomposition of $[1]^{|Q|}$. To find such combinations and the corresponding characteristic equations, it suffices to consider $|Q|$-strand braids. It is natural to begin with the cases $|Q|=2$ and $|Q|=3$ and then try to generalize the method.

Because the R-matrices split into blocks corresponding to different irreducible representations, their characteristic equations can be written in the form

$$\prod_{Q\vdash 1^{|Q|}} F_Q(R) = 0,$$

where F_Q determines the characteristic equation $F_Q(P_Q R)=0$ of the block corresponding to the representation Q.

The case $|Q|=2$. For $|Q|=2$, we have one R-matrix satisfying the characteristic equation

$$F_2(R_1)F_{11}(R_1) \equiv (R_1-q)(R_1+q^{-1}) = 0. \tag{4.10}$$

This equation contains sufficient information to construct both projectors:

$$P_2 = \frac{R_1+q^{-1}}{q+q^{-1}}, \qquad P_{11} = \frac{R_1-q}{-q^{-1}-q}. \tag{4.11}$$

This result coincides with the previously obtained formulas (4.2).

The case $|Q|=3$. For $|Q|=3$, we have two matrices R_1 and R_2 satisfying the same characteristic equation (3.5). The projectors obtained from the characteristic equation for the first R-matrix allow "distinguishing" the representations [3] and $\underline{[21]}$ symmetric in the first pair of strands from the representations $\overline{[21]}$ and [111] antisymmetric in the first pair of strands. The projectors obtained from the equation for the second R-matrix allow distinguishing the analogous groups of representations with respect to the second pair of strands. This is insufficient for constructing the projectors onto each of the representations [3], [21], and [111]. Therefore, we must construct combinations of R-matrices that would allow finding all necessary projectors. The combination $(R_1-R_2)^2$ is an example of an appropriate combination in the case $|Q|=3$. Using the explicit expressions for R_1 and R_2, we can establish that this combination satisfies the characteristic equation

$$(R_1-R_2)^2\big((R_1-R_2)^2-(q^2+1+q^{-2})\big) = 0,$$

where the representation [3] and also the representation [111] correspond to the eigenvalue 0, while both representations [21] correspond to the eigenvalue q^2+1+q^{-2}. We can hence obtain the projector P_{21}:

$$P_{21} = \frac{(R_1 - R_2)^2}{q^2+1+q^{-2}}. \tag{4.12}$$

Also using (4.10), we can obtain all four projectors:

$$\begin{aligned}
P_3 &= \frac{(R_1+q^{-1})\big((q^2+1+q^{-2})-(R_1-R_2)^2\big)}{(q+q^{-1})(q^2+1+q^{-2})},\\
P_{\underline{21}} &= \frac{(R_1+q^{-1})(R_1-R_2)^2}{(q+q^{-1})(q^2+1+q^{-2})},\\
P_{\overline{21}} &= \frac{(q-R_1)(R_1-R_2)^2}{(q+q^{-1})(q^2+1+q^{-2})},\\
P_{111} &= \frac{(q-R_1)\big((q^2+1+q^{-2})-(R_1-R_2)^2\big)}{(q+q^{-1})(q^2+1+q^{-2})}.
\end{aligned} \tag{4.13}$$

4.2.4. Torus products of the *R*-matrices. An unresolved question remains. Is there a universal method for constructing projectors onto arbitrary representations using the characteristic equations? This would be possible if we could find a combination of R-matrices whose eigenvalues were known for an arbitrary representation and had sufficiently many distinct eigenvalues to distinguish all irreducible representations at the corresponding level. An obvious candidate for such a combination is the product

$$\mathfrak{R}_{|Q|} \equiv \prod_{i=1}^{|Q|-1} R_{|Q|-i}.$$

Precisely this product appears in studying torus knots. It is known from the Rosso–Jones formula that its eigenvalues are

$$\lambda_{Q,j} = q^{\varkappa_Q/|Q|}\Lambda_{Q,j}, \quad j=1,\dots,N_Q,$$

where the coefficients $\Lambda_{Q,j}$ are q independent numbers implicitly defined by the Adams rule [58]–[61]. These coefficients are in fact the $|Q|$th-degree roots of unity, $\Lambda_{Q,j}=e^{2\pi ik/|Q|}$, and the multiplicity of each eigenvalue depends nontrivially on Q. Hence,

$$|\lambda_{Q,j}| = q^{\varkappa_Q/|Q|}, \quad j=1,\dots,N_Q.$$

For such products of R-matrices, the sets of the eigenvalues corresponding to different representations are nonintersecting up to the level $|Q|=5$. Therefore, the projectors onto all the irreducible representations with $|Q|$ boxes in the Young diagram can be obtained from the single equation for $\mathfrak{R}_{|Q|}$. This method stops working at the level $|Q|=6$, where $\varkappa_{411}=\varkappa_{33}=3$, and additional equations are hence required. Moreover, even if all the eigenvalues are different, their explicit forms for an arbitrary representation are unknown. We must determine the eigenvalues from the Adams rule or directly calculate them for each $|Q|$, as was done in [86], [90]. Furthermore, even in the simplest cases these eigenvalues are quite complicated and include fractional degrees of q and roots of unity. Because of these problems, it is unclear how to construct the projector onto an arbitrary representation using the torus product $\mathfrak{R}_{|Q|}$. We restrict ourself to several examples of using such products.

Because the case $|Q| = 2$ is trivial, we begin with the case $|Q| = 3$: we have

$$F_3(R_2R_1)F_{21}(R_2R_1)F_{111}(R_2R_1) \equiv (R_2R_1 - q^2)\big((R_2R_1)^2 + R_2R_1 + 1\big)(q^2R_2R_1 - 1) = 0. \tag{4.14}$$

The corresponding projectors onto symmetric and antisymmetric representations are obtained directly:

$$\begin{aligned} P_3 &= \frac{\big((R_2R_1)^2 + R_2R_1 + 1\big)(R_2R_1 - q^{-2})}{(q^4 + q^2 + 1)(q^2 - q^{-2})}, \\ P_{111} &= \frac{(R_2R_1 - q^2)\big((R_2R_1)^2 + R_2R_1 + 1\big)}{(q^{-2} - q^2)(q^{-4} + q^{-2} + 1)}. \end{aligned} \tag{4.15}$$

But for the remaining representation, a simple construction like (4.8) gives the operator $P \equiv (R_2R_1 - q^2)(q^2R_2R_1 - 1)$, which does not satisfy the condition $P^2 = \text{const} \cdot P$. It is therefore not a projector, even up to normalization, but a linear operator that annihilates representations [3] and [111] and somehow rotates two copies of representation [21]. To obtain the projector according to (4.9), we must further decompose (4.14) as

$$(R_2R_1 - q^2)(R_2R_1 - e^{2\pi i/3})(R_2R_1 - e^{4\pi i/3})(q^2R_2R_1 - 1) = 0.$$

The projector P_{21} can then be obtained as a sum

$$\begin{aligned} P_{21} = &\frac{(R_2R_1 - q^2)(q^2R_2R_1 - 1)}{e^{2\pi i/3} - e^{4\pi i/3}} \times \\ &\times \left(\frac{R_2R_1 - e^{4\pi i/3}}{(e^{2\pi i/3} - q^2)(q^2e^{2\pi i/3} - 1)} - \frac{R_2R_1 - e^{2\pi i/3}}{(e^{4\pi i/3} - q^2)(q^2e^{4\pi i/3} - 1)} \right) = \\ = &-\frac{(R_2R_1 - q^2)(q^2R_2R_1 - 1)(R_2R_1 + 1)}{q^4 + q^2 + 1}. \end{aligned} \tag{4.16}$$

The summands are rank-one projectors onto each of the isomorphic representations [21], but these copies are not $\underline{[21]}$ and $\overline{[21]}$ in standard basis (3.2) used here. It can be shown using the properties of R-matrices described in Sec. 3.3 that relations (4.5), relations (4.15) and (4.16), and relation (4.13) give three equivalent answers for the projectors.

In the case of $|Q| = 4$, the relevant product now includes three R-matrices, $\mathfrak{R}_3 \equiv R_3R_2R_1$, and satisfies the characteristic equation

$$F_4(\mathfrak{R}_3)F_{31}(\mathfrak{R}_3)F_{22}(\mathfrak{R}_3)F_{211}(\mathfrak{R}_3)F_{1111}(\mathfrak{R}_3) = 0,$$

where

$$\begin{aligned} &F_4(\mathfrak{R}_3) = q^3 - \mathfrak{R}_3, \qquad F_{31}(\mathfrak{R}_3) = (q + \mathfrak{R}_3)(q^2 + \mathfrak{R}_3^2), \qquad F_{22}(\mathfrak{R}_3) = \mathfrak{R}_3^2 - 1, \\ &F_{211}(\mathfrak{R}_3|q) = F_{31}(\mathfrak{R}_3|-q^{-1}), \qquad F_{1111}(\mathfrak{R}_3|q) = F_4(\mathfrak{R}_3|-q^{-1}) \end{aligned}$$

(in the last two equalities, we explicitly showed that $\mathfrak{R}_3$ depends on q and characteristic equations for different representations related to the change of q to $-q^{-1}$). Hence, the entire characteristic equation is

$$(q^3 - \mathfrak{R}_3) \cdot (q + \mathfrak{R}_3)(q^2 + \mathfrak{R}_3^2) \cdot (\mathfrak{R}_3^2 - 1) \cdot (1 - q\mathfrak{R}_3)(1 + q^2\mathfrak{R}_3^2) \cdot (1 - q^3\mathfrak{R}_3) = 0.$$

Here, the dots separate factors corresponding to different irreducible representations. According to (4.8) and (4.9), the projectors can be written as

$$P_4 = \prod_{\substack{Q_i\vdash 1^4,\\ Q_i\neq 4}} \frac{F_{Q_i}(\mathfrak{R}_3)}{F_{Q_i}(q^3)} = \frac{(q+\mathfrak{R}_3)(q^2+\mathfrak{R}_3^2)(\mathfrak{R}_3^2-1)(q\mathfrak{R}_3-1)(q^2\mathfrak{R}_3^2+1)(q^3\mathfrak{R}_3+1)}{q^3(q^2+1)(q^{12}-1)(q^{16}-1)},$$

$$P_{31} = \prod_{\substack{Q_i\vdash 1^4,\\ Q_i\neq 31}} F_{Q_i}(\mathfrak{R}_3)\Bigg(\frac{\mathfrak{R}_3+q^2}{F'_{41}(-q)\prod_{Q_i\vdash 1^4,\,Q_i\neq 31}F_{Q_i}(-q)} +$$

$$+\frac{(\mathfrak{R}_3+q)(\mathfrak{R}_3+iq)}{F'_{41}(iq)\prod_{Q_i\vdash 1^4,\,Q_i\neq 31}F_{Q_i}(iq)} + \frac{(\mathfrak{R}_3+q)(\mathfrak{R}_3-iq)}{F'_{41}(-iq)\prod_{Q_i\vdash 1^4,\,Q_i\neq 31}F_{Q_i}(-iq)}\Bigg) =$$

$$= \frac{(\mathfrak{R}_3-q^3)(\mathfrak{R}_3^2-1)}{q^3(q^2+1)(q^4-1)(q^{16}-1)}\times$$

$$\times(q^8\mathfrak{R}_3^2-q^4\mathfrak{R}_3^2+\mathfrak{R}_3^2-q^5\mathfrak{R}_3+q\mathfrak{R}_3+q^2)(q\mathfrak{R}_3-1)(q^2\mathfrak{R}_3^2+1)(q^3\mathfrak{R}_3^2+1),$$

$$P_{22} = \prod_{\substack{Q_i\vdash 1^4,\\ Q_i\neq 22}} F_{Q_i}(\mathfrak{R}_3)\Bigg(\frac{\mathfrak{R}_3+1}{F'_{22}(1)\prod_{Q_i\vdash 1^4,\,Q_i\neq 22}F_{Q_i}(1)} + \frac{\mathfrak{R}_3-1}{F'_{22}(-1)\prod_{Q_i\vdash 1^4,\,Q_i\neq 22}F_{Q_i}(-1)}\Bigg) =$$

$$= \frac{(q^3-\mathfrak{R}_3)(q+\mathfrak{R}_3)(q^2+\mathfrak{R}_3^2)(1-q\mathfrak{R}_3)(1+q^2\mathfrak{R}_3^2)(1-q^3\mathfrak{R}_3)}{(q^4-1)^2(q^4+q^2+1)}.$$

The same procedure can be repeated for $|Q|=5$ and $|Q|=7$. But this method does not work for $|Q|=6$ and most higher representations: in these cases, different representations can have the same $\varkappa_Q$, as, for example, happens with the representations [411] and [33] for $|Q|=6$. Consequently, the eigenvalues of the torus product for these representations also coincide, namely, $|\lambda_{411}|=|\lambda_{33}|=q$. As a result, the projectors constructed in the way described above do not "distinguish" these two representations. Indeed,

$$F_{411}(\mathfrak{R}_3) = (\mathfrak{R}_3^2-q^2)^2(\mathfrak{R}_3^2-q\mathfrak{R}_3+q^2)(\mathfrak{R}_3^4+q^2\mathfrak{R}_3^2+q^4),$$

$$F_{33}(\mathfrak{R}_3) = (\mathfrak{R}_3^2-q^2)(q^2-q\mathfrak{R}_3+\mathfrak{R}_3^2).$$

Because all the eigenvalues of the representation [33] are also eigenvalues of the representation [411], even the rank-one projector onto any one of the isomorphic representations [33] cannot be constructed using these formulas. This means that other equations are needed. It would be natural to consider equations of the same type from lower levels, i.e., for $|Q|\leq 6$, and thus use the already known answers for the projectors onto the lower representations. But torus products cannot be used directly in this way, because the eigenvectors of $\mathfrak{R}_{|Q|}$ are not eigenvectors of $\mathfrak{R}_{|Q|+1}$ in general.

4.2.5. Link products of the *R*-matrices. There is one more class of R-matrix products for which the eigenvalues are known in the general case: it is the operators that we call link products,

$$\mathfrak{R}_{n,m}\mathfrak{R}^{\dagger}_{n,m} = \Bigg(\prod_{i=1}^{m}\prod_{j=1}^{n} R_{n+i-j}\Bigg)\Bigg(\prod_{i=1}^{n}\prod_{j=1}^{m} R_{m+i-j}\Bigg). \tag{4.17}$$

These products arise in studying two-strand links. If we place a representation of size n at the first strand and of size m at the second strand, then each pair $\mathcal{R}_{Q_1Q_2}\mathcal{R}_{Q_2Q_1}$ should be replaced with product (4.17) in the cabling procedure. As we discuss in detail in Sec. 6, every such product satisfies the characteristic equation

$$\prod_{\substack{T_1\vdash 1^m,\, T_2\vdash 1^n,\\ Q\vdash 1^{m+n}}} (\mathfrak{R}_{n,m}\mathfrak{R}^{\dagger}_{n,m} - q^{2\varkappa_Q-2\varkappa_{T_1}-2\varkappa_{T_2}}) = 0.$$

These link products are longer than torus products but are more convenient for some reasons. First, their eigenvalues are explicitly known for an arbitrary representation, and the corresponding expressions are simpler than in the torus product case: they have only integer degrees of q. Moreover, standard basis (3.2) used here consists of eigenvectors of the link product. The described properties allow constructing a recursive procedure for calculating projectors.

The simplest case is $T_1 = [1]$. Then $\mathfrak{R}_{1,|T_2|}$ is just a torus product:

$$\mathfrak{R}_{1,|T_2|}\mathfrak{R}^{\dagger}_{1,|T_2|} \equiv \mathfrak{R}_{|T_2|}\mathfrak{R}^{\dagger}_{|T_2|}.$$

As before, we start with some examples and then discuss the generalization possibilities.

In the case $|Q| = 2$, the characteristic equation is a quite simple deformation of its analogue written above: we have

$$F_2 = \mathfrak{R}_2\mathfrak{R}^{\dagger}_2 - q^{2\varkappa_2} = \mathfrak{R}_2\mathfrak{R}^{\dagger}_2 - q^2, \qquad F_{11} = \mathfrak{R}_2\mathfrak{R}^{\dagger}_2 - q^{2\varkappa_{11}} = \mathfrak{R}_2\mathfrak{R}^{\dagger}_2 - q^{-2},$$

and therefore

$$F_2F_{11} = (\mathfrak{R}_2\mathfrak{R}^{\dagger}_2 - q^2)(\mathfrak{R}_2\mathfrak{R}^{\dagger}_2 - q^{-2}) = 0,$$

whence we obtain expressions for the projectors

$$P_2 = \frac{\mathfrak{R}_2\mathfrak{R}^{\dagger}_2 - q^{-2}}{q^2 - q^{-2}}, \qquad P_{11} = \frac{\mathfrak{R}_2\mathfrak{R}^{\dagger}_2 - q^{2}}{q^{-2} - q^{2}}.$$

These expressions lead to answer (4.11), obtained using other methods described here, if we apply the identity

$$\mathfrak{R}_2\mathfrak{R}^{\dagger}_2 = R_1^2 = (q - q^{-1})R_1 + 1.$$

In the first nontrivial case, $|Q| = 3$, we have

$$\begin{aligned}
&F_3 = \mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{2\varkappa_3-2\varkappa_2} = \mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^4,\\
&F_{21} = (\mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{2\varkappa_{21}-2\varkappa_3})(\mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{2\varkappa_{21}-2\varkappa_{111}}) = (\mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{-2})(\mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{2}),\\
&F_{111} = \mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{2\varkappa_{111}-2\varkappa_{11}} = \mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{-4}
\end{aligned}$$

in the characteristic equation. Therefore,

$$F_3F_{21}F_{111} = (\mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^4)(\mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{-2})(\mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{2})(\mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{-4}) = 0.$$

The rank-one projectors onto the fully symmetric and the fully antisymmetric representations are

$$P_3 = P^{(3)}_{q^4} = \frac{(\mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{-2})(\mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{2})(\mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{-4})}{(q^4 - q^{-2})(q^4 - q^2)(q^4 - q^{-4})},$$

$$P_{111} = P^{(3)}_{q^{-4}} = \frac{(\mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{4})(\mathfrak{R}_3\mathfrak{R}^{\dagger}_3 - q^{-2})(q^{-4} - q^{2})}{(q^{-4} - q^4)(q^{-4} - q^{-2})(q^{-4} - q^2)},$$

where $P_\lambda^{(m)}$ is the projector onto the eigenvalue λ obtained from the m-strand equation. The remaining projector is of rank two and is equal to the sum of the two rank-one projectors:

$$P_{21} = P_{q^2}^{(3)} + P_{q^{-2}}^{(3)} = \frac{(\mathfrak{R}_3\mathfrak{R}_3^\dagger - q^4)(\mathfrak{R}_3\mathfrak{R}_3^\dagger - q^{-2})(\mathfrak{R}_3\mathfrak{R}_3^\dagger - q^{-4})}{(q^2-q^4)(q^2-q^{-2})(q^2-q^{-4})} +$$

$$+ \frac{(\mathfrak{R}_3\mathfrak{R}_3^\dagger - q^4)(\mathfrak{R}_3\mathfrak{R}_3^\dagger - q^{2})(\mathfrak{R}_3\mathfrak{R}_3^\dagger - q^{-4})}{(q^{-2}-q^4)(q^{-2}-q^{2})(q^{-2}-q^{-4})} =$$

$$= -\frac{(\mathfrak{R}_3\mathfrak{R}_3^\dagger - q^4)(\mathfrak{R}_3\mathfrak{R}_3^\dagger - q^{-4})(\mathfrak{R}_3\mathfrak{R}_3^\dagger - q^2 - q^{-2})}{(q-q^{-1})(q^3-q^{-3})}.$$

For $|Q| = 4$, the characteristic equation is determined by the functions

$$F_4 = \mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{2\varkappa_4 - 2\varkappa_3} = \mathfrak{R}_4\mathfrak{R}_4^\dagger - q^6,$$

$$F_{31} = (\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{2\varkappa_{31} - 2\varkappa_3})(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{2\varkappa_{31} - 2\varkappa_{21}})^2 = (\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{-2})(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{2})^2,$$

$$F_{22} = (\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{2\varkappa_{22} - 2\varkappa_{21}})^2 = (\mathfrak{R}_4\mathfrak{R}_4^\dagger - 1)^2,$$

$$F_{211} = (\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{2\varkappa_{211} - 2\varkappa_{21}})^2(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{2\varkappa_{211} - 2\varkappa_{111}}) = (\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{2})(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{2})^{-2},$$

$$F_{1111} = \mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{2\varkappa_4 - 2\varkappa_3} = \mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{-6},$$

and therefore

$$F_4F_{31}F_{22}F_{211}F_{1111} = (\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^6)(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^2)(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{-2})(\mathfrak{R}_4\mathfrak{R}_4^\dagger - 1)(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{-6}) = 0. \tag{4.18}$$

In this case, the one written equation is insufficient for finding all the projectors because the same set of eigenvalues can correspond to different irreducible representations. Namely, both irreducible representations [31] and [211] correspond to the pair of eigenvalues $q^{\pm 2}$. We can resolve this problem by noting that each factor in (4.18) corresponds to a certain path, i.e., to a sequence of irreducible representations at the preceding levels. For example, the double factor with q^2 in F_{31} corresponds to the pair of paths $[1] \to [\,\cdot\,] \to [21] \to [31]$, where $[\,\cdot\,] = [2]$ or $[11]$, and the factor with q^{-2} corresponds to the path $[1] \to [2] \to [3] \to [31]$. In F_{211}, the factor with q^2 corresponds to the path $[1] \to [11] \to [21] \to [31]$, and the double factor with q^{-2} corresponds to the pair of paths $[1] \to [\,\cdot\,] \to [21] \to [211]$, where $[\,\cdot\,] = [2]$ or $[11]$. This means that we can find the sought projectors using the already constructed projectors onto representations of the preceding levels:

$$P_{31} = P_{q^{-2}}^{(4)}P_3 + P_{q^2}^{(4)}P_{21} =$$

$$= \frac{(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^6)(\mathfrak{R}_4\mathfrak{R}_4^\dagger - 1)(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{-6})}{(q^4-q^{-4})(q^2-q^{-2})^2(q-q^{-1})}\big(q^3(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^2)P_3 + q^{-3}(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{-2})P_{21}\big),$$

$$P_{211} = P_{q^{-2}}^{(4)}P_{21} + P_{q^2}^{(4)}P_{111} =$$

$$= \frac{(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^6)(\mathfrak{R}_4\mathfrak{R}_4^\dagger - 1)(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{-6})}{(q^4-q^{-4})(q^2-q^{-2})^2(q-q^{-1})}\big(q^3(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^2)P_{21} + q^{-3}(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{-2})P_{111}\big).$$

All the remaining projectors are constructed just as in the preceding cases:

$$P_4 = P^{(4)}_{q^6} = \frac{(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^6)(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^2)(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{-2})(\mathfrak{R}_4\mathfrak{R}_4^\dagger - 1)(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{-6})}{(q^6-q^2)(q^6-q^{-2})(q^6-1)(q^6-q^{-6})},$$

$$P_{22} = P^{(4)}_{1} = \frac{(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^6)(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^2)(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{-2})(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{-6})}{(1-q^6)(1-q^2)(1-q^{-2})(1-q^{-6})},$$

$$P_{1111} = P^{(4)}_{q^{-6}} = \frac{(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^6)(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^2)(\mathfrak{R}_4\mathfrak{R}_4^\dagger - q^{-2})(\mathfrak{R}_4\mathfrak{R}_4^\dagger - 1)}{(q^{-6}-q^6)(q^{-6}-q^2)(q^{-6}-q^{-2})}.$$

4.2.6. Arbitrary representations. In principle, the system of equations for the link products $\mathfrak{R}_{|Q|}\mathfrak{R}^\dagger_{|Q|}$ suffices for constructing a projector onto any representation. For example, let a representation Q be obtained from a representation T by adding one box (i,j) to the Young diagram. Then $\varkappa_Q = \varkappa_T + j - i$. Consequently, for any T, all the representations Q thus obtained have different $\varkappa_Q$. If we choose the eigenvectors of the matrix $\mathfrak{R}_{|Q|}\mathfrak{R}^\dagger_{|Q|}$ as Q, then the projectors are obtained from the equation

$$\prod_{i,j}(P_T\mathfrak{R}_{|Q|}\mathfrak{R}^\dagger_{|Q|}P_T - q^{2j-2i}) = 0$$

and have the form

$$P_Q\big|_{Q=T\cup(k,l)} = \sum_{i,j}\prod_{(i,j)\neq(k,l)}\frac{P_T\mathfrak{R}_{|Q|}\mathfrak{R}^\dagger_{|Q|}P_T - q^{2j-2i}}{q^{2l-2k}-q^{2j-2i}} =$$

$$= \sum_{i,j}\prod_{(i,j)\neq(k,l)}\frac{\mathfrak{R}_{|Q|}\mathfrak{R}^\dagger_{|Q|} - q^{2j-2i}}{q^{2l-2k}-q^{2j-2i}}P_T,$$

where all the products are calculated over all additions of a box (with the coordinates (i,j)) to the Young diagram T that again yield a Young diagram. This formula allows calculating the projector onto the entire space of the representations Q simultaneously arising in the decompositions of $1^{|Q|}$ and $1\otimes Q$. To obtain the projector onto one of the representations Q in the decomposition of $1^{|Q|}$, we must substitute the projector onto one of the representations of T in the decomposition of $1^{|T|}$ for P_T. In contrast, to obtain the projector onto the entire space of representations Q in the decomposition of $1^{|Q|}$, we must calculate the sum over all possible T:

$$P_Q = \sum_{T:\,|T|=|Q|-1} P_Q\big|_{Q=T\cup(k,l)}.$$

The presented formulas give a recursive procedure for constructing a projector onto any representation.

4.2.7. (Anti-)symmetric representations. The method described above allows calculating the projector onto an arbitrary representation. But it is unclear how to write a general formula for the projectors. Such a formula can nevertheless be written for the simplest class of representations, namely, for the fully symmetric and fully antisymmetric representations. These projectors have the form

$$P_{[r]} = P_{[r-1]}\left(1 + \sum_{j=1}^{k} q^j \prod_{i=1}^{j} R_{k-i+1}\right) = \prod_{k=1}^{r-1}\left(1 + \sum_{j=1}^{k} q^j \prod_{i=1}^{j} R_{k-i+1}\right)$$

or

$$P_{[r]} = \sum_{k=1}^{r!} q^{l_k}\,\Xi_k,$$

where the sum is over all $r!$ independent elements Ξ_k (see Eqs. (3.6) and (3.7)) of the polynomial ring and l_k is the number of R-matrices in the Ξ_k.

The projector onto the fully antisymmetric representations is analogous:

$$P_{[1^r]}(q) = P_{[r]}(-q^{-1}).$$

5. Framing in the cabling procedure

The $\mathcal{R}$-matrices are defined up to a common factor because it would not affect the Yang–Baxter equation. This common factor can be chosen variously, and the answer for the HOMFLY polynomials depends on this choice. From the standpoint of the Chern–Simons theory, this means that Wilson loops correspond to framed knots [16], [100], and the choice of the common factor corresponds to the choice of the framing. Hence, we call the choice of the common factor the framing. It can be shown that different framings should differ from one another by the factor $q^{\varkappa_T+N|T|/2}$ [48].

The framing can be chosen differently in concrete cases. One common way comes from the representation theory, where the eigenvalues of the $\mathcal{R}$-matrix are equal to $q^{\varkappa_Q-N|Q|/2-\varkappa_{T_1}+N|T_1|/2-\varkappa_{T_2}+N|T_2|/2}$, T_1 and T_2 correspond to representations placed on two crossing strands, and Q is one of the irreducible representation in the decomposition of $T_1\otimes T_2$ (the number of different representations Q is equal to the number of distinct eigenvalues). With such a framing, the eigenvalues of the $\mathcal{R}$-matrix are equal to q and $-q^{-1}$ in the fundamental representation and q^4, -1, and q^{-2} in the first symmetric representation. Such a framing is said to be vertical, and the cabling procedure in it gives the same answers as in a direct calculation in terms of colored $\mathcal{R}$-matrices.

We must choose another framing to ensure topological invariance (it is called the topological framing and was used, e.g., in [86] to evaluate knot polynomials). The eigenvalues of the $\mathcal{R}$-matrices are then given by $q^{\varkappa_Q-4\varkappa_T}A^{-|T|} = q^{\varkappa_Q-4\varkappa_T-N|T|}$ ($T_1 = T_2 = T$). The topological framing differs from the vertical framing by $q^{-2\varkappa_T-N|T|}$. In the topological framing, the eigenvalues of the $\mathcal{R}$-matrix are equal to q/A and $-1/qA$ in the fundamental representation and q^2/A^2, $1/A^2q^2$, and $1/A^2q^4$ in the first symmetric representation.

The topological framing is considered only in the case $T_1 = T_2$, which corresponds to the sense of this framing. The topological framing should give the same answer for all braids representing a given knot and, generally speaking, having different numbers of crossings. Precisely this is achieved by the special choice of the common factor (presented above). If two strands carry different representations, then they necessarily belong to different components of the link. As is known from topology [100], the algebraic number of crossings between different components of a link cannot change under any smooth deformations. Therefore, the topological framing must not include any factors for crossings between different components of the link. The case where different components carry the same representation is not an exception: in the topological framing, only self-crossings of connected components contribute.

6. Cabling procedure for two-strand knots and links

In the case of two-strand knots, the colored HOMFLY polynomials are known for an arbitrary representation because only diagonal $\mathcal{R}$-matrices are needed for calculating them. The two-strand case is therefore convenient for verifying the consistency of the expressions for projectors and $\mathcal{R}$-matrix eigenvalues in different representations. The general formula is

$$H^{T[2,n]}_{T_1T_2} = \mathrm{Tr}_{T_1\otimes T_2}((\mathcal{R}_{T_1T_2}\mathcal{R}_{T_2T_1})^{n/2}) = \frac{1}{q^{n\varkappa_{T_1}+n\varkappa_{T_2}}}\sum_{Q_i\vdash T_1\otimes T_2}(\pm q^{\varkappa_{Q_i}})^n N^{Q_i}_{T_1T_2}S^*_{Q_i}, \qquad (6.1)$$

where $q^{\varkappa_Q}$ are eigenvalues of the two-strand $\mathcal{R}$-matrix as described in Sec. 3.1, the sum is taken over all irreducible representations Q_i in the expansion of the tensor product $T_1\otimes T_2$, and Q_i has the multiplicity $N^{Q_i}_{T_1T_2}$. The coefficient before the sum leads to the answer in the vertical framing (see Sec. 5).

The cabling procedure gives the formula

$$H_{T_1T_2}^{T[2,n]} = \mathrm{Tr}_{1^{\otimes(|T_1|+|T_2|)}}\left(P_{T_1T_2}\left(\prod_{i=1}^{T_2}\prod_{j=1}^{T_1} R_{|T_1|+i-j}\prod_{i=1}^{T_1}\prod_{j=1}^{T_2} R_{|T_2|+i-j}\right)^{n/2}\right).$$

It is necessary to separate two cases: the two-strand knots and the two-strand links. Studying links allows determining the form of the eigenvalues in the case where different representations are on different link components; we note that this is impossible when studying knots. On the other hand, because the two-strand knots always contain an even number of crossings, the signs of the eigenvalues cannot be found from the cabling procedure for the links: two-strand knots are needed for this.

6.1. Two-strand links. In the case of two-strand links, there are two $\mathcal{R}$-matrices, $\mathcal{R}_{T_1T_2}$ and $\mathcal{R}_{T_2T_1}$. They differ as linear operators because they act in different spaces:

$$\mathcal{R}_{T_1T_2}\colon \quad T_1\otimes T_2 \to T_2\otimes T_1, \qquad \mathcal{R}_{T_2T_1}\colon \quad T_2\otimes T_1 \to T_1\otimes T_2.$$

But their matrices coincide up to transposition: $\mathcal{R}_{T_1T_2} = \mathcal{R}^{\dagger}_{T_2T_1}$. Expression (6.1) for links with $2n$ crossings can be rewritten as

$$H_{T_1T_2}^{T[2,2n]} = (q^{\varkappa_{T_1}+\varkappa_{T_2}})^{-2n}\sum_{Q_i\vdash T_1\otimes T_2} N_{T_1T_2}^{Q_i} q^{2n\varkappa_{Q_i}} S^*_{Q_i} = \sum_{Q_i\vdash T_1\otimes T_2} H_{T_1T_2|Q_i}^{[2,2n]} S^*_{Q_i}.$$

It is impossible to determine the signs of eigenvalues using the cabling procedure for two-strand links because all the powers of the eigenvalues in this formula are even. We can obtain exactly the same answer using the cabling procedure, even for individual coefficients $H_{T_1Q_2|Q_i}^{[2,2n]}$ in the character expansion for the HOMFLY polynomial:

$$H_{T_1T_2|Q_i}^{T[2,2n]} = N_{T_1T_2}^{Q_i} q^{2n\varkappa_{Q_i}-2n\varkappa_{T_1}-2n\varkappa_{T_2}}$$

(it suffices to verify this equality for $2\le 2n\le \min(\mathrm{rank}\,P_{T_1},\mathrm{rank}\,P_{T_2})$). Using the cabling procedure, we can see that both the level–rank duality

$$H_{T_1T_2|Q_i}^{T[2,2n]}(q) = H_{\widetilde{T}_1\widetilde{T}_2|\widetilde{Q}_i}^{T[2,2n]}(-q^{-1}),$$

where $\widetilde{T}$ denotes the representation whose Young diagram is the transposed diagram for the representation T, and the symmetry under permutations of the representations

$$H_{T_2T_1|Q_i}^{T[2,2n]} = H_{T_1T_2|Q_i}^{T[2,2n]}$$

hold. These properties were described, for example, in [86], [87]. They hold in this case because $N_{T_1T_2}^{Q_i} = N_{T_2T_1}^{Q_i} = N_{\widetilde{T}_1\widetilde{T}_2}^{\widetilde{Q}_i}$ and $\varkappa_Q = -\varkappa_{\widetilde{Q}}$. It can also be verified that the choice of different representations arising in the decomposition of $1^{\otimes|T_1|+|T_2|}$ and isomorphic T_1 and T_2 does not change the answer.

The absolute values of the eigenvalues of the colored $\mathcal{R}$-matrices are presented in Appendix B in [98]. The multiplicities of different irreducible representations are easily determined using representation theory, namely from the property of the characters that

$$S_{T_1}S_{T_2} = \sum_{Q_i\vdash T_1\otimes T_2} N_{T_1T_2}^{Q_i} S_{Q_i}.$$

For example,

$$S_{31}S_{31} = S_{62} + S_{611} + S_{53} + 2S_{521} + S_{5111} + S_{44} + 2S_{431} + S_{422} + S_{4211} + S_{332} + S_{3311}.$$

Almost all of the studied representations have trivial multiplicities 0 or 1. Nontrivial multiplicities for the considered two-strand knots are encountered beginning with representations of size three. The list of all representations with nontrivial multiplicities for the representations T_1 and T_2 of sizes 3 and 4 was given in Appendix B in [98].

The decomposition of $1^{\otimes|T_1|+|T_2|}$ includes some irreducible representations that do not appear in the decomposition of $T_1 \otimes T_2$. For all such representations, the cabling procedure should somehow give zero coefficients in calculating the answer. Because we place the projectors described in Sec. 4.1 on each side of each R-matrix, zero appears automatically for all representations not contained in the decomposition of $T_1 \otimes 1^{\otimes|T_2|}$. The remaining nonzero blocks of the R-matrices turn out to be degenerate, and all representations that are not contained in $1^{\otimes|T_1|} \otimes T_2$ vanish, as they should.

6.2. Two-strand links. As already mentioned, the cabling procedure for two-strand knots allows finding not only the absolute values of the eigenvalues of the colored $\mathcal{R}$-matrix but also their signs. The drawback is that we can study only the case where $T_1 = T_2$ (because the strand in the knot can carry only one representation). With the cabling method, we can verify that in all multiplicity-free cases (we verified this up to $|T_1| = |T_2| = 4$), the eigenvalues of a colored two-strand $\mathcal{R}$-matrix satisfy the rule that the eigenvalue with the highest power of q has a plus sign, the eigenvalue with the next-to-highest power has a minus sign, and the signs alternate after that. This rule was used, for example, in [89]. Exceptions to this rule occur when multiplicities appear in the decomposition of the tensor product of two irreducible representations. Here is the simplest example of this phenomenon:

$$[21] \times [21] = [42] + [411] + [33] + 2[321] + [3111] + [222] + [2211].$$

In this case, the cabling procedure gives the eigenvalues

$$q^5, \quad -q^3, \quad -q^3, \quad 1, \quad -1, \quad q^{-3}, \quad q^{-3}, \quad -q^{-5}.$$

In the next-to-simplest case

$$[31] \times [31] = [62] + [611] + [53] + 2[521] + [5111] + [44] + 2[431] + [422] + [4211] + [332] + [3311],$$

the eigenvalues calculated using the cabling procedure are

$$q^{10}, \quad -q^8, \quad -q^6, \quad q^3, \quad -q^3, \quad 1, \quad q^4, \quad 1, \quad -1, \quad q^{-2}, \quad -q^{-4}, \quad -q^{-4}, \quad q^{-6}.$$

Hence, the general rule for the sign choice in the two-strand $\mathcal{R}$-matrix is unclear.

The obtained results for the eigenvalues were presented in the extended electronic version of this paper [98].

As a check, we can substitute the calculated eigenvalues in expression (6.1) and verify that the following two properties of the HOMFLY polynomial hold. First, the HOMFLY polynomial must indeed be a polynomial. Second, the polynomial of the two-strand knot with a single crossing must be equal to the polynomial of the unknot up to the framing factor. Indeed, such a knot is transformed into the unknot by the first Reidemeister move.

Hence, the eigenvalues must satisfy the system of equations $S^*_R = \sum \lambda_Q S^*_Q$; moreover, $\sum (\lambda_Q)^{2n+1} S^*_Q$ is a multiple of S^*_R. All the eigenvalue signs are probably determined by these equations, but the way to find the signs from these equations is unclear.

7. Cabling procedure for three- and four-strand knots

7.1. Three-strand knots. The HOMFLY polynomials for knots represented as the closure of three-strand braids in the representations [2] (the first symmetric representation) and [11] (the first antisymmetric representation) were previously obtained in [87]. The same results can be obtained using the cabling methods. For this, we must replace each colored $\mathcal{R}$-matrix in a three-strand braid with a combination of the fundamental matrices in a six-strand braid:

$$\mathcal{R}_{(1^2\otimes 1^2)\otimes 1^2} = R_2R_1R_3R_2, \qquad \mathcal{R}_{1^2\otimes(1^2\otimes 1^2)} = R_4R_3R_5R_4. \tag{7.1}$$

We must then introduce the projectors onto the corresponding representations. As already stated in Sec. 4, any number of projectors can be inserted (but not less than one on each connected component), and any form of those described in Sec. 4 can be used. The most convenient way to simplify the calculations is to use the path description of projectors and to insert two of them for each $\mathcal{R}$-matrix, surrounding each combination of operators in (7.1) with them.

For level-three representations, i.e., for [3], [21], and [111], the three-strand braid must be replaced with a nine-strand braid with the corresponding replacement for the $\mathcal{R}$-matrices:

$$\mathcal{R}_{(1^3\otimes 1^3)\otimes 1^3} = R_3R_2R_1R_4R_3R_2R_5R_4R_3, \qquad \mathcal{R}_{1^3\otimes(1^3\otimes 1^3)} = R_6R_5R_4R_7R_6R_5R_8R_7R_6.$$

All the needed projectors are described in Sec. 4. The HOMFLY polynomials for three-strand knots in representations [3] and [111] were presented in [89], and the answers for representation [21] can be found in Appendix C in [98]. The same procedure can be repeated in the case of higher representations, but more computational power is needed.

7.2. Four-strand knots. In the case of knots represented as the closure of four-strand braids, the consideration is essentially analogous to that in the preceding subsection. For example, for the representations [2] and [11] (the first symmetric and first antisymmetric representations), the colored $\mathcal{R}$-matrices must be replaced with the fundamental matrices:

$$\mathcal{R}_{(1^2\otimes 1^2)\otimes 1^2\otimes 1^2} = R_2R_1R_3R_2, \qquad \mathcal{R}_{1^2\otimes(1^2\otimes 1^2)\otimes 1^2} = R_4R_3R_5R_4,$$

$$\mathcal{R}_{1^2\otimes 1^2\otimes(1^2\otimes 1^2)} = R_6R_5R_7R_6.$$

In addition, the corresponding projectors in Sec. 4 must be inserted. The HOMFLY polynomials for four-strand knots for the level-two representations were presented in Appendix D in [98].

8. Multicolored three-strand links

Colored links are much more complicated to study than colored knots because links have several connected components and each component can have its own representation. The invariants of links with the same representation on all components are calculated analogously to the invariants of knots, and we therefore concentrate on links with different representations on the components here. Knots and links even in the fundamental representation have one more property: orientation. While knots can be oriented in only one way (in fact, there are two ways, but they are interchanged by a symmetry transformation), different components with different relative orientations can be considered in the case of links. If a link is represented in the form of a braid, then the orientation is given automatically: differently oriented links have different braid representations. We also note that HOMFLY polynomials for links are not polynomials in fact: they always contain some denominator depending on the chosen representations.

Here, we consider the simplest and mostly known examples to illustrate the approach under investigation.

8.1. Method of colored $\mathcal{R}$-matrices. The basic problem of calculations in the Reshetikhin–Turaev framework in this case is as follows. If all strands in the braid have the same color, then all the $\mathcal{R}$-matrices are labeled with the ordinal numbers of the crossing strands, namely, $\mathcal{R}$ corresponds to the crossing of the kth and $(k+1)$th strands of the braid. If different strands have different colors, then the situation is different. In this case, the $\mathcal{R}$-matrices are given both by the crossing strands and by their colors (see the examples dissected below). We must also take into account that $\mathcal{R}$-matrices contain the permutation operators and hence change the placement of strands in the braid:

$$\mathcal{R}_{T_1\ldots(T_iT_{i+1})\ldots T_{m-1}}:\quad T_1\otimes\cdots\otimes(T_i\otimes T_{i+1})\otimes\cdots\otimes T_{m-1}\to$$

$$\to T_1\otimes\cdots\otimes(T_{i+1}\otimes T_i)\otimes\cdots\otimes T_{m-1},$$

$$\mathcal{U}_{T_1\ldots(T_{i-1}T_iT_{i+1})\ldots T_{m-1}}:\quad T_1\otimes\cdots\otimes\big((T_{i-1}\otimes T_i)\otimes T_{i+1}\big)\otimes\cdots\otimes T_{m-1}\to$$

$$\to T_1\otimes\cdots\otimes\big(T_{i-1}\otimes(T_i\otimes T_{i+1})\big)\otimes\cdots\otimes T_{m-1}.$$

Using this rule, we can find all possible sequences of $\mathcal{R}$-matrices that can be encountered in a given colored braid.

The eigenvalues of the $\mathcal{R}$-matrices are known: they are the same as for the diagonal $\mathcal{R}$-matrices described in Sec. 3.1 and are equal to $\mathcal{R}_{T_1T_2|Q}=\pm q^{\varkappa_Q-\varkappa_{T_1}-\varkappa_{T_2}}$ in the vertical framing. The eigenvalue signs are essential only for crossings of strands of the same color because the number of the crossings between every pair of strands of different colors is always even. In the considered examples, these signs simply alternate in each $\mathcal{R}$-matrix, i.e., the eigenvalue with the highest power of q has a plus sign, the next has a minus sign, and so on. Nondiagonal matrices can be constructed from the diagonal matrices using Racah matrices [86], which can be calculated by representation theory methods [99].

We consider three particular cases where the braids representing the links are colored as $[1]\otimes[1]\otimes[2]$, $[1]\otimes[2]\otimes[2]$, and $[1]\otimes[2]\otimes[3]$. In all cases, the representation theory of the $SU_q(2)$ group suffices for calculating the Racah matrices.

8.2. The case $[1]\otimes[1]\otimes[2]$.

8.2.1. Calculations using the colored $\mathcal{R}$-matrices. The tensor product of the representations decomposes as

$$[1]\otimes[1]\otimes[2]=[4]+2\;[31]+[22]+[211].$$

For the singlets, there are no mixing matrices, and the corresponding components of the $\mathcal{R}$-matrices are entirely described by their eigenvalues:

$$\mathcal{R}_{(1\otimes2)\otimes1|4}=\mathcal{R}_{(2\otimes1)\otimes1|4}=\mathcal{R}_{1(12)|4}=\mathcal{R}_{1\otimes(2\otimes1)|4}=q^{\varkappa_3-\varkappa_2}=q^2,$$

$$\mathcal{R}_{(1\otimes2)\otimes1|22}=\mathcal{R}_{(2\otimes1)\otimes1|22}=\mathcal{R}_{1(12)|22}=\mathcal{R}_{1\otimes(2\otimes1)|22}=q^{\varkappa_{21}-\varkappa_2}=-q^{-1},$$

$$\mathcal{R}_{(1\otimes2)\otimes1|211}=\mathcal{R}_{(2\otimes1)\otimes1|211}=\mathcal{R}_{1(12)|211}=\mathcal{R}_{1\otimes(2\otimes1)|211}=q^{\varkappa_{21}-\varkappa_2}=-q^{-1},$$

$$\mathcal{R}_{(1\otimes1)\otimes2|4}=\mathcal{R}_{2\otimes(1\otimes1)|4}=q^{\varkappa_2}=q,$$

$$\mathcal{R}_{(1\otimes1)\otimes2|22}=\mathcal{R}_{2\otimes(1\otimes1)|22}=q^{\varkappa_2}=q,$$

$$\mathcal{R}_{(1\otimes1)\otimes2|211}=\mathcal{R}_{2\otimes(1\otimes1)|211}=-q^{\varkappa_{11}}=-q^{-1}.$$

For the doublets, both the diagonal and the nondiagonal $\mathcal{R}$-matrices are needed, and the latter are calculated using mixing matrices. The doublet components of the diagonal $\mathcal{R}$-matrices are

$$\mathcal{R}_{(1\otimes1)\otimes2|31} = \mathrm{diag}(q^{\varkappa_2}, -q^{\varkappa_{11}}) = \mathrm{diag}(q, -q^{-1}),$$
$$\mathcal{R}_{(1\otimes2)\otimes1|31} = \mathcal{R}^{\dagger}_{(2\otimes1)\otimes1|31} = \mathrm{diag}(q^{\varkappa_3-\varkappa_2}, -q^{\varkappa_{21}-\varkappa_2}) = \mathrm{diag}(q^2, -q^{-1}).$$

The mixing matrices calculated using the representation theory for the group $SU_q(2)$ are[4]

$$\mathcal{U}_{1\otimes2\otimes1|31} = \begin{pmatrix} \dfrac{1}{[3]_q} & \dfrac{\sqrt{[2]_q[4]_q}}{[3]_q} \\ \dfrac{\sqrt{[2]_q[4]_q}}{[3]_q} & -\dfrac{1}{[3]_q} \end{pmatrix},$$

$$\mathcal{U}_{1\otimes1\otimes2|31} = \mathcal{U}_{2\otimes1\otimes1|31} = \begin{pmatrix} \dfrac{1}{\sqrt{[3]_q}} & \sqrt{\dfrac{[4]_q}{[2]_q[3]_q}} \\ \sqrt{\dfrac{[4]_q}{[2]_q[3]_q}} & -\dfrac{1}{\sqrt{[3]_q}} \end{pmatrix}.$$

As a result, we obtain the nondiagonal $\mathcal{R}$-matrices

$$\mathcal{R}_{2\otimes(1\otimes1)|31} = \mathcal{U}_{2\otimes1\otimes1|31} \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix} \mathcal{U}^{\dagger}_{2\otimes1\otimes1|31} = \begin{pmatrix} \dfrac{1}{q^3[3]_q} & \dfrac{\sqrt{[2]_q[4]_q}}{[3]_q} \\ \dfrac{\sqrt{[2]_q[4]_q}}{[3]_q} & -\dfrac{q^3}{[3]_q} \end{pmatrix},$$

$$\mathcal{R}_{1\otimes(1\otimes2)|31} = \mathcal{R}^{\dagger}_{1\otimes(2\otimes1)|31} =$$
$$= \mathcal{U}_{1\otimes1\otimes2|31} \begin{pmatrix} q^2 & 0 \\ 0 & -q^{-1} \end{pmatrix} \mathcal{U}^{\dagger}_{1\otimes2\otimes1|31} = \begin{pmatrix} -\dfrac{1}{q\sqrt{[3]_q}} & \dfrac{q\sqrt{[4]_q}}{\sqrt{[2]_q[3]_q}} \\ \dfrac{\sqrt{[4]_q}}{\sqrt{[2]_q[3]_q}} & \frac{q^3}{\sqrt{[3]_q}} \end{pmatrix}.$$

As an example of applying these formulas, we compute the HOMFLY polynomials for several simplest two-component three-strand links in the representation $[1]\otimes[1]\otimes[2]$. To study this case, we must place the representation [1] on the two-strand component of the link and [2] on the one-strand component. The simplest link of this type is a pair of split unknots in the form of a three-strand braid with the braid word σ_1. The corresponding HOMFLY polynomial in the vertical framing is equal to

$$H^{\bigcirc^2}_{1\otimes2} = \mathrm{Tr}_{1\otimes1\otimes2}\,\mathcal{R}_{(1\otimes1)\otimes2} = qS^*_4 + (q-q^{-1})S^*_{31} + qS^*_{22} - q^{-1}S^*_{211} = S^*_1 S^*_2.$$

The example next in simplicity is the representation of the torus link $T[2,4]$ in the form of a three-strand braid with the braid word $\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1$ or $\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1$. The corresponding HOMFLY polynomial has the form

$$H^{T[2,4]}_{2\otimes1\otimes1} = \mathrm{Tr}_{2\otimes1\otimes1}\,\mathcal{R}_{(2\otimes1)\otimes1}\mathcal{R}_{1\otimes(2\otimes1)}\mathcal{R}_{(1\otimes1)\otimes2}\mathcal{R}_{1\otimes(1\otimes2)}\mathcal{R}_{(1\otimes2)\otimes1} =$$
$$= q^9 S^*_4 + (q-q^3)S^*_{31} + q^3 S^*_{22} - q^{-5}S^*_{311}.$$

[4]See, e.g., [86] for a detailed derivation of the mixing matrices from the representation theory.

The obtained answer coincides with the answer for the two-strand representation of the same link (if the topological framing is used, which differs from the vertical framing by the factor A^{-1}):

$$A^{-1}H_{2\otimes 1\otimes 1}^{T[2,4]-3\text{-strand}}=H_{1\otimes 2}^{T[2,4]-2\text{-strand}}=q^{4\varkappa_3-8\varkappa_2}S_3^*+q^{4\varkappa_{21}-8\varkappa_2}S_{21}^*=q^8S_3^*+q^{-4}S_{21}^*.$$

The simplest nontorus three-strand two-component link is 5_1^2 or the Whitehead link[5] represented as a braid with the braid word $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}$. It differs from the torus link $T[2,4]$ by inverting all $\mathcal{R}$-matrices acting on the first pair of strands. The HOMFLY polynomial of the Whitehead link has the form

$$\begin{aligned}H_{2\otimes 1\otimes 1}^{5_1^2}=&\ \mathrm{Tr}_{2\otimes 1\otimes 1}\ \mathcal{R}^{-1}_{(2\otimes 1)\otimes 1}\mathcal{R}_{1\otimes(2\otimes 1)}\mathcal{R}^{-1}_{(1\otimes 1)\otimes 2}\mathcal{R}_{1\otimes(1\otimes 2)}\mathcal{R}_{(1\otimes 2)\otimes 1}=\\&=q^{-1}S_4^*+(-q^7+q^5+q^3-2q+2q^{-1}-q^{-3}-q^{-5}+q^{-7})S_{31}^*+q^{-1}S_{22}^*-qS_{221}^*=\\&=\frac{S_2^*}{q^2-1}\big((-q^3+q^{-1}-q^{-3})A^{-2}+\\&\quad+(q^7-q^5-q^3+3q-q^{-1}-q^{-3}+q^{-5})+(-q^5+q^3+q-q^{-1})A^2\big).\end{aligned}$$

8.2.2. Calculations using the cabling procedure. To calculate the same polynomials by the cabling method, we must replace each colored $\mathcal{R}$-matrix with the corresponding product of fundamental matrices, in this case acting on the four-strand braid (because $1+1+2=4$):

$$\begin{aligned}&\mathcal{R}_{(1\otimes 1)\otimes 2}\to R_1, &&\mathcal{R}_{(1\otimes 2)\otimes 1}=\mathcal{R}^\dagger_{(2\otimes 1)\otimes 1}\to R_2R_1,\\&\mathcal{R}_{1\otimes(1\otimes 2)}=\mathcal{R}^\dagger_{1\otimes(2\otimes 1)}\to R_3R_2, &&\mathcal{R}_{2\otimes(1\otimes 1)}\to R_3.\end{aligned}\tag{8.1}$$

The corresponding replacement for the inverse $\mathcal{R}$-matrices is

$$\begin{aligned}&\mathcal{R}^{-1}_{(1\otimes 1)\otimes 2}\to R_1^{-1}, &&\mathcal{R}^{-1}_{(1\otimes 2)\otimes 1}=\mathcal{R}^\dagger_{(2\otimes 1)\otimes 1}\to R_2^{-1}R_1^{-1},\\&\mathcal{R}^{-1}_{1\otimes(1\otimes 2)}=\mathcal{R}^\dagger_{1\otimes(2\otimes 1)}\to R_3^{-1}R_2^{-1}, &&\mathcal{R}^{-1}_{2\otimes(1\otimes 1)}\to R_3^{-1}.\end{aligned}$$

In addition, we must insert projectors. We should use one of three projectors depending on where the projector is placed in the braid:[6]

$$P_{2\otimes 1\otimes 1}=\frac{1+qR_1}{1+q^2},\qquad P_{1\otimes 2\otimes 1}=\frac{1+qR_2}{1+q^2},\qquad P_{1\otimes 1\otimes 2}=\frac{1+qR_3}{1+q^2}.\tag{8.2}$$

The answers for particular links can be obtained by applying rule (8.1) to the projectors (8.2) corresponding to the particular braid and then calculating the HOMFLY polynomial for the obtained braid in the fundamental representation. The answers thus obtained coincide with the answers obtained using colored $\mathcal{R}$-matrices.

In the case of two split unknots in the form of a three-strand braid with the braid word σ_1, the HOMFLY polynomial obtained using the cabling procedure is equal to

$$H_{1\otimes 2}^{\bigcirc^2}=\mathrm{Tr}_{1^4}\,P_{1\otimes 1\otimes 2}R_1=qS_4^*+(q-q^{-1})S_{31}^*+qS_{22}^*-q^{-1}S_{211}^*=A\ S_1^*S_2^*.$$

[5]The Rolfsen notation c_k^n for a link generalizes the Rolfsen notation c_k for a knot. In this notation, c is the crossing number, i.e., the minimum number of intersections in the planar diagram of the knot or link, and n is the number of components.

[6]As described in Sec. 4, different placements of the projectors and different numbers of projectors can be used. In all calculations here, we use one projector placed at the beginning of the braid.

The torus link $T[2,4]$ represented as a three-strand braid has the HOMFLY polynomial

$$H_{2\otimes 1\otimes 1}^{T[2,4]-3\text{-strand}} = \mathrm{Tr}_{1^4}\, P_{2\otimes 1\otimes 1} R_1 R_2 \cdot R_2 R_3 \cdot R_1 \cdot R_3 R_2 \cdot R_2 R_1 =$$
$$= q^9 S_4^* + (q - q^3) S_{31}^* + q^3 S_{22}^* - q^{-5} S_{311}^* = A H_{1\otimes 2}^{T[2,4]-2\text{-strand}}.$$

Finally, for the HOMFLY polynomial for the Whitehead link, the cabling procedure gives

$$H_{2\otimes 1\otimes 1}^{5_1^2} = \mathrm{Tr}_{2\otimes 1\otimes 1}\, P_{2\otimes 1\otimes 1} R_1^{-1} R_2^{-1} \cdot R_2 R_3 \cdot R_1^{-1} \cdot R_3 R_2 \cdot R_2 R_1 =$$
$$= q^{-1} S_4^* + (-q^7 + q^5 + q^3 - 2q + 2q^{-1} - q^{-3} - q^{-5} + q^{-7}) S_{31}^* + q^{-1} S_{22}^* - q S_{221}^*.$$

8.3. The case $[2]\otimes[2]\otimes[1]$.

8.3.1. Calculations using colored $\mathcal{R}$-matrices. In the case $[2]\otimes[2]\otimes[1]$, the tensor product of representations decomposes as

$$[2]\otimes[2]\otimes[1] = [5] + 2\,[41] + 2\,[32] + [311] + [221].$$

The singlet components of the colored $\mathcal{R}$-matrices are

$$\mathcal{R}_{(1\otimes 2)\otimes 2|5} = \mathcal{R}_{(2\otimes 1)\otimes 2|5} = \mathcal{R}_{2\otimes(1\otimes 2)|5} = \mathcal{R}_{2\otimes(2\otimes 1)|5} = q^{\varkappa_3 - \varkappa_2} = q^2,$$
$$\mathcal{R}_{(1\otimes 2)\otimes 2|311} = \mathcal{R}_{(2\otimes 1)\otimes 2|311} = \mathcal{R}_{2\otimes(1\otimes 2)|311} = \mathcal{R}_{2\otimes(2\otimes 1)|311} = q^{\varkappa_{21} - \varkappa_2} = -q^{-1},$$
$$\mathcal{R}_{(1\otimes 2)\otimes 2|221} = \mathcal{R}_{(2\otimes 1)\otimes 2|221} = \mathcal{R}_{2\otimes(1\otimes 2)|221} = \mathcal{R}_{2\otimes(2\otimes 1)|221} = q^{\varkappa_{21} - \varkappa_2} = -q^{-1},$$
$$\mathcal{R}_{(2\otimes 2)\otimes 1|5} = \mathcal{R}_{1\otimes(2\otimes 2)|5} = q^{\varkappa_4 - 2\varkappa_2} = q^4,$$
$$\mathcal{R}_{(2\otimes 2)\otimes 1|311} = \mathcal{R}_{1\otimes(2\otimes 2)|311} = -q^{\varkappa_{31} - 2\varkappa_2} = -1,$$
$$\mathcal{R}_{(2\otimes 2)\otimes 1|221} = \mathcal{R}_{1\otimes(2\otimes 2)|221} = q^{\varkappa_{22} - 2\varkappa_2} = q^{-2}.$$

The doublet components of the diagonal colored $\mathcal{R}$-matrices in this case are

$$\mathcal{R}_{(1\otimes 2)\otimes 2|41} = \mathcal{R}_{(1\otimes 2)\otimes 2|32} = \mathcal{R}^\dagger_{(2\otimes 1)\otimes 2|41} = \mathcal{R}^\dagger_{(2\otimes 1)\otimes 2|32} =$$
$$= \mathrm{diag}(q^{\varkappa_3 - \varkappa_2}, -q^{\varkappa_{21} - \varkappa_2}) = \mathrm{diag}(q^2, -q^{-1}),$$
$$\mathcal{R}_{(2\otimes 2)\otimes 1|41} = \mathrm{diag}(q^{\varkappa_4 - 2\varkappa_2}, -q^{\varkappa_{31} - 2\varkappa_2}) = \mathrm{diag}(q^4, -1),$$
$$\mathcal{R}_{(2\otimes 2)\otimes 1|32} = \mathrm{diag}(-q^{\varkappa_{31} - 2\varkappa_2}, q^{\varkappa_{22} - 2\varkappa_2}) = \mathrm{diag}(-1, q^{-2}).$$

The corresponding Racah matrices are

$$\mathcal{U}_{2\otimes 2\otimes 1|41} = \mathcal{U}_{1\otimes 2\otimes 2|41} = \begin{pmatrix} \sqrt{\dfrac{[2]_q}{[3]_q[4]_q}} & \sqrt{\dfrac{[2]_q[5]_q}{[3]_q[4]_q}} \\ \sqrt{\dfrac{[2]_q[5]_q}{[3]_q[4]_q}} & -\sqrt{\dfrac{[2]_q}{[3]_q[4]_q}} \end{pmatrix},$$

$$\mathcal{U}_{2\otimes 2\otimes 1|32} = \mathcal{U}_{1\otimes 2\otimes 2|32} = \begin{pmatrix} \dfrac{1}{\sqrt{[3]_q}} & \sqrt{\dfrac{[4]_q}{[2]_q[3]_q}} \\ \sqrt{\dfrac{[4]_q}{[2]_q[3]_q}} & -\dfrac{1}{\sqrt{[3]_q}} \end{pmatrix},$$

$$\mathcal{U}_{2\otimes 1\otimes 2|41}=\begin{pmatrix}\dfrac{[2]_q}{[3]_q} & \dfrac{\sqrt{[5]_q}}{[3]_q}\\ \dfrac{\sqrt{[5]_q}}{[3]_q} & -\dfrac{[2]_q}{[3]_q}\end{pmatrix},$$

$$\mathcal{U}_{2\otimes 1\otimes 2|32}=\begin{pmatrix}\dfrac{1}{[3]_q} & \dfrac{\sqrt{[2]_q[4]_q}}{[3]_q}\\ \dfrac{\sqrt{[2]_q[4]_q}}{[3]_q} & -\dfrac{1}{[3]_q}\end{pmatrix}.$$

As a result, we obtain the doublet components of the nondiagonal colored $\mathcal{R}$-matrices:

$$\mathcal{R}_{2\otimes(1\otimes 2)|41}=\mathcal{R}^{\dagger}_{2\otimes(1\otimes 2)|41}=$$

$$=\mathcal{U}_{2\otimes 1\otimes 2|41}\begin{pmatrix}q^2 & 0\\ 0 & -q^{-1}\end{pmatrix}\mathcal{U}^{\dagger}_{2\otimes 2\otimes 1|41}=\begin{pmatrix}-\dfrac{\sqrt{[2]_q}}{q^3\sqrt{[3]_q[4]_q}} & \dfrac{q\sqrt{[2]_q[5]_q}}{\sqrt{[3]_q[4]_q}}\\ \dfrac{\sqrt{[2]_q[5]_q}}{\sqrt{[3]_q[4]_q}} & \dfrac{q^4\sqrt{[2]_q}}{\sqrt{[3]_q[4]_q}}\end{pmatrix},$$

$$\mathcal{R}_{2\otimes(1\otimes 2)|32}=\mathcal{R}^{\dagger}_{2\otimes(1\otimes 2)|32}=$$

$$=\mathcal{U}_{2\otimes 1\otimes 2|32}\begin{pmatrix}q^2 & 0\\ 0 & -q^{-1}\end{pmatrix}\mathcal{U}^{\dagger}_{2\otimes 2\otimes 1|32}=\begin{pmatrix}-\dfrac{1}{q^2\sqrt{[3]_q}} & \dfrac{\sqrt{[4]_q}}{\sqrt{[2]_q[3]_q}}\\ \dfrac{q\sqrt{[4]_q}}{\sqrt{[2]_q[3]_q}} & \dfrac{q^3}{\sqrt{[3]_q}}\end{pmatrix},$$

$$\mathcal{R}_{1\otimes(2\otimes 2)|41}=\mathcal{U}_{2\otimes 2\otimes 1|41}\begin{pmatrix}q^4 & 0\\ 0 & -1\end{pmatrix}\mathcal{U}^{\dagger}_{2\otimes 2\otimes 1|41}=\begin{pmatrix}-\dfrac{[2]_q}{q[3]_q} & \dfrac{q^2\sqrt{[5]_q}}{[3]_q}\\ \dfrac{q^2\sqrt{[5]_q}}{[3]_q} & \dfrac{q^5[2]_q}{[3]_q}\end{pmatrix},$$

$$\mathcal{R}_{1\otimes(2\otimes 2)|32}=\mathcal{U}_{2\otimes 2\otimes 1|32}\begin{pmatrix}-1 & 0\\ 0 & -q^{-2}\end{pmatrix}\mathcal{U}^{\dagger}_{2\otimes 2\otimes 1|32}=\begin{pmatrix}\dfrac{1}{q^4[3]_q} & -\dfrac{\sqrt{[2]_q[4]_q}}{q[3]_q}\\ -\dfrac{\sqrt{[2]_q[4]_q}}{q[3]_q} & -\dfrac{q^2}{[3]_q}\end{pmatrix}.$$

As examples, we consider the same links as in Sec. 8.2 but now with the representation [2] placed on the two-strand component and the representation [1] on the one-strand component.

For two split unknots, the colored $\mathcal{R}$-matrix approach gives

$$H^{\bigcirc^2}_{1\otimes 2}=q^4S^*_5+(q^4-1)S^*_{41}+(q^{-2}-1)S^*_{32}-S^*_{311}+q^{-2}S^*_{221}=A^2q^2S^*_1S^*_2.$$

The factor A^2q^2 in this expression relates the vertical and the topological framings (see Sec. 5). For the torus link $T[2,4]$ represented by a three-strand braid, we obtain

$$\begin{aligned}\mathcal{H}^{T[2,4]-3\text{-strand}}_{1\otimes 2\otimes 2}&=\mathrm{Tr}_{1\otimes 2\otimes 2}\,\mathcal{R}_{(1\otimes 2)\otimes 2}\mathcal{R}_{2\otimes(1\otimes 2)}\mathcal{R}_{(2\otimes 2)\otimes 1}\mathcal{R}_{2\otimes(2\otimes 1)}\mathcal{R}^{\dagger}_{(2\otimes 1)\otimes 2}=\\ &=q^{12}S^*_5+(q^2-q^6)S^*_{41}+(q^2-1)S^*_{32}-q^{-4}S^*_{311}+q^{-6}S^*_{221}=\\ &=A^2q^2H^{T[2,4]-2\text{-strand}}_{2\otimes 1}=A^2q^2(q^{4\varkappa_3-8\varkappa_2}S^*_3+q^{4\varkappa_{21}-8\varkappa_2}S^*_{21})=A^2q^{10}S^*_3+A^2q^{-2}S^*_{21}.\end{aligned}$$

For the Whitehead link, we obtain the HOMFLY polynomial

$$H^{5_1^2}_{1\otimes 2\otimes 2} = \mathrm{Tr}_{1\otimes 2\otimes 2}\, \mathcal{R}^{-1}_{(1\otimes 2)\otimes 2}\mathcal{R}_{2\otimes(1\otimes 2)}\mathcal{R}^{-1}_{(2\otimes 2)\otimes 1}\mathcal{R}_{2\otimes(2\otimes 1)}\mathcal{R}^{\dagger}_{(2\otimes 1)\otimes 2} =$$

$$= q^{-4}S^*_5 + (-q^6+q^4+q^2-2+2q^{-4}-q^{-6}-q^{-8}+q^{-10})S^*_{41} +$$

$$+(-q^6+q^4+q^2-2+2q^{-4}-q^{-6}-q^{-8}+q^{-10})S^*_{32} - S^*_{311} + q^2 S^*_{221} =$$

$$= A^{-1}q^{-1}H^{5_1^2}_{2\otimes 1\otimes 1},$$

and the polynomials $H^{5_1^2}_{2\otimes 1\otimes 1}$ and $H^{5_1^2}_{1\otimes 2\otimes 2}$ are hence equal up to a factor. This factor appears because of the difference between the vertical framing, used in these calculations, and the topological framing, in which the answers for these two polynomials should coincide. The relation $H^{5_1^2}_{2\otimes 1\otimes 1} = H^{5_1^2}_{1\otimes 2\otimes 2}$ must be satisfied in the topological framing because the Whitehead link can be transformed using Reidemeister moves such that the components of the link exchange places.

One more consistency check is that the answer must have the factorization property [61], [102], [103] (see Sec. 11 for the details). The untied components of the Whitehead link are isomorphic to the unknots, which agrees with the formula

$$\left.\frac{H^{5_1^2}_{1\otimes 2\otimes 2}}{S^*_1 S^*_2}\right|_{q\to 1} = A^{-2}.$$

8.3.2. Calculations using the cabling procedure. The cabling procedure in this case is analogous to the one described in Sec. 8.2.2. The colored $\mathcal{R}$-matrices are replaced with the combinations

$$\mathcal{R}_{(2\otimes 2)\otimes 1} \to R_2R_1R_3R_2, \qquad \mathcal{R}_{(1\otimes 2)\otimes 2} = \mathcal{R}^{\dagger}_{(2\otimes 1)\otimes 2} \to R_1R_2,$$

$$\mathcal{R}_{2\otimes(1\otimes 2)} = \mathcal{R}^{\dagger}_{2\otimes(2\otimes 1)} \to R_3R_4, \qquad \mathcal{R}_{1\otimes(2\otimes 2)} \to R_3R_2R_4R_3.$$

Products of the inverse R-matrices are substituted for the inverse crossings. The corresponding projectors are

$$\mathcal{P}_{1\otimes 2\otimes 2} = \frac{1+qR_2}{1+q^2}\frac{1+qR_4}{1+q^2}, \qquad \mathcal{P}_{2\otimes 1\otimes 2} = \frac{1+qR_1}{1+q^2}\frac{1+qR_4}{1+q^2}, \qquad \mathcal{P}_{2\otimes 2\otimes 1} = \frac{1+qR_1}{1+q^2}\frac{1+qR_3}{1+q^2}.$$

The HOMFLY polynomial for two split unknots is

$$H^{\bigcirc^2}_{1\otimes 2} = \mathrm{Tr}_{1^5}\, P_{2\otimes 2\otimes 1}R_1 = qS^*_4 + (q-q^{-1})S^*_{31} + qS^*_{22} - q^{-1}S^*_{211} = AS^*_1S^*_2.$$

For the torus link $T^{2,4}$ represented as a three-strand braid, we have

$$\mathcal{H}^{T[2,4]-3\text{-strand}}_{1\otimes 2\otimes 2} = \mathrm{Tr}_{1^5}\, P_{1\otimes 2\otimes 2}R_1R_2\cdot R_3R_4\cdot R_2R_1R_3R_2\cdot R_4R_3\cdot R_2R_1 =$$

$$= q^{12}S^*_5 + (q^2-q^6)S^*_{41} + (q^2-1)S^*_{32} - q^{-4}S^*_{311} + q^{-6}S^*_{221}.$$

For the nontorus Whitehead link, the HOMFLY polynomial is written as

$$\mathcal{H}^{5_1^2}_{1\otimes 2\otimes 2} = \mathrm{Tr}_{1\otimes 2\otimes 2}\, P_{1\otimes 2\otimes 2}R_1^{-1}R_2^{-1}\cdot R_3R_4\cdot R_2^{-1}R_1^{-1}R_3^{-1}R_2^{-1}\cdot R_4R_3\cdot R_2R_1 =$$

$$= q^{-4}S^*_5 + (-q^6+q^4+q^2-2+2\,q^{-4}-q^{-6}-q^{-8}+q^{-10})S^*_{41} +$$

$$+(q^8-q^6-q^4+2q^2-2+q^{-2}+q^{-4}-q^{-6})S^*_{32} - S^*_{311} + q^2S^*_{221}.$$

All these answers coincide with the expressions previously obtained using colored $\mathcal{R}$-matrices.

8.4. The case $[1]\otimes[2]\otimes[3]$.

8.4.1. Calculations using colored $\mathcal{R}$-matrices. The tensor product of the representations $[1]\otimes[2]\otimes[3]$ decomposes as

$$[1]\otimes[2]\otimes[3]=[6]+2\,[51]+2\,[42]+[411]+[33]+[321].$$

The singlet components of the $\mathcal{R}$-matrices are

$$\begin{aligned}
&\mathcal{R}_{(1\otimes2)\otimes3|6}=\mathcal{R}_{(2\otimes1)\otimes3|6}=\mathcal{R}_{3\otimes(1\otimes2)|6}=\mathcal{R}_{3\otimes(2\otimes1)|6}=q^{\varkappa_3-\varkappa_2}=q^2,\\
&\mathcal{R}_{(1\otimes2)\otimes3|411}=\mathcal{R}_{(2\otimes1)\otimes3|411}=\mathcal{R}_{3\otimes(1\otimes2)|411}=\mathcal{R}_{3\otimes(2\otimes1)|411}=-q^{\varkappa_{21}-\varkappa_2}=-q^{-1},\\
&\mathcal{R}_{(1\otimes2)\otimes3|33}=\mathcal{R}_{(2\otimes1)\otimes3|33}=\mathcal{R}_{3\otimes(1\otimes2)|33}=\mathcal{R}_{3\otimes(2\otimes1)|33}=q^{\varkappa_3-\varkappa_2}=q^2,\\
&\mathcal{R}_{(1\otimes2)\otimes3|321}=\mathcal{R}_{(2\otimes1)\otimes3|321}=\mathcal{R}_{3\otimes(1\otimes2)|321}=\mathcal{R}_{3\otimes(2\otimes1)|321}=-q^{\varkappa_{21}-\varkappa_2}=-q^{-1},\\
&\mathcal{R}_{(1\otimes3)\otimes2|6}=\mathcal{R}_{(3\otimes1)\otimes2|6}=\mathcal{R}_{2\otimes(1\otimes3)|6}=\mathcal{R}_{2\otimes(3\otimes1)|6}=q^{\varkappa_4-\varkappa_3}=q^3,\\
&\mathcal{R}_{(1\otimes3)\otimes2|411}=\mathcal{R}_{(3\otimes1)\otimes2|411}=\mathcal{R}_{2\otimes(1\otimes3)|411}=\mathcal{R}_{2\otimes(3\otimes1)|411}=-q^{\varkappa_{31}-\varkappa_3}=-q^{-1},\\
&\mathcal{R}_{(1\otimes3)\otimes2|33}=\mathcal{R}_{(3\otimes1)\otimes2|33}=\mathcal{R}_{2\otimes(1\otimes3)|33}=\mathcal{R}_{2\otimes(3\otimes1)|33}=-q^{\varkappa_{31}-\varkappa_3}=-q^{-1},\\
&\mathcal{R}_{(1\otimes3)\otimes2|321}=\mathcal{R}_{(3\otimes1)\otimes2|321}=\mathcal{R}_{2\otimes(1\otimes3)|321}=\mathcal{R}_{2\otimes(3\otimes1)|321}=q^{\varkappa_{31}-\varkappa_3}=q^{-3},\\
&\mathcal{R}_{(2\otimes3)\otimes1|6}=\mathcal{R}_{(3\otimes2)\otimes1|6}=\mathcal{R}_{1\otimes(2\otimes3)|6}=\mathcal{R}_{1\otimes(3\otimes2)|6}=q^{\varkappa_5-\varkappa_2-\varkappa_3}=q^6,\\
&\mathcal{R}_{(2\otimes3)\otimes1|411}=\mathcal{R}_{(3\otimes2)\otimes1|411}=\mathcal{R}_{1\otimes(2\otimes3)|411}=\mathcal{R}_{1\otimes(3\otimes2)|411}=-q^{\varkappa_{41}-\varkappa_2-\varkappa_3}=-q,\\
&\mathcal{R}_{(2\otimes3)\otimes1|33}=\mathcal{R}_{(3\otimes2)\otimes1|33}=\mathcal{R}_{1\otimes(2\otimes3)|33}=\mathcal{R}_{1\otimes(3\otimes2)|33}=q^{\varkappa_{32}-\varkappa_2-\varkappa_3}=q^{-2},\\
&\mathcal{R}_{(2\otimes3)\otimes1|321}=\mathcal{R}_{(3\otimes2)\otimes1|321}=\mathcal{R}_{1\otimes(2\otimes3)|321}=\mathcal{R}_{1\otimes(3\otimes2)|321}=q^{\varkappa_{32}-\varkappa_2-\varkappa_3}=q^{-2}.
\end{aligned}$$

The doublet components of the diagonal $\mathcal{R}$-matrices are

$$\begin{aligned}
&\mathcal{R}_{(1\otimes2)\otimes3|51}=\mathcal{R}_{(1\otimes2)\otimes3|42}=\mathcal{R}^\dagger_{(2\otimes1)\otimes3|51}=\mathcal{R}^\dagger_{(2\otimes1)\otimes3|42}=\\
&\qquad=\mathrm{diag}(q^{\varkappa_3-\varkappa_2},-q^{\varkappa_{21}-\varkappa_2})=\mathrm{diag}(q^2,-q^{-1}),\\
&\mathcal{R}_{(1\otimes3)\otimes2|51}=\mathcal{R}_{(1\otimes3)\otimes2|42}=\mathcal{R}^\dagger_{(3\otimes1)\otimes2|51}=\mathcal{R}^\dagger_{(3\otimes1)\otimes2|42}=\\
&\qquad=\mathrm{diag}(q^{\varkappa_4-\varkappa_3},-q^{\varkappa_{31}-\varkappa_3})=\mathrm{diag}(q^3,-q^{-1}),\\
&\mathcal{R}_{(2\otimes3)\otimes1|51}=\mathcal{R}^\dagger_{(3\otimes2)\otimes1|51}=\mathrm{diag}(q^{\varkappa_5-\varkappa_2-\varkappa_3},-q^{\varkappa_{41}-\varkappa_2-\varkappa_3})=\mathrm{diag}(q^6,-q),\\
&\mathcal{R}_{(2\otimes3)\otimes1|42}=\mathcal{R}^\dagger_{(3\otimes2)\otimes1|42}=\mathrm{diag}(q^{\varkappa_{41}-\varkappa_2-\varkappa_3},-q^{\varkappa_{32}-\varkappa_2-\varkappa_3})=\mathrm{diag}(q,-q^2).
\end{aligned}$$

The corresponding Racah matrices are

$$\mathcal{U}_{1\otimes2\otimes3|51}=\mathcal{U}_{3\otimes2\otimes1|51}=\begin{pmatrix}\dfrac{1}{\sqrt{[5]_q}} & \sqrt{\dfrac{[2]_q[6]_q}{[3]_q[5]_q}}\\ \sqrt{\dfrac{[2]_q[6]_q}{[3]_q[5]_q}} & -\dfrac{1}{\sqrt{[5]_q}}\end{pmatrix},$$

$$\mathcal{U}_{1\otimes2\otimes3|42}=\mathcal{U}_{3\otimes2\otimes1|42}=\begin{pmatrix}\frac{[2]_q}{[3]_q} & \frac{\sqrt{[5]_q}}{[3]_q}\\ \frac{\sqrt{[5]_q}}{[3]_q} & -\frac{[2]_q}{[3]_q}\end{pmatrix},$$

$$\mathcal{U}_{1\otimes3\otimes2|51}=\mathcal{U}_{2\otimes3\otimes1|51}=\begin{pmatrix}\sqrt{\frac{[2]_q}{[4]_q[5]_q}} & \sqrt{\frac{[3]_q[6]_q}{[4]_q[5]_q}}\\ \sqrt{\frac{[3]_q[6]_q}{[4]_q[5]_q}} & -\sqrt{\frac{[2]_q}{[4]_q[5]_q}}\end{pmatrix},$$

$$\mathcal{U}_{1\otimes3\otimes2|42}=\mathcal{U}_{2\otimes3\otimes1|42}=\begin{pmatrix}\sqrt{\frac{[2]_q}{[3]_q[4]_q}} & \sqrt{\frac{[2]_q[5]_q}{[3]_q[4]_q}}\\ \sqrt{\frac{[2]_q[5]_q}{[3]_q[4]_q}} & -\sqrt{\frac{[2]_q}{[3]_q[4]_q}}\end{pmatrix},$$

$$\mathcal{U}_{2\otimes1\otimes3|51}=\mathcal{U}_{3\otimes1\otimes2|51}=\begin{pmatrix}\sqrt{\frac{[2]_q}{[4]_q}} & \sqrt{\frac{[6]_q}{[3]_q[4]_q}}\\ \sqrt{\frac{[6]_q}{[3]_q[4]_q}} & -\sqrt{\frac{[2]_q}{[4]_q}}\end{pmatrix},$$

$$\mathcal{U}_{2\otimes1\otimes3|42}=\mathcal{U}_{3\otimes1\otimes2|42}=\begin{pmatrix}\sqrt{\frac{[2]_q}{[3]_q[4]_q}} & \sqrt{\frac{[2]_q[5]_q}{[3]_q[4]_q}}\\ \sqrt{\frac{[2]_q[5]_q}{[3]_q[4]_q}} & -\sqrt{\frac{[2]_q}{[3]_q[4]_q}}\end{pmatrix}.$$

This leads to the formulas for the doublet components of the nondiagonal $\mathcal{R}$-matrices:

$$\mathcal{R}_{3\otimes(1\otimes2)|51}=\mathcal{R}^{\dagger}_{3\otimes(2\otimes1)|51}=$$

$$=\mathcal{U}_{3\otimes1\otimes2|51}\begin{pmatrix}q^2 & 0\\ 0 & -q^{-1}\end{pmatrix}\mathcal{U}^{\dagger}_{3\otimes2\otimes1|51}=\begin{pmatrix}-\frac{\sqrt{[2]_q}}{q^4\sqrt{[4]_q[5]_q}} & \frac{q\sqrt{[3]_q[6]_q}}{\sqrt{[4]_q[5]_q}}\\ \frac{\sqrt{[3]_q[6]_q}}{\sqrt{[4]_q[5]_q}} & \frac{q^5\sqrt{[2]_q}}{\sqrt{[4]_q[5]_q}}\end{pmatrix},$$

$$\mathcal{R}_{3\otimes(1\otimes2)|42}=\mathcal{R}^{\dagger}_{3\otimes(2\otimes1)|42}=$$

$$=\mathcal{U}_{3\otimes1\otimes2|42}\begin{pmatrix}q^2 & 0\\ 0 & -q^{-1}\end{pmatrix}\mathcal{U}^{\dagger}_{3\otimes2\otimes1|42}=\begin{pmatrix}-\frac{\sqrt{[2]_q}}{q^3\sqrt{[3]_q[4]_q}} & \frac{\sqrt{[2]_q[5]_q}}{\sqrt{[3]_q[4]_q}}\\ \frac{q\sqrt{[2]_q[5]_q}}{\sqrt{[3]_q[4]_q}} & \frac{q^4\sqrt{[2]_q}}{\sqrt{[3]_q[4]_q}}\end{pmatrix},$$

$$\mathcal{R}_{2\otimes(1\otimes3)|51}=\mathcal{R}^{\dagger}_{2\otimes(3\otimes1)|51}=$$

$$=\mathcal{U}_{2\otimes1\otimes3|51}\begin{pmatrix}q^3 & 0\\ 0 & -q^{-1}\end{pmatrix}\mathcal{U}^{\dagger}_{2\otimes3\otimes1|51}=\begin{pmatrix}-\frac{1}{q^3\sqrt{[5]_q}} & \frac{q^2\sqrt{[2]_q[6]_q}}{\sqrt{[3]_q[5]_q}}\\ \frac{\sqrt{[2]_q[6]_q}}{\sqrt{[3]_q[5]_q}} & \frac{q^5}{\sqrt{[5]_q}}\end{pmatrix},$$

$$\mathcal{R}_{2\otimes(1\otimes3)|42} = \mathcal{R}^\dagger_{2\otimes(3\otimes1)|42} =$$

$$= \mathcal{U}_{2\otimes1\otimes3|42}\begin{pmatrix} q^3 & 0 \\ 0 & -q^{-1}\end{pmatrix}\mathcal{U}^\dagger_{2\otimes3\otimes1|42} = \begin{pmatrix} -\dfrac{1}{q^2[3]_q} & \dfrac{q\sqrt{[5]_q}}{[3]_q} \\ \dfrac{q\sqrt{[5]_q}}{[3]_q} & \dfrac{q^4}{[3]_q}\end{pmatrix},$$

$$\mathcal{R}_{1\otimes(2\otimes3)|51} = \mathcal{R}^\dagger_{1\otimes(3\otimes2)|51} =$$

$$= \mathcal{U}_{1\otimes2\otimes3|51}\begin{pmatrix} q^6 & 0 \\ 0 & -q\end{pmatrix}\mathcal{U}^\dagger_{1\otimes3\otimes2|51} = \begin{pmatrix} -\dfrac{\sqrt{[2]_q}}{\sqrt{[4]_q}} & \dfrac{q^4\sqrt{[6]_q}}{\sqrt{[3]_q[4]_q}} \\ \dfrac{q^3\sqrt{[6]_q}}{\sqrt{[3]_q[4]_4}} & \dfrac{q^7\sqrt{[2]_q}}{\sqrt{[4]_q}}\end{pmatrix},$$

$$\mathcal{R}_{1\otimes(2\otimes3)|42} = \mathcal{R}^\dagger_{1\otimes(3\otimes2)|42} =$$

$$= \mathcal{U}_{1\otimes2\otimes3|42}\begin{pmatrix} q & 0 \\ 0 & -q^2\end{pmatrix}\mathcal{U}^\dagger_{1\otimes3\otimes2|42} = \begin{pmatrix} -\dfrac{\sqrt{[2]_q}}{\sqrt{q^4[3]_q[4]_q}} & \dfrac{\sqrt{[2]_q[5]_q}}{\sqrt{[3]_q[4]_q}} \\ \dfrac{\sqrt{[2]_q[5]_q}}{q\sqrt{[3]_q[4]_q}} & \dfrac{q^3\sqrt{[2]_q}}{\sqrt{[3]_q[4]_q}}\end{pmatrix}.$$

We now consider several examples of calculating the HOMFLY polynomials for the simplest three-strand three-component links. The simplest nontrivial three-component link is the split union of the Hopf link and the unknot represented by the three-strand braid with the braid word σ_1^2. There are six (3!=6) different ways to place three representations on a three-strand braid, which obviously reduce to three different links because each link of this type has two braid representations differing in permutations of the $\mathcal{R}$-matrices in the trace. Taking this into account, we have three different links differing in the choice of the representation for the unknot:

$$\operatorname{Tr}\mathcal{R}_{(1\otimes2)\otimes3}\mathcal{R}_{(2\otimes1)\otimes3} = H^{T[2,2]}_{1\otimes2}\,S^*_3, \qquad \operatorname{Tr}\mathcal{R}_{(1\otimes3)\otimes2}\mathcal{R}_{(3\otimes1)\otimes2} = H^{T[2,2]}_{1\otimes3}\,S^*_2,$$
$$\operatorname{Tr}\mathcal{R}_{(2\otimes3)\otimes1}\mathcal{R}_{(3\otimes2)\otimes1} = H^{T[2,2]}_{2\otimes3}\,S^*_1.$$

We can verify that the answers for the HOMFLY polynomial for the Hopf link in the expression calculated for the three-strand braid coincide with the answers for the Hopf link represented by a two-strand braid:

$$H^{T[2,2]}_{1\otimes2} = q^{\varkappa_3-\varkappa_2}S^*_3 + q^{\varkappa_{21}-\varkappa_2}S^*_{21} = q^4S^*_3 + q^{-2}S^*_{21},$$
$$H^{T[2,2]}_{1\otimes3} = q^{\varkappa_4-\varkappa_3}S^*_4 + q^{\varkappa_{31}-\varkappa_3}S^*_{21} = q^6S^*_4 + q^{-2}S^*_{21},$$
$$H^{T[2,2]}_{2\otimes3} = q^{\varkappa_5-\varkappa_3-\varkappa_2}S^*_5 + q^{\varkappa_{41}-\varkappa_3-\varkappa_2}S^*_{41} + q^{\varkappa_{32}-\varkappa_3-\varkappa_2}S^*_{32} = q^{12}S^*_5 + q^2S^*_{41} + q^{-4}S^*_{32}.$$

The next simplest example is the "double Hopf" link, i.e., the composite link represented by the braid with the braid word $\sigma_1\sigma_1\sigma_2\sigma_2$. As in the preceding case, six ways to place the representations on the braid strands lead to three different HOMFLY polynomials:

$$\operatorname{Tr}_{1\otimes2\otimes3}\mathcal{R}_{1\otimes(2\otimes3)}\mathcal{R}_{(1\otimes3)\otimes2}\mathcal{R}_{(3\otimes1)\otimes2}\mathcal{R}_{1\otimes(3\otimes2)} =$$
$$= \operatorname{Tr}_{2\otimes1\otimes3}\,\mathcal{R}_{2\otimes(1\otimes3)}\mathcal{R}_{(2\otimes3)\otimes1}\mathcal{R}_{(3\otimes2)\otimes1}\mathcal{R}_{2\otimes(3\otimes1)} = \frac{H^{T[2,2]}_{2\otimes3}H^{T[2,2]}_{1\otimes3}}{S^*_3},$$

$$\mathrm{Tr}_{3\otimes1\otimes2}\mathcal{R}_{3\otimes(1\otimes2)}\mathcal{R}_{(3\otimes2)\otimes1}\mathcal{R}_{(2\otimes3)\otimes1}\mathcal{R}_{3\otimes(2\otimes1)} = \\ = \mathrm{Tr}_{1\otimes3\otimes2}\,\mathcal{R}_{1\otimes(3\otimes2)}\mathcal{R}_{(1\otimes2)\otimes3}\mathcal{R}_{(2\otimes1)\otimes3}\mathcal{R}_{1\otimes(2\otimes3)} = \frac{H^{T[2,2]}_{1\otimes2}H^{T[2,2]}_{2\otimes3}}{S^*_2},$$

$$\mathrm{Tr}_{3\otimes2\otimes1}\mathcal{R}_{3\otimes(2\otimes1)}\mathcal{R}_{(3\otimes1)\otimes2}\mathcal{R}_{(1\otimes3)\otimes2}\mathcal{R}_{3\otimes(1\otimes2)} = \\ = \mathrm{Tr}_{2\otimes3\otimes1}\,\mathcal{R}_{2\otimes(3\otimes1)}\mathcal{R}_{(2\otimes1)\otimes3}\mathcal{R}_{(1\otimes2)\otimes3}\mathcal{R}_{2\otimes(1\otimes3)} = \frac{H^{T[2,2]}_{1\otimes2}H^{T[2,2]}_{1\otimes3}}{S^*_1}.$$

It is known [100] that the HOMFLY polynomial of a composite knot or link is proportional to the product of the HOMFLY polynomial of its constituents, which indeed holds in the considered case.

The simplest prime three-strand three-component link is the torus link $T[3,3] = 6^3_3$ in the form of a braid with the braid word $\sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2$. For it, we can apply the evolution method [61], [86], [93], which allows finding an answer not for a separate link but for the whole series at once. In this case, the series is $T[3,3n]$. In the corresponding calculation, we must find the eigenvalues of the product

$$\mathfrak{R} \equiv \mathcal{R}_{(1\otimes2)\otimes3}\mathcal{R}_{2\otimes(1\otimes3)}\mathcal{R}_{(2\otimes3)\otimes1}\mathcal{R}_{3\otimes(2\otimes1)}\mathcal{R}_{(3\otimes1)\otimes2}\mathcal{R}_{1\otimes(3\otimes2)}.$$

The coefficients in the character expansion of the HOMFLY polynomial are then expressed in terms of the nth powers of these eigenvalues:

$$H^{T[3,3n]}_{1\otimes2\otimes3} = \mathrm{Tr}_{1\otimes2\otimes3}\,\mathfrak{R}^n = q^{22n}S^*_6 + 2q^{10n}S^*_{51} + 2q^{2n}S^*_{42} + q^{-2n}S^*_{411} + q^{-2n}S^*_{33} + q^{-8n}S^*_{321}.$$

This answer is obviously invariant under permutations of the representations on different strands because it corresponds to cyclic permutations of the $\mathcal{R}$-matrices in the trace.

The simplest nontorus three-component three-strand link is the link 6^3_2 (Borromean rings). It is represented as the braid $\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2\sigma_1^{-1}\sigma_2$. The corresponding HOMFLY polynomial is

$$H^{6^3_2}_{1\otimes2\otimes3} = \mathrm{Tr}_{1\otimes2\otimes3}\,\mathcal{R}^{-1}_{(1\otimes2)\otimes3}\mathcal{R}_{2\otimes(1\otimes3)}\mathcal{R}^{-1}_{(2\otimes3)\otimes1}\mathcal{R}_{3\otimes(2\otimes1)}\mathcal{R}^{-1}_{(3\otimes1)\otimes2}\mathcal{R}_{1\otimes(3\otimes2)} = \\ = S^*_6 + (-q^{12} + q^{10} + q^8 - q^4 - q^2 + 4 - q^{-2} - q^{-4} + q^{-8} + q^{-10} - q^{-12})S^*_{51} + \\ + (-q^{10} + q^8 + 2q^6 - 2q^4 - q^2 + 4 - q^{-2} - 2q^{-4} + 2q^{-6} + q^{-8} - q^{-10})S^*_{42} + \\ + S^*_{411} + S^*_{33} + S^*_{321} = \\ = \frac{S^*_3}{(q^2-1)^2(q^4-1)}\big((q^{15} - 2q^{13} + q^9 + q^7 - q^5 - 2q^3 + q)A^3 + \\ + (-q^{17} + 2q^{15} - 2q^{11} + 2q^7 + 3q^5 - 2q^3 - q + 2q^{-1} + q^{-3} - q^{-5})A + \\ + (q^{13} - q^{11} - 2q^9 + q^7 + 2q^5 - 3q^3 - 2q + 2q^{-3} - 2q^{-7} + q^{-9})A^{-1} + \\ + (-q^7 + 2q^5 + q^3 - q - q^{-1} + 2q^{-5} - q^{-7})A^{-3}\big).$$

The answer is again symmetric under permutations of the representations.

The Borromean rings link consists of three intertwined unknots. Hence, the factorization property [61], [102], [103] (also see Sec. 11 here) must give

$$\left.\frac{H^{6^3_2}_{1\otimes2\otimes3}}{S^*_1S^*_2S^*_3}\right|_{q\to1} = 1.$$

This relation is indeed satisfied.

8.4.2. Calculations using the cabling procedure. In this case, we have six different substitutions corresponding to different colored crossings:

$$\mathcal{R}_{(1\otimes 2)\otimes 3}=\mathcal{R}^\dagger_{(2\otimes 1)\otimes 3}\to R_1R_2,\qquad \mathcal{R}_{(1\otimes 3)\otimes 2}=\mathcal{R}^\dagger_{(3\otimes 1)\otimes 2}\to R_1R_2R_3,$$

$$\mathcal{R}_{(2\otimes 3)\otimes 1}=\mathcal{R}^\dagger_{(3\otimes 2)\otimes 1}\to R_2R_1R_3R_2R_4R_3,\qquad \mathcal{R}_{1\otimes(2\otimes 3)}=\mathcal{R}^\dagger_{1\otimes(3\otimes 2)}\to R_3R_2R_4R_3R_5R_4,$$

$$\mathcal{R}_{2\otimes(1\otimes 3)}=\mathcal{R}^\dagger_{2\otimes(3\otimes 1)}\to R_3R_4R_5,\qquad \mathcal{R}_{3\otimes(1\otimes 2)}=\mathcal{R}^\dagger_{3\otimes(2\otimes 1)}\to R_4R_5.$$

The inverse crossings should be replaced with the corresponding products of the inverted R-matrices. There are six projectors corresponding to the six (3!=6) permutations of the representations placed on the strands:

$$P_{1\otimes 2\otimes 3}=\frac{1+qR_2}{1+q^2}\frac{1+qR_4+qR_5+qR_4R_5+qR_5R_4+q^2R_4R_5R_4}{(1+q^2)(1+q^2+q^4)},$$

$$P_{2\otimes 1\otimes 3}=\frac{1+qR_1}{1+q^2}\frac{1+qR_4+qR_5+qR_4R_5+qR_5R_4+q^2R_4R_5R_4}{(1+q^2)(1+q^2+q^4)},$$

$$P_{1\otimes 3\otimes 2}=\frac{1+qR_5}{1+q^2}\frac{1+qR_2+qR_3+qR_2R_3+qR_3R_2+q^2R_2R_3R_2}{(1+q^2)(1+q^2+q^4)},$$

$$P_{3\otimes 1\otimes 2}=\frac{1+qR_5}{1+q^2}\frac{1+qR_1+qR_2+qR_1R_2+qR_2R_1+q^2R_1R_2R_1}{(1+q^2)(1+q^2+q^4)},$$

$$P_{2\otimes 3\otimes 1}=\frac{1+qR_1}{1+q^2}\frac{1+qR_3+qR_4+qR_3R_4+qR_4R_3+q^2R_3R_4R_3}{(1+q^2)(1+q^2+q^4)},$$

$$P_{3\otimes 2\otimes 1}=\frac{1+qR_4}{1+q^2}\frac{1+qR_1+qR_2+qR_1R_2+qR_2R_1+q^2R_1R_2R_1}{(1+q^2)(1+q^2+q^4)}.$$

Using these formulas, we can calculate the HOMFLY polynomials for three-strand three-component links in the representation $[1]\otimes[2]\otimes[3]$. For the split union of the Hopf link and of the unknot, the answer is

$$\mathrm{Tr}_{1^6}\,P_{1\otimes 2\otimes 3}R_2R_1\cdot R_1R_2=H^{T[2,2]}_{1\otimes 2}S^*_3,$$

$$\mathrm{Tr}_{1^6}\,P_{1\otimes 3\otimes 2}R_3R_2R_1\cdot R_1R_2R_3=H^{T[2,2]}_{1\otimes 3}S^*_2,$$

$$\mathrm{Tr}_{1^6}\,P_{2\otimes 3\otimes 1}R_2R_1R_3R_2R_4R_3\cdot R_3R_4R_2R_3R_1R_2=H^{T[2,2]}_{2\otimes 3}S^*_1.$$

For the "double Hopf link," we have

$$\mathrm{Tr}_{1^6}\,P_{1\otimes 2\otimes 3}R_3R_2R_4R_3R_5R_4\cdot R_3R_2R_1\cdot R_1R_2R_3\cdot R_4R_5R_3R_4R_2R_3=$$

$$=\mathrm{Tr}_{1^6}\,P_{2\otimes 1\otimes 3}R_5R_4R_3\cdot R_2R_1R_3R_2R_4R_3\cdot R_3R_4R_2R_3R_1R_2\cdot R_3R_4R_5=\frac{H^{T[2,2]}_{2\otimes 3}H^{T[2,2]}_{1\otimes 3}}{S^*_3},$$

$$\mathrm{Tr}_{1^6}\,P_{3\otimes 1\otimes 2}R_4R_5\cdot R_3R_4R_2R_3R_1R_2\cdot R_2R_1R_3R_2R_4R_3\cdot R_4R_5=$$

$$=\mathrm{Tr}_{1^6}\,P_{1\otimes 3\otimes 2}R_3R_2R_4R_3R_5R_4\cdot R_2R_1\cdot R_1R_2\cdot R_4R_5R_3R_4R_2R_3=\frac{H^{T[2,2]}_{1\otimes 2}H^{T[2,2]}_{2\otimes 3}}{S^*_2},$$

$$\mathrm{Tr}_{1^6}\,P_{3\otimes 2\otimes 1}R_5R_4\cdot R_1R_2R_3\cdot R_3R_2R_1\cdot R_4R_5=$$

$$=\mathrm{Tr}_{1^6}\,P_{2\otimes 3\otimes 1}R_3R_4R_5\cdot R_1R_2\cdot R_2R_1\cdot R_5R_4R_3=\frac{H^{T[2,2]}_{1\otimes 2}H^{T[2,2]}_{1\otimes 3}}{S^*_1}.$$

For the series of torus links, we have

$$H^{T[3,3n]}_{1\otimes2\otimes3} = \mathrm{Tr}_{1^6}\, P_{1\otimes2\otimes3}(R_2R_1\,\cdot\,R_5R_4R_3\,\cdot\,R_2R_1R_3R_2R_4R_3\,\cdot\,R_4R_5\,\cdot\,R_3R_2R_1\,\cdot\,R_4R_5R_3R_4R_2R_3)^n =$$

$$= q^{22n}S^*_6 + 2q^{10n}S^*_{51} + 2q^{2n}S^*_{42} + q^{-2n}S^*_{411} + q^{-2n}S^*_{33} + q^{-8n}S^*_{321}.$$

Finally, for the Borromean rings link, the HOMFLY polynomial is written as

$$\begin{aligned} H^{6^3_2}_{1\otimes2\otimes3} = {} & \mathrm{Tr}_{1^6}\, P_{1\otimes2\otimes3}R_2^{-1}R_1^{-1}\cdot R_5R_4R_3\cdot R_2^{-1}R_1^{-1}R_3^{-1}R_2^{-1}R_4^{-1}R_3^{-1}\times \\ & \times R_4R_5\cdot R_3^{-1}R_2^{-1}R_1^{-1}\cdot R_4R_5R_3R_4R_2R_3 = \\ = {} & S^*_6 + (-q^{12}+q^{10}+q^8-q^4-q^2+4-q^{-2}-q^{-4}+q^{-8}+q^{-10}-q^{-12})S^*_{51} + \\ & + (-q^{10}+q^8+2q^6-2q^4-q^2+4-q^{-2}-2q^{-4}+2q^{-6}+q^{-8}-q^{-10})S^*_{42} + \\ & + S^*_{411}+S^*_{33}+S^*_{321}. \end{aligned}$$

In all cases, the answers coincide with the calculations using colored $\mathcal{R}$-matrices.

9. Cabling procedure from the representation theory standpoint

In the preceding sections, we described the cabling procedure and all the quantities needed for it and also considered a series of examples. In doing so, we regarded the procedure itself as a postulate. In this section, we discuss why this cabling procedure should work from the representation theory standpoint. We emphasize that the cabling procedure arises not from topology but from representation theory. Hence, the cabling procedure can be used to calculate not only topological invariants, i.e., the HOMFLY polynomials, but also objects that are closely related to them but are not topologically invariant themselves, such as extended HOMFLY polynomials [85] and $\mathcal{R}$-matrices.

In representation theory, higher representations are introduced using the coproduct operation. The coproduct yields the action of an algebra on the tensor product of the representations and thus determines the decomposition of the tensor product of representations into irreducible representations. This is exactly what the cabling approach does. We consider the two-strand $\mathcal{R}$-matrix corresponding to crossing strands with representations T_1 and T_2. We describe it using fundamental R-matrices acting in the $|T_1|+|T_2|$-strand braid. The corresponding cabling procedure can be schematically described by two operations: first, the colored $\mathcal{R}$-matrix is replaced with the product of fundamental representations,

$$\mathcal{R}_{T_1\otimes T_2} \;\rightarrow\; \prod_{i=1}^{|T_2|}\prod_{j=1}^{|T_1|} R_{|T_1|+i-j}, \tag{9.1}$$

and, second, the projectors P_R are inserted.

First operation (9.1) is based on the equality

$$\mathcal{R}_{1^{|T_1|}\otimes 1^{|T_2|}} = \prod_{i=1}^{|T_2|}\prod_{j=1}^{|T_1|} R_{|T_1|+i-j}, \tag{9.2}$$

which in turn is a consequence of the definition of the coproduct for $\mathcal{R}$-matrices [104]–[106]

$$\begin{aligned} \mathcal{R}_{(T_1\otimes T_2)\otimes T_3} &= (I_{T_1}\otimes\mathcal{R}_{T_2\otimes T_3})\cdot(\mathcal{R}_{T_1\otimes T_3}\otimes I_{T_2}), \\ \mathcal{R}_{T_1\otimes(T_2\otimes T_3)} &= (\mathcal{R}_{T_1\otimes T_2}\otimes I_{T_3})\cdot(I_{T_2}\otimes\mathcal{R}_{T_1\otimes T_3}), \end{aligned} \tag{9.3}$$

where T_1, T_2, and T_3 are arbitrary, not necessarily irreducible, representations. Relation (9.2) follows from (9.3) by induction. We apply the first relation in (9.3) $|T_1|$ times,

$$\mathcal{R}_{1^{m+1}\otimes 1} = \mathcal{R}_{(1^m\otimes 1)\otimes 1} = (I_{1^m} \otimes \mathcal{R}_{1\otimes 1}) \cdot (\mathcal{R}_{1^m\otimes 1} \otimes I_1) =$$

$$= R_{m+1}(\mathcal{R}_{1^m\otimes 1} \otimes I_1) = \prod_{i=1}^{m+1} R_{m-i+1},$$

and then similarly apply the second relation in (9.3) $|T_2|$ times.

The second operation is based on the definition of projectors. This definition relies on the expansion of the tensor power of the fundamental (or higher) representation into the irreducible representations: $1^m = \sum_{T\vdash 1^m} T$ (see Sec. 4). After the projectors are defined, the second step is based on the known identity in linear algebra

$$\mathrm{Tr}_{1^{|T_1|}\otimes 1^{|T_2|}}\, P_{T_1} P_{T_2} \cdot (\,\cdot\,) = \mathrm{Tr}_{T_1\otimes T_2} \cdot (\,\cdot\,).$$

Summarizing, we note that the cabling procedure is based on the claim that *a colored $\mathcal{R}$-matrix is equal to the product of the fundamental R-matrices and the projectors.*[7] In particular, this means that in the basis where the corresponding projectors are diagonal, the "cabling" product of the fundamental R-matrices splits into "colored blocks" corresponding to different irreducible representations. This statement is nontrivial from the standpoint in Sec. 3, where the fundamental and colored R-matrices are considered two independent quantities defined via their eigenvalues.

The main technical obstacle here is to pass from standard basis (3.2), where the form of the fundamental R-matrices is known, to the basis

$$(1\otimes 1\otimes\cdots\otimes 1)\otimes(1\otimes 1\otimes\cdots 1)\otimes\cdots\otimes(1\otimes 1\otimes\cdots\otimes 1),$$

where the colored R-matrices should be calculated. The Racah coefficients [99] directly describing this transition are unknown in the general case. But the necessary transition matrix can be found in several cases. To simplify the problem of passing from one basis to the other, we can try to diagonalize the projectors P_Q,

$$1\otimes 1\otimes\cdots\otimes(1\otimes 1\otimes\cdots\otimes 1)\otimes\cdots\otimes 1 \to 1\otimes 1\otimes\cdots\otimes Q\otimes\cdots\otimes 1, \tag{9.4}$$

for which expressions are known in the form of polynomials of the fundamental R-matrices (see Sec. 4). In turn, the fundamental R-matrices are known explicitly (see Sec. 3.3).

9.1. Colored $\mathcal{R}$-matrices from fundamental ones. As described in Sec. 3.3, fundamental R-matrices are built from blocks of sizes 1×1 and 2×2. The 1×1 blocks are just the eigenvalues $\pm q^{\pm 1}$, and the 2×2 blocks have the form

$$\begin{pmatrix} -\dfrac{1}{q^k[k]} & \dfrac{\sqrt{[k-1][k+1]}}{[k]} \\ \dfrac{\sqrt{[k-1][k+1]}}{[k]} & \dfrac{q^k}{[k]} \end{pmatrix}$$

and can be diagonalized using

$$\begin{pmatrix} \sqrt{\dfrac{[k-1]}{[2][k]}} & -\sqrt{\dfrac{[k+1]}{[2][k]}} \\ \sqrt{\dfrac{[k+1]}{[2][k]}} & \sqrt{\dfrac{[k-1]}{[2][k]}} \end{pmatrix}.$$

Hence, it is easy to diagonalize each $\mathcal{R}$-matrix separately. We can thus obtain expressions for $\mathcal{R}$-matrices for level-two representations from the fundamental R-matrices.

[7] The exact equality holds in the vertical framing (see Sec. 5). Under another framing choice, framing factors must also be taken into account.

9.1.1. The case $|Q| = 2$. Colored $\mathcal{R}$-matrices for the level-two representations (i.e., the representations [2] and [11]) are diagonal in a basis of the form

$$(1 \otimes 1) \otimes (1 \otimes 1) \otimes (1 \otimes 1) \otimes \dots. \tag{9.5}$$

In such a basis, all the projectors

$$P_2^{(j)}\colon \quad (1 \otimes 1) \otimes (1 \otimes 1) \cdots \otimes (1 \otimes 1) \otimes \cdots \to (1 \otimes 1) \otimes (1 \otimes 1) \cdots \otimes 2 \otimes \dots,$$

$$P_{11}^{(j)}\colon \quad (1 \otimes 1) \otimes (1 \otimes 1) \cdots \otimes (1 \otimes 1) \otimes \cdots \to (1 \otimes 1) \otimes (1 \otimes 1) \cdots \otimes 11 \otimes \dots,$$

where j denotes the number of projected pairs of fundamental representations, are diagonal.

We have (see Sec. 4)

$$P_2^{(j)} = \frac{1 + qR_{2j-1}}{1 + q^2}, \qquad P_{11}^{(j)} = \frac{q^2 - qR_{2j-1}}{1 + q^2},$$

and diagonalizing all the projectors $P_{2,11}^{(j)}$ (which is equivalent to diagonalizing all the $P_2^{(j)}$) means diagonalizing all the "odd" fundamental R_{2j-1}-matrices. In fact, this can be done because the only noncommuting R-matrices are adjacent pairs, i.e., R_i and R_{i+1}. For R-matrices of the form described above, this means that where there is a 2×2 block in one of the matrices, there must be either a diagonal block with equal diagonal elements or the same 2×2 block in all the other matrices.

9.1.2. The case $|Q| = 3$. A basis in which the $\mathcal{R}$-matrices for level-three representations are diagonal has the form

$$(1 \otimes 1 \otimes 1) \otimes (1 \otimes 1 \otimes 1) \otimes \cdots \otimes (1 \otimes 1 \otimes 1). \tag{9.6}$$

In such a basis, all the projectors

$$P_3^{(j)}\colon \quad (1 \otimes 1) \otimes (1 \otimes 1) \cdots \otimes (1 \otimes 1) \otimes \cdots \to (1 \otimes 1) \otimes (1 \otimes 1) \cdots \otimes 2 \otimes \dots,$$

$$P_{21}^{(j)}\colon \quad (1 \otimes 1) \otimes (1 \otimes 1) \cdots \otimes (1 \otimes 1) \otimes \cdots \to (1 \otimes 1) \otimes (1 \otimes 1) \cdots \otimes 11 \otimes \dots,$$

$$P_{111}^{(j)}\colon \quad (1 \otimes 1) \otimes (1 \otimes 1) \cdots \otimes (1 \otimes 1) \otimes \cdots \to (1 \otimes 1) \otimes (1 \otimes 1) \cdots \otimes 11 \otimes \dots.$$

are diagonal. As in what follows, it suffices to diagonalize only the projectors $P_{21}^{(j)}$. As follows from (4.12), the corresponding formulas for projectors expressed in terms of R-matrices have the form

$$P_{21}^{(j)} = \frac{(R_{3j-2} - R_{3j-1})^2}{q^{-2} + 1 + q^2}.$$

Basis (9.6) is not well defined because there is a freedom to choose the basis in the two-dimensional space of the representation [21] arising in expanding the product in each bracket: $[1] \otimes [1] \otimes [1] = [3] + 2[21] + [111]$.[8] This ambiguity can be eliminated by fixing the order of multiplying the fundamental representations in each bracket, for example, as

$$\big(((1 \otimes 1) \otimes 1\big) \otimes \big((1 \otimes 1) \otimes 1\big) \otimes \cdots\big) \otimes \big((1 \otimes 1) \otimes 1\big). \tag{9.7}$$

All the projectors ${}^*P_2^{(j)} = (1 + qR_{3j-2})/(1 + q^2)$ are also diagonal in this basis. They can be diagonalized in the same way as the projectors $P_2^{(j)}$ were diagonalized above. We can then calculate the matrices of $P_{21}^{(j)}$

[8] There is also a more general ambiguity in choosing the basis because the order of multiplying the brackets is not defined. In all cases, we choose the basis $\big(\big(\big((\cdot) \otimes (\cdot)\big) \otimes (\cdot)\big) \cdots\big) \otimes \dots$, which is a generalization of standard basis (3.2).

in the same basis. Nondiagonal elements appear in this matrix only where there are unit or zero blocks in the already diagonalized projectors $P_{21}^{(j)}$ and ${}^*P_{21}^{(l)}$ because these projectors commute for all j and l. In the six-strand case, all the $P_{21}^{(j)}$ consist of 2×2 blocks with the single exception of the 3×3 block appearing several times in the component $P_{21|321}^{(j)}$ corresponding to the representation [321]. The universal form of the blocks in the projectors onto level-three representations, in contrast to the blocks in the fundamental R-matrices and the projectors onto level-two representations, is unknown. We calculated these blocks in several particular cases and presented explicit expressions in Appendix F in [98].

9.2. The permutation operators in $\mathcal{R}$- and U-matrices. We can define the operators $\mathcal{U}$ (mixing matrices) in one of two ways. In the first way, these operators are defined as the matrices of transition from one basis (e.g., where the $\mathcal{R}$-matrix corresponding to the crossing between the first pair of strands is diagonal) to another basis (e.g., where the $\mathcal{R}$-matrix corresponding to the crossing between the second pair of strands is diagonal). All the $\mathcal{U}$-matrices described in Sec. 8 are defined this way. The approach that uses mixing matrices of this type assumes that the diagonal form of the $\mathcal{R}$-matrix is known irrespective of the location of the corresponding crossing strands. There is another way to define the operators $\mathcal{U}$. The main difference is that the operator $\mathcal{U}$ now includes a permutation operator and consequently changes the placement of strands in the braid. We let U denote these matrices. The operators U discussed in this section allow defining an arbitrary $\mathcal{R}$-matrix using only diagonal $\mathcal{R}$-matrices corresponding to the crossing of the first and second strands. The way to define the operators U is shown below (here and hereafter, m denotes the number of strands):

For $m = 2$,

$$T_1 \otimes T_2 \xrightarrow{\quad \mathcal{R}_{T_1T_2} \quad} T_2 \otimes T_1,$$

For $m = 3$,

$$\begin{array}{ccc}
(T_1 \otimes T_2) \otimes T_3 & \xrightarrow{\quad \mathcal{R}_{(T_1T_2)T_3} \quad} & (T_2 \otimes T_1) \otimes T_3 \\
\Big\downarrow U_{T_1T_2T_3} & & \Big\downarrow U_{T_2T_1T_3}\,, \\
T_3 \otimes (T_2 \otimes T_1) & \xrightarrow{\quad \mathcal{R}_{T_3(T_2T_1)} \quad} & T_3 \otimes (T_1 \otimes T_2)
\end{array}$$

and

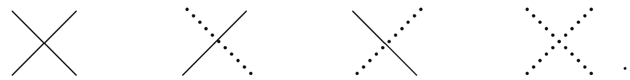
$$\mathcal{R}_{T_3(T_2T_1)} = U_{T_1T_2T_3} \mathcal{R}_{(T_1T_2)T_3}\ U^{\dagger}_{T_2T_1T_3}.$$

In what follows, we consider some concrete examples and describe the form of the products of the fundamental $\mathcal{R}$-matrices corresponding to colored $\mathcal{R}$-matrices. In all the examples discussed below, just as everywhere in this paper, $\mathcal{R}$-matrices are written in the vertical framing.

9.3. The case $|T| = 2$, $m = 2$. In the case where $|T| = 2$ and $m = 2$, we have a single $\mathcal{R}$-matrix, which according to (9.1) should be replaced with the product

$$\mathcal{R}_1 \to \mathfrak{R}_1 = R_2R_1R_3R_2.$$

In basis (9.5), $\mathfrak{R}$ splits into blocks corresponding to the colored $\mathfrak{R}$-matrices of the type $\mathcal{R}_{QQ'}$, where $|Q| = |Q'| = 2$ (in the drawing, solid lines denote the symmetric representation, and dashed lines denote the antisymmetric representation):



$\mathcal{R}_{2\,2}$ $\quad$ $\mathcal{R}_{11\,2}$ $\quad$ $\mathcal{R}_{2\,11}$ $\quad$ $\mathcal{R}_{11\,11}$

In this case, the solid lines correspond to the representation [2], and the dashed lines correspond to [11].

For $Q = [31]$, we have the following. The representation [31] appears in the decompositions of $2 \otimes 2$, $2 \otimes 11$, and $11 \otimes 2$ and does not appear in the decomposition of $11 \otimes 11$. Hence, the $\mathfrak{R}$-matrix contains three of the four possible blocks:

$$\mathfrak{R}_{1|31} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & q^2 \\ 0 & q^2 & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{2\otimes 2|31} & 0 & 0 \\ 0 & 0 & \mathcal{R}_{2\otimes 11|31} \\ 0 & \mathcal{R}_{11\otimes 2|31} & 0 \end{pmatrix}.$$

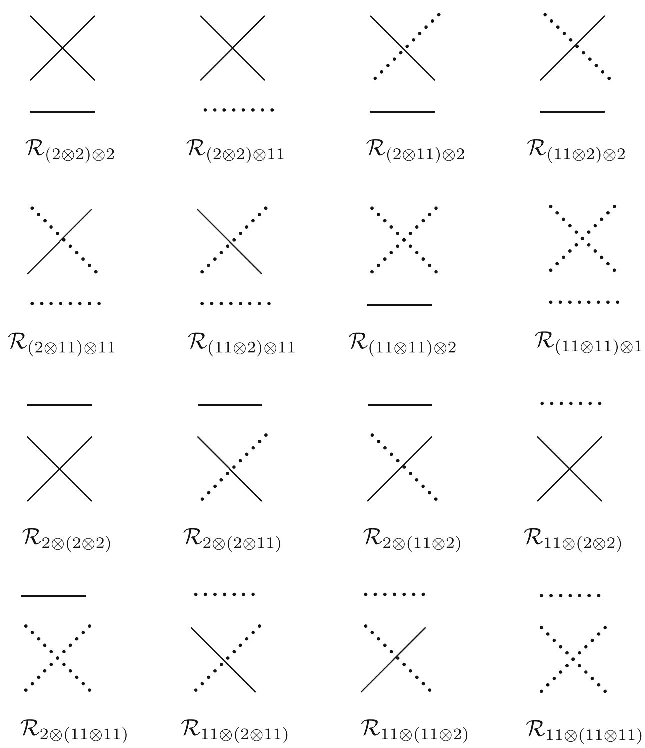
For $Q = [22]$, we have the following. The representation [22] appears in the decompositions of $2 \otimes 2$ and $11 \otimes 11$ and does not appear in the decompositions of $2 \otimes 11$ and $11 \otimes 2$. Hence, the $\mathfrak{R}$-matrix contains two of the four possible blocks:

$$\mathfrak{R}_{1|22} = \begin{pmatrix} q^{-2} & 0 \\ 0 & q^2 \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{2\otimes 2|22} & 0 \\ 0 & \mathcal{R}_{11\otimes 11|22} \end{pmatrix}.$$

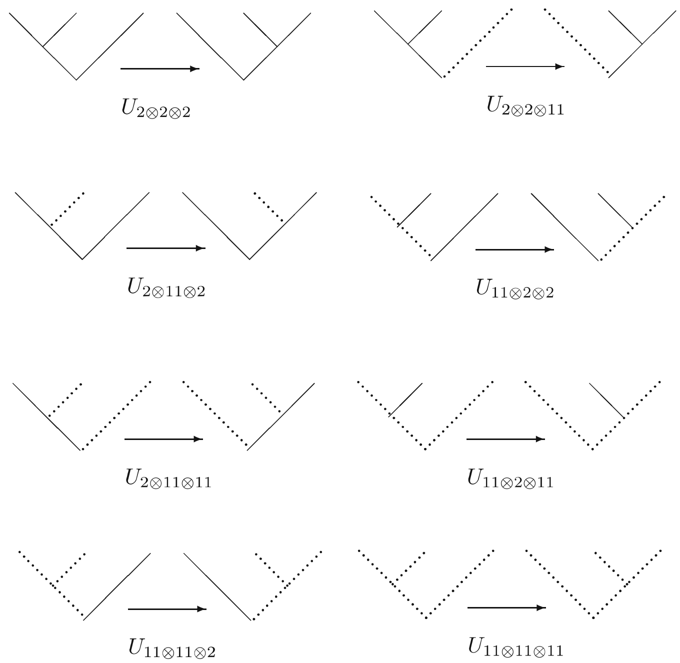
9.4. The case $|T| = 2$, $m = 3$. In the case of three-strand knots, there are two different $\mathcal{R}$-matrices:

$$\mathcal{R}_1 \to \mathfrak{R}_1 \equiv R_2 R_1 R_3 R_2, \qquad \mathcal{R}_2 \to \mathfrak{R}_2 \equiv R_4 R_3 R_5 R_4.$$

The colored mixing matrix U should also be replaced with a certain product of the fundamental matrices, $\mathfrak{U} = UVWY \cdot UVW$ (the notation is as in [86], [90]). There are eight different types of crossings corresponding to the $\mathfrak{R}$-matrices:



The eight possible transitions between them are described by the operators $\mathfrak{U}$ given by



In the basis $\big((1\otimes1)\otimes(1\otimes1)\big)\otimes(1\otimes1)$, the matrices $\mathfrak{R}$ and $\mathfrak{U}$ have a block structure:

$$\mathfrak{R}_1 = \begin{pmatrix} \mathcal{R}_{(2\otimes2)\otimes2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{R}_{(2\otimes2)\otimes11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{R}_{(2\otimes11)\otimes2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{R}_{(11\otimes2)\otimes2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{R}_{(2\otimes11)\otimes11} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{R}_{(11\otimes2)\otimes11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{R}_{(11\otimes11)\otimes2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{R}_{(11\otimes11)\otimes11} \end{pmatrix},$$

$$\mathfrak{R}_2 = \begin{pmatrix} \mathcal{R}_{2\otimes(2\otimes2)} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{R}_{2\otimes(2\otimes11)} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathcal{R}_{2\otimes(11\otimes2)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{R}_{11\otimes(2\otimes2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathcal{R}_{2\otimes(11\otimes11)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{R}_{11\otimes(2\otimes11)} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathcal{R}_{11\otimes(11\otimes2)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathcal{R}_{11\otimes(11\otimes11)} \end{pmatrix},$$

$$\mathfrak{U} = \begin{pmatrix} U_{2\otimes2\otimes2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & U_{11\otimes2\otimes2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & U_{2\otimes11\otimes2} & 0 & 0 & 0 & 0 \\ 0 & U_{2\otimes2\otimes11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & U_{11\otimes11\otimes2} & 0 \\ 0 & 0 & 0 & 0 & U_{11\otimes2\otimes11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & U_{2\otimes11\otimes11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & U_{11\otimes11\otimes11} \end{pmatrix}.$$

This means that the equality $\mathfrak{R}_2 = \mathfrak{U}\,\mathfrak{R}_1\mathfrak{U}^\dagger$ splits into separate parts:

$$\mathcal{R}_{2\otimes(2\otimes2)} = U_{2\otimes2\otimes2}\mathcal{R}_{(2\otimes2)\otimes2}U^\dagger_{2\otimes2\otimes2},$$

$$\mathcal{R}_{11\otimes(11\otimes11)} = U_{11\otimes11\otimes11}\mathcal{R}_{(11\otimes11)\otimes11}U^{\dagger}_{11\otimes11\otimes11},$$

$$\mathcal{R}_{11\otimes(2\otimes2)} = U_{2\otimes2\otimes11}\mathcal{R}_{(2\otimes2)\otimes11}U^{\dagger}_{11\otimes2\otimes2},$$

$$\mathcal{R}_{2\otimes(11\otimes2)} = U_{2\otimes11\otimes2}\mathcal{R}_{(2\otimes11)\otimes2}U^{\dagger}_{2\otimes11\otimes2},$$

$$\mathcal{R}_{2\otimes(2\otimes11)} = U_{11\otimes2\otimes2}\mathcal{R}_{(11\otimes2)\otimes2}U^{\dagger}_{2\otimes11\otimes11},$$

$$\mathcal{R}_{2\otimes(11\otimes11)} = U_{11\otimes11\otimes2}\mathcal{R}_{(11\otimes11)\otimes2}U^{\dagger}_{2\otimes11\otimes11},$$

$$\mathcal{R}_{11\otimes(2\otimes11)} = U_{11\otimes2\otimes11}\mathcal{R}_{(11\otimes2)\otimes11}U^{\dagger}_{11\otimes2\otimes11},$$

$$\mathcal{R}_{2\otimes(11\otimes11)} = U_{11\otimes11\otimes2}\mathcal{R}_{(11\otimes11)\otimes2}U^{\dagger}_{2\otimes11\otimes11}.$$

The explicit form of the blocks was presented in Appendix G in [98].

9.5. The case $|T| = 3$, $m = 2$. Because of the technical complexity of the calculations, we restrict ourself to only calculating two-strand knots for representations of size three. This case requires calculating for six (2·3=6) strands in the fundamental representation (as in the case of three-strand knots in the representations of size two). We should take a basis of the form $(1\otimes1\otimes1)\otimes(1\otimes1\otimes1)$.

According to (9.1), we must replace each $\mathcal{R}$-matrix with the product of nine fundamental R-matrices:

$$\mathcal{R}_1 \to \mathfrak{R} = R_3R_2R_1R_4R_3R_2R_5R_4R_3.$$

In basis (9.7), this matrix decomposes into blocks corresponding to the irreducible representations arising in the decomposition $(1\otimes1)\otimes1 = 3 + \underline{21} + \overline{21} + 111$. We have

$$\mathfrak{R} = \mathrm{diag}(\mathfrak{r}_{3\otimes111}, \mathfrak{r}_{3\otimes21}, \mathfrak{r}_{21\otimes111}, \mathfrak{r}_{21\otimes21}),$$

where

$$\mathfrak{r}_{3\otimes111} = \begin{pmatrix} \mathcal{R}_{3\otimes3} & 0 & 0 & 0 \\ 0 & 0 & \mathcal{R}_{3\otimes111} & 0 \\ 0 & \mathcal{R}_{111\otimes3} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{R}_{111\otimes111} \end{pmatrix}, \qquad \mathfrak{r}_{3\otimes21} = \begin{pmatrix} 0 & \mathcal{R}_{3\otimes\underline{21}} & 0 & 0 \\ \mathcal{R}_{\underline{21}\otimes3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{R}_{3\otimes\overline{21}} \\ 0 & 0 & \mathcal{R}_{\overline{21}\otimes3} & 0 \end{pmatrix},$$

$$\mathfrak{r}_{21\otimes111} = \begin{pmatrix} 0 & \mathcal{R}_{111\otimes\underline{21}} & 0 & 0 \\ \mathcal{R}_{\underline{21}\otimes111} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{R}_{111\otimes\overline{21}} \\ 0 & 0 & \mathcal{R}_{\overline{21}\otimes111} & 0 \end{pmatrix}, \qquad \mathfrak{r}_{21\otimes21} = \begin{pmatrix} \mathcal{R}_{\underline{21}\otimes\underline{21}} & 0 & 0 & 0 \\ 0 & 0 & \mathcal{R}_{\underline{21}\otimes\overline{21}} & 0 \\ 0 & \mathcal{R}_{\overline{21}\otimes\underline{21}} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{R}_{\overline{21}\otimes\overline{21}} \end{pmatrix}.$$

The explicit form of the blocks was presented in Appendix H in [98].

10. Cabling procedure and the eigenvalue conjecture

It was proposed in [89] that a certain relation exists between the eigenvalues of an $\mathcal{R}$-matrix and the corresponding Racah coefficients. It was asserted that all three-strand U-matrices depend only on the (somehow normalized) eigenvalues of the $\mathcal{R}$-matrix; it was further supposed that all three strands are in the same representation. Studying the cabling procedure, we can explain why the U-matrix elements are expressed in terms of the eigenvalues of R-matrices.

10.1. Constraints on the $\mathcal{R}$-matrix elements from the cabling procedure. The existence of the cabling procedure implies that there are severe constraints on the form of $\mathcal{R}$-matrices. As already discussed in Sec. 9, this procedure in fact describes the fusion (coproduct) rule for $\mathcal{R}$-matrices. This means that the specific products of $\mathcal{R}$-matrices in a certain basis split into blocks corresponding to the $\mathcal{R}$-matrices in higher representations (see examples in Sec. 9). Hence, coproduct rule (9.3) can be considered a system of constraints on the form of $\mathcal{R}$-matrices:

$$\mathcal{U}_{xxx}\mathcal{R}_{(xx)x}\mathcal{U}^{\dagger}_{xxx}\mathcal{R}_{(xx)x}\mathcal{U}_{xxx} = \mathrm{Diag}(\mathcal{R}_{wx})_{w\vdash x\otimes x}, \tag{10.1}$$

where the diagonalized $\mathcal{R}$-matrix is in the right-hand side; each possible w corresponds to one of the eigenvalues of the matrix. The appearance of the three transition matrices instead of two in this relation can be explained as follows: the $\mathcal{R}$-matrix $\mathcal{R}_{wx}$ is diagonal if it acts from the space $w\otimes x$ to the space $x\otimes w$. Therefore, if the matrix in the right-hand side of (10.1) is required to be diagonal, then three transition matrices are needed. Relation (10.1) as always splits into separate colored blocks, one for each space of irreducible representations $Q \vdash x\otimes x\otimes x$ corresponding to the same Young diagram. We keep in mind that all relations in this section are written for separate colored blocks $\mathcal{U}_{xxx|Q}$ although the index Q is omitted. In particular, the index w in (10.1) ranges not all irreducible representations from the decomposition of $x\otimes x$ but only those that contain Q in the decomposition of $w\otimes x$.

If all elements of the diagonal matrix $\mathcal{R}_{xx}$ are known, then system (10.1) imposes severe constraints on the elements of the Racah matrix U and on the eigenvalues of the matrix $\mathcal{R}_{wx}$ for the representations w such that $|w| > |x|$. At first glance, system (10.1) even seems overdetermined: the condition that the off-diagonal elements in the left-hand side vanish leads to N^2-N relations, while there are only $(N^2-N)/2$ free parameters in the orthogonal matrix $\mathcal{U}$. The seeming contradiction is resolved because the Racah matrix $\mathcal{U}$ is not only orthogonal but also satisfies $\mathcal{U}\sigma = \sigma\mathcal{U}^{\dagger}$ for some matrix σ such that $\sigma_{ij}=0$ for $i\neq j$ and $\sigma_{ii}=\pm 1$. We say that such matrices $\mathcal{U}$ are pseudosymmetric. Assuming that $\mathcal{U}$ is pseudosymmetric, we can see that the number of independent equations in (10.1) decreases. In concrete cases, we show that the number of independent equations is exactly equal to the number of variables. In fact, it suffices to consider only symmetric matrices because each pseudosymmetric solution $\mathcal{U}$ of (10.1) corresponds to a symmetric solution.[9]

10.2. Expressions for $\mathcal{U}$-matrix elements in terms of $\mathcal{R}$-matrix eigenvalues. For the further analysis, we slightly simplify expression (10.1). It follows from (10.1) and the orthogonal matrix property $\det\mathcal{U} = \pm 1$ that $\det^2\mathcal{R}_{(xx)x} = \det\mathcal{R}_{wx}$. Hence, a direct substitution shows that (10.1) still holds if the $\mathcal{R}$-matrices in both sides of the equation are rescaled as

$$\mathcal{R} \to \frac{\mathcal{R}}{(\pm\det\mathcal{R})^{1/n}}, \tag{10.2}$$

where n is the size of the matrices and the sign of the determinant is chosen such that the expression in the brackets is positive for positive real q (this choice slightly simplifies the subsequent formulas). We can then rewrite (10.1) in components as

$$\sum_{j,k=1}^{n} \mathcal{U}_{ij}\,\mathcal{U}_{jk}\,\mathcal{U}_{lk}\,\xi_j\xi_k = \delta_{il}\eta_i, \tag{10.3}$$

where ξ and η are the normalized eigenvalues introduced in [89]:[10]

$$\frac{\mathcal{R}_{(xx)x}}{(\pm\det\mathcal{R})^{1/n}} \equiv \mathrm{diag}(\xi_1,\ldots,\xi_n), \qquad \frac{\mathrm{Diag}(\mathcal{R}_{wx})_{w\vdash x\otimes x}}{(\pm\det\mathcal{R})^{1/n}} \equiv \mathrm{diag}(\eta_1,\ldots,\eta_n).$$

[9]If the matrix $\mathcal{U}$ is pseudosymmetric, i.e., $\mathcal{U}\sigma = \sigma\mathcal{U}^{\dagger}$, then the matrix $U\sigma$ is symmetric: $(U\sigma)^{\dagger} = U\sigma$. Hence, a symmetric solution can be obtained from a pseudosymmetric solution by replacing $\mathcal{U}$ with $\mathcal{U}\sigma$ in (10.1).

[10]Normalized eigenvalues were denoted by $\tilde{\xi}$ in [89] while ξ was used for ordinary eigenvalues. We use ξ for the normalized eigenvalues because ordinary eigenvalues do not appear in the calculations in this section.

Equations (10.3) together with the orthogonality condition $\sum_{k=1}^{n} \mathcal{U}_{ik}\mathcal{U}_{lk} = \delta_{il}$ establish a system of equations for the $\mathcal{U}$-matrix. At first glance, it is impossible to find an explicit solution of the system because it is substantially nonlinear. But this system reduces to the linear form in some particular cases.

Because we assume that $\mathcal{U}$ is orthogonal and symmetric, we can express it in terms of a symmetric linear projector $\mathcal{P}$ as

$$\mathcal{U} = 1 - 2\mathcal{P}, \qquad \mathcal{P}^2 = \mathcal{P}, \quad \mathcal{P}^\dagger = \mathcal{P}, \tag{10.4}$$

and consequently $\mathcal{U}^2 = 1$, $\mathcal{U}^\dagger = \mathcal{U}$. In the simplest case, $\operatorname{rank}\mathcal{P} = 1$, and then

$$\mathcal{P} = u_i u_j, \qquad \mathcal{U}_{ij} = \delta_{ij} - 2u_i u_j. \tag{10.5}$$

Substituting these equations in (10.3) in this case gives

$$\delta_{il}\xi_i^2 - 2u_i u_l(\xi_l^2 + \xi_i^2 + \xi_i\xi_l) + 4u_i u_l\left((\xi_i + \xi_l)\sum_{j=1}^{n} u_j^2\xi_j + \sum_{j=1}^{n} u_j^2\xi_j^2\right) - 8u_i u_l\left(\sum_{j=1}^{n} u_j^2\xi_j\right)^2 = \delta_{ij}\eta_i. \tag{10.6}$$

As a result, we obtain

$$-(\xi_l^2 + \xi_i^2 + \xi_i\xi_l) + 2\left((\xi_i + \xi_l)\sum_{j=1}^{n} u_j^2\xi_j + \sum_{j=1}^{n} u_j^2\xi_j^2\right) - 4\left(\sum_{j=1}^{n} u_j^2\xi_j\right)^2 = 0, \quad i \neq l, \tag{10.7}$$

for the off-diagonal components. Condition (10.4) then reduces to

$$\sum_{i=1}^{n} u_i^2 = 1. \tag{10.8}$$

The projector in form (10.5) is the only possible choice for the mixing matrix blocks of size $n = 2$. For $n = 3$, the case $\operatorname{rank}\mathcal{P} = 2$ is also possible, which is reducible to the preceding case using the relations

$$\mathcal{U} = 1 - 2\mathcal{P} = 2(\mathcal{P} - 1) - 1, \qquad (1 - \mathcal{P})^2 = 1 - \mathcal{P}, \quad \operatorname{rank}(1 - \mathcal{P}) = n - \operatorname{rank}\mathcal{P}.$$

We obtain explicit expressions for the $\mathcal{U}$-matrix for $n = 2, 3$ in Secs. 10.2.1 and 10.2.2. For $n \geq 4$ and $\operatorname{rank}\mathcal{P} = 1$, system of equations (10.7), (10.8) turns out to be incompatible, and the case $\operatorname{rank}\mathcal{P} = 2$ is not reducible to (10.5). Hence, there is no solution of form (10.5) for $n \geq 4$.

After the $\mathcal{U}$-matrices are obtained, the normalized eigenvalues η for the higher representations are immediately determined from (10.6). Another way is to determine η_i directly from (10.1) by supposing that $\mathcal{U}^\dagger = \mathcal{U}$. Indeed, introducing the notation $\widetilde{\mathcal{R}} \equiv \operatorname{Diag}\big(\mathcal{R}_{wx}\big)_{w\vdash x\otimes x}$, we rewrite (10.6) as $\mathcal{URURU} = \widetilde{\mathcal{R}}$. Consequently,

$$\mathcal{URU} = \widetilde{\mathcal{R}}\mathcal{U}\mathcal{R}^{-1}, \qquad \mathcal{URU} = \mathcal{R}^{-1}\mathcal{U}\widetilde{\mathcal{R}},$$

and hence $\widetilde{\mathcal{R}}\mathcal{U}\mathcal{R}^{-1} - \mathcal{R}^{-1}\mathcal{U}\widetilde{\mathcal{R}} = 0$.

If $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are renormalized according to (10.2), then the last equation rewritten in components is $(\xi_i^{-1}\eta_j - \xi_j^{-1}\eta_i)\mathcal{U}_{ij} = 0$, whence we have $\xi_i\eta_i = C$ if $\mathcal{U}_{ij} \neq 0$ for all i and j (which holds for all known $\mathcal{U}$-matrices). It follows from the relation for the determinants of the matrices in (10.1) that $C^3 = 1$. Moreover, we can set $C = 1$ because the common phase factor is inessential for determining the normalized eigenvalues. Therefore, $\eta_i = \xi_i^{-1}$. The same relation is obtained using the general formula $\mathcal{R}_{xy|Q} = q^{\varkappa_Q - \varkappa_x - \varkappa_y}$ in accordance with (3.1).

10.2.1. The 2×2 mixing matrices. Conditions (10.7) and (10.8) give two equations for u_1^2 and u_2^2, one quadratic and one linear:

$$
\begin{aligned}
&u_1^2\xi_1^4 + 2\xi_1\xi_2 u_1^2 u_2^2 + u_2^2\xi_2^4 - (2\xi_1^2 + \xi_1\xi_2)u_1^2 - (2\xi_2^2 + \xi_1\xi_2)u_1^2 + \xi_1^2 + \xi_1\xi_2 + \xi_2^2 = 0,\\
&u_1^2 + u_2^2 = 1.
\end{aligned}
$$

Eliminating u_2^2, we obtain $\big((\xi_1 - \xi_2)u_1^2 - \xi_1\big)^2 = 0$. Hence, system (10.7), (10.8) has a unique solution, and this solution is rational in u_1^2 and u_2^2. Substituting this solution in (10.4) yields the $\mathcal{U}$-matrix

$$
\begin{pmatrix} \mathcal{U}_{11} & \mathcal{U}_{12} \\ \mathcal{U}_{12} & -\mathcal{U}_{11} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\xi_2 - \xi_1} & \dfrac{\sqrt{\xi_1^2 + 1 + \xi_2^2}}{\xi_1 - \xi_2} \\ \dfrac{\sqrt{\xi_1^2 + 1 + \xi_2^2}}{\xi_1 - \xi_2} & \dfrac{1}{\xi_1 - \xi_2} \end{pmatrix}.
$$

This expression coincides with the one presented in [89].

10.2.2. The 3×3 mixing matrices. For $n > 2$, Eq. (10.7) gives not one but several relations (one for each off-diagonal element of the symmetric matrix), and the difference of any two of them gives a linear equation for u_i^2:

$$
-\big(\xi_l^2 - \xi_m^2 + \xi_i(\xi_l - \xi_m)\big) + 2(\xi_l - \xi_m)\sum_{j=1}^{n} u_j^2\xi_j = 0, \quad i \neq l \neq m.
$$

For $n = 3$, this equation leads to three different equations, each of which factors as

$$
\begin{aligned}
&i = 1,\ l = 2,\ m = 3: \quad (\xi_2 - \xi_3)(\xi_1 u_1^2 + \xi_2 u_2^2 + \xi_3 u_3^2 - \xi_1 - \xi_2 - \xi_3) = 0,\\
&i = 2,\ l = 1,\ m = 3: \quad (\xi_1 - \xi_3)(\xi_1 u_1^2 + \xi_2 u_2^2 + \xi_3 u_3^2 - \xi_1 - \xi_2 - \xi_3) = 0, \qquad (10.9)\\
&i = 3,\ l = 1,\ m = 2: \quad (\xi_2 - \xi_3)(\xi_1 u_1^2 + \xi_2 u_2^2 + \xi_3 u_3^2 - \xi_1 - \xi_2 - \xi_3) = 0.
\end{aligned}
$$

The factor depending on u (the second factor) coincides in all three expressions. This factor is linear in u_1^2, u_2^2, and u_3^2, and the same holds for (10.8), which for $n = 3$ has the form

$$
u_1^2 + u_2^2 + u_3^2 = 1. \qquad (10.10)
$$

Expressing u_2^2 and u_3^2 from (10.9) and (10.10) and substituting the results in (10.7) for $i = 1$ and $l = 2$, for example, we obtain

$$
\begin{aligned}
&\xi_1^2 u_1^4 + \xi_2^2 u_2^4 + \xi_3^2 u_3^4 + 2\xi_1\xi_2 u_1^2 u_2^2 + 2\xi_1\xi_3 u_1^2 u_3^2 + 2\xi_2\xi_3 u_2^2 u_3^2 - (2\xi_1^2 + \xi_1\xi_2)u_1^2 -\\
&\qquad - (2\xi_2^2 + \xi_1\xi_2)u_2^2 - (\xi_3^2 + \xi_1\xi_3 + \xi_2\xi_3)u_3^2 + (\xi_1^2 + \xi_2^2 + \xi_1\xi_2) = 0.
\end{aligned}
$$

All this leads to a linear equation for u_1^2:

$$
(\xi_1^2 - \xi_1\xi_2 - \xi_1\xi_3 + \xi_2\xi_3)u_1^2 - \xi_1^2 = 0.
$$

The solutions of this equation together with the corresponding solutions for u_2^2 and u_3^2 after substitution in (10.4) reproduce the formula in [89] for the 3×3 mixing block:

$$
\mathcal{U}_{11} = -\frac{\xi_1(\xi_2 + \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)}, \qquad \mathcal{U}_{22} = -\frac{\xi_2(\xi_1 + \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)}, \qquad \mathcal{U}_{33} = -\frac{\xi_3(\xi_1 + \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)},
$$

$$
\mathcal{U}_{12}^2 = -\frac{(\xi_1^2 - 1)(\xi_2^2 - 1)}{\xi_1\xi_2(\xi_1 - \xi_2)^2(\xi_1 - \xi_3)(\xi_2 - \xi_3)}, \qquad \mathcal{U}_{23}^2 = -\frac{(\xi_2^2 - 1)(\xi_3^2 - 1)}{\xi_2\xi_3(\xi_2 - \xi_3)^2(\xi_3 - \xi_2)(\xi_3 - \xi_1)},
$$

$$
\mathcal{U}_{13}^2 = -\frac{(\xi_1^2 - 1)(\xi_3^2 - 1)}{\xi_1\xi_3(\xi_1 - \xi_3)^2(\xi_1 - \xi_2)(\xi_3 - \xi_2)}.
$$

10.3. Eigenvalue conjecture in the case of different representations. Using the methods described in Sec. 10.1, we can try to generalize the eigenvalue conjecture to the case with different representations on the different link components. Such a generalization potentially leads to the possibility of calculating invariants of colored links by a direct application of the Reshetikhin–Turaev formalism. The generalized eigenvalue conjecture that we propose and illustrate below is that the mixing matrices for a given colored braid are completely determined by the normalized eigenvalues of those $\mathcal{R}$-matrices that can appear in the given braid.

10.3.1. Two different representations.

Conjecture 1. *Each of the two mixing matrices* $\mathcal{U}_{QSQ}$ *and* $\mathcal{U}_{QQS} = \mathcal{U}^\dagger_{SQQ}$ *is expressible in terms of the two sets of normalized eigenvalues of the two matrices* $\mathcal{R}_{QQ}$ *and* $\mathcal{R}_{SQ}$.

We can then write two relations instead of (10.1):

$$\mathcal{U}_{xyx}\mathcal{R}_{(yx)x}\mathcal{U}_{xxy}\mathcal{R}_{(xx)y}\mathcal{U}_{xxy} = \mathcal{R}_{wx}, \qquad \mathcal{U}_{xxy}\mathcal{R}_{(xy)x}\mathcal{U}_{xyx}\mathcal{R}_{(xy)x}\mathcal{U}_{yxx} = \mathcal{R}_{wy}. \tag{10.11}$$

But the first of them already suffices to determine both the mixing matrices and the eigenvalues in the right-hand side.

We present the form of the 2×2 mixing matrix. Writing $\mathcal{R}$ and $\mathcal{U}$ in matrix form and substituting them in (10.11), we obtain the two matrix equations

$$\begin{pmatrix} -c_x & s_x \\ s_x & c_x \end{pmatrix}\begin{pmatrix} \xi_{yx} & 0 \\ 0 & -\xi_{yx}^{-1} \end{pmatrix}\begin{pmatrix} -c_y & s_y \\ s_y & c_y \end{pmatrix}\begin{pmatrix} \xi_{yy} & 0 \\ 0 & -\xi_{yy}^{-1} \end{pmatrix}\begin{pmatrix} -c_y & s_y \\ s_y & c_y \end{pmatrix} = \begin{pmatrix} \xi_{wy} & 0 \\ 0 & -\xi_{wy}^{-1} \end{pmatrix},$$

$$\begin{pmatrix} -c_y & s_y \\ s_y & c_y \end{pmatrix}\begin{pmatrix} \xi_{xy} & 0 \\ 0 & -\xi_{xy}^{-1} \end{pmatrix}\begin{pmatrix} -c_x & s_x \\ s_x & c_x \end{pmatrix}\begin{pmatrix} \xi_{yx} & 0 \\ 0 & -\xi_{yx}^{-1} \end{pmatrix}\begin{pmatrix} -c_y & s_y \\ s_y & c_y \end{pmatrix} = \begin{pmatrix} \xi_{wx} & 0 \\ 0 & -\xi_{wx}^{-1} \end{pmatrix}.$$

It turns out to be quite simple to solve this system. The compatibility condition for the off-diagonal components of the first equation is reducible to an equation linear in c_y^2 and $s_y^2 = 1 - c_y^2$. We can find c_y^2 from this equation. After c_y^2 is found, we can obtain c_x and s_x as solutions of the corresponding degenerate linear system. Substituting the obtained c_x^2 and c_y^2 back in (10.11), we find ξ_{xx} and ξ_{xy}. The solutions thus obtained are

$$\begin{aligned} c_x^2 &= \frac{\xi_{xx}^2 - \xi_{xy}^4}{(1-\xi_{xy}^4)(1+\xi_{xx}^2)}, & c_y^2 &= \frac{\xi_{xy}^2(1-\xi_{xx}^2)^2}{\xi_{xx}^2(1-\xi_{yx}^4)^2}, \\ s_x^2 &= \frac{1-\xi_{xx}^2\xi_{xy}^4}{(1-\xi_{xy}^4)(1+\xi_{xx}^2)}, & s_y^2 &= \frac{(\xi_{xx}^2-\xi_{yx}^4)(1-\xi_{xy}^4\xi_{xx}^2)}{\xi_{xx}^2(1-\xi_{yx}^4)^2}, \\ \xi_{wx} &= -\frac{1}{\xi_{yx}}, & \xi_{wy} &= -\frac{1}{\xi_{xx}}. \end{aligned}$$

10.3.2. Three different representations.

Conjecture 2. *Each of the three mixing matrices* $\mathcal{U}_{zxy} = \mathcal{U}^\dagger_{yxz}$, $\mathcal{U}_{xzy} = \mathcal{U}^\dagger_{yxz}$, *and* $\mathcal{U}_{zyx} = \mathcal{U}^\dagger_{xyz}$ *is expressible in terms of normalized eigenvalues of the three matrices* $\mathcal{R}_{yz}$, $\mathcal{R}_{xz}$, *and* $\mathcal{R}_{yx}$.

In this case, we have three equations instead of one (10.1):

$$\begin{aligned}
&\mathcal{U}_{xyz}\mathcal{R}_{(yz)x}\mathcal{U}_{xzy}\mathcal{R}_{(xz)y}\mathcal{U}_{zxy} = \mathcal{R}_{wz},\\
&\mathcal{U}_{xzy}\mathcal{R}_{(zy)x}\mathcal{U}_{xyz}\mathcal{R}_{(xy)z}\mathcal{U}_{yxz} = \mathcal{R}_{wy},\\
&\mathcal{U}_{zyx}\mathcal{R}_{(yx)z}\mathcal{U}_{zxy}\mathcal{R}_{(zx)y}\mathcal{U}_{xzy} = \mathcal{R}_{wx}.
\end{aligned} \tag{10.12}$$

In fact, any two of them suffice to determine all the mixing matrices and eigenvalues in the right-hand side.

We write the 2×2 mixing matrices. The first equation in (10.12) in this case becomes

$$\begin{pmatrix} -c_y & s_y \\ s_y & c_y \end{pmatrix}\begin{pmatrix} \xi_{zy} & 0 \\ 0 & -\xi_{zy}^{-1} \end{pmatrix}\begin{pmatrix} -c_z & s_z \\ s_z & c_z \end{pmatrix}\begin{pmatrix} \xi_{xz} & 0 \\ 0 & -\xi_{xz}^{-1} \end{pmatrix}\begin{pmatrix} -c_x & s_x \\ s_x & c_x \end{pmatrix} = \begin{pmatrix} \xi_{wz} & 0 \\ 0 & -\xi_{wz}^{-1} \end{pmatrix}. \tag{10.13}$$

In all, we have four equations for the off-diagonal components of two equations in system (10.12). We can find the solutions of this system by studying the compatibility conditions for these four equations (these conditions are reducible to linear equations for squared parameters of the mixing matrices). The solution has the form

$$c_z^2 = \frac{(\xi_{xz}^2 - \xi_{xy}^2\xi_{zy}^2)(\xi_{zy}^2 - \xi_{xz}^2\xi_{xy}^2)}{\xi_{xy}^2(1-\xi_{zy}^4)(1-\xi_{xz}^4)}, \qquad s_z^2 = \frac{(\xi_{xy}^2 - \xi_{xz}^2\xi_{zy}^2)(1 - \xi_{zy}^2\xi_{xz}^2\xi_{xy}^2)}{\xi_{xy}^2(1-\xi_{zy}^4)(1-\xi_{xz}^4)}, \qquad \xi_{wz} = -\frac{1}{\xi_{yx}}.$$

The signs of the solutions c_z and s_z can be recovered by substituting them in (10.13).

10.4. Blocks in the nondiagonal $\mathcal{R}$-matrices. Following the approach in Secs. 10.1–10.3, we can also determine the form of $\mathcal{R}$-matrix blocks, which were presented in Sec. 3.2. From the consistency condition for the cabling procedure, we can derive the relations directly for the elements of the R-matrices:

$$\mathrm{Diag}(\mathcal{R}^2_{ux|w})_{\substack{w\vdash u\otimes x,\\ w\otimes x\dashv Q}} = \mathcal{R}_{x^{m+1}\otimes x}\mathcal{R}_{x\otimes x^{m+1}}. \tag{10.14}$$

It then follows from relation (10.1) and coproduct formula (9.3) that

$$\begin{aligned}
\mathrm{Diag}(\mathcal{R}^2_{ux|w})_{\substack{w\vdash u\otimes x,\\ w\otimes x\dashv Q}} &= \mathcal{R}_{x^m\otimes(x\otimes x)}\,\mathcal{R}_{(x^m\otimes x)\otimes x}\,\mathcal{R}_{(x\otimes x^m)\otimes x}\,\mathcal{R}_{x^m\otimes(x\otimes x)} =\\
&= \mathcal{R}_{x^m\otimes(x\otimes x)}\,\mathrm{Diag}(\mathcal{R}^2_{wx|Q})_{\substack{w\vdash u\otimes x,\\ w\otimes x\dashv Q}}\mathcal{R}_{x^m\otimes(x\otimes x)},
\end{aligned}$$

where u is an irreducible representation such that $u \vdash x^{m-1}$ and $Q \vdash u\otimes x^2$. Different u never appear in the same mixing block because of the properties of $\mathcal{R}$-matrices. The size n of a nondiagonal block is equal to the number of representations w that satisfy both $w \vdash u\otimes x$ and $w\otimes x \dashv Q$. Below, we find the explicit form of the blocks in nondiagonal $\mathcal{R}$ matrices in two cases where the mixing blocks are of size $n = 2$: for the fundamental representation $x = \Box$ and in the case where the representations x, u, Q, and hence all w are described by hook diagrams.

Introducing the notation r_{11}, r_{12}, and r_{22} for the elements of the unknown block and λ_1 and λ_2 for the corresponding eigenvalues of $\mathcal{R}_{ux|w}$, we can rewrite (10.14) for the 2×2 block as

$$\begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix}\begin{pmatrix} \lambda_1^2 & 0 \\ 0 & \lambda_2^2 \end{pmatrix}\begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix} = \begin{pmatrix} \tilde\lambda_1^2 & 0 \\ 0 & \tilde\lambda_2^2 \end{pmatrix}.$$

The equations for the off-diagonal elements are written as $r_{12}(\lambda_1^2 r_{11} + \lambda_2^2 r_{22}) = 0$. We also use the obvious property of the 2×2 matrix

$$r_{11} + r_{22} = \mu_1 + \mu_2, \qquad r_{11}r_{22} - r_{12}^2 = \mu_1\mu_2,$$

where μ_1 and μ_2 are the eigenvalues of the nondiagonal block, which we assume to be known. The solutions of these equations are

$$r_{11} = -\frac{(\mu_1+\mu_2)\lambda_2^2}{\lambda_1^2-\lambda_2^2}, \qquad r_{22} = -\frac{(\mu_1+\mu_2)\lambda_1^2}{\lambda_1^2-\lambda_2^2}, \qquad r_{12}^2 = \frac{(\mu_1\lambda_1^2+\mu_2\lambda_2^2)(\mu_1\lambda_2^2+\mu_2\lambda_1^2)}{(\lambda_1^2-\lambda_2^2)^2}. \tag{10.15}$$

We consider the fundamental representation. If $x = \Box$, then representation Q is obtained from u by adding the pair of boxes (i_1, j_1) and (i_2, j_2). Consequently, there are two possible representations w obtained from u by adding one or the other of these boxes. In accordance with general formula (3.1), we have

$$\mu_1 = q, \qquad \mu_2 = -q^{-1}, \qquad \lambda_{1,2} = q^{\varkappa_{w_{1,2}} - \varkappa_u} = q^{j_{1,2}-i_{1,2}}.$$

Substituting these expressions in (10.15) gives the formula in Sec. 3.2 for the blocks in the fundamental R-matrices:

$$\mathcal{R}_{1^m\otimes(1\otimes1)|Q} = \begin{pmatrix} -\dfrac{q-q^{-1}}{q^n(q^n-q^{-n})} & \dfrac{\sqrt{(q^{n+1}-q^{-n-1})(q^{n-1}-q^{1-n})}}{q^n-q^{-n}} \\ \dfrac{\sqrt{(q^{n+1}-q^{-n-1})(q^{n-1}-q^{1-n})}}{q^n-q^{-n}} & \dfrac{q^n(q-q^{-1})}{q^n-q^{-n}} \end{pmatrix},$$

where $Q = w_1 \cup (i_1, j_1) = w_2 \cup (i_2, j_2)$ and $n = j_2 - i_2 + j_1 - i_1$.

We now consider hook-type representations. If $x = [r_1, 1^{s_1}]$ and $u = [r_2, 1^{s_2}]$ are hook-type representations, then Q can be one of three hook-type representations because

$$\begin{aligned}
&[r_2, 1^{s_2}] \times [r_1, 1^{s_1}] \times [r_1, 1^{s_1}] = \\
&\qquad = ([r_2 + r_1, 1^{s_2+s_1}] + [r_2 + r_1 - 1, 1^{s_2+s_1+1}] + \cdots) \times [r_1, 1^{s_1}] = \\
&\qquad = [2r_1 + r_2, 1^{2s_1+s_2}] + 2[2r_1 + r_2 - 1, 1^{2s_1+s_2+1}] + [2r_1 + r_2 - 2, 1^{2s_1+s_2+2}] + \ldots,
\end{aligned}$$

where the ellipsis denotes representations not having the hook form. The three possible hook-type representations are

$$Q = [2r_1 + r_2, 1^{2s_1+s_2}], \qquad Q' = [2r_1 + r_2 - 1, 1^{2s_1+s_2+1}], \qquad Q'' = [2r_1 + r_2 - 2, 1^{2s_1+s_2+2}],$$

where the first and the last representations are singlets, the second is a doublet, and the intermediate representations are

$$w_1 = [r_1 + r_2, 1^{s_1+s_2}], \qquad w_2 = [r_1 + r_2 - 1, 1^{s_1+s_2+1}].$$

Because any path that ends at the hook-type representation Q goes only through hook-type representations, the corresponding matrix block $\mathcal{R}_{x^m\otimes x\otimes x}$ corresponds to one of the two representations y arising in the decomposition of $x \otimes x$:

$$y_1 = [2r_1, 1^{2s_1}], \qquad y_2 = [2r_1 - 1, 1^{2s_1+1}].$$

For these representations, eigenvalues in (10.15) are given by (3.1):

$$\mu_1 = q^{\varkappa_{y_1}-\varkappa_x} = q^{r_1(2r_1-1)-s_1(2s_1+1)} \equiv q^{r_1+s_1}\mu, \qquad \mu_2 = -q^{\varkappa_{y_2}-\varkappa_x} = -q^{-r_1-s_1}\mu,$$

$$\lambda_1 = q^{\varkappa_{w_1}-\varkappa_u} \equiv q^{r_2+s_2}\lambda, \qquad \lambda_2 = q^{\varkappa_{w_2}-\varkappa_u} = q^{-r_2-s_2}\lambda.$$

Substituting these eigenvalues in (10.15), we obtain the explicit form of the blocks:

$$\mathcal{R}_{x^m\otimes(x\otimes x)|Q} = q^{2r_1(r_1-1)-2s_1(s_1+1)} \times$$
$$\times \begin{pmatrix} -\dfrac{q^r-q^{-r}}{q^{rn}(q^{rn}-q^{-rn})} & \dfrac{\sqrt{(q^{rn+r}-q^{-rn-r})(q^{rn-r}-q^{r-rn})}}{q^{rn}-q^{-rn}} \\ \dfrac{\sqrt{(q^{rn+r}-q^{-rn-r})(q^{rn-r}-q^{r-rn})}}{q^{rn}-q^{-rn}} & \dfrac{q^{rn}(q^r-q^{-r})}{q^{rn}-q^{-rn}} \end{pmatrix}, \qquad (10.16)$$

where $n = s_2 - r_2 + 1$, $r = r_1 + s_1$, $x = [r_1, 1^{s_1}]$, and $Q = [2r_1 + r_2 - 1, 1^{2s_1+s_2+1}]$.

11. Special polynomials

We consider special polynomials describing the double scaling limit of HOMFLY polynomials at large N and small $\hbar$ (i.e., $q \equiv q^{\hbar} \to 1$ and $A = q^N = \text{const}$). The series into which a HOMFLY polynomial expands in this limit is called a genus expansion [95], [96]. But we are here more interested in the zeroth-order term of this expansion, which is called a special polynomial [61],

$$\sigma_Q^{\mathcal{K}}(A) = H_Q^{\mathcal{K}}(A,q)\big|_{A=q^N=\text{const},\, q=1}.$$

The behavior of a HOMFLY polynomial in this limit, obtained in [61] and studied in [95], [96], [102]–[108], relates fundamental and colored polynomials:

$$\sigma_Q^{\mathcal{K}} = \sigma_1^{\mathcal{K}^{|Q|}} = (\sigma_q^K)^{|Q|}. \qquad (11.1)$$

This relation is called the factorization property. It follows from the basic properties of the HOMFLY polynomials: from the skein relations, shown in Fig. 7, from the existence of the cabling procedure, from the factorization of a HOMFLY polynomial for a split union of knots or links, and from the theorem that the fundamental unreduced HOMFLY polynomial (not divided by an unknot) for an n-component link diverges as $(q-q^{-1})^{-n}$ in the $q{\to}1$ limit [20].

It follows from the cabling procedure that the colored HOMFLY polynomial in representation Q diverges in the limit $q \to 1$ as the fundamental HOMFLY polynomial for the $|Q|$-component link, i.e., as $(q-q^{-1})^{-|Q|}$. Indeed, there cannot be more than $|Q|$ components, and a link with $|Q|$ components is always obtained. Hence, $\sigma_Q^{\mathcal{K}} \sim (q-q^{-1})^{-|Q|}$. Further, using the skein relations, we can transform the $|Q|$-component link into the split union of its $|Q|$ components. In this transformation, the $q{\to}1$ limit of the HOMFLY polynomial does not change. Indeed, if the skein relations are applied to a crossing of two different components, then the term corresponding to resolving this crossing contains one connected component fewer and hence diverges as $(q-q^{-1})^{-|Q|+1}$. With the coefficient $(q-q^{-1})^{-1}$ before this term taken into account, the corresponding expression diverges as $(q-q^{-1})^{-|Q|+2}$, i.e., is less singular as $q \to 1$ than the two remaining terms and can be set to zero. As a result, we obtain an identity meaning that the HOMFLY polynomials of two links with their linking numbers differing by unity are equal in the limit $q \to 1$. Consequently, $\sigma_1^{\mathcal{K}^{|Q|}} = (\sigma_q^K)^{|Q|}$. We note that the variable A does not appear in the skein relations in the vertical framing and therefore cannot appear explicitly in relation (11.1).

Our arguments relate to HOMFLY polynomials calculated in the vertical framing (see Sec. 5). But it is easy to show that identity (11.1) also holds in the topological framing.

12. Alexander polynomials

We consider Alexander polynomials, which are HOMFLY polynomials in the limit $A \to 1$,

$$A_Q^{\mathcal{K}}(q) = H_Q^{\mathcal{K}}(A,q)\big|_{A=q^N=1,q=\text{const}}.$$

For representations Q corresponding to hook-type diagrams, there is a conjecture describing the dependence of an Alexander polynomial on the representation [61], [87]. This conjecture asserts that

$$A_Q^{\mathcal{K}}(q) = A_1^{\mathcal{K}}(q^{|Q|}). \tag{12.1}$$

This conjecture can be proved by a method analogous to the method described in Sec. 3. The answers for the R-matrices presented in that section are generalizable to hook-type diagrams.

In the limit $A \to 1$, only the factor $A - A^{-1}$ in the Schur polynomials is essential. The degree of this term in S_Q^* is equal to the number of hooks in the diagram Q. This means that if we calculate the polynomial in the representation with the hook-type diagram $[r, 1^s]$, then all the diagrams in the character expansion for HOMFLY polynomials that are not hook-type diagrams do not contribute to the answer for the reduced HOMFLY polynomial in the limit $A \to 1$. For all the diagrams appearing in the answer, the $\mathcal{R}$-matrices consist of blocks of a size not exceeding 2×2 (see Sec. 10.4), similarly to what happens in the fundamental case. These blocks have form (10.16), and it can be directly verified that they satisfy the following relation with respect to the substitution $q \to q^{r+s}$:

$$\mathcal{R}^i_{\square} \xrightarrow{q \to q^{r+s}} \mathcal{R}^i_{[r,1^s]}.$$

This means that property (12.1) is satisfied for the individual coefficients of the Schur polynomials in character expansion (1.2). Moreover, it can be verified that the characters (Schur polynomials) themselves satisfy the needed relation in the limit $A \to 1$. Consequently, the Alexander polynomials must also satisfy (12.1).

13. Conclusion

We have considered a method for calculating colored HOMFLY polynomials using the HOMFLY polynomials in the fundamental representation. This method is called the cabling procedure and consists of three steps. The first step is to construct the cabled knot from the initial knot. This step can be clarified by a simple picture (see Sec. 2). The second step is to find the projector, which describes the combination of fundamental HOMFLY polynomials into which the colored HOMFLY polynomial decomposes. The third step is to calculate the colored HOMFLY polynomials.

To calculate the HOMFLY polynomials in the fundamental representation in the framework of the Reshetikhin–Turaev approach (the approach we used), we must calculate the $\mathcal{R}$-matrices in the fundamental representation. The corresponding expressions are described by a quite simple path formula (see Sec. 3). The obtained results allow constructing the HOMFLY polynomial of an arbitrary knot. The basic problem is that the complexity of the calculation of a knot polynomial increases rapidly as the minimum number of strands in the braid representing the knot increases.

We can also construct the matrix of the projector onto an arbitrary representation in terms of the path (see Sec. 4). Although the obtained answer describes only the projector placed in the first cable, it generally suffices for calculating the colored HOMFLY polynomial for an arbitrary knot in an arbitrary representation. The basic problem is again the complexity of the calculation. Because a colored HOMFLY polynomial in a representation Q is expressed in the cabling procedure in terms of HOMFLY polynomials for knots containing $|Q|$ times more strands than in the braid representation of the initial knot, the calculations become much more cumbersome in the case of colored polynomials. Currently, explicit calculations have been done up to 12 strands in the braid. This allowed finding colored HOMFLY polynomials in various representations for a series of knots and links. The answers were presented in Appendices C, D, and E in [98].

We discussed the description of the cabling procedure in the language of representation theory. This description allows obtaining the form of $\mathcal{R}$-matrices in the fundamental representation in Sec. 3. Other applications of the cabling procedure include explaining the conjectures formulated in previous papers: the eigenvalue conjecture [89] and the dependence of the Alexander and special polynomials on the representation [61].

Although there is a method that in principle allows constructing an arbitrary colored HOMFLY polynomial, much remains to be studied. The internal structure of HOMFLY polynomials is still not clarified. There are several examples where a general formula (or at least more general than one concrete HOMFLY polynomial) is already known: torus knots [58]–[61], twist knots [88], [49], and double-braid knots [93]. But the majority of examples are completely mysterious. Obtaining general formulas for a series of knots and representations would be extremely useful not only for investigating the HOMFLY polynomials themselves but also for the studying some related topics such as difference equations and τ-functions [71]–[77], [85], [95], [96], superpolynomials [54]–[57], [61]–[70] and Khovanov homologies [109]–[115].

From the standpoint of obtaining general formulas for HOMFLY polynomials, the eigenvalue conjecture [89] may turn out to be very fruitful. This conjecture in fact asserts that a HOMFLY polynomial is fully described by the $\mathcal{R}$-matrix eigenvalues although several $\mathcal{R}$-matrices that do not mutually commute are involved in the direct calculation. The description of the eigenvalue conjecture in the language of the cabling procedure (Sec. 10) is a possible means for studying this question further.

Another possible direction of research is the complexification of the topology of the space (see, e.g., [100] and the references therein). Nearly all the known answers relate to the theory on the S^3 manifold, and almost nothing is known about the case $S^1 \times S^2$ next in complexity.

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