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DARBOUX TRANSFORMATIONS AND RECURSION OPERATORS FOR DIFFERENTIAL–DIFFERENCE EQUATIONS

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We review two concepts directly related to the Lax representations of integrable systems: Darboux transformations and recursion operators. We present an extensive list of integrable differential–difference equations with their Hamiltonian structures, recursion operators, nontrivial generalized symmetries, and Darboux–Lax representations. The new results include multi-Hamiltonian structures and recursion operators for integrable Volterra-type equations and integrable discretizations of derivative nonlinear Schrödinger equations such as the Kaup–Newell, Chen–Lee–Liu, and Ablowitz–Ramani–Segur (Gerdjikov–Ivanov) lattices. We also compute the weakly nonlocal inverse recursion operators.

Keywords: symmetry, recursion operator, bi-Hamiltonian structure, Darboux transformation, Lax representation, integrable equation

1. Introduction

Our aim in this paper is to give a comprehensive account of multi-Hamiltonian structures, recursion operators, and Darboux–Lax representations for a wide class of integrable differential–difference equations. Some of these results are well known but scattered in the literature. In many cases, we have completed the picture by providing explicit expressions for the Hamiltonian, symplectic, and recursion operators and Darboux–Lax representations. The Lax representations of nonlinear differential and difference equations play a central role in the theory of integrable systems. They allow using the inverse scattering transform to construct exact solutions and study the asymptotics of the initial value problem. Moreover, they allow constructing a recursion operator, which generates infinite hierarchies of symmetries and conservation laws. Currently, there is not any general method for finding the Lax representation for a given equation. The most successful approach is the Wahlquist–Estabrook prolongation procedure [1]. Mikhailov and coauthors recently tackled this problem from a different angle [2]–[5]. They studied possible reductions of a general Lax representation using the reduction group approach [6]–[8] and further leading to a classification of Lax representations and the corresponding integrable equations.

The concept of Darboux transformations originated from classical differential geometry [9]. Applying Darboux transformations to the corresponding Lax representation leads to Bäcklund transformations and to generation of new exact solutions for the integrable system. Bäcklund transformations can be regarded as an integrable system of differential–difference equations in their own right. These differential–difference equations play the role of infinitesimal symmetries for the integrable partial difference equations that can be obtained from the condition of the Bianchi commutativity of the Darboux transformations.

Differential–difference systems are the main object of our study. Our notation is standard. We illustrate the notation with the well-known example of the Volterra equation [10] (see Sec. 4.1 for more algebraic

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properties of this equation), which we can write in the form

$$u_t = u(u_1 - u_{-1}). \tag{1}$$

Here, we assume that the dependent variable u is a function $u(n,t)$ of a lattice variable $n \in \mathbb{Z}$ and a continuous variable t, and we set

$$u_t = \partial_t u(n,t), \qquad u_j = u(n+j,t).$$

We omit the subscript index zero and use u instead of u_0. Volterra equation (1) encodes an infinite sequence of differential equations

$$\partial_t u(n,t) = u(n,t)(u(n+1,t) - u(n-1,t)), \quad n \in \mathbb{Z}.$$

Equation (1) has an infinite hierarchy of symmetries; in other words, it is compatible with an infinite sequence of evolutionary equations of the form

$$u_{t_m} = K_m(u_m, u_{m-1}, \dots, u_{1-m}, u_{-m}), \quad m \in \mathbb{N},$$

where $t_1 = t$, $K_1 = u(u_1 - u_{-1})$, $K_2 = u(u_1 u_2 + u_1^2 + u u_1 - u u_{-1} - u_{-1}^2 - u_{-1} u_{-2})$, and K_m are certain polynomials in the variables $u_m, u_{m-1}, \dots, u_{1-m}, u_{-m}$. The compatibility means that $\partial_{t_m}(\partial_t u) = \partial_t(\partial_{t_m} u)$ or vanishing of the Lie bracket $[K_1, K_m]$, defined as

$$[K_1, K_m] := K_{m\star}(K_1) - K_{1\star}(K_m),$$

where $K_{m\star}$ denotes the Fréchet derivative

$$K_{m\star} = \sum_{p=-m}^{m} \frac{\partial K_m}{\partial u_p} \mathcal{S}^p,$$

and $\mathcal{S}$ denotes the shift operator such that $\mathcal{S}^j(u_k) = u_{k+j}$ and we have $\mathcal{S}^j(f(u_{k_1}, \dots, u_{k_2})) = f(u_{k_1+j}, \dots, u_{k_2+j})$ for any function $f(u_{k_1}, \dots, u_{k_2})$.

Symmetries of the Volterra equation can be generated by the recursion operator

$$\mathcal{R} = u\mathcal{S} + u + u_1 + u\mathcal{S}^{-1} + u(u_1 - u_{-1})(\mathcal{S} - 1)^{-1}\frac{1}{u}, \tag{2}$$

namely,

$$K_{m+1} = \mathcal{R}^m(K_1).$$

The vector fields corresponding to the Volterra equation and its symmetries are difference polynomials, i.e., elements of the difference polynomial ring $R = [\mathbb{C}, u, \mathcal{S}]$, which is a ring without zero divisors over the field of complex numbers $\mathbb{C}$ with an infinite number of indeterminates u_k, $k \in \mathbb{Z}$, and equipped with the automorphism $\mathcal{S}$. The corresponding field of fractions $\mathcal{F} = (\mathbb{C}, u, \mathcal{S})$ is a difference field of rational functions of indeterminates u_k over $\mathbb{C}$, which inherits the automorphism $\mathcal{S}$. Difference operators are defined as finite sums of the form $\sum a_k \mathcal{S}^k$, where $a_k \in \mathcal{F}$ (the above defined Fréchet derivative $K_{m\star}$ is an example of a difference operator). Difference operators act naturally on elements of $\mathcal{F}$. As in the differential case, the elements of the ring R, field $\mathcal{F}$, and difference operators are respectively called local polynomials, local functions, and local operators.

The recursion operator $\mathcal{R}$ given by (2) is not a local operator. It contains a local part and the term $u(u_1 - u_{-1})(\mathcal{S}-1)^{-1}u^{-1}$. The action of $\mathcal{R}$ can be defined on those elements of $\mathcal{F}$ that belong to the image of the difference operator $u(\mathcal{S}-1)$. We can directly verify that $K_1, K_2 \in \operatorname{Im}(u(\mathcal{S}-1))$. Similarly to [11], it can be shown that all $K_{m+1} = \mathcal{R}^m(K_1)$, $m \in \mathbb{N}$, are difference polynomials. The action of $\mathcal{R}$ is not uniquely defined on the elements of $\operatorname{Im}(u(\mathcal{S}-1))$. Indeed, the base field $\mathbb{C}$ is the kernel space of the difference operator $u(\mathcal{S}-1)$; therefore, for $a = u(\mathcal{S}-1)(b)$, $b \in \mathcal{F}$, we have $(\mathcal{S}-1)^{-1}(u^{-1}a) = b + \alpha$, where $\alpha \in \mathbb{C}$ is an arbitrary constant. Obviously, acting on K_1 with the recursion operator $\mathcal{R}$ defined by (2), we obtain

$$\mathcal{R}(u(u_1 - u_{-1})) = u(u_1u_2 + u_1^2 + uu_1 - uu_{-1} - u_{-1}^2 - u_{-1}u_{-2}) + \alpha u(u_1 - u_{-1}).$$

The action of $\mathcal{R}$ is well defined on the sequence of the quotient linear spaces

$$\mathcal{K}_1 = \operatorname{Span}_{\mathbb{C}}(K_1), \qquad \mathcal{K}_m = \operatorname{Span}_{\mathbb{C}}(K_1, K_2, \dots, K_m) / \operatorname{Span}_{\mathbb{C}}(K_1, K_2, \dots, K_{m-1}),$$

and describing the result of the action of a recursion operator on a symmetry in what follows, we give one representative from the corresponding coset.

The recursion operator $\mathcal{R}$ given by (2) is a pseudodifference operator. It can be represented in the form $\mathcal{R} = BA^{-1}$, where A and B are difference operators. For example, we can take $A = \mathcal{H}_1$ and $B = \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hamiltonian operators for the Volterra equation (see Sec. 4.1). We note that the pseudodifference operator $\mathcal{R}$ is a sum of a local (difference) operator and a nonlocal term of the form $P(\mathcal{S}-1)^{-1}Q$. We say that a pseudodifference operator is *weakly nonlocal* if it can be represented in the form

$$A + \sum_i P_i(\mathcal{S}-1)^{-1}Q_i,$$

where A, P_i, and Q_i are difference operators and the sum is finite (a similar terminology was first introduced in the study of pseudodifferential Hamiltonian operators [12]). Therefore, the recursion operator $\mathcal{R}$ given by (2) is a weakly nonlocal pseudodifference operator. Moreover, it is easy to show that $\mathcal{R}^m$, $m \in \mathbb{N}$ is a weakly nonlocal operator. In the majority of cases studied in this paper, the recursion operators are weakly nonlocal. Exceptions include the recursion operator for the Narita–Itoh–Bogoyavlensky lattice [13] (see Sec. 4.5).

We note that in the case of multicomponent systems of integrable difference equations with weakly nonlocal recursion operators, surprisingly, the inverse recursion operator is often also weakly nonlocal. This allows generating infinitely many local symmetries corresponding to the inverse flows. For example, the Heisenberg ferromagnet lattice [14] (see Sec. 4.17 for more algebraic properties of this equation)

$$\begin{aligned} u_t &= (u-v)(u-u_1)(u_1-v)^{-1}, \\ v_t &= (u-v)(v_{-1}-v)(u-v_{-1})^{-1} \end{aligned}$$

has the recursion operator

$$\mathcal{R} = \begin{pmatrix} \dfrac{(u-v)^2}{(u_1-v)^2}\mathcal{S} - \dfrac{2(u-u_1)(v-v_{-1})}{(u-v_{-1})(u_1-v)} & -\dfrac{(u-u_1)^2}{(u_1-v)^2} \\ \dfrac{(v-v_{-1})^2}{(u-v_{-1})^2} & \dfrac{(u-v)^2}{(u-v_{-1})^2}\mathcal{S}^{-1} \end{pmatrix} + 2K^{(1)}(\mathcal{S}-1)^{-1}Q^{(1)},$$

where

$$K^{(1)} = \begin{pmatrix} \dfrac{(u-v)(u-u_1)}{u_1-v} \\ \dfrac{(u-v)(v_{-1}-v)}{u-v_{-1}} \end{pmatrix}, \qquad Q^{(1)} = \left(\frac{v-v_{-1}}{(u-v)(u-v_{-1})}, \quad \frac{u-u_1}{(u-v)(u_1-v)} \right).$$

The operator $\mathcal{R}$ is weakly nonlocal and has a weakly nonlocal inverse

$$\mathcal{R}^{-1}=\begin{pmatrix} \dfrac{(u-v)^2}{(u_{-1}-v)^2}\mathcal{S}^{-1} & \dfrac{(u-u_{-1})^2}{(u_{-1}-v)^2} \\ -\dfrac{(v-v_1)^2}{(u-v_1)^2} & \dfrac{(u-v)^2}{(u-v_1)^2}\mathcal{S}-\dfrac{2(u-u_{-1})(v-v_1)}{(u-v_1)(u_{-1}-v)} \end{pmatrix}-2K^{(-1)}(\mathcal{S}-1)^{-1}Q^{(-1)},$$

where

$$K^{(-1)}=\begin{pmatrix} \dfrac{(u-v)(u_{-1}-u)}{u_{-1}-v} \\ \dfrac{(u-v)(v-v_1)}{u-v_1} \end{pmatrix}, \qquad Q^{(-1)}=\left(\frac{v-v_1}{(u-v)(u-v_1)},\quad \frac{u-u_{-1}}{(u-v)(u_{-1}-v)}\right).$$

Hence, the Heisenberg ferromagnet lattice has infinitely many local symmetries $\mathcal{R}^l(K^{(1)})$ and $\mathcal{R}^{-l}(K^{(-1)})$ for all $l \in \mathbb{N}$. This phenomenon was explored for the Ablowitz–Ladik lattice and the Bruschi–Ragnisco lattice in [15]. Here, we compute the weakly nonlocal inverse recursion operators for all multicomponent integrable differential–difference equations if they exist. Such inverses do not exist for scalar nonlinear integrable differential–difference equations. For a given weakly nonlocal difference operator, whether there exists a weakly nonlocal inverse operator is still an open problem.

This paper is arranged as follows. We first review two closely related topics concerning Lax representations: the Darboux transformations of the Lax representation, from which we derive the integrable differential–difference equations, and the derivation of the recursion operator for the resulting equations using the Darboux transformation. We illustrate the methods with two typical examples: the well-known nonlinear Schrödinger (NLS) equation and a deformation of the derivative NLS equation corresponding to the dihedral reduction group $\mathbb{D}_2$.

We complete the paper with a long list of integrable differential–difference equations, where we list the equations themselves, their Hamiltonian structures, recursion operators, nontrivial generalized symmetries, and Lax representations. We also include partial results on their master symmetries. For some equations, we add further notes concerning the links with other known equations and the weakly nonlocal inverses of recursion operators if they exist. The list is far from complete. A similar list for 1+1 integrable evolutionary equations can be found in [16].

We mainly rely on sources with results about integrable systems useful for our work although we also made some attempts to find the original contributions. In compiling the list, we verified the objects collected from the vast literature and made them consistent. Our list also includes several new results (to the best of our knowledge):

1. the Hamiltonian operators, symplectic operators, and recursion operators (Sec. 4.4) for Eqs. (68)–(70), and the relations between them,
2. the Hamiltonian operators, symplectic operators, and recursion operators for the Kaup–Newell lattice (Sec. 4.14), the Chen–Lee–Liu lattice (Sec. 4.15), and the Ablowitz–Ramani–Segur (Gerdjikov–Ivanov) lattice (Sec. 4.16), and
3. all weakly nonlocal inverse recursion operators if they exist (except those, already known, for the Ablowitz–Ladik and Bruschi–Ragnisco lattices).

2. Lax representations and Darboux matrices

With a system of evolutionary nonlinear partial differential equations

$$\mathbf{u}_t=\mathbf{F}(\mathbf{u},\mathbf{u}_x,\dots,\mathbf{u}_{x,\dots,x}), \quad \mathbf{u}\in\mathbb{C}^m, \tag{3}$$

solvable by the spectral transform method [17]–[19], we associate a pair of linear operators

$$L = D_x - U(\mathbf{u};\lambda), \qquad A = D_t - V(\mathbf{u};\lambda),$$

which is conventionally called the Lax pair. Here, U and V are square matrices whose elements are functions of the dependent variable $\mathbf{u}$ and its x-derivatives and certain rational (in some cases elliptic) functions of the spectral parameter λ such that Eq. (3) is equivalent to the commutativity condition for these operators

$$[L, A] = D_t(U) - D_x(V) + [U, V] = 0. \tag{4}$$

This equation is often called a zero-curvature representation or Lax representation of Eq. (3). Here, we mainly consider U and V 2×2 matrices whose elements are rational in the spectral parameter λ.

Generally speaking, symmetries of an evolutionary equation are its compatible evolutionary equations. Integrable equation (3) has an infinite sequence of commuting symmetries

$$\mathbf{u}_{t_k} = \mathbf{F}^k(\mathbf{u}, \mathbf{u}_x, \ldots, \mathbf{u}_{x,\ldots,x}), \quad k \in \mathbb{N}, \tag{5}$$

which can be associated with a commutative algebra of linear operators

$$A^k = D_{t_k} - V^k(\mathbf{u};\lambda), \qquad [A^i, A^j] = 0. \tag{6}$$

Similar to Eq. (3), system (5) is equivalent to $[L, A^k] = 0$. The operator A and Eq. (3) can be considered members of the respective sequence of the operators $\{A^k\}$ and symmetries (5) for particular values of k. The commutativity of operators can be seen as a compatibility condition for the infinite sequence of linear problems

$$D_x(\Psi) = U(\mathbf{u};\lambda)\Psi, \qquad D_{t_k}(\Psi) = V^k(\mathbf{u};\lambda)\Psi, \tag{7}$$

i.e., the condition for the existence of a common fundamental solution Ψ of all these problems, $\det\Psi \neq 0$.

We regard a Darboux transformation as a linear map $\mathcal{S}$ acting on a fundamental solution

$$\mathcal{S}\colon \Psi \mapsto \overline{\Psi} = M\Psi, \quad \det M \neq 0, \tag{8}$$

such that the matrix function $\overline{\Psi}$ is a fundamental solution of the linear problems

$$D_x(\overline{\Psi}) = U(\overline{\mathbf{u}};\lambda)\overline{\Psi}, \qquad D_{t_k}(\overline{\Psi}) = V^k(\overline{\mathbf{u}};\lambda)\overline{\Psi}, \tag{9}$$

with the new "potentials" $\overline{\mathbf{u}}$. The matrix M is often called the Darboux matrix. The elements of the Darboux matrix M are rational (elliptic) functions of the spectral parameter λ. As a function of λ, the determinant of M can vanish only at a finite set of points on the Riemann sphere (the parallelogram of periods). The Darboux matrix M depends on $\mathbf{u}$ and $\overline{\mathbf{u}}$ and can also depend on some auxiliary functions $\mathbf{g}$ (or parameters, if $\mathbf{g}$ is a constant). (The matrices in (18) are examples of such $\mathbf{g}$.) We therefore let $M = M(\mathbf{u}, \overline{\mathbf{u}}, \mathbf{g};\lambda)$ denote the Darboux matrix. It follows from the compatibility of (8) and (9) that

$$D_x(M) = U(\overline{\mathbf{u}};\lambda)M - MU(\mathbf{u};\lambda), \tag{10}$$

$$D_{t_k}(M) = V^k(\overline{\mathbf{u}};\lambda)M - MV^k(\mathbf{u};\lambda). \tag{11}$$

Equations (10) and (11) are differential equations relating the two solutions $\mathbf{u}$ and $\overline{\mathbf{u}}$ of (3) and (5). They are also often called Bäcklund transformations of (3) in the literature.

A Darboux transformation maps one compatible system (7) into another system (9). It defines a Darboux map $\mathcal{S}\colon \mathbf{u} \mapsto \overline{\mathbf{u}}$. Map (8) is invertible ($\det M \neq 0$) and can be iterated:

$$\ldots \xrightarrow{\mathcal{S}} \Psi = M(\underline{\mathbf{u}}, \mathbf{u}, \underline{\mathbf{g}}; \lambda)\underline{\Psi} \xrightarrow{\mathcal{S}} \overline{\Psi} = M(\mathbf{u}, \overline{\mathbf{u}}, \mathbf{g}; \lambda)\Psi \xrightarrow{\mathcal{S}}$$

$$\xrightarrow{\mathcal{S}} \overline{\overline{\Psi}} = M(\overline{\mathbf{u}}, \overline{\overline{\mathbf{u}}}, \overline{\mathbf{g}}; \lambda)\overline{\Psi} \xrightarrow{\mathcal{S}} \overline{\overline{\overline{\Psi}}} = M(\overline{\overline{\mathbf{u}}}, \overline{\overline{\overline{\mathbf{u}}}}, \overline{\overline{\mathbf{g}}}; \lambda)\overline{\overline{\Psi}} \xrightarrow{\mathcal{S}} \ldots .$$

It suggests the notation

$$\begin{array}{llllll}
\ldots, & \Psi_{-1} = \underline{\Psi}, & \Psi_0 = \Psi, & \Psi_1 = \overline{\Psi}, & \Psi_2 = \overline{\overline{\Psi}}, & \ldots, \\
\ldots, & \mathbf{u}_{-1} = \underline{\mathbf{u}}, & \mathbf{u}_0 = \mathbf{u}, & \mathbf{u}_1 = \overline{\mathbf{u}}, & \mathbf{u}_2 = \overline{\overline{\mathbf{u}}}, & \ldots, \\
\ldots, & \mathbf{g}_{-1} = \underline{\mathbf{g}}, & \mathbf{g}_0 = \mathbf{g}, & \mathbf{g}_1 = \overline{\mathbf{g}}, & \mathbf{g}_2 = \overline{\overline{\mathbf{g}}}, & \ldots .
\end{array}$$

With a vertex k of the one-dimensional lattice $\mathbb{Z}$, we associate the variables Ψ_k and $\mathbf{u}_k$; with the edges joining the vertices k and $k+1$, we associate the auxiliary functions (parameters) $\mathbf{g}_k$ and the matrix $M_k = M(\mathbf{u}_k, \mathbf{u}_{k+1}, \mathbf{g}_k; \lambda)$. In this notation, the Darboux maps $\mathcal{S}$ and $\mathcal{S}^{-1}$ increase and decrease the subscript index by one, and we therefore call it the $\mathcal{S}$-shift or the shift operator $\mathcal{S}$. In what follows, we often omit zero in the subscript index and write $\mathbf{u}$ and $\mathbf{g}$ instead of $\mathbf{u}_0$ and $\mathbf{g}_0$.

In this notation, Eq. (10) and sequence (11) are a hierarchy of compatible systems of differential–difference equations. When the resulting equations from (10) and (11) are in the evolutionary form, they constitute an infinite-dimensional Lie algebra of commuting symmetries. The existence of an infinite algebra of commuting symmetries is often taken as a definition of the integrability of an equation (and of the whole hierarchy of symmetries) [20]–[23].

To illustrate this construction, we consider two examples: the well-known example of the NLS equation and new results on differential–difference equations corresponding to the dihedral reduction group $\mathbb{D}_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ (the Klein group). We use these examples to illustrate the derivation of recursion operators.

2.1. The nonlinear Schrödinger equation. The NLS equation

$$\begin{aligned} 2p_t &= p_{xx} - 8p^2 q, \\ 2q_t &= -q_{xx} + 8q^2 p \end{aligned} \tag{12}$$

has zero-curvature representation (4), where [24]

$$U(\mathbf{u}; \lambda) = \begin{pmatrix} 0 & 2p \\ 2q & 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{u} = \begin{pmatrix} p \\ q \end{pmatrix}, \tag{13}$$

$$V(\mathbf{u}; \lambda) = \begin{pmatrix} -2pq & p_x \\ -q_x & 2pq \end{pmatrix} + \lambda \begin{pmatrix} 0 & 2p \\ 2q & 0 \end{pmatrix} + \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

The NLS equation has an infinite hierarchy of commuting symmetries. The matrix part of the corresponding linear operators $A^k = D_{t_k} - V^k$ has the form

$$V^0 = J, \qquad V^{k+1} = \lambda V^k + B^k(\mathbf{u}), \tag{15}$$

where $J = \mathrm{diag}(1,-1)$ and $B^k(\mathbf{u})$ are traceless matrices with elements depending on p, q, and their x-derivatives. The matrices $B^k(\mathbf{u})$ can be found recursively [25]. In particular, we have

$$
\begin{aligned}
&V^1 = U(\mathbf{u};\lambda), \qquad V^2 = V(\mathbf{u};\lambda), \\
&V^3 = \lambda V(\mathbf{u};\lambda) + \frac{1}{2}\begin{pmatrix} 2pq_x - 2qp_x & p_{xx} - 8p^2q \\ q_{xx} - 8q^2p & 2qp_x - 2pq_x \end{pmatrix}.
\end{aligned} \tag{16}
$$

The symmetries corresponding to A^0, A^1, A^2, and A^3 are

$$
\begin{aligned}
&p_{t_0} = 2p, \qquad p_{t_1} = p_x, \qquad p_{t_2} = \frac{1}{2}p_{xx} - 4p^2q, \qquad p_{t_3} = \frac{1}{4}p_{xxx} - 6pqp_x, \\
&q_{t_0} = -2q, \qquad q_{t_1} = q_x, \qquad q_{t_2} = -\frac{1}{2}q_{xx} + 4q^2p, \qquad q_{t_3} = \frac{1}{4}q_{xxx} - 6pqq_x.
\end{aligned} \tag{17}
$$

It is known (see [26], [27]) that any Darboux matrix for the NLS equation is a composition of the three elementary Darboux matrices

$$
\begin{aligned}
&M(\mathbf{u},\mathbf{u}_1,f;\lambda) = \begin{pmatrix} \lambda + f & p \\ q_1 & 1 \end{pmatrix}, \qquad N(\mathbf{u},\mathbf{u}_1,h;\lambda) = \begin{pmatrix} \lambda + h & p \\ p^{-1} & 0 \end{pmatrix}, \\
&K(\mathbf{u},\mathbf{u}_1,\alpha;\lambda) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}
\end{aligned} \tag{18}
$$

and their inverses. For each of these elementary Darboux matrices, we derive the Darboux map and the corresponding differential–difference equation.

($\boldsymbol{K}$): The map corresponding to the Darboux matrix K is independent of the spectral parameter λ and the variables $\mathbf{u}$ and $\mathbf{u}_1$. Moreover, it follows from (10) that α is a constant (is independent of x), $p_1 = \alpha^2 p$, and $q_1 = \alpha^{-2}q$. In this case, the Darboux map is therefore a gauge transformation corresponding to a point symmetry of Eq. (12) and its hierarchy of symmetries.

($\boldsymbol{N}$): Substituting $N(\mathbf{u},\mathbf{u}_1,h;\lambda)$ in (10) yields

$$
q_1 = \frac{1}{p}, \qquad h_x = 2p_1q_1 - 2pq, \qquad p_x = -2hp, \qquad q_{1,x} = 2hq_1, \tag{19}
$$

and we therefore obtain the Darboux map in the explicit form

$$
p_1 = p^2q - \frac{p}{4}\left(\frac{p_x}{p}\right)_x, \qquad q_1 = \frac{1}{p}.
$$

After the change of the variables $p = e^{\phi}$ (and hence $q = e^{-\phi_{-1}}$), Eqs. (19) yields the system of evolutionary equations

$$
\phi_x = -2h, \qquad h_x = 2e^{\phi_1 - \phi} - 2e^{\phi - \phi_{-1}}, \tag{20}
$$

which after the elimination of h becomes the Toda lattice

$$
\phi_{xx} = 4e^{\phi - \phi_{-1}} - 4e^{\phi_1 - \phi}.
$$

It is an infinite chain of differential equations for the dependent variables ϕ_n, $n \in \mathbb{Z}$.

We note that $\phi = \log p$ and $h = -\phi_x/2$. Using (19) to eliminate x-derivatives from (17), we can obtain symmetries of Toda chain (20):

$$\begin{aligned}
&\phi_{t_0} = 2, \qquad \phi_{t_1} = -2h, \qquad \phi_{t_2} = 2h^2 - 2(\mathcal{S}+1)e^{\phi-\phi_{-1}},\\
&h_{t_0} = 0, \qquad h_{t_1} = 2(\mathcal{S}-1)e^{\phi-\phi_{-1}}, \qquad h_{t_2} = -2(\mathcal{S}-1)(e^{\phi-\phi_{-1}}(h_{-1}+h)),\\
&\phi_{t_3} = -2h^3 + 2e^{\phi-\phi_{-1}}(2h+h_{-1}) + 2e^{\phi_1-\phi}(2h+h_1),\\
&h_{t_3} = 2(\mathcal{S}-1)((h_{-1}^2 + h_{-1}h + h^2)e^{\phi-\phi_{-1}} + e^{2\phi-2\phi_{-1}} + (\mathcal{S}+1)e^{\phi-\phi_{-2}}).
\end{aligned} \tag{21}$$

Symmetries (21) have the Darboux–Lax representation

$$N_{t_k} - \mathcal{S}(U^k)N + NU^k = 0, \qquad N = \begin{pmatrix} \lambda + h & e^{\phi} \\ e^{-\phi} & 0 \end{pmatrix}, \tag{22}$$

and the matrices $U^0, \ldots, U^3$ are obtained from $V^0, \ldots, V^3$ given by (15) and (16) by eliminating x-derivatives using (19). In Sec. 3, we derive a recursion operator for generating symmetries of the Toda lattice.

(*M*): Substituting $M(\mathbf{u}, \mathbf{u}_1, f; \lambda)$ in (10) leads to the system of differential–difference equations

$$f_x = 2p_1q_1 - 2pq, \qquad p_x = 2p_1 - 2fp, \qquad q_x = -2q_{-1} + 2f_{-1}q. \tag{23}$$

Symmetries of this system can be found from Lax Darboux representations (11), where V^k are given by (15) and (16) after elimination of x-derivatives using (23):

$$\begin{aligned}
&p_{t_0} = 2p, \qquad p_{t_1} = 2p_1 - 2fp, \qquad p_{t_2} = 2(f^2p - fp_1 - f_1p_1 - p^2q - pp_1q_1 + p_2),\\
&q_{t_0} = -2q, \qquad q_{t_1} = -2q_{-1} + 2f_{-1}q, \qquad q_{1,t_2} = 2(f_{-1}q + fq - f^2q_1 + pqq_1 + p_1q_1^2 - q_{-1}),\\
&f_{t_0} = 0, \qquad f_{t_1} = 2p_1q_1 - 2pq, \qquad f_{t_2} = 2(\mathcal{S}-1)(pq_{-1} + p_1q - (f_{-1}+f)pq).
\end{aligned} \tag{24}$$

System (23) and its symmetries (24) have the first integral $\Phi = f - pq_1$, and hence $\Phi_{t_k} = 0$. Indeed, $\det M(\mathbf{u}, \mathbf{u}_1, f; \lambda) = \lambda + f - pq_1$ should be a constant (is independent of x and t_k) because the matrices U and V^k are traceless (Abel's theorem). Therefore, we can set $f_k = p_kq_{k+1} + \alpha_k$, where $\alpha_k \in \mathbb{C}$ is a constant. We can eliminate f from system (23), and this leads to

$$p_x = 2p_1 - 2p^2q_1 - 2\alpha p, \qquad q_x = -2q_{-1} + 2p_{-1}q^2 + 2\alpha_{-1}q, \tag{25}$$

whose symmetries can be obtained from (24) by the same elimination of f. When $\alpha_k = 0$, Eq. (25) becomes the Merola–Ragnisco–Tu lattice under an invertible transformation $x = t/2$, $p = u$, and $q_1 = v$ listed in Sec. 4.10.

2.2. Equations corresponding to the dihedral reduction group $\mathbb{D}_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. The integrability of the system

$$\begin{aligned}
&2p_t = p_{xx} + 4q_x - 8(p^2q)_x,\\
&2q_t = -q_{xx} + 4p_x - 8(q^2p)_x
\end{aligned} \tag{26}$$

was established in [21]. This equation can be seen as a nontrivial inhomogeneous deformation of the derivative NLS equation. The corresponding Lax pair

$$L(\mathbf{u}; \lambda) = D_x - V^1(\mathbf{u}; \lambda), \qquad A(\mathbf{u}; \lambda) = D_t - V^2(\mathbf{u}; \lambda)$$

has the matrix part of the form

$$V^1(\mathbf{u};\lambda) = 2p\mathbf{a}_1(\lambda) + 2q\mathbf{a}_2(\lambda) + 2\mathbf{a}_3(\lambda), \tag{27}$$

$$V^2(\mathbf{u};\lambda) = w(\lambda)V^1(\mathbf{u};\lambda) + \frac{p_x - 4p^2q}{2}\mathbf{a}_1(\lambda) - \frac{q_x + 4q^2p}{2}\mathbf{a}_2(\lambda) - 2pq\mathbf{a}_3(\lambda), \tag{28}$$

where

$$\mathbf{a}_1(\lambda) = \begin{pmatrix} 0 & \lambda \\ \lambda^{-1} & 0 \end{pmatrix}, \qquad \mathbf{a}_2(\lambda) = \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda & 0 \end{pmatrix}, \qquad \mathbf{a}_3(\lambda) = \frac{\lambda^2 - \lambda^{-2}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{29}$$

The matrices $V^k(\mathbf{u};\lambda)$ are invariant under the group (reduction group [8]) generated by the transformations

$$V^k(\mathbf{u};\lambda) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} V^k(\mathbf{u};-\lambda) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$V^k(\mathbf{u};\lambda) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V^k(\mathbf{u};\lambda^{-1}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is isomorphic to the dihedral group $\mathbb{D}_2 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ (the Klein group). The function

$$w(\lambda) = \frac{\lambda^2 + \lambda^{-2}}{2} \tag{30}$$

is a primitive automorphic function of the corresponding Möbius group ($w(\lambda) = w(-\lambda) = w(\lambda^{-1})$), and any rational automorphic function of this group is a rational function of $w(\lambda)$ [3]. A hierarchy of higher symmetries of Eq. (26) can be generated by the Lax operators $A^k = D_{t_k} - V^k$, where the matrices V^k have the form

$$V^{k+1} = w(\lambda)V^k + r_1^k\mathbf{a}_1(\lambda) + r_2^k\mathbf{a}_2(\lambda) + r_3^k\mathbf{a}_3(\lambda), \tag{31}$$

and the coefficients r_1^k, r_2^k, and r_3^k are polynomials in p, q, and their x-derivatives and can be found recursively.

It was shown in [28] that an elementary Darboux matrix for Lax operator (27) can be written in the form

$$M(p, q_1, f, g; \lambda) = f(w(\lambda)\mathbf{I} + \mathbf{a}_3(\lambda) + p\mathbf{a}_1(\lambda) + q_1\mathbf{a}_2(\lambda) + g\mathbf{I}), \tag{32}$$

where $\mathbf{I}$ is the unit matrix. The matrices V^k are all traceless, and the determinant

$$\det M(p, q_1, f, g; \lambda) = f^2(2w(\lambda)(g - pq_1) + 1 + g^2 - p^2 - q_1^2) = 2w(\lambda)\Phi^1 + \Phi^2$$

is therefore independent of x and t_k. We hence have two invariants

$$\Phi^1 = f^2(g - pq_1), \qquad \Phi^2 = f^2(1 + g^2 - p^2 - q_1^2). \tag{33}$$

Choosing an appropriate scaling $M \to \gamma M$, we can make $\Phi^1 = 1$ (if $\Phi^1 \neq 0$) or $\Phi^2 = 1$ (if $\Phi^2 \neq 0$).

There are three essentially different cases [28]:

1. $\Phi^1 = 0$ and $\Phi^2 = 1$. In this case, the determinant is a constant (is independent of λ).
2. $\Phi^1 = 1$ and $\Phi^2 = \pm 2$. In this case, the determinant has two double zeroes and is a square of a rational function of λ.

3. $\Phi^1 = 1$ and $\Phi^2 = 2\alpha$. The determinant has four distinct zeros in the complex plane λ (assuming that $\alpha \neq \pm 1$).

In case 1, we have

$$g = pq_1, \tag{34}$$

and f can be found as a solution of the equation $f^2(1-p^2)(1-q_1^2) = 1$. We can therefore determine the matrix $M(p, q_1, f, pq_1; \lambda)$ in (32). Substituting it and $V^1(\mathbf{u}; \lambda)$ given by (27) instead of $U(\mathbf{u}; \lambda)$ in (10) leads to the system

$$p_x = 2(1-p^2)(q_1 - q), \qquad q_x = 2(1-q^2)(p - p_{-1}), \tag{35}$$

which can be written as a scalar equation for one function $v_{2n} = p_n$, $v_{2n-1} = q_n$:

$$v_x = (1 - v^2)(v_1 - v_{-1}). \tag{36}$$

In case 2, we take $2\Phi^1 + \Phi^2 = 0$ (the other choice of the sign would eventually lead to a point-equivalent system). We have

$$2\Phi^1 + \Phi^2 = f^2(1 + g + p + q_1)(1 + g - p - q_1) = 0.$$

We choose $1 + g + p + q_1 = 0$ (the second choice $1 + g - p - q_1 = 0$ would lead to a point-equivalent system). Substituting $M(p, q_1, f, -p - q_1 - 1; \lambda)$ and $V^1(\mathbf{u}; \lambda)$ given by (27) instead of $U(\mathbf{u}; \lambda)$ in (10) then leads to the system

$$p_x = 2(1+p)(\mathcal{S} - 1)(q - p - pq), \qquad q_{1,x} = 2(1 + q_1)(\mathcal{S} - 1)(p - q - pq). \tag{37}$$

In general case 3 of the elementary Darboux transformation, we have

$$\Phi^2 - 2\alpha\Phi^1 = (g^2 - 2\alpha g - p^2 - q_1^2 + 2\alpha p q_1 + 1)f^2 = 0, \qquad f = \frac{1}{\sqrt{g - pq_1}}. \tag{38}$$

Substituting $M(p, q_1, f, g; \lambda)$ and $V^1(\mathbf{u}; \lambda)$ given by (27) instead of $U(\mathbf{u}; \lambda)$ in (10) leads to the system

$$\begin{aligned}
p_x &= 2g(p_1 - p) + 2q_1 - 2q - 2p(p_1q_1 - pq), \\
q_{1,x} &= 2g(q_1 - q) + 2p_1 - 2p - 2q_1(p_1q_1 - pq), \\
g_x &= 2p(p_1 - p) + 2q_1(q_1 - q) - 2g(p_1q_1 - pq), \\
f_x &= 2f(p_1q_1 - pq).
\end{aligned}$$

The invariants Φ^1 and Φ^2 are first integrals of this system. The functions f and g can therefore be eliminated using first integrals (38). We note that the parameter α in (38) is constant in x and t_k but can depend on the shift variable, and hence $\mathcal{S}(\alpha) = \alpha_1$.

3. Recursion operators for differential–difference equations

In this section, we show how to derive a recursion operator using a Darboux–Lax representation for a differential–difference equation. Our construction is a natural generalization of the method used in the theory of integrable PDEs [25], [29]–[32]. The main idea of the method is based on the fact that the matrices $V^k(\mathbf{u}, \lambda)$ of the operators $A^k = D_{t_k} - V^k(\mathbf{u}, \lambda)$ corresponding to a hierarchy can be related as

$$V^{k+1}(\mathbf{u}, \lambda) = \mu(\lambda)V^k(\mathbf{u}, \lambda) + B^k(\mathbf{u}, \lambda), \tag{39}$$

where $\mu(\lambda)$ is a rational multiplier (elliptic in the case of the Landau–Lifshitz equation) and $B^k(\mathbf{u}, \lambda)$ is a rational matrix with a fixed (i.e., k-independent) divisor of poles. If the system and its Lax representation is obtained as a result of a reduction with a reduction group G, then the multiplier $\mu(\lambda)$ is a primitive automorphic function [3] of a finite reduction group or in the elliptic case is one of the generators of the G-invariant subring of the coordinate ring [30]. The matrix $B^k(\mathbf{u}, \lambda)$ also depends on the dependent variables $\mathbf{u}$ and their x-derivatives.

Substituting (39) in the Lax representation $[L, A^k] = 0$ (see (6)) results in

$$D_{t_{k+1}}(L) = \mu(\lambda)D_{t_k}(L) - B^k L + LB^k. \tag{40}$$

We can use Eq. (40) to find B^k in terms of the variables $\mathbf{u}_{t_k}$, $\mathbf{u}_{t_{k+1}}$, and x-derivatives of $\mathbf{u}$. We can then regard Eq. (40) as a recurrence relation $\mathbf{u}_{t_{k+1}} = \mathcal{R}(\mathbf{u}_{t_k})$, where $\mathcal{R}$ is a linear pseudodifferential recursion operator mapping a symmetry to a new symmetry. A recursion operator can be related to a bi-Hamiltonian structure. Indeed, if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two compatible Hamiltonian operators, then $\mathcal{R} = \mathcal{H}_1\mathcal{H}_2^{-1}$ is a Nijenhuis recursion operator [33], [11]. The sufficient condition for $\mathcal{R}$ to be a recursion operator for Eq. (3) is [34]

$$D_t(\mathcal{R}) = [\mathbf{F}_*, \mathcal{R}],$$

where $\mathbf{F}_*$ is the Fréchet derivative of $\mathbf{F}$.

A similar construction can be used in the differential–difference case [13]. Substituting (39) in Darboux–Lax representation (11) results in

$$D_{t_{k+1}}(M) = \mu(\lambda)D_{t_k}(M) - \mathcal{S}(B^k)M + MB^k. \tag{41}$$

Equation (41) allows expressing the elements of the matrix B^k in terms of $\mathbf{u}$, $\mathbf{u}_{t_k}$, $\mathbf{u}_{t_{k+1}}$, and their $\mathcal{S}$-shifts. It allows finding a linear pseudodifference operator $\mathcal{R}$ such that $\mathbf{u}_{t_{k+1}} = \mathcal{R}(\mathbf{u}_{t_k})$, i.e., a recursion operator for a differential–difference hierarchy of commuting symmetries. As in the differential case, if we know two compatible Hamiltonian operators $\mathcal{H}_1$ and $\mathcal{H}_2$ or a compatible pair of a Hamiltonian operator $\mathcal{H}$ and a symplectic operator $\mathcal{J}$, then $\mathcal{R} = \mathcal{H}_1\mathcal{H}_2^{-1}$ and $\widetilde{\mathcal{R}} = \mathcal{H}\mathcal{J}$ are recursion operators. It follows from this construction that a Darboux matrix M and a multiplier $\mu(\lambda)$ define a recursion operator completely and uniquely.

In this section, we illustrate this construction with a few examples. In Sec. 4, we present an extensive list (but far from complete) of integrable differential–difference equations with recursion operators, multi-Hamiltonian structures, and Darboux–Lax representations.

3.1. Differential–difference equations from the NLS equation. We illustrate the construction using the Darboux matrices M and N (see (18)) related to the NLS equation. In this case, the multiplier $\mu(\lambda) = \lambda$ (see (15)), and the matrix B^k is independent of the spectral parameter λ.

We construct a recursion operator for Toda lattice (20) using the Darboux matrix N (see (22)) and the multiplier $\mu(\lambda) = \lambda$. We substitute

$$B = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \tag{42}$$

in the equation

$$D_{t_{k+1}}(N) = \lambda D_{t_k}(N) - \mathcal{S}(B)N + NB. \tag{43}$$

The part of (43) linear in λ leads to the system of equations

$$(\mathcal{S} - 1)a = h_{t_k}, \qquad b = -\phi_{t_k}e^{\phi}, \qquad c_1 = -\phi_{t_k}e^{-\phi}.$$

Hence, we can find all the elements $a = (\mathcal{S}-1)^{-1}h_{t_k}$, $b = -e^{\phi}\phi_{t_k}$, and $c = -e^{-\phi_{-1}}\mathcal{S}^{-1}\phi_{t_k}$ of the matrix B. Then the λ-independent part of (43) reduces to the equations

$$\phi_{t_{k+1}} = bh - a - a_1 = -(\mathcal{S}+1)(\mathcal{S}-1)^{-1}h_{t_k} - h\phi_{t_k},$$
$$h_{t_{k+1}} = (a-a_1)h + ce^{\phi} - b_1e^{-\phi} = -hh_{t_k} - e^{\phi-\phi_{-1}}\mathcal{S}^{-1}\phi_{t_k} + e^{\phi_1-\phi}\mathcal{S}\phi_{t_k},$$

which leads to the recurrence relation

$$\begin{pmatrix}\phi_{t_{k+1}}\\ h_{t_{k+1}}\end{pmatrix} = \mathcal{R}\begin{pmatrix}\phi_{t_k}\\ h_{t_k}\end{pmatrix}$$

with the pseudodifference operator

$$\mathcal{R} = \begin{pmatrix} -h & -(\mathcal{S}+1)(\mathcal{S}-1)^{-1}\\ e^{\phi_1-\phi}\mathcal{S} - e^{\phi-\phi_{-1}}\mathcal{S}^{-1} & -h\end{pmatrix}. \tag{44}$$

Starting from the seed symmetry $\phi_{t_0} = -2$, $h_{t_0} = 0$, we can recursively produce the hierarchy of symmetries (21).

Similarly to the case of scalar discrete equations [11], [35], the canonical series of the densities of local conservation laws can be found by taking residues $\rho_k = \operatorname{res}\mathcal{R}^k$. In the case of a matrix pseudodifference operator $\mathcal{A}$, the residue $\operatorname{res}\mathcal{A}$ is defined as follows. Any pseudodifference operator $\mathcal{A}$ can be uniquely represented by its Laurent series

$$\mathcal{A} = \sum_{k=0}^{\infty}\mathcal{A}^{m-k}\mathcal{S}^{m-k}.$$

The residue is then defined as $\operatorname{res}\mathcal{A} = \operatorname{trace}(\mathcal{A}^0)$. For example, we rewrite recursion operator (44) as

$$\mathcal{R} = \begin{pmatrix}0 & 0\\ e^{\phi_1-\phi} & 0\end{pmatrix}\mathcal{S} + \begin{pmatrix}-h & -1\\ 0 & -h\end{pmatrix} + \begin{pmatrix}0 & -2\\ -e^{\phi-\phi_{-1}} & 0\end{pmatrix}\mathcal{S}^{-1} + \dots.$$

It follows that

$$\rho_1 = \operatorname{res}\mathcal{R} = -2h, \qquad \rho_2 = \operatorname{res}\mathcal{R}^2 = 2h^2 - 2e^{\phi_1-\phi} - 2e^{\phi-\phi_{-1}}, \qquad \dots$$

are conserved densities for Toda lattice (20). Indeed, we have

$$D_x\rho_1 = -4(\mathcal{S}-1)e^{\phi-\phi_{-1}},$$
$$D_x\rho_2 = 4(\mathcal{S}-1)(e^{\phi-\phi_{-1}}(h_{-1}+h)).$$

We now take the Darboux matrix M (see (18)) and the matrix B of form (42). From the terms in the equation

$$D_{t_{k+1}}(M) = \lambda D_{t_k}(M) - \mathcal{S}(B)M + MB \tag{45}$$

that are linear in λ, it follows that

$$a = (\mathcal{S}-1)^{-1}f_{t_k}, \qquad b = -p_{t_k}, \qquad c = q_{t_k}.$$

The λ-independent part of(45) leads to the recurrence relation for system (23)

$$\begin{pmatrix} f_{t_{k+1}} \\ p_{t_{k+1}} \\ q_{t_{k+1}} \end{pmatrix} = \begin{pmatrix} -ff_{t_k} + \mathcal{S}(qp_{t_k}) + pq_{t_k} \\ -2p(\mathcal{S}-1)^{-1} f_{t_k} - f_{t_k} p - fp_{t_k} + \mathcal{S}(p_{t_k}) \\ 2q(\mathcal{S}-1)^{-1} f_{t_k} - q\mathcal{S}^{-1}(f_{t_k}) + \mathcal{S}^{-1} q_{t_k} - f_{-1} q_{t_k} \end{pmatrix} = \mathcal{R} \begin{pmatrix} f_{t_k} \\ p_{t_k} \\ q_{t_k} \end{pmatrix}, \tag{46}$$

where the pseudodifference recursion operator has the form

$$\mathcal{R} = \begin{pmatrix} -f & q_1\mathcal{S} & p \\ -p & -f+\mathcal{S} & 0 \\ -q\mathcal{S}^{-1} & 0 & -f_{-1}+\mathcal{S}^{-1} \end{pmatrix} + \begin{pmatrix} 0 \\ -2p \\ 2q \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

Using the first integral of system (23) and eliminating $f_k = p_k q_{k+1} + \alpha_k$, we obtain the corresponding recurrence relation and operator for system (25):

$$\begin{pmatrix} p_{t_{k+1}} \\ q_{t_{k+1}} \end{pmatrix} = \mathcal{R}' \begin{pmatrix} p_{t_k} \\ q_{t_k} \end{pmatrix}, \tag{47}$$

where

$$\mathcal{R}' = \begin{pmatrix} \mathcal{S} - 2pq_1 - \alpha & -p^2\mathcal{S} - 2pp_{-1} \\ -q^2\mathcal{S}^{-1} & \mathcal{S}^{-1} - \alpha_{-1} \end{pmatrix} + \begin{pmatrix} -2p \\ 2q \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} q_1, & p_{-1} \end{pmatrix}.$$

As already noted, Eq. (25) is related to the Merola–Ragnisco–Tu lattice; we can verify that we obtain the same operator from the recursion operator for the Merola–Ragnisco–Tu lattice in Sec. 4.10 using formula (58).

3.2. Difference equation corresponding to the dihedral reduction group. We show how to compute a recursion operator for Eq. (37), whose Lax representation is invariant under the dihedral reduction group.

After the simple change of variables $p \to p-1$, $q \to q-1$, Eq. (37) becomes

$$p_x = 2p(\mathcal{S}-1)(2q-pq), \qquad q_{1,x} = 2q_1(\mathcal{S}-1)(2p-pq). \tag{48}$$

This equation is the relativistic Volterra lattice in Sec. 4.9 under the scaling transformation

$$p = -2v, \qquad q = -2u, \qquad x = -\frac{t}{4}. \tag{49}$$

The corresponding Darboux matrix and the multiplier are

$$M = f \begin{pmatrix} \lambda^2 + 1 - p - q_1 & \lambda(p-1) + \lambda^{-1}(q_1 - 1) \\ \lambda(q_1-1) + \lambda^{-1}(p-1) & \lambda^{-2} + 1 - p - q_1 \end{pmatrix}, \qquad \mu(\lambda) = \frac{1}{2}(\lambda^2 + \lambda^{-2}). \tag{50}$$

This information suffices for finding a recursion operator, Eq. (48) itself, and its hierarchy of local symmetries. It follows from

$$\det M = -(\lambda - \lambda^{-1})^2 f^2 pq_1$$

that f^2pq_1 is independent of x (Abel's theorem), and we can therefore set $f = (pq_1)^{-1/2}$.

Expression (31) suggests that

$$B = a\mathbf{a}_1 + b\mathbf{a}_2 + c\mathbf{a}_3,$$

where the coefficients a, b, and c are independent of λ. In Eq. (41), the right- and left-hand sides are now rational matrix functions in λ. The left-hand side has only simple poles. Requiring that the coefficients of the third-order poles in λ^{-3} in the right-hand side be zero is equivalent to

$$\begin{aligned} a &= \frac{1}{2}(c+c_1)(p-1) + \frac{(p-1)p\mathcal{S}q_{t_k} - (p+1)q_1 p_{t_k}}{4pq_1}, \\ b_1 &= \frac{1}{2}(c+c_1)(q_1-1) + \frac{(q_1+1)p\mathcal{S}q_{t_k} + (1-q_1)q_1 p_{t_k}}{4pq_1}. \end{aligned} \tag{51}$$

The second-order poles vanish under the condition

$$c - c_1 = \frac{q_1 p_{t_k} + pq_{1,t_k}}{2pq_1}. \tag{52}$$

Using (52), we can now simplify (51) by eliminating c_1 and c_{-1} from a and b:

$$\begin{aligned} a &= c(p-1) - \frac{1}{2}p_{t_k}, \\ b &= c(q-1) + \frac{1}{2}q_{t_k}. \end{aligned} \tag{53}$$

The residues in the left- and right-hand sides in Eq. (41) lead to the equations

$$\begin{aligned} p_{t_{k+1}} &= cp(\mathcal{S}-1)(pq-2q) + \left(p - q_1 - 1 - \frac{1}{2}p_1 q_1\right)p_{t_k} - \frac{1}{2}pq_1\mathcal{S}p_{t_k} + \\ &\quad + \left(p - \frac{1}{2}p^2\right)q_{t_k} + \left(p - \frac{1}{2}pp_1\right)\mathcal{S}q_{t_k}, \\ q_{t_{k+1}} &= cq(\mathcal{S}-1)(p_{-1}q_{-1} - 2p_{-1}) + \left(q + p_{-1} - 1 - \frac{1}{2}p_{-1}q_{-1}\right)q_{t_k} - \\ &\quad - \frac{1}{2}p_{-1}q\mathcal{S}^{-1}q_{t_k} + \left(q - \frac{1}{2}q^2\right)p_{t_k} + \left(q - \frac{1}{2}qq_{-1}\right)\mathcal{S}^{-1}p_{t_k}. \end{aligned} \tag{54}$$

The λ-independent part of Eq. (41) is satisfied after (52)–(54) are substituted in (41).

Equation (54) together with (52) is a recurrence relation. The differential–difference equation (a seed symmetry) $p_x = 2K_p^{(1)}$, $q_x = 2K_q^{(1)}$ (see (48)) can be recovered from this recursion by taking the derivative of the right-hand side of Eqs. (54) with respect to c, i.e.,

$$(K_p^{(1)}, K_q^{(1)}) = \big(p(\mathcal{S}-1)(pq-2q),\, q(\mathcal{S}-1)(p_{-1}q_{-1} - 2p_{-1})\big).$$

This is not surprising. Indeed, to solve Eq. (52) for c, we must invert the difference operator $\mathcal{S}-1$, whose kernel is the field of constants. As a result, the vector p_x, q_x can contribute to $p_{t_{k+1}}, q_{t_{k+1}}$ with an arbitrary constant coefficient.

We now write the recursion operator corresponding to recurrence relation (54) explicitly as

$$\begin{aligned} \mathcal{R} = &\begin{pmatrix} p + q_1 - 1 - \frac{1}{2}p_1 q_1 - \frac{1}{2}pq_1\mathcal{S} & p - \frac{1}{2}p^2 + \left(p - \frac{1}{2}pp_1\right)\mathcal{S} \\ q - \frac{1}{2}q^2 + \left(q - \frac{1}{2}qq_{-1}\right)\mathcal{S}^{-1} & q + p_{-1} - 1 - \frac{1}{2}p_{-1}q_{-1} - \frac{1}{2}p_{-1}q\mathcal{S}^{-1} \end{pmatrix} - \\ &- \begin{pmatrix} p(\mathcal{S}-1)(pq-2q) \\ q(\mathcal{S}-1)(p_{-1}q_{-1} - 2p_{-1}) \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{1}{2p} & \frac{1}{2q_1}\mathcal{S} \end{pmatrix}. \end{aligned} \tag{55}$$

Obviously, $\mathcal{R}+I$, where I is the 2×2 identity matrix, is also a recursion operator, which can be recovered from the recursion operator for the relativistic Volterra lattice listed in Sec. 4.9 using transformation (49).

4. A list of integrable differential–difference equations

In this section, we present a long list of integrable differential–difference equations with their Hamiltonian structures, recursion operators, nontrivial generalized symmetries, and Lax representations. To be self-contained, we introduce the notation and recall some definitions of the objects in our list in terms of the Lie derivatives. The theoretical background and detailed definitions of the Hamiltonian and symplectic operators can be found in [36], [37]. Here, we mention that there are some recent developments in the theory of nonlocal Hamiltonian structures for nonlinear partial differential equations in [38].

Let $\mathbf{u}=(u^1,\dots,u^N)$ be a vector-valued function of $n\in\mathbb{Z}$ and the time t. An evolutionary differential–difference equation for the dependent variable $\mathbf{u}$ has the form

$$\mathbf{u}_t=\mathbf{K}[\mathbf{u}], \tag{56}$$

where $\mathbf{K}[\mathbf{u}]$ means that the smooth vector-valued function $\mathbf{K}$ depends on $\mathbf{u}$ and its shifts $\mathbf{u}_i=\mathcal{S}^i\mathbf{u}$. In all our examples, after an appropriate point change of variables, we can consider $\mathbf{K}[\mathbf{u}]\in\mathcal{F}^N$, where $\mathcal{F}=(\mathbb{C},\mathbf{u},\mathcal{S})$ is a difference field of rational functions of $\{u_k^i\mid k\in\mathbb{Z},\ i=1,\dots,N\}$. The Fréchet derivative $a_\star$ of $a\in\mathcal{F}$ is defined as the row vector of the difference operators

$$a_\star=\sum_{k\in\mathbb{Z}}\left(\frac{\partial a}{\partial u_k^1},\dots,\frac{\partial a}{\partial u_k^N}\right)\mathcal{S}^k.$$

Hence, the Fréchet derivative of $\mathbf{K}[\mathbf{u}]$ is a difference operator with square matrix coefficients, and the elements of the coefficient matrices are elements of $\mathcal{F}$.

A variational derivative of $a\in\mathcal{F}$ is a column vector

$$\delta_{\mathbf{u}}(a):=a_\star^\dagger(1)=\left(\frac{\partial}{\partial u^1},\dots,\frac{\partial}{\partial u^N}\right)^{\mathrm{T}}\sum_{k\in\mathbb{Z}}\mathcal{S}^k(a).$$

If Eq. (56) is Hamiltonian, then it can be written in the form

$$\mathbf{u}_t=\mathcal{H}(\delta_{\mathbf{u}}(f)),$$

where $\mathcal{H}$ denotes a Hamiltonian (pseudo)difference operator and f is a Hamiltonian function (or the Hamiltonian of the system).

Definition 1. Given differential–difference evolutionary equation (56), we say that

1. $\mathbf{G}$ is its symmetry if $L_{\mathbf{K}}\mathbf{G}:=[\mathbf{K},\mathbf{G}]:=\mathbf{G}_\star(\mathbf{K})-\mathbf{K}_\star(\mathbf{G})=0$,
2. $\mathcal{H}$ is its Hamiltonian operator if $L_{\mathbf{K}}\mathcal{H}:=\mathcal{H}_\star[\mathbf{K}]-\mathbf{K}_\star\mathcal{H}-\mathcal{H}\mathbf{K}_\star^\dagger=0$ is satisfied by the Hamiltonian operator $\mathcal{H}$,
3. $\mathcal{J}$ is its symplectic operator if the symplectic operator $\mathcal{J}$ satisfies $L_{\mathbf{K}}\mathcal{J}:=\mathcal{J}_\star[\mathbf{K}]+\mathbf{K}_\star^\dagger\mathcal{J}+\mathcal{J}\mathbf{K}_\star=0$, and
4. $\mathcal{R}$ is its recursion operator if $L_{\mathbf{K}}\mathcal{R}:=\mathcal{R}_\star[\mathbf{K}]-\mathbf{K}_\star\mathcal{R}+\mathcal{R}\mathbf{K}_\star=0$.

Here, $L_{\mathbf{K}}$ denotes the Lie derivative, $\star$ denotes the Fréchet derivative, and $\dagger$ denotes the formal conjugation of the difference operator.

If a symmetry of Eq. (56) is explicitly dependent on $\mathbf{u}_i$ with $i \neq 0$, then we call it a generalized symmetry. An equation is integrable if it has infinitely many generalized symmetries depending on finite sets of variables $\mathbf{u}_i$, whose sizes increase. Symmetries of an integrable systems can be generated by a recursion operator, which is often nonlocal. Sufficient conditions on nonlocal pseudodifference recursion operators that guarantee producing an infinite hierarchy of commuting local symmetries were discussed in [11]. In the list, we also give partial results on master symmetries (see [39], [40] for more details on master symmetries).

We now consider how the recursion, Hamiltonian, and symplectic operators change under transformations (difference substitutions). If evolutionary difference equation (56) is related by a difference substitution $\mathbf{u} = \mathbf{F}[\mathbf{v}]$ to another equation of the form

$$\mathbf{v}_t = \mathbf{G}[\mathbf{v}], \tag{57}$$

then the recursion, Hamiltonian, and symplectic operators $\widehat{\mathcal{R}}$, $\widehat{\mathcal{H}}$, and $\widehat{\mathcal{J}}$ for Eq. (57) can be expressed in terms of the corresponding operators $\mathcal{R}$, $\mathcal{H}$, and $\mathcal{J}$ for Eq. (56):

$$\widehat{\mathcal{R}} = \mathbf{F}_\star^{-1} \circ \mathcal{R}|_{\mathbf{u}=\mathbf{F}[\mathbf{v}]} \circ \mathbf{F}_\star, \qquad \widehat{\mathcal{H}} = \mathbf{F}_\star^{-1} \circ \mathcal{H}|_{\mathbf{u}=\mathbf{F}[\mathbf{v}]} \circ {\mathbf{F}_\star^\dagger}^{-1}, \qquad \widehat{\mathcal{J}} = \mathbf{F}_\star^\dagger \circ \mathcal{J}|_{\mathbf{u}=\mathbf{F}[\mathbf{v}]} \circ \mathbf{F}_\star, \tag{58}$$

where $\circ$ denotes the composition of operators.

Example 1. It is known that the Volterra equation $u_t = u(u_1 - u_{-1})$ (Sec. 4.1) can be related to the modified Volterra equation

$$v_t = v^2(v_1 - v_{-1}) \tag{59}$$

by a difference substitution (a Miura-type transformation) $u = F[v] = vv_1$. The operator $\mathcal{H}_1 = u(\mathcal{S} - \mathcal{S}^{-1})u$ is a Hamiltonian operator for the Volterra equation. We note that the Fréchet derivative for vv_1 is $F_\star = v_1 + v\mathcal{S}$, and we have $F_\star^{-1}u = v(1+\mathcal{S})^{-1}$ and $u{F_\star^\dagger}^{-1} = \mathcal{S}(1+\mathcal{S})^{-1}v$. We can now find the Hamiltonian operator $\widehat{\mathcal{H}}_1$ for modified Volterra equation (59):

$$\begin{aligned}\widehat{\mathcal{H}}_1 &= (v_1 + v\mathcal{S})^{-1} vv_1 (\mathcal{S} - \mathcal{S}^{-1}) vv_1 (v_1 + v\mathcal{S})^{-1} = \\ &= v(1+\mathcal{S})^{-1}(\mathcal{S}^2 - 1)(\mathcal{S}+1)^{-1}v = v(\mathcal{S}-1)(\mathcal{S}+1)^{-1}v.\end{aligned}$$

In the same way, we can compute the second Hamiltonian operator $\widehat{\mathcal{H}}_2$ and the recursion operator $\widehat{\mathcal{R}}$ for the modified Volterra equation (cf. Sec. 4.2).

In Sec. 2, we discussed that the compatibility of Darboux map (8) with Lax operator (9)

$$\mathcal{S}(\Phi) = M\Phi, \qquad D_t(\Phi) = U(\mathbf{u};\lambda)\Phi \tag{60}$$

yields (10),

$$D_t(M) = \mathcal{S}(U)M - MU, \tag{61}$$

which is equivalent to an integrable system of differential–difference equations. Compatibility condition (61) is often called a zero-curvature representation or Lax representation of Eq. (56) in the literature. Because it involves a Darboux matrix and a Lax operator, it is more appropriate to call it a Darboux–Lax representation. But we still call it a Lax representation for consistency with the literature. In the following list, we simply give the expressions for both the matrices M and U for Lax representations.

There are many publications where an integrable system emerges as a compatibility condition of two linear problems with scalar linear difference operators L and A,

$$L\phi = \lambda\phi, \qquad \phi_t = A\phi, \tag{62}$$

where ϕ is an eigenfunction of L corresponding to the eigenvalue λ and $\lambda_t = 0$. Equation (56) is equivalent to the compatibility condition

$$D_t(L) = [A, L] = AL - LA. \tag{63}$$

This approach in fact resembles the original Lax formulation, where differential operators are just replaced with difference operators. In the theory of ordinary differential equations, a scalar higher-order equation can be represented as a system of first-order equations. We can do similarly in the case of higher-order scalar difference equations and represent them as first-order difference systems. We can thus rewrite scalar representation (62), (63) as first-order matrix Darboux–Lax representation (60), (61).

Example 2. We obtain the matrix Lax representation for the Volterra chain listed in Sec. 4.1 from the scalar Lax representation with $L = \mathcal{S} + u\mathcal{S}^{-1}$ and $A = \mathcal{S}^2 + u_1 + u$.

Let $\Phi = (\phi^1, \phi^2)^{\mathrm{T}} = (\phi, -\mathcal{S}^{-1}\phi)^{\mathrm{T}}$. We can rewrite $L\phi = \lambda\phi$ in (62) as

$$\mathcal{S}(\Phi) = \mathcal{S}\begin{pmatrix} \phi \\ -\phi_{-1} \end{pmatrix} = \begin{pmatrix} \mathcal{S}\phi \\ -\phi \end{pmatrix} = \begin{pmatrix} \lambda & u \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \phi \\ -\phi_{-1} \end{pmatrix},$$

i.e., $\mathcal{S}(\phi^1) = \lambda\phi^1 + u\phi^2$ and $\mathcal{S}(\phi^2) = -\phi^1$. We can now rewrite $\phi_t = A\phi$ as

$$\begin{aligned} \phi^1_t &= \mathcal{S}^2\phi^1 + (u_1 + u)\phi^1 = \mathcal{S}(\lambda\phi^1 + u\phi^2) + (u_1 + u)\phi^1 = \\ &= \lambda\mathcal{S}\phi^1 + u\phi^1 = (\lambda^2 + u)\phi^1 + \lambda u\phi^2, \\ \phi^2_t &= -\mathcal{S}^{-1}\phi^1_t = -(\lambda\phi^1 + u\phi^2 + (u_{-1} + u)\mathcal{S}^{-1}\phi^1) = -\lambda\phi^1 + u_{-1}\phi^2. \end{aligned}$$

Hence,

$$D_t(\Phi) = \begin{pmatrix} \phi^1_t \\ \phi^2_t \end{pmatrix} = \begin{pmatrix} \lambda^2 + u & \lambda u \\ -\lambda & u_{-1} \end{pmatrix}\begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix}.$$

We present either the scalar or the matrix Lax representation in our list.

The following should be kept in mind:

1. Relations between nonlocal difference operators (e.g., (67)) should be understood as identities in the noncommutative field of pseudodifference Laurent series.

2. In the computations with pseudodifference operators, we often use identities similar to "integration by parts":
$$(\mathcal{S} - 1)^{-1}(f_1 - f)(\mathcal{S} - 1)^{-1} = f(\mathcal{S} - 1)^{-1} - (\mathcal{S} - 1)^{-1}f_1.$$

3. Because we have $(\mathcal{S} - 1)c = 0$ for any constant $c \in \mathbb{C}$, the action of operators involving $(\mathcal{S} - 1)^{-1}$ is not uniquely defined. The corresponding results given in the list are up to these "integration constants."

4.1. The Volterra chain.

- Equation [10]:

$$u_t = u(u_1 - u_{-1}). \tag{64}$$

- Hamiltonian structure [41], [42]: $u_t = H_i \delta_u f_i$,

$$\mathcal{H}_1 = u(\mathcal{S} - \mathcal{S}^{-1})u, \qquad f_1 = u,$$

$$\mathcal{H}_2 = u(\mathcal{S}u\mathcal{S} + u\mathcal{S} + \mathcal{S}u - u\mathcal{S}^{-1} - \mathcal{S}^{-1}u - \mathcal{S}^{-1}u\mathcal{S}^{-1})u, \qquad f_2 = \frac{1}{2}\log u.$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}_1^{-1} = u\mathcal{S} + u + u_1 + u\mathcal{S}^{-1} + u(u_1 - u_{-1})(\mathcal{S} - 1)^{-1}\frac{1}{u} =$$

$$= u(\mathcal{S} - \mathcal{S}^{-1})u\left(\frac{1}{u}(\mathcal{S} - 1)^{-1} + \mathcal{S}(\mathcal{S} - 1)^{-1}\frac{1}{u}\right).$$

- Nontrivial symmetry [42], [43]:

$$\mathcal{R}(u_t) = u(u_1u_2 + u_1^2 + uu_1 - uu_{-1} - u_{-1}^2 - u_{-1}u_{-2}).$$

- Master symmetry [42], [39], [44]:

$$\mathcal{R}(u) = nu_t + u(2u_1 + u + u_{-1}).$$

- Lax representation [45]:

$$L = \mathcal{S} + u\mathcal{S}^{-1}, \qquad A = \mathcal{S}^2 + u_1 + u,$$

which can also be written in the matrix form

$$M = \begin{pmatrix} \lambda & u \\ -1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} \lambda^2 + u & \lambda u \\ -\lambda & u_{-1} \end{pmatrix}. \tag{65}$$

This equation is also known as the Lotka–Volterra model, the Kac–van Moerbeke lattice, or the Langmuir lattice [46]. The so-called Kac–van Moerbeke–Langmuir equation [41]

$$w_\tau = w(w_1^\epsilon - w_{-1}^\epsilon), \quad \epsilon \neq 0 \text{ constant},$$

is related to (64) by the point transformation $u = w^\epsilon$ and $t = \epsilon\tau$. This equation is also written as

$$w_t = e^{w+w_1} - e^{w+w_{-1}},$$

which can be transformed into (64) by the transformation $u = e^{w+w_1}$.

4.2. Modified Volterra equation.

- Equation [47], [43]:

$$u_t = u^2(u_1 - u_{-1}).$$

- Hamiltonian structure [48], [43]:

$$\mathcal{H}_1 = u(\mathcal{S} - 1)(\mathcal{S} + 1)^{-1}u, \qquad f_1 = uu_1,$$
$$\mathcal{H}_2 = u^2(\mathcal{S} - \mathcal{S}^{-1})u^2, \qquad f_2 = \log u.$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}_1^{-1} = u^2\mathcal{S} + 2uu_1 + u^2\mathcal{S}^{-1} + 2u^2(u_1 - u_{-1})(\mathcal{S} - 1)^{-1}\frac{1}{u}.$$

- Nontrivial symmetry [43]:

$$\mathcal{R}(u_t) = u^2u_1^2(u_2 + u) - u^2u_{-1}^2(u + u_{-2}).$$

- Master symmetry [43]:

$$\mathcal{R}\left(\frac{u}{2}\right) = nu_t + \frac{u^2}{2}(3u_1 + u_{-1}).$$

- Lax representation [26]:

$$M = \begin{pmatrix} 0 & u \\ -u & \lambda \end{pmatrix}, \qquad U = \begin{pmatrix} uu_{-1} & \lambda u_{-1} \\ -\lambda u & \lambda^2 + uu_{-1} \end{pmatrix}.$$

The modified Volterra equation is also known as the discrete modified Korteweg–de Vries equation. Under the Miura transformation $w = uu_1$, it can be transformed into the Volterra chain $w_t = w(w_1 - w_{-1})$ as in Sec. 4.1.

4.3. Yamilov's discretization of the Krichever–Novikov equation.

- Equation [49] (V4, $\nu = 0$ in Sec. 4.4):

$$u_t = \frac{R(u_1, u, u_{-1})}{u_1 - u_{-1}} := K^{(1)},$$

where R is a polynomial with the constant coefficients $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{C}$ defined by

$$R(u, v, w) = (\alpha v^2 + 2\beta v + \gamma)uw + (\beta v^2 + \lambda v + \delta)(u + w) + \gamma v^2 + 2\delta v + \epsilon. \tag{66}$$

- Two nontrivial symmetries [50]–[53], [35], [11]:

$$K^{(2)} = \frac{R(u, u_{-1}, u)R(u_1, u, u_1)}{(u_1 - u_{-1})^2}\left(\frac{1}{u_2 - u} + \frac{1}{u - u_{-2}}\right),$$
$$K^{(3)} = \frac{R(u_1, u, u_1)R(u, u_{-1}, u)}{(u_1 - u_{-1})^2}\left(\frac{\mathcal{S}^2K^{(1)}}{(u_2 - u)^2} + \frac{\mathcal{S}^{-2}K^{(1)}}{(u - u_{-2})^2}\right) +$$
$$+ K^{(1)}K^{(2)}\left(\frac{1}{u_2 - u} + \frac{1}{u - u_{-2}}\right).$$

- Hamiltonian structure [11], [53]:

$$\mathcal{H} = A\mathcal{S} - \mathcal{S}^{-1}A + 2K^{(1)}(\mathcal{S}-1)^{-1}\mathcal{S}K^{(2)} + 2K^{(2)}(\mathcal{S}-1)^{-1}K^{(1)},$$

$$\widehat{\mathcal{H}} = \widehat{A}\mathcal{S}^2 - \mathcal{S}^{-2}\widehat{A} + \widehat{B}\mathcal{S} - \mathcal{S}^{-1}\widehat{B} + K^{(2)}(\mathcal{S}-1)^{-1}(\mathcal{S}+1)K^{(2)} +$$
$$+ 2K^{(1)}(\mathcal{S}-1)^{-1}\mathcal{S}K^{(3)} + 2K^{(3)}(\mathcal{S}-1)^{-1}K^{(1)},$$

where

$$A = \frac{R(u_2,u_1,u_2)R(u_1,u,u_1)R(u,u_{-1},u)}{(u_1-u_{-1})^2(u_2-u)^2},$$

$$\widehat{A} = \frac{R(u_3,u_2,u_3)R(u_2,u_1,u_2)R(u_1,u,u_1)R(u,u_{-1},u)}{(u_1-u_{-1})^2(u_2-u)^2(u_3-u_1)^2},$$

$$\widehat{B} = 2A\left(\frac{K^{(1)}}{u-u_{-2}} - \frac{\partial_u R(u_1,u,u_1)}{2(u_1-u_{-1})} + \frac{\partial^2 R(u_1,u,u_1)}{4\partial u\,\partial u_1}\right) + \frac{2R(u,u_{-1},u)}{(u_1-u_{-1})^2}\mathcal{S}(K^{(1)}K^{(2)}).$$

- Symplectic operator:

$$\mathcal{J} = \frac{1}{R(u_1,u,u_1)}\mathcal{S} - \mathcal{S}^{-1}\frac{1}{R(u_1,u,u_1)}.$$

- Recursion operator [11], [53]:

$$\mathcal{R} = \mathcal{H}\mathcal{J} \qquad \text{and} \qquad \widehat{\mathcal{R}} = \widehat{\mathcal{H}}\mathcal{J}.$$

The recursion operators $\mathcal{R}$ and $\widehat{\mathcal{R}}$ satisfy the algebraic equation

$$(2\widehat{\mathcal{R}} - I_3)^2 = 4(\mathcal{R}+I_2)^3 - g_2(\mathcal{R}+I_2) - g_3, \tag{67}$$

where I_2, I_3, g_2, and g_3 are the relative and modular invariants related to $h = R(u_1,u,u_1)$ and a quartic polynomial $f(u) = (\partial_{u_1}h)^2 - 2h\partial_{u_1}^2 h$ defined by

$$g_2 = \frac{1}{48}(2ff^{IV} - 2f'f''' + (f'')^2),$$

$$g_3 = \frac{1}{3456}(12ff''f^{IV} - 9(f')^2f^{IV} - 6f(f''')^2 + 6f'f''f''' - 2(f'')^3),$$

$$I_2 = \frac{1}{6}(h\,\partial_u^2\,\partial_{u_1}^2 h - (\partial_u h)(\partial_u\,\partial_{u_1}^2 h) - (\partial_{u_1}h)(\partial_u^2\,\partial_{u_1}h) + (\partial_u^2 h)(\partial_{u_1}^2 h)) + \frac{1}{12}(\partial_u\,\partial_{u_1}h)^2,$$

$$I_3 = \frac{1}{4}\det\begin{pmatrix} h & \partial_u h & \partial_u^2 h \\ \partial_{u_1}h & \partial_{u_1}\partial_u h & \partial_{u_1}\partial_u^2 h \\ \partial_{u_1}^2 h & \partial_{u_1}^2\partial_u h & \partial_{u_1}^2\partial_u^2 h \end{pmatrix}.$$

4.4. Integrable Volterra-type equations. The classification of integrable Volterra type equations of the form

$$u_t = f(u_{-1},u,u_1),$$

where f is a smooth function of all its variables was obtained by Yamilov using the symmetry approach. In his remarkable review paper [43], he presented the following complete list of integrable Volterra-type equations (with higher-order conservation laws) up to point transformations:

$$\text{V1:} \quad u_t = P(u)(u_1 - u_{-1}), \tag{68}$$

$$\text{V2:}\quad u_t = P(u^2)\left(\frac{1}{u_1+u} - \frac{1}{u+u_{-1}}\right), \tag{69}$$

$$\text{V3:}\quad u_t = Q(u)\left(\frac{1}{u_1-u} + \frac{1}{u-u_{-1}}\right), \tag{70}$$

$$\text{V4:}\quad u_t = \frac{R(u_1,u,u_{-1}) + \nu R(u_1,u,u_1)^{1/2}R(u_{-1},u,u_{-1})^{1/2}}{u_1-u_{-1}}, \quad \nu\in\{0,\pm1\}, \tag{71}$$

$$\text{V5:}\quad u_t = y(u_1-u)+y(u-u_{-1}), \quad y' = P(y), \tag{72}$$

$$\text{V6:}\quad u_t = y(u_1-u)y(u-u_{-1})+\mu, \qquad y' = \frac{P(y)}{y}, \quad \mu\in\mathbb{C}, \tag{73}$$

$$\text{V7:}\quad u_t = \frac{1}{y(u_1-u)+y(u-u_{-1})}+\mu, \qquad y' = P(y^2), \quad \mu\in\mathbb{C}, \tag{74}$$

$$\text{V8:}\quad u_t = \frac{1}{y(u_1+u)-y(u+u_{-1})}, \qquad y' = Q(y), \tag{75}$$

$$\text{V9:}\quad u_t = \frac{y(u_1+u)-y(u+u_{-1})}{y(u_1+u)+y(u+u_{-1})}, \qquad y' = \frac{P(y^2)}{y}, \tag{76}$$

$$\text{V10:}\quad u_t = \frac{y(u_1+u)+y(u+u_{-1})}{y(u_1+u)-y(u+u_{-1})}, \qquad y' = \frac{Q(y)}{y}, \tag{77}$$

$$\text{V11:}\quad u_t = \frac{(1-y(u_1-u))(1-y(u-u_{-1}))}{y(u_1-u)+y(u-u_{-1})}+\mu, \qquad y' = \frac{P(y^2)}{1-y^2}, \quad \mu\in\mathbb{C}, \tag{78}$$

where P and Q are polynomials with the constant coefficients α, β, γ, δ, and ϵ defined by

$$P(u) = \alpha u^2 + \beta u + \gamma, \tag{79}$$

$$Q(u) = \alpha u^4 + \beta u^3 + \gamma u^2 + \delta u + \epsilon \tag{80}$$

and the polynomial R is defined by (66). As stated in [43], the problem of constructing the generalized symmetries for all equations V1–V11 remains open although the master symmetries for some forms of equations in the list are known [42], [39]. We know that the Miura transformation $\tilde{u} = y(u_1-u)$ transforms equations V5 and V6 into V1 and equations V7 and V11 into V2, and the Miura transformation $\tilde{u} = y(u_1+u)$ transforms equation V9 into V2 and equations V8 and V10 into V3 [43]. We present the recursion operators, Hamiltonian operators, and master symmetries for the first four equations. The corresponding operators for other equations can be obtained via the Miura transformations.

4.4.1. Equation V1 (see (68)).

- Hamiltonian structure:

$$\mathcal{H} = P(u)(\mathcal{S}-\mathcal{S}^{-1})P(u).$$

- Symplectic operator:

$$\alpha(\mathcal{S}-\mathcal{S}^{-1}) + (\alpha u_1+\beta+\alpha u_{-1})\mathcal{S}(\mathcal{S}-1)^{-1}\frac{P'(u)}{P(u)} + \frac{P'(u)}{P(u)}(\mathcal{S}-1)^{-1}(\alpha u_1+\beta+\alpha u_{-1}).$$

- Recursion operator:

$$\mathcal{R} = P(u)\mathcal{S} + 2\alpha u u_1 + \beta(u+u_1) + P(u)\mathcal{S}^{-1} + u_t(\mathcal{S}-1)^{-1}\frac{P'(u)}{P(u)}.$$

- Nontrivial symmetry :

$$\mathcal{R}(u_t) = P(u)(P(u_1)u_2 + \alpha u u_1^2 + \beta(u+u_1)u_1 - P(u_{-1})u_{-2} - \alpha u u_{-1}^2 - \beta(u+u_{-1})u_{-1}).$$

- Master symmetry:

$$nu_t + P(u)\left(cu_1 + \frac{\beta}{\alpha} + (2-c)u_{-1}\right), \quad c \in \mathbb{C}, \quad \text{if } \alpha \neq 0,$$

$$nu_t + P(u)(cu_1 + u + (3-c)u_{-1}), \quad c \in \mathbb{C}, \qquad \text{if } \alpha = 0.$$

- Lax representation:

The case $\alpha = \beta = 0$ yields a linear equation.

The case $\alpha = 0$ and $\beta \neq 0$ reduces to Volterra equation (64) via the linear substitution $u \mapsto \beta^{-1}(u-\gamma)$. It hence has Lax representation (65).

In the case $\alpha \neq 0$, the linear substitution $t \mapsto \alpha^{-1}t$ and $u \mapsto u - \beta/2$ transforms the polynomial $P(u)$ in V1 into the form $P(u) = u^2 + c$, where $c = \gamma/\alpha - \beta^2/4\alpha^2$. The corresponding Lax representation has the form [26]

$$M = \begin{pmatrix} c\lambda^{-1} & u \\ -u & \lambda \end{pmatrix}, \qquad U = \begin{pmatrix} c^2\lambda^{-2} + uu_{-1} & c\lambda^{-1}u + \lambda u_{-1} \\ -c\lambda^{-1}u_{-1} - \lambda u & \lambda^2 + uu_{-1} \end{pmatrix}.$$

This equation includes both the Volterra chain in Sec. 4.1 and the modified Volterra equation in Sec. 4.2. The Hamiltonian f for a nonlinear equation, i.e., $\alpha\beta \neq 0$, depends on the coefficients in the polynomial P defined by (79). If $\alpha \neq 0$, then we take $f = (1/2\alpha)\log P(u)$, and if $\alpha = 0$, then we take $f = u/\beta$.

The Hamiltonian operator, symplectic operator, and recursion operator satisfy the unexpected relation

$$\mathcal{H}\mathcal{J} = \alpha\mathcal{R}^2 + \beta^2\mathcal{R} + 2\gamma(\beta^2 - 2\alpha\gamma).$$

4.4.2. Equation V2 (see (69)).

- Hamiltonian structure:

$$\mathcal{H} = \frac{P(u^2)}{u_1+u}\mathcal{S}\frac{P(u^2)}{u+u_{-1}} - \frac{P(u^2)}{u+u_{-1}}\mathcal{S}^{-1}\frac{P(u^2)}{u_1+u} - u_t(\mathcal{S}+1)(\mathcal{S}-1)^{-1}u_t.$$

- Symplectic operator:

$$\mathcal{J} = \frac{1}{(u+u_1)^2}\mathcal{S} - \mathcal{S}^{-1}\frac{1}{(u+u_1)^2} + \delta_u\rho(\mathcal{S}+1)(\mathcal{S}-1)^{-1}\delta_u\rho - $$
$$- (\beta^2 - 4\alpha\gamma)\frac{u}{P(u^2)}(\mathcal{S}-1)(\mathcal{S}+1)^{-1}\frac{u}{P(u^2)},$$

where $\rho = (1/2)\log((u+u_1)^2/P(u^2))$ and hence

$$\delta_u\rho = \frac{1}{u_1+u} + \frac{1}{u+u_{-1}} - \frac{P(u^2)'}{2P(u^2)}, \qquad P(u^2)' = 2u(2\alpha u^2 + \beta).$$

- Recursion operator:

$$\mathcal{R} = \frac{P(u^2)}{(u_1+u)^2}\mathcal{S} + P(u^2)\left(\frac{1}{(u+u_1)^2} + \frac{2}{(u+u_1)(u+u_{-1})} - \frac{1}{(u+u_{-1})^2}\right) + $$
$$+ P(u^2)'\left(\frac{1}{2u} - \frac{1}{u+u_1}\right) + \frac{P(u^2)}{(u+u_{-1})^2}\mathcal{S}^{-1} + 2u_t(\mathcal{S}-1)^{-1}\delta_u\rho.$$

- Nontrivial symmetry :

$$\mathcal{R}(u_t) = \frac{P(u^2)P(u_1^2)}{(u+u_1)^2}\left(\frac{1}{u_1+u_2} - \frac{1}{u_1-u_{-1}}\right) - \\ - \frac{P(u^2)P(u_{-1}^2)}{(u+u_{-1})^2}\left(\frac{1}{u_{-1}+u_{-2}} - \frac{1}{u_{-1}-u_1}\right) + \\ + 2P(u^2)\frac{\alpha u_1^2 u_{-1}^2 + \beta u_1 u_{-1} + \gamma}{(u+u_1)(u_1-u_{-1})(u+u_{-1})}.$$

- Master symmetry:

$$nu_t + \frac{P(u^2)}{u+u_{-1}} - \alpha u^3 - \beta u \quad \text{if } \gamma = 0,$$

$$nu_t + \frac{P(u^2)}{u+u_{-1}} - \alpha u^3 - \frac{\beta}{2}u \quad \text{if } \beta^2 - 4\alpha\gamma = 0,$$

$$nu_t + \frac{P(u^2)}{u+u_{-1}} \quad \text{if } \alpha = 0.$$

Here, we found the master symmetries only in some special cases. For the Hamiltonian operator given above, we have $\mathcal{H}\delta_u\rho = \mathcal{R}(u_t)$. Moreover, the Hamiltonian operator, symplectic operator, and recursion operator satisfy the relation

$$\mathcal{H}\mathcal{J} = \mathcal{R}^2 - 2\beta\mathcal{R} + \beta^2 - 4\alpha\gamma.$$

The Calogero–Degasperis lattice [39]

$$u_t = \frac{1}{4}(1-u^2)(b^2 - a^2u^2)\left(\frac{1}{u_1+u} - \frac{1}{u+u_{-1}}\right)$$

is a special case of equation V2. The authors of [39] gave a different form of its master symmetry by introducing a time dependence for the coefficients a and b.

4.4.3. Equation V3 (see (70)).

- Hamiltonian structure:

$$\mathcal{H} = \frac{Q(u)}{u_1-u}\mathcal{S}\frac{Q(u)}{u-u_{-1}} - \frac{Q(u)}{u-u_{-1}}\mathcal{S}^{-1}\frac{Q(u)}{u_1-u} + u_t(\mathcal{S}+1)(\mathcal{S}-1)^{-1}u_t.$$

- Symplectic operator:

$$\mathcal{J} = \frac{1}{(u_1-u)^2}\mathcal{S} - \mathcal{S}^{-1}\frac{1}{(u_1-u)^2} - \delta_u\rho(\mathcal{S}+1)(\mathcal{S}-1)^{-1}\delta_u\rho - \\ - \left(2\alpha\gamma - \frac{\beta^2}{4}\right)\frac{u^2}{Q(u)}(\mathcal{S}-1)(\mathcal{S}+1)^{-1}\frac{u^2}{Q(u)} - \beta\delta\frac{u}{Q(u)}(\mathcal{S}-1)(\mathcal{S}+1)^{-1}\frac{u}{Q(u)} - \\ - \left(2\gamma\epsilon - \frac{\delta^2}{4}\right)\frac{1}{Q(u)}(\mathcal{S}-1)(\mathcal{S}+1)^{-1}\frac{1}{Q(u)} + \\ + (2\alpha\delta + \beta\gamma)\left(\frac{u^2}{Q(u)}(\mathcal{S}+1)^{-1}\frac{u}{Q(u)} - \frac{u}{Q(u)}\mathcal{S}(\mathcal{S}+1)^{-1}\frac{u^2}{Q(u)}\right) - \\ - \left(\frac{\beta\delta}{2} - \gamma^2 - 4\alpha\epsilon\right)\left(\frac{u^2}{Q(u)}(\mathcal{S}+1)^{-1}\frac{1}{Q(u)} - \frac{1}{Q(u)}\mathcal{S}(\mathcal{S}+1)^{-1}\frac{u^2}{Q(u)}\right) + \\ + (\gamma\delta + 2\beta\epsilon)\left(\frac{u}{Q(u)}(\mathcal{S}+1)^{-1}\frac{1}{Q(u)} - \frac{1}{Q(u)}\mathcal{S}(\mathcal{S}+1)^{-1}\frac{u}{Q(u)}\right),$$

where

$$\rho = \frac{1}{2}\log\frac{Q(u)}{(u_1-u)^2},$$

and hence

$$\delta_u\rho = \frac{1}{u_1-u} - \frac{1}{u-u_{-1}} + \frac{Q'(u)}{2Q(u)}.$$

- Recursion operator:

$$\mathcal{R} = \frac{Q(u)}{(u_1-u)^2}\mathcal{S} + Q(u)\left(\frac{1}{(u-u_{-1})^2} + \frac{2}{(u_1-u)(u-u_{-1})} - \frac{1}{(u_1-u)^2}\right) -$$
$$- \frac{Q'(u)}{u_1-u} - 2\alpha u^2 - \beta u - \gamma + \frac{Q(u)}{(u-u_{-1})^2}\mathcal{S}^{-1} - 2u_t(\mathcal{S}-1)^{-1}\delta_u\rho.$$

- Nontrivial symmetry:

$$\mathcal{R}(u_t) = \frac{Q(u)Q(u_1)}{(u_1-u)^2(u_2-u_1)} + \frac{Q(u)Q(u_{-1})}{(u-u_{-1})^2(u_{-1}-u_{-2})} + \alpha Q(u)(u_1-u_{-1}) +$$
$$+ Q(u)\left(\frac{Q(u)}{(u_1-u)(u-u_{-1})} + 2\alpha u^2 + \beta u\right)\left(\frac{1}{u_1-u} + \frac{1}{u-u_{-1}}\right).$$

- Master symmetry:

$$nu_t - \frac{Q(u)}{u+u_{-1}} + \alpha u^3 + \beta u^2 + \gamma u \quad \text{if } \delta = \epsilon = 0,$$
$$nu_t - \frac{Q(u)}{u+u_{-1}} \qquad \text{if } \alpha = \beta = 0.$$

Similarly to equation V2 in Sec. 4.4.2, we have not found the master symmetry for the polynomial Q in (80) with arbitrary coefficients. But if we seek a master symmetry of the form

$$nu_t - \frac{Q(u)}{u+u_{-1}} + \sum_{i=0}^{3} c_i u^i, \quad c_i \in \mathbb{C},$$

then we can determine the constants c_i for certain polynomials Q. Two examples are listed above.

We note that $\mathcal{H}\delta_u\rho = \mathcal{R}(u_t)$ and the product of the Hamiltonian operator and symplectic operator yields the square of the recursion operator, i.e.,

$$\mathcal{H}\mathcal{J} = \mathcal{R}^2.$$

4.5. The Narita–Itoh–Bogoyavlensky lattice.

- Equation [45], [54], [55]:

$$u_t = u\left(\sum_{k=1}^{p} u_k - \sum_{k=1}^{p} u_{-k}\right), \quad p \in \mathbb{N}.$$

- Hamiltonian structure [15]:

$$\mathcal{H} = u\left(\sum_{i=1}^{p}\mathcal{S}^i - \sum_{i=1}^{p}\mathcal{S}^{-i}\right)u, \qquad f = u.$$

- Recursion operator [13]:

$$\mathcal{R} = u(\mathcal{S} - \mathcal{S}^{-p})(\mathcal{S} - 1)^{-1} \prod_{i=1}^{\to p} (\mathcal{S}^{p+1-i}u - u\mathcal{S}^{-i})(\mathcal{S}^{p-i}u - u\mathcal{S}^{-i})^{-1},$$

where the notation $\prod_{i=1}^{\to p}$ indicates the order of the value i, from 1 to p, i.e., $\prod_{i=1}^{\to p} a_i = a_1 a_2 \cdots a_p$.

- Nontrivial symmetry:

$$\mathcal{R}(u_t) = u(1 - \mathcal{S}^{-(p+1)})\mathcal{S}^{1-p} \sum_{0 \le i \le j \le 2p-1} u_j u_{i+p}.$$

- Master symmetry [13]: $\mathcal{R}(u)$.
- Lax representation [45]:

$$L = \mathcal{S} + u\mathcal{S}^{-p}, \qquad A = (L^{(p+1)})_{\ge 0},$$

where $(\,\cdot\,)_{\ge 0}$ means taking the terms with a nonnegative power of $\mathcal{S}$ in $L^{(p+1)}$.

For $p = 1$, 2, or 3, a few higher-order symmetries were given explicitly in [15], where the authors also studied their Hamiltonian operator, recursion operator, and master symmetry for $p = 1, 2$.

The Narita–Itoh–Bogoyavlensky lattice is known as an integrable discretization of the Korteweg–de Vries equation. It can also be represented as

$$v_t = v\left(\prod_{k=1}^{p} v_k - \prod_{k=1}^{p} v_{-k}\right),$$

which is related to the Narita–Itoh–Bogoyavlensky lattice via the transformation $u = \prod_{k=0}^{p-1} v_k$ for fixed p.

Taking $p = 1$, we obtain the well-known Volterra chain in Sec. 4.1. These chains can therefore also be considered a generalization of the Volterra chain.

Let $u = \prod_{k=0}^{p} w_k$. Then w satisfies the so-called modified Bogoyavlensky chain

$$w_t = w^2\left(\prod_{k=1}^{p} w_k - \prod_{k=1}^{p} w_{-k}\right).$$

The recursion operator given above for the Narita–Itoh–Bogoyavlensky lattice is highly nonlocal (so is the master symmetry). Recently, Svinin [56] derived explicit formulas for its generalized symmetries in terms of a family of homogeneous difference polynomials. The properties of these homogeneous difference polynomials [57] allow proving the locality of its infinitely many symmetries [13].

A family of integrable lattice hierarchies associated with fractional Lax operators was introduced by Adler and Postnikov [58], [59]. One simple example is

$$u_t = u^2\left(\prod_{k=1}^{p} u_k - \prod_{k=1}^{p} u_{-k}\right) - u\left(\prod_{k=1}^{p-1} u_k - \prod_{k=1}^{p-1} u_{-k}\right), \quad 2 \le p \in \mathbb{N}, \tag{81}$$

which is an integrable discretization of the Sawada–Kotera equation. It can be considered an inhomogeneous generalization of Bogoyavlensky-type lattices. The problem of constructing the Hamiltonian structure and recursion operator for such a family of equations is still open.

Even in the scalar case, the classification of higher-order integrable evolutionary differential–difference equations is still open.

4.6. The Toda lattice.

- Equation [60]:

$$q_{tt} = e^{q_1 - q} - e^{q - q_{-1}}.$$

In the Manakov–Flaschka coordinates [61], [46] defined by $u = e^{q_1 - q}$ and $v = q_t$, it can be rewritten as the two-component evolution system

$$\begin{aligned} u_t &= u(v_1 - v), \\ v_t &= u - u_{-1}. \end{aligned} \tag{82}$$

- Hamiltonian structure [41], [39], [36]:

$$\mathcal{H}_1 = \begin{pmatrix} 0 & u(\mathcal{S} - 1) \\ (1 - \mathcal{S}^{-1})u & 0 \end{pmatrix}, \qquad f_1 = u + \frac{v^2}{2},$$

$$\mathcal{H}_2 = \begin{pmatrix} u(\mathcal{S} - \mathcal{S}^{-1})u & u(\mathcal{S} - 1)v \\ v(1 - \mathcal{S}^{-1})u & u\mathcal{S} - \mathcal{S}^{-1}u \end{pmatrix}, \qquad f_2 = v.$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}_2 \mathcal{H}_1^{-1} = \begin{pmatrix} v_1 + u(v_1 - v)(\mathcal{S} - 1)^{-1}\frac{1}{u} & u\mathcal{S} + u \\ 1 + \mathcal{S}^{-1} + (u - u_{-1})(\mathcal{S} - 1)^{-1}\frac{1}{u} & v \end{pmatrix} =$$

$$= \begin{pmatrix} v_1 & u\mathcal{S} + u \\ 1 + \mathcal{S}^{-1} & v \end{pmatrix} + \begin{pmatrix} u(v_1 - v) \\ u - u_{-1} \end{pmatrix} (\mathcal{S} - 1)^{-1} \begin{pmatrix} \frac{1}{u} & 0 \end{pmatrix}.$$

- Nontrivial symmetry:

$$\mathcal{R}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u(v_1^2 - v^2 + u_1 - u_{-1}) \\ u(v_1 + v) - u_{-1}(v_{-1} + v) \end{pmatrix}.$$

- Master symmetry [39]:

$$\mathcal{R}\begin{pmatrix} u \\ \frac{v}{2} \end{pmatrix} = \begin{pmatrix} nu_t + \frac{3}{2}uv_1 + \frac{1}{2}uv \\ nv_t + u + u_{-1} + \frac{v^2}{2} \end{pmatrix}.$$

- Lax representation:

$$M = \begin{pmatrix} \lambda + v_1 & u \\ -1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & -u \\ 1 & \lambda + v \end{pmatrix}.$$

The Hirota nonlinear equation [62]

$$\begin{aligned} u_t &= v_1 - v, \\ v_t &= v(u - u_{-1}) \end{aligned}$$

is related to Toda lattice (82) by a simple invertible transformation. Namely, let $u = q$ and $v = p_{-1}$. Then the variables p and q satisfy the Toda equation. All its properties can be obtained from those of the Toda lattice.

4.7. A relativistic Toda system.

- Equation [63]:

$$q_{t,t} = q_t q_{-1,t}\frac{e^{q_{-1}-q}}{1+e^{q_{-1}-q}} - q_t q_{1,t}\frac{e^{q-q_1}}{1+e^{q-q_1}}.$$

We introduce the dependent variables [64], [65]

$$u = \frac{q_t e^{q-q_1}}{1+e^{q-q_1}}, \qquad v = \frac{q_t}{1+e^{q-q_1}}.$$

The equation can then be written as

$$\begin{aligned} u_t &= u(u_{-1} - u_1 + v - v_1),\\ v_t &= v(u_{-1} - u). \end{aligned} \tag{83}$$

- Hamiltonian structure [65]:

$$\mathcal{H}_1 = \begin{pmatrix} 0 & u(1-\mathcal{S}) \\ (\mathcal{S}^{-1}-1)u & u\mathcal{S} - \mathcal{S}^{-1}u \end{pmatrix}, \qquad f_1 = \frac{1}{2}(u^2+v^2) + uv + u_1 u + uv_1,$$

$$\mathcal{H}_2 = \begin{pmatrix} u(\mathcal{S}^{-1}-\mathcal{S})u & u(1-\mathcal{S})v \\ v(\mathcal{S}^{-1}-1)u & 0 \end{pmatrix}, \qquad f_2 = u + v.$$

- Recursion operator [65]:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}_1^{-1} =$$

$$= \begin{pmatrix} u\mathcal{S} + u + v_1 + u_1 + u\mathcal{S}^{-1} - u(v - v_1 + u_{-1} - u_1)(\mathcal{S}-1)^{-1}\dfrac{1}{u} & u\mathcal{S} + u \\ v + v\mathcal{S}^{-1} - v(u_{-1}-u)(\mathcal{S}-1)^{-1}\dfrac{1}{u} & v \end{pmatrix} =$$

$$= \begin{pmatrix} u\mathcal{S} + u + v_1 + u_1 + u\mathcal{S}^{-1} & u\mathcal{S} + u \\ v + v\mathcal{S}^{-1} & v \end{pmatrix} - \begin{pmatrix} u_t \\ v_t \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \dfrac{1}{u} & 0 \end{pmatrix}.$$

- Nontrivial symmetry:

$$\begin{pmatrix} uu_{-1}(u + u_{-1} + u_{-2} + 2v + v_{-1}) - \\ \quad - uu_1(u_2 + u_1 + u + 2v_1 + v_2) + u^2(v - v_1) + u(v^2 - v_1^2) \\ vu_{-1}(u_{-2} + u_{-1} + v + v_{-1}) - uv(u_1 + u + v_1 + v) \end{pmatrix}.$$

- Master symmetry [65]:

$$\mathcal{R}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -nu_t + u(v + 2v_1 + u + 2u_1 + u_{-1}) \\ -nv_t + v(u + v + u_{-1}) \end{pmatrix}, \qquad (\mathcal{S}-1)^{-1}1 = n.$$

- Lax representation:

$$M = \begin{pmatrix} \lambda v - \lambda^{-1} & u_{-1} \\ -1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} -\lambda^{-2} - u_{-1} & \lambda^{-1}u_{-1} \\ -\lambda^{-1} & -u_{-2} - v_{-1} \end{pmatrix}.$$

As noted in [65], the inverse of this recursion operator $\mathcal{R}$ is also weakly nonlocal:

$$\mathcal{R}^{-1} = \mathcal{H}_1\mathcal{H}_2^{-1} = \begin{pmatrix} \dfrac{1}{v_1} & -\dfrac{u}{v_1^2}\mathcal{S} + \dfrac{u}{v^2} - \dfrac{2u}{vv_1} \\ -\mathcal{S}^{-1}\dfrac{1}{v} - \dfrac{1}{v_1} & \dfrac{u}{v_1^2}\mathcal{S} + \mathcal{S}^{-1}\dfrac{u}{v^2} + \dfrac{2u}{vv_1} + \dfrac{1}{v} \end{pmatrix} + \\ + \begin{pmatrix} \dfrac{u}{v_1} - \dfrac{u}{v} \\ \dfrac{u_{-1}}{v_{-1}} - \dfrac{u}{v_1} \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \dfrac{1}{u} & -\dfrac{2}{v} \end{pmatrix}.$$

But the recursion operators $\mathcal{R}$ and $\mathcal{R}^{-1}$ have different starting symmetries (seeds). The right-hand side of Eq. (83) is the seed for $\mathcal{R}$, while the seed for $\mathcal{R}^{-1}$ is

$$\sigma = \begin{pmatrix} \dfrac{u}{v_1} - \dfrac{u}{v} \\ \dfrac{u_{-1}}{v_{-1}} - \dfrac{u}{v_1} \end{pmatrix}.$$

Moreover, $\mathcal{R}$ acting on σ and $\mathcal{R}^{-1}$ acting on the right-hand side of the equation do not yield new symmetries.

There is a Miura transformation $u = -u'_{-1}/v'v'_{-1}$, $v = -1/v'_{-1}$ between the flow corresponding to σ and Eq. (83), where u' and v' denote dependent variables for σ.

Other integrable equations related to the relativistic Toda lattice were studied in [64]. For example, the equation

$$q_{t,t} = q_{-1,t}e^{q_{-1}-q} - e^{2q_{-1}-2q} - q_{1,t}e^{q-q_1} + e^{2q-2q_1}$$

can also be rewritten as system (83) by setting

$$u = e^{q-q_1}, \qquad v = q_t - e^{q_{-1}-q} - e^{q-q_1}.$$

4.8. Two-component Volterra lattice.

- Equation [41]:

$$\begin{aligned} u_t &= u(v_1 - v), \\ v_t &= v(u - u_{-1}). \end{aligned} \tag{84}$$

- Hamiltonian structure [41]:

$$\mathcal{H}_1 = \begin{pmatrix} 0 & u(\mathcal{S}-1)v \\ v(1-\mathcal{S}^{-1})u & 0 \end{pmatrix}, \qquad f_1 = u + v,$$

$$\mathcal{H}_2 = \begin{pmatrix} u(\mathcal{S}v - v\mathcal{S}^{-1})u & u(u\mathcal{S} - u + \mathcal{S}v - v)v \\ v(u - \mathcal{S}^{-1}u + v - v\mathcal{S}^{-1})u & v(u\mathcal{S} - \mathcal{S}^{-1}u)v \end{pmatrix}, \qquad f_2 = \log u.$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}_1^{-1} = \\ = \begin{pmatrix} u + v_1 + u(v_1 - v)(\mathcal{S}-1)^{-1}\dfrac{1}{u} & u\mathcal{S} + \dfrac{uv_1}{v} + u(v_1 - v)(\mathcal{S}-1)^{-1}\dfrac{1}{v} \\ v + v\mathcal{S}^{-1} + v(u - u_{-1})(\mathcal{S}-1)^{-1}\dfrac{1}{u} & u + v + v(u - u_{-1})(\mathcal{S}-1)^{-1}\dfrac{1}{v} \end{pmatrix} = \\ = \begin{pmatrix} u + v_1 & u\mathcal{S} + \dfrac{uv_1}{v} \\ v + v\mathcal{S}^{-1} & u + v \end{pmatrix} + \begin{pmatrix} u(v_1 - v) \\ v(u - u_{-1}) \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \dfrac{1}{u} & \dfrac{1}{v} \end{pmatrix}.$$

- Nontrivial symmetry:

$$\mathcal{R}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u^2(v_1 - v) + u(v_1^2 - v^2 + v_1 u_1 - v u_{-1}) \\ v^2(u - u_{-1}) + v(u^2 - u_{-1}^2 + u v_1 - u_{-1} v_{-1}) \end{pmatrix}.$$

- Master symmetry [39]:

$$\mathcal{R}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2nu_t + u^2 + 3uv_1 \\ 2nv_t + vu_{-1} + 2uv + v^2 \end{pmatrix}.$$

- Lax representation [66]:

$$L = \lambda \mathcal{S}^{-1} + v + u_{-1} + \lambda^{-1} uv\mathcal{S}, \qquad A = \lambda^{-1} uv\mathcal{S}.$$

This system comes from the Volterra chain in Sec. 4.1 written in the variable w, i.e.,

$$w_t = w(w_1 - w_{-1}), \tag{85}$$

by renaming $u(n,t) = w(2n,t)$ and $v(n,t) = w(2n-1,t)$. It is related to Toda equation (82), written in the variables $\bar{u}$ and $\bar{v}$, by the Miura transformation [67]

$$\bar{u} = uv, \qquad \bar{v} = u_{-1} + v. \tag{86}$$

In fact, the (master) symmetries, conservation laws, and local Hamiltonian structures of this system can be easily obtained from the Volterra chain in the same way. For instance, we can derive the first Hamiltonian operator $\mathcal{H}_1$ as follows. A symmetry flow of Volterra chain (85) is

$$w_\tau = w(\mathcal{S} - \mathcal{S}^{-1})wQ[n] = ww_1 Q[n+1] - ww_{-1}Q[n-1],$$

where $Q[n]$ is the variational derivative of a conserved density for (85). We now write both even and odd chains and rename them for the variables u and v accordingly. We have

$$u_\tau = uv_1 Q[2n+1] - uvQ[2n-1], \qquad v_\tau = vuQ[2n] - vu_{-1}Q[2n-2],$$

i.e.,

$$\begin{pmatrix} u_\tau \\ v_\tau \end{pmatrix} = \begin{pmatrix} 0 & u(\mathcal{S}-1)v \\ v(1-\mathcal{S}^{-1})u & 0 \end{pmatrix} \begin{pmatrix} Q[2n] \\ Q[2n-1] \end{pmatrix}.$$

Using the same method, we can derive $\mathcal{H}_2$ in the list. Such a construction holds for all scalar equations. It is tricky if the operator is nonlocal.

4.9. The relativistic Volterra lattice.

- Equation [66], [67], [28]:

$$u_t = u(v - v_{-1} + uv - u_{-1}v_{-1}),$$

$$v_t = v(u_1 - u + u_1 v_1 - uv).$$

- Hamiltonian structure [66]:

$$\mathcal{H}_1 = \begin{pmatrix} 0 & u(1-\mathcal{S}^{-1})v \\ v(\mathcal{S}-1)u & 0 \end{pmatrix}, \qquad f_1 = u + v + uv,$$

$$\mathcal{H}_2 = \begin{pmatrix} uv(1+u)\mathcal{S}u - u\mathcal{S}^{-1}uv(1+u) & uv(u+v+uv) - \\ & \quad -u(\mathcal{S}^{-1}uv\mathcal{S}^{-1} + u\mathcal{S}^{-1} + \mathcal{S}^{-1}v)v \\ v(\mathcal{S}uv\mathcal{S} + v\mathcal{S} + \mathcal{S}u)u - uv(u+v+uv) & v\mathcal{S}uv(1+v) - uv(1+v)\mathcal{S}^{-1}v \end{pmatrix},$$

$f_2 = \log u \qquad \text{or} \qquad f_2 = \log v.$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}_1^{-1} = \begin{pmatrix} uv_{-1}\mathcal{S}^{-1} + u + v + uv & u(1+u_{-1})\mathcal{S}^{-1} + u(1+u) \\ v(1+v_1)\mathcal{S} + \dfrac{u_1 v(1+v_1)}{u} & u_1 v\mathcal{S} + u_1 + v + u_1 v_1 \end{pmatrix} +$$

$$+ \begin{pmatrix} u(v - v_{-1} + uv - u_{-1}v_{-1}) \\ v(u_1 - u + u_1 v_1 - uv) \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \dfrac{1}{u} & \dfrac{1}{v} \end{pmatrix}.$$

- Nontrivial symmetry:

$$\mathcal{R}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} uv(1+u)(u + u_1 + u_1 v_1) + uv^2(1+u)^2 - \\ \quad -uv_{-1}^2(1+u_{-1})^2 - u^2 v_{-1} - u_{-1}uv_{-1}(1 + u + v_{-2} + u_{-1} + u_{-2}v_{-2}) \\ u_1 v v_1(1 + 2u_1 + u_2 v_2 + u_2) + u_1^2 v + u_1 v v_1^2(1+u_1) + \\ \quad +v^2 u_1(1+v_1) - uv(1+v)(v + v_{-1} + u_{-1}v_{-1}) - u^2 v(1+v)^2 \end{pmatrix}.$$

- Lax representation

$$M = \begin{pmatrix} \lambda^2 + 2u_1 + 2v + 1 & -\lambda(2v+1) - \lambda^{-1}(2u_1+1) \\ -\lambda(2u_1+1) - \lambda^{-1}(2v+1) & \lambda^{-2} + 2u_1 + 2v + 1 \end{pmatrix},$$

$$U = \begin{pmatrix} -\dfrac{\lambda^2 - \lambda^{-2}}{8} + uv + \dfrac{u}{2} + \dfrac{v}{2} & \lambda\dfrac{(2v+1)}{4} + \lambda^{-1}\dfrac{(2u+1)}{4} \\ \lambda\dfrac{(2u+1)}{4} + \lambda^{-1}\dfrac{(2v+1)}{4} & \dfrac{\lambda^2 - \lambda^{-2}}{8} + uv + \dfrac{u}{2} + \dfrac{v}{2} \end{pmatrix}.$$

In [66], the Lax representation is given in the form

$$U_t = UC - AU, \qquad W_t = WB - CW, \qquad V_t = VB - AV,$$

where the difference operators U, V, W, A, B, and C are given by

$$\begin{aligned} U &= u + \lambda\mathcal{S}^{-1}, & A &= u + u_{-1}v_{-1} + v_{-1} + \lambda\mathcal{S}^{-1}, \\ W &= 1 + \lambda^{-1}v\mathcal{S}, & B &= u + uv + v_{-1} + \lambda\mathcal{S}^{-1}, \\ V &= 1 - \lambda^{-1}uv\mathcal{S}, & C &= u + uv + v + \lambda\mathcal{S}^{-1}. \end{aligned}$$

The recursion operator $\mathcal{R}$ has a weakly nonlocal inverse:

$$\mathcal{R}' = \begin{pmatrix} \dfrac{uv}{(u_1+v+1)^2}\mathcal{S} + \\ \quad + \dfrac{uv(u+1)+((1+v_{-1})^2+u)(u_1+1)}{(u+v_{-1}+1)^2(u_1+v+1)} & -\dfrac{u(u+1)}{(u+v_{-1}+1)^2}\mathcal{S}^{-1} - \\ & \quad -\dfrac{u(u_1+1)}{(u_1+v+1)^2} \\ -\dfrac{v(1+v)}{(u_1+v+1)^2}\mathcal{S} - & \dfrac{uv}{(u+v_{-1}+1)^2}\mathcal{S}^{-1} + \\ \quad -\dfrac{v(u^2+u^2v+2u_1v_{-1}+u_1+u_1v_{-1}^2)}{u(u+v_{-1}+1)(u_1+v+1)} & \quad + \dfrac{(1+u_1)(1+v)}{(u_1+v+1)^2} \end{pmatrix} +$$

$$+ \begin{pmatrix} -\dfrac{uv_{-1}}{u+v_{-1}+1} + \dfrac{uv}{u_1+v+1} \\ -\dfrac{uv}{u+v_{-1}+1} + \dfrac{u_1v}{u_1+v+1} \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \dfrac{2}{u+v_{-1}+1} - \dfrac{1}{u} & \dfrac{2}{u_1+v+1} - \dfrac{1}{v} \end{pmatrix}.$$

In fact, it commutes with $\mathcal{R}$ and satisfies $\mathcal{R}'(\mathcal{R}+\mathrm{id}) = \mathrm{id}$. It is related to relativistic Toda equation (83) written in the variables $\bar{u}$ and $\bar{v}$ by the Miura transformation $\bar{u} = -uv$ and $\bar{v} = -(u+v_{-1}+1)$ [67]. This transformation is similar to (86), which explains the name of this equation.

4.10. The Merola–Ragnisco–Tu lattice.

- Equation [15], [68]:

$$u_t = u_1 - u^2 v,$$
$$v_t = -v_{-1} + v^2 u.$$

- Hamiltonian structure [15]:

$$\mathcal{H} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad f = u_1 v - \frac{u^2v^2}{2}.$$

- Recursion operator [15]:

$$\mathcal{R} = \begin{pmatrix} \mathcal{S} - 2uv & -u^2 \\ v^2 & \mathcal{S}^{-1} \end{pmatrix} + 2\begin{pmatrix} -u \\ v \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} v & u \end{pmatrix}.$$

- Nontrivial symmetry [15]:

$$\mathcal{R}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} u_2 - u_1^2 v_1 - u^2 v_{-1} - 2uvu_1 + u^3v^2 \\ -v_{-2} + v_{-1}^2 u_{-1} + v^2 u_1 + 2uvv_{-1} - u^2v^3 \end{pmatrix}.$$

- Master symmetry [15]:

$$\mathcal{R}\begin{pmatrix} (n+1)u \\ -nv \end{pmatrix} = \begin{pmatrix} nu_t + 2u_1 - 2u^2v - 2u(\mathcal{S}-1)^{-1}uv \\ nv_t + v_{-1} + uv^2 + 2v(\mathcal{S}-1)^{-1}uv \end{pmatrix}.$$

- Lax representation:

$$M = \begin{pmatrix} -1 & v \\ u & -2\lambda - uv \end{pmatrix}, \qquad U = \begin{pmatrix} -\lambda & -v_{-1} \\ -u & \lambda \end{pmatrix}.$$

The recursion operator $\mathcal{R}$ has a weakly nonlocal inverse:

$$\mathcal{R}^{-1} = \begin{pmatrix} \dfrac{1}{(u_{-1}v+1)^2}\mathcal{S}^{-1} & \dfrac{u_{-1}^2}{(u_{-1}v+1)^2} \\ -\dfrac{v_1^2}{(uv_1+1)^2} & \dfrac{1}{(uv_1+1)^2}\mathcal{S} - \dfrac{2u_{-1}v_1}{(u_{-1}v+1)(uv_1+1)} \end{pmatrix} + \\ + 2\begin{pmatrix} \dfrac{u_{-1}}{u_{-1}v+1} \\ -\dfrac{v_1}{uv_1+1} \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \dfrac{v_1}{uv_1+1} & \dfrac{u_{-1}}{u_{-1}v+1} \end{pmatrix}.$$

The symmetry $(u, -v)^{\mathrm{T}}$ is a seed for both $\mathcal{R}$ and $\mathcal{R}^{-1}$.

Under the invertible transformation $t \mapsto -t$, $u \mapsto -u$, and $v \mapsto v_1$, this lattice transforms into

$$u_t = -u_1 - u^2 v_1,$$
$$v_t = v_{-1} + v^2 u_{-1},$$

which is related to the NLS system $u_t = u_{xx} + 2u^2v$, $-v_t = v_{xx} + 2v^2u$, presented in [69].

4.11. The Kaup lattice.

- Equation [69]:

$$u_t = (u+v)(u_1 - u),$$
$$v_t = (u+v)(v - v_{-1}).$$

- Hamiltonian structure [69]:

$$\mathcal{H} = \begin{pmatrix} 0 & u+v \\ -(u+v) & 0 \end{pmatrix}, \qquad f = u_1 v - uv.$$

- Recursion operator:

$$\mathcal{R} = \begin{pmatrix} (u+v)\mathcal{S} + u_1 - u & 0 \\ 0 & (u+v)\mathcal{S}^{-1} + u_1 - u \end{pmatrix} + \begin{pmatrix} u_t \\ v_t \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \dfrac{1}{u+v} & \dfrac{1}{u+v} \end{pmatrix} + \\ + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathcal{S}(\mathcal{S}-1)^{-1} \begin{pmatrix} v_{-1} - v & u_1 - u \end{pmatrix}.$$

- Nontrivial symmetry:

$$\mathcal{R}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} (u+v)(uu_1 + uv_{-1} + u_1v_1 - u_1u_2 - u_2v_1 - u_1v_{-1}) \\ (u+v)(u_{-1}v_{-2} + v_{-2}v_{-1} - u_1v - u_{-1}v_{-1} + u_1v_{-1} - v_{-1}v) \end{pmatrix}.$$

- Lax representation [69]:

$$M = \begin{pmatrix} u-\lambda & uv + \lambda(u+v) + \lambda^2 \\ 1 & v - \lambda \end{pmatrix}, \qquad U = \begin{pmatrix} u & (u+\lambda)(v_{-1}+\lambda) \\ 1 & v_{-1} \end{pmatrix}.$$

There exists another weakly nonlocal recursion operator

$$\mathcal{R}' = \begin{pmatrix} \dfrac{u+v}{(u_{-1}+v)^2}\mathcal{S}^{-1} & -\dfrac{(u-u_{-1})}{(u_{-1}+v)^2} \\ -\dfrac{(v_1-v)}{(u+v_1)^2} & \dfrac{u+v}{(u+v_1)^2}\mathcal{S} + \dfrac{u-u_{-1}-v_1+v}{(u_{-1}+v)(u+v_1)} \end{pmatrix} -$$

$$- \begin{pmatrix} 1 \\ -1 \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \dfrac{1}{u+v_1} - \dfrac{1}{u+v} & \dfrac{1}{u_{-1}+v} - \dfrac{1}{u+v} \end{pmatrix} -$$

$$- \begin{pmatrix} \dfrac{u-u_{-1}}{u_{-1}+v} \\ \dfrac{v_1-v}{u+v_1} \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \dfrac{2}{u+v_1} - \dfrac{1}{u+v} & \dfrac{2}{u_{-1}+v} - \dfrac{1}{u+v} \end{pmatrix}.$$

The symmetry $(1,-1)^{\mathrm{T}}$ is a seed for both $\mathcal{R}$ and $\mathcal{R}'$.

4.12. The Ablowitz–Ladik lattice.

- Equation [70]:

$$\left.\begin{aligned} u_t &= (1-uv)(\alpha u_1 - \beta u_{-1}) \\ v_t &= (1-uv)(\beta v_1 - \alpha v_{-1}) \end{aligned}\right\} := \alpha K_1 + \beta K_{-1}.$$

- Hamiltonian structure [15]:

$$\mathcal{H} = \begin{pmatrix} 0 & 1-uv \\ -(1-uv) & 0 \end{pmatrix}, \qquad f = (\alpha u_1 - \beta u_{-1})v.$$

- Recursion operator [15], [71], [72]:

$$\mathcal{R} = \begin{pmatrix} (1-uv)\mathcal{S} - u_1 v - u v_{-1} & -u u_1 \\ v v_{-1} & (1-uv)\mathcal{S}^{-1} \end{pmatrix} + \begin{pmatrix} -u \\ v \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} v_{-1} & u_1 \end{pmatrix} -$$

$$- \begin{pmatrix} (1-uv)u_1 \\ -(1-uv)v_{-1} \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} \dfrac{v}{1-uv} & \dfrac{u}{1-uv} \end{pmatrix}.$$

- Nontrivial symmetry [15]:

$$\mathcal{R} \begin{pmatrix} (1-uv)u_1 \\ -(1-uv)v_{-1} \end{pmatrix} = \begin{pmatrix} (1-uv)((1-u_1v_1)u_2 - v u_1^2 - u u_1 v_{-1}) \\ (1-uv)(-(1-u_{-1}v_{-1})v_{-2} + u v_{-1}^2 + u_1 v_{-1} v) \end{pmatrix}.$$

- Master symmetry [15]:

$$\mathcal{R} \begin{pmatrix} nu \\ -nv \end{pmatrix} = \begin{pmatrix} (n+1)(1-uv)u_1 - u^2 v_{-1} - u(\mathcal{S}-1)^{-1} u v_{-1} \\ (1-n)(1-uv)v_{-1} + u v v_{-1} + v(\mathcal{S}-1)^{-1} u v_{-1} \end{pmatrix}.$$

- Lax representation:

$$M=\begin{pmatrix}\lambda & u\\ v & \dfrac{1}{\lambda}\end{pmatrix},\qquad U=\alpha\begin{pmatrix}\lambda^2-uv_{-1} & \lambda u\\ \lambda v_{-1} & 0\end{pmatrix}+\beta\begin{pmatrix}0 & \lambda^{-1}u_{-1}\\ \lambda^{-1}v & \lambda^{-2}-u_{-1}v\end{pmatrix}.$$

The coefficients of α and β, namely, K_1 and K_{-1}, are commuting symmetries for the equation. The inverse of the recursion operator $\mathcal{R}$ is weakly nonlocal:

$$\mathcal{R}^{-1}=\begin{pmatrix}(1-uv)\mathcal{S}^{-1} & uu_{-1}\\ -vv_1 & (1-uv)\mathcal{S}-uv_1-u_{-1}v\end{pmatrix}+\begin{pmatrix}u\\ -v\end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix}v_1 & u_{-1}\end{pmatrix}+$$
$$+\begin{pmatrix}(1-uv)u_{-1}\\ -(1-uv)v_1\end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix}\dfrac{v}{1-uv} & \dfrac{u}{1-uv}\end{pmatrix}.$$

Both $\mathcal{R}$ and $\mathcal{R}^{-1}$ share the common seed $\sigma=\left(\begin{smallmatrix}u\\ -v\end{smallmatrix}\right)$. Starting from it, we can generate the commuting symmetries $\mathcal{R}^{-i}(\sigma)$ and $\mathcal{R}^{i}(\sigma)$ for $i\in\mathbb{N}$. More Lie algebra structure among the symmetries and master symmetries can be found in [15].

4.13. The Bruschi–Ragnisco lattice.

- Equation [73], [66], [41]:

$$u_t=u_1v-uv_{-1},$$
$$v_t=v(v-v_{-1}).$$

- Hamiltonian structure [73], [15]:

$$\mathcal{H}_1=\begin{pmatrix}0 & (1-\mathcal{S}^{-1})v\\ v(\mathcal{S}-1) & 0\end{pmatrix},\qquad f_1=u_1v,$$
$$\mathcal{H}_2=\begin{pmatrix}v\mathcal{S}u-u\mathcal{S}^{-1}v & v(\mathcal{S}-1)v\\ v(1-\mathcal{S}^{-1})v & 0\end{pmatrix},\qquad f_2=u.$$

- Recursion operator [41], [15]:

$$\mathcal{R}=\mathcal{H}_2\mathcal{H}_1^{-1}=\begin{pmatrix}v\mathcal{S} & u_1+u\mathcal{S}^{-1}+(u_1v-uv_{-1})(\mathcal{S}-1)^{-1}\dfrac{1}{v}\\ 0 & v\mathcal{S}^{-1}+v(v-v_{-1})(\mathcal{S}-1)^{-1}\dfrac{1}{v}\end{pmatrix}=$$
$$=\begin{pmatrix}v\mathcal{S} & u_1+u\mathcal{S}^{-1}\\ 0 & v\mathcal{S}^{-1}\end{pmatrix}+\begin{pmatrix}u_t\\ v_t\end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix}0 & \dfrac{1}{v}\end{pmatrix}.$$

- Nontrivial symmetry [15]:

$$\mathcal{R}\begin{pmatrix}u_t\\ v_t\end{pmatrix}=\begin{pmatrix}vv_1u_2-v_{-2}v_{-1}u\\ v(vv_{-1}-v_{-1}v_{-2})\end{pmatrix}.$$

- Master symmetry [15]:

$$\mathcal{R}\begin{pmatrix}u\\ v\end{pmatrix}=\begin{pmatrix}nu_t+2u_1v+u_{-1}u\\ nv_t+vv_{-1}\end{pmatrix}.$$

The recursion operator $\mathcal{R}$ has a weakly nonlocal inverse:

$$\mathcal{R}^{-1} = \mathcal{H}_1\mathcal{H}_2^{-1} = \begin{pmatrix} \mathcal{S}^{-1}\dfrac{1}{v} & -\mathcal{S}^{-1}\dfrac{u}{v^2} - \dfrac{u}{v^2} + \left(\dfrac{u_{-1}}{v_{-1}} - \dfrac{u}{v}\right)(\mathcal{S}-1)^{-1}\dfrac{1}{v} \\ 0 & v\mathcal{S}\dfrac{1}{v^2} - \dfrac{1}{v} + \dfrac{1}{v_1} + \left(\dfrac{v}{v_1} - 1\right)(\mathcal{S}-1)^{-1}\dfrac{1}{v} \end{pmatrix} =$$

$$= \begin{pmatrix} \mathcal{S}^{-1}\dfrac{1}{v} & -\mathcal{S}^{-1}\dfrac{u}{v^2} - \dfrac{u}{v^2} \\ 0 & v\mathcal{S}\dfrac{1}{v^2} - \dfrac{1}{v} + \dfrac{1}{v_1} \end{pmatrix} + \begin{pmatrix} \dfrac{u_{-1}}{v_{-1}} - \dfrac{u}{v} \\ \dfrac{v}{v_1} - 1 \end{pmatrix} (\mathcal{S}-1)^{-1} \begin{pmatrix} 0 & \dfrac{1}{v} \end{pmatrix}.$$

But the recursion operators $\mathcal{R}$ and $\mathcal{R}^{-1}$ have different seeds similar to those in the relativistic Toda system in Sec. 4.7. The seed for $\mathcal{R}$ is the equation itself, while the seed for $\mathcal{R}^{-1}$ is

$$\sigma = \begin{pmatrix} \dfrac{u_{-1}}{v_{-1}} - \dfrac{u}{v} \\ \dfrac{v}{v_1} - 1 \end{pmatrix}.$$

The operator $\mathcal{R}^{-1}$ acting on the equation itself and $\mathcal{R}$ acting on σ do not generate new symmetries. More Lie algebra structure among the symmetries and master symmetries can be found in [15].

In fact, the Bruschi–Ragnisco lattice is trivially solvable, and so are its symmetry flows. The equation for the second component v_t is independent of the first component u. Moreover, the scalar lattice $v_t = v(v - v_{-1})$ and $v_\tau = v/v_1 - 1$ can be respectively linearized into $w_t = w_{-1}$ and $w_t = w_1$ by the transformation $v = -w_{-1}/w$. Once it is solved, the equation for u is then linear.

4.14. The Kaup–Newell lattice.

- Equation [74]:

$$\left.\begin{aligned} u_t &= a\left(\frac{u_1}{1-u_1v_1} - \frac{u}{1-uv}\right) + b\left(\frac{u}{1+uv_1} - \frac{u_{-1}}{1+u_{-1}v}\right) \\ v_t &= a\left(\frac{v}{1-uv} - \frac{v_{-1}}{1-u_{-1}v_{-1}}\right) + b\left(\frac{v_1}{1+uv_1} - \frac{v}{1+u_{-1}v}\right) \end{aligned}\right\} := aK_1 + bK_{-1}.$$

- Hamiltonian structure [74]:

$$\mathcal{H} = \begin{pmatrix} 0 & \mathcal{S}-1 \\ 1-\mathcal{S}^{-1} & 0 \end{pmatrix}, \qquad f = -a\log(1-uv) + b\log(1+uv_1).$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}\mathcal{J} = \begin{pmatrix} -\dfrac{1}{(1-u_1v_1)^2}\mathcal{S} + \dfrac{1}{(1-uv)^2} - \dfrac{2u_1v}{(1-u_1v_1)(1-uv)} & -\dfrac{u_1^2}{(1-u_1v_1)^2}\mathcal{S} + \dfrac{u^2}{(1-uv)^2} - \dfrac{2uu_1}{(1-uv)(1-u_1v_1)} \\ -\dfrac{v_{-1}^2}{(1-u_{-1}v_{-1})^2}\mathcal{S}^{-1} - \dfrac{v^2}{(1-uv)^2} & -\dfrac{1}{(1-u_{-1}v_{-1})^2}\mathcal{S}^{-1} + \dfrac{1-2uv}{(1-uv)^2} \end{pmatrix} -$$

$$-2K_1(\mathcal{S}-1)^{-1}\begin{pmatrix} \dfrac{v}{1-uv} & \dfrac{u}{1-uv} \end{pmatrix},$$

where the symplectic operator $\mathcal{J}$ is defined by

$$\mathcal{J} = \begin{pmatrix} 0 & \frac{1}{1-uv} \\ -\frac{1}{1-uv} & 0 \end{pmatrix} - \begin{pmatrix} \frac{v}{1-uv} \\ \frac{u}{1-uv} \end{pmatrix} (\mathcal{S}+1)(\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{v}{1-uv} & \frac{u}{1-uv} \end{pmatrix}.$$

- Nontrivial symmetry:

$$\mathcal{R}(K_1) = \begin{pmatrix} \frac{1}{(1-u_1v_1)^2}\left(u_1 - \frac{u_2}{1-u_2v_2} - \frac{u_1^2 v}{1-uv}\right) - \\ \qquad - \frac{1}{(1-uv)^2}\left(u - \frac{u_1}{1-u_1v_1} - \frac{u^2 v_{-1}}{1-u_{-1}v_{-1}}\right) \\ \frac{1}{(1-uv)^2}\left(v - \frac{u_1 v^2}{1-u_1v_1} - \frac{v_{-1}}{1-u_{-1}v_{-1}}\right) - \\ \qquad - \frac{1}{(1-u_{-1}v_{-1})^2}\left(v_{-1} - \frac{uv_{-1}^2}{1-uv} - \frac{v_{-2}}{1-u_{-2}v_{-2}}\right) \end{pmatrix}.$$

- Lax representation [74]:

$$M = \begin{pmatrix} \lambda + (1-\lambda)uv & u \\ (1-\lambda)v & 1 \end{pmatrix},$$

$$U = a \begin{pmatrix} \frac{\lambda-1}{2} & \frac{u}{1-uv} \\ \frac{(1-\lambda)v_{-1}}{1-u_{-1}v_{-1}} & -\frac{\lambda-1}{2} \end{pmatrix} + $$

$$+ \frac{b}{2\lambda(1+u_{-1}v)} \begin{pmatrix} (\lambda-1)(1-u_{-1}v) & 2u_{-1} \\ 2(1-\lambda)v & (1-\lambda)(1-u_{-1}v) \end{pmatrix}.$$

The recursion operator $\mathcal{R}$ has the seed $\sigma = \binom{-u}{v}$ and $\mathcal{R}(\sigma) = K_1$ with zero taken as the integration constant. As with the Ablowitz–Ladik lattice in Sec. 4.12, the coefficients of a and b, namely, K_1 and K_{-1}, are commuting symmetries of the equation. Indeed, there exists another weakly nonlocal recursion operator

$$\mathcal{R}' = \mathcal{H}\mathcal{J}' = \begin{pmatrix} \frac{1}{(1+u_{-1}v)^2}\mathcal{S}^{-1} - \frac{1+2uv_1}{(1+uv_1)^2} & -\frac{u^2}{(1+uv_1)^2}\mathcal{S} + \frac{u_{-1}^2}{(1+u_{-1}v)^2} - \\ & \qquad - \frac{2uu_{-1}}{(1+u_{-1}v)(1+uv_1)} \\ -\frac{v^2}{(1+u_{-1}v)^2}\mathcal{S}^{-1} - \frac{v_1^2}{(1+uv_1)^2} & \frac{1}{(1+uv_1)^2}\mathcal{S} - \frac{1}{(1+u_{-1}v)^2} - \\ & \qquad - \frac{2u_{-1}v_1}{(1+u_{-1}v)(1+uv_1)} \end{pmatrix} -$$

$$- 2K_{-1}(\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{v_1}{1+uv_1} & \frac{u_{-1}}{1+u_{-1}v} \end{pmatrix},$$

where the symplectic operator $\mathcal{J}'$ is

$$\mathcal{J}' = \begin{pmatrix} 0 & \frac{1}{1+uv_1}(\mathcal{S}-u_{-1}v_1)\frac{1}{1+u_{-1}v} \\ \frac{1}{1+u_{-1}v}(u_{-1}v_1-\mathcal{S}^{-1})\frac{1}{1+uv_1} & 0 \end{pmatrix} -$$

$$- \begin{pmatrix} \frac{v_1}{1+uv_1} \\ \frac{u_{-1}}{1+u_{-1}v} \end{pmatrix} (\mathcal{S}+1)(\mathcal{S}-1)^{-1} \begin{pmatrix} \frac{v_1}{1+uv_1} & \frac{u_{-1}}{1+u_{-1}v} \end{pmatrix}.$$

Again, σ is a seed for $\mathcal{R}'$ and $\mathcal{R}'(\sigma) = K_{-1}$ taking

$$(\mathcal{S}-1)^{-1}\left(-\frac{uv_1}{1+uv_1} + \frac{u_{-1}v}{1+u_{-1}v}\right) = -\frac{u_{-1}v}{1+u_{-1}v}.$$

By direct calculation, we have $\mathcal{J}'\mathcal{R} = \mathcal{J}\mathcal{R}' = \mathcal{J}' - \mathcal{J}$. Hence, these two recursion operators satisfy the relations $\mathcal{R}'\mathcal{R} = \mathcal{R}\mathcal{R}' = \mathcal{R}' - \mathcal{R}$, i.e., $(\mathcal{R}'-\mathrm{id})(\mathcal{R}-\mathrm{id}) = (\mathcal{R}-\mathrm{id})(\mathcal{R}'-\mathrm{id}) = \mathrm{id}$.

4.15. The Chen–Lee–Liu lattice.

- Equation [74]:

$$\left.\begin{aligned} u_t &= a(1+uv)(u_1-u) + b(1+u_{-1}v)^{-1}(u-u_{-1}) \\ v_t &= a(1+uv)(v-v_{-1}) + b(1+uv_1)^{-1}(v_1-v) \end{aligned}\right\} := aK_1 + bK_{-1}.$$

- Hamiltonian structure:

$$\mathcal{H} = \begin{pmatrix} 0 & 1+uv \\ -(1+uv) & 0 \end{pmatrix}, \qquad f = a(uv_{-1}-uv) + b\log\frac{1+uv}{1+uv_1}.$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}^{-1} = \begin{pmatrix} (1+uv)\mathcal{S} - 2uv + u_1v + uv_{-1} & uu_1 - u^2 \\ v^2 - vv_{-1} & (1+uv)\mathcal{S}^{-1} \end{pmatrix} +$$

$$+ K_1(\mathcal{S}-1)^{-1}\begin{pmatrix} \frac{v}{1+uv} & \frac{u}{1+uv} \end{pmatrix} - \begin{pmatrix} u \\ -v \end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix} v-v_{-1} & u-u_1 \end{pmatrix},$$

where the Hamiltonian operator $\mathcal{H}_2$ is given by

$$\mathcal{H}_2 = \begin{pmatrix} 0 & (1+uv)(\mathcal{S}(1+uv) - uv + u_1v) \\ (uv - u_1v - (1+uv)\mathcal{S}^{-1})(1+uv) & 0 \end{pmatrix} -$$

$$- K_1(\mathcal{S}-1)^{-1}\begin{pmatrix} u & -v \end{pmatrix} - \begin{pmatrix} u \\ -v \end{pmatrix}\mathcal{S}(\mathcal{S}-1)^{-1}K_1^{\mathrm{T}}.$$

- Nontrivial symmetry :

$$\mathcal{R}(K_1) = \begin{pmatrix} (1+uv)(u_1u_2v_1 + u_2 + u_1^2v + uu_1v_{-1} + \\ \qquad + u^2v - u_1 - u^2v_{-1} - 2uu_1v - u_1^2v_1) \\ (1+uv)(u_1v^2 + 2uv_{-1}v + u_{-1}v_{-1}^2 + v_{-1} - v_{-2} - \\ \qquad - uv^2 - uv_{-1}^2 - u_1v_{-1}v - u_{-1}v_{-2}v_{-1}) \end{pmatrix}.$$

- Lax representation [74]:

$$M = \begin{pmatrix} \lambda + uv & u \\ (1-\lambda)v & 1 \end{pmatrix},$$

$$U = a\begin{pmatrix} \lambda - 1 + uv_{-1} & u \\ (1-\lambda)v_{-1} & 0 \end{pmatrix} + \frac{b}{\lambda(1+u_{-1}v)}\begin{pmatrix} u_{-1}v & u_{-1} \\ (1-\lambda)v & 1-\lambda \end{pmatrix}.$$

The coefficients for a and b, namely, K_1 and K_{-1}, are commuting symmetries for the equation. The above recursion operator $\mathcal{R}$ has a seed $\sigma = \binom{-u}{v}$ and $\mathcal{R}(\sigma) = K_1$.

There exists another weakly nonlocal recursion operator

$$\mathcal{R}' = \mathcal{H}_2'\mathcal{H}^{-1} = \begin{pmatrix} \dfrac{1+uv}{(1+u_{-1}v)^2}\mathcal{S}^{-1} & -\dfrac{u_{-1}(u-u_{-1})}{(1+u_{-1}v)^2} \\ -\dfrac{v_1(v_1-v)}{(1+uv_1)^2} & \dfrac{1+uv}{(1+uv_1)^2}\mathcal{S} + \dfrac{v_1u - 2u_{-1}v_1 + u_{-1}v}{(1+u_{-1}v)(1+uv_1)} \end{pmatrix} -$$

$$- K_{-1}(\mathcal{S}-1)^{-1}\begin{pmatrix} \dfrac{2v_1}{1+uv_1} - \dfrac{v}{1+uv} & \dfrac{2u_{-1}}{1+u_{-1}v} - \dfrac{u}{1+uv} \end{pmatrix} -$$

$$- \begin{pmatrix} -u \\ v \end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix} \dfrac{v_1}{1+uv_1} - \dfrac{v}{1+uv} & \dfrac{u_{-1}}{1+u_{-1}v} - \dfrac{u}{1+uv} \end{pmatrix},$$

where the Hamiltonian operator $\mathcal{H}_2'$ is

$$\mathcal{H}_2' = \begin{pmatrix} 0 & \dfrac{1+uv}{1+u_{-1}v}\left(\mathcal{S}^{-1}\dfrac{1+uv}{1+uv_1} + \dfrac{v_1(u-u_{-1})}{1+uv_1}\right) \\ -\left(\dfrac{1+uv}{1+uv_1}\mathcal{S} + \dfrac{v_1(u-u_{-1})}{1+uv_1}\right)\dfrac{1+uv}{1+u_{-1}v} & 0 \end{pmatrix} -$$

$$- K_{-1}(\mathcal{S}+1)(\mathcal{S}-1)^{-1}K_{-1}^{\mathrm{T}} -$$

$$- K_{-1}\mathcal{S}(\mathcal{S}-1)^{-1}\begin{pmatrix} -u & v \end{pmatrix} - \begin{pmatrix} -u \\ v \end{pmatrix}(\mathcal{S}-1)^{-1}K_{-1}^{\mathrm{T}}.$$

Again, σ is a seed for $\mathcal{R}'$.

4.16. The Ablowitz–Ramani–Segur (Gerdjikov–Ivanov) lattice.

- Equation [74]:

$$\left.\begin{aligned} u_t &= (au_1 - bu_{-1})(1+uv)(1-uv_1) \\ v_t &= (bv_1 - av_{-1})(1+uv)(1-u_{-1}v) \end{aligned}\right\} := aK_1 + bK_{-1}.$$

- Symplectic operator:

$$\mathcal{J} = \begin{pmatrix} 0 & \dfrac{1}{1-uv_1}\mathcal{S} - \dfrac{1}{1+uv} \\ \dfrac{1}{1+uv} - \mathcal{S}^{-1}\dfrac{1}{1-uv_1} & 0 \end{pmatrix},$$

and we have

$$\mathcal{J}(aK_1 + bK_{-1}) = \delta_{(u,v)}(a(uv_{-1} - uv - uvu_1v_1) + b(u_{-1}v_1 - uv_1 + u_{-1}uvv_1)).$$

- Hamiltonian structure:

$$\mathcal{H} = \begin{pmatrix} 0 & (1+uv)(\mathcal{S}(1+uv) + uv_{-1})(1-u_{-1}v) \\ -(1-u_{-1}v)((1+uv)\mathcal{S}^{-1} + uv_{-1})(1+uv) & 0 \end{pmatrix} -$$

$$- K_1\mathcal{S}(\mathcal{S}-1)^{-1}\begin{pmatrix} u & -v \end{pmatrix} - \begin{pmatrix} u \\ -v \end{pmatrix}(\mathcal{S}-1)^{-1}K_1^{\mathrm{T}}.$$

- Recursion operator:

$$\mathcal{R} = \begin{pmatrix} (1+uv)(1-uv_1)\mathcal{S} + u_1v - u_1v_1 + \\ \quad + uv_{-1} - uv(1+u_{-1}v_{-1} + 2u_1v_1) & -uu_1(1+uv)\mathcal{S} - u^2(1+u_{-1}v_{-1}) + \\ & \quad + \dfrac{1-uv_1}{1-u_{-1}v}u_1(u - u_{-1} - 2uu_{-1}v) \\ -(1-u_{-1}v)v_{-1}v - (1+uv)v_{-1}v\mathcal{S}^{-1} & (1+uv)(1-u_{-1}v)\mathcal{S}^{-1} + uvu_{-1}v_{-1} \end{pmatrix} +$$

$$+ \begin{pmatrix} u_1(1+uv)(1-uv_1) \\ -v_{-1}(1+uv)(1-u_{-1}v) \end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix} \dfrac{v}{1+uv} - \dfrac{v_1}{1-uv_1} & \dfrac{u}{1+uv} - \dfrac{u_{-1}}{1-u_{-1}v} \end{pmatrix} -$$

$$- \begin{pmatrix} u \\ -v \end{pmatrix}(\mathcal{S}-1)^{-1}\begin{pmatrix} v - v_{-1} + u_{-1}v_{-1}v + u_1vv_1 & u - u_1 + uu_{-1}v_{-1} + uu_1v_1 \end{pmatrix} =$$

$$= \mathcal{H}\mathcal{J} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

- Nontrivial symmetry:

$$\mathcal{R}(K_1) = \begin{pmatrix} (1+uv)(1-uv_1)((1+u_1v_1)(1-u_1v_2)u_2 - u_1^2v_1(1+uv) + \\ \quad + uu_1v_{-1}(1-u_{-1}v) - u_1v(u-u_1)) \\ (1+uv)(u_{-1}v-1)((1+u_{-1}v_{-1})(1-u_{-2}v_{-1})v_{-2} - \\ \quad - u_{-1}v_{-1}^2(1+uv) + u_1v_{-1}v(1-uv_1) + uv_{-1}(v_{-1}-v)) \end{pmatrix}.$$

- Lax representation [74]:

$$M = \begin{pmatrix} \lambda + uv & u \\ (1-\lambda)(1-uv_1)v & 1-uv_1 \end{pmatrix},$$

$$U = a\begin{pmatrix} \lambda + uv_{-1}(1-u_{-1}v) & u \\ (1-\lambda)(1-u_{-1}v)v_{-1} & -uv \end{pmatrix} +$$

$$+ \frac{b}{\lambda}\begin{pmatrix} u_{-1}v - \lambda & u_{-1} \\ (1-\lambda)(1-u_{-1}v)v & (1-\lambda)(1-u_{-1}v) - \lambda u_{-1}v_1(1+uv) \end{pmatrix}.$$

The equation given in [74] is

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = (a-b)\begin{pmatrix} -u \\ v \end{pmatrix} + aK_1 + bK_{-1}.$$

Because the vector $\sigma = \left(\begin{smallmatrix} -u \\ v \end{smallmatrix}\right)$ commutes with both K_1 and K_{-1}, we remove this term in our consideration. There exists another weakly nonlocal recursion operator

$$\mathcal{R}' = \begin{pmatrix} (1+uv)(1-uv_1)\mathcal{S}^{-1} + uvu_{-1}v_1 & \begin{matrix} uu_{-1}(1+uv)\mathcal{S} + uu_{-2}(1+u_{-1}v_{-1}) - \\ - \dfrac{1-uv_1}{1-u_{-1}v}u_{-1}(u-u_{-1}-2uu_{-1}v) \end{matrix} \\ (1-u_{-1}v)v_1v + (1+uv)v_1v\mathcal{S}^{-1} & \begin{matrix} (1+uv)(1-u_{-1}v)\mathcal{S} + uv_1 - 2u_{-1}uvv_1 - \\ - u_{-1}v_1 + u_{-1}v - u_{-2}v(1+u_{-1}v_{-1}) \end{matrix} \end{pmatrix} +$$

$$+ \begin{pmatrix} -u_{-1}(1+uv)(1-uv_1) \\ v_1(1+uv)(1-u_{-1}v) \end{pmatrix} (\mathcal{S}-1)^{-1} \times$$

$$\times \begin{pmatrix} \dfrac{v}{1+uv} - \dfrac{v_1}{1-uv_1} & \dfrac{u}{1+uv} - \dfrac{u_{-1}}{1-u_{-1}v} \end{pmatrix} + \begin{pmatrix} u \\ -v \end{pmatrix} (\mathcal{S}-1)^{-1} \times$$

$$\times \begin{pmatrix} v_2(1+u_1v_1) + v_1(u_{-1}v-1) & u_{-2}(1+u_{-1}v_{-1}) + u_{-1}(uv_1-1) \end{pmatrix} =$$

$$= \mathcal{H}'\mathcal{J} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where the Hamiltonian operator $\mathcal{H}'$ is

$$\mathcal{H}' = \begin{pmatrix} 0 & -(1+uv)(1-u_{-1}v) \\ (1+uv)(1-u_{-1}v) & 0 \end{pmatrix} -$$

$$- K_{-1}\mathcal{S}(\mathcal{S}-1)^{-1}\begin{pmatrix} u & -v \end{pmatrix} - \begin{pmatrix} u \\ -v \end{pmatrix}(\mathcal{S}-1)^{-1}K_{-1}^{\mathrm{T}}.$$

The operator $\mathcal{R}'$ is the inverse operator of $\mathcal{R}$. The vector σ is the seed for both of them and, $\mathcal{R}'(K_{-1})$ is

$$\mathcal{R}'(K_{-1}) = \begin{pmatrix} \begin{matrix} (1-uv_1)(1+uv)((1+u_{-1}v_{-1})(1-u_{-1}v)u_{-2} - \\ - uu_{-1}v_2(1+u_1v_1) - u_{-1}^2v_1(1+uv) + u_{-1}(u_{-1}v+uv_1)) \end{matrix} \\ \begin{matrix} (1-u_{-1}v)(1+uv)(-(1+u_1v_1)(1-uv_1)v_2 + \\ + u_{-2}vv_1(1+u_{-1}v_{-1}) + u_{-1}v_1(v_1vu+v_1-v) - uv_1^2) \end{matrix} \end{pmatrix}.$$

4.17. The Heisenberg ferromagnet lattice.

- Equation [14]:

$$u_t = (u-v)(u-u_1)(u_1-v)^{-1},$$

$$v_t = (u-v)(v_{-1}-v)(u-v_{-1})^{-1}.$$

- Hamiltonian structure:

$$\mathcal{H} = \begin{pmatrix} 0 & (u-v)^2 \\ -(u-v)^2 & 0 \end{pmatrix}, \qquad f = \log(u-v) - \log(u-v_{-1}).$$

- Symplectic operator:

$$\mathcal{J} = \begin{pmatrix} 0 & -(u-v_{-1})^{-2}\mathcal{S}^{-1} + \dfrac{(u-u_1)(v-v_{-1})}{(u-v)^2(u-v_{-1})(u_1-v)} \\ (u_1-v)^{-2}\mathcal{S} - \dfrac{(u-u_1)(v-v_{-1})}{(u-v)^2(u-v_{-1})(u_1-v)} & 0 \end{pmatrix} -$$

$$- \begin{pmatrix} \dfrac{v-v_{-1}}{(u-v)(u-v_{-1})} \\ \dfrac{u-u_1}{(u-v)(u_1-v)} \end{pmatrix} (\mathcal{S}+1)(\mathcal{S}-1)^{-1} \left(\dfrac{v-v_{-1}}{(u-v)(u-v_{-1})} \quad \dfrac{u-u_1}{(u-v)(u_1-v)} \right).$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}\mathcal{J} = \begin{pmatrix} \dfrac{(u-v)^2}{(u_1-v)^2}\mathcal{S} - \dfrac{2(u-u_1)(v-v_{-1})}{(u-v_{-1})(u_1-v)} & -\dfrac{(u-u_1)^2}{(u_1-v)^2} \\ \dfrac{(v-v_{-1})^2}{(u-v_{-1})^2} & \dfrac{(u-v)^2}{(u-v_{-1})^2}\mathcal{S}^{-1} \end{pmatrix} -$$

$$- 2 \begin{pmatrix} u_t \\ v_t \end{pmatrix} (\mathcal{S}-1)^{-1} \left(\dfrac{v-v_{-1}}{(u-v)(u-v_{-1})} \quad \dfrac{u-u_1}{(u-v)(u_1-v)} \right).$$

- Nontrivial symmetry:

$$\mathcal{R}\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \dfrac{u-v}{(u_1-v)^2}\left(\dfrac{(u-v)(u_1-v_1)(u_1-u_2)}{u_2-v_1} + \dfrac{(u-u_1)^2(v_{-1}-v)}{u-v_{-1}} \right) \\ \dfrac{u-v}{(u-v_{-1})^2}\left(\dfrac{(u-v)(u_{-1}-v_{-1})(v_{-2}-v_{-1})}{u_{-1}-v_{-2}} + \dfrac{(v-v_{-1})^2(u-u_1)}{u_1-v} \right) \end{pmatrix}.$$

- Lax representation:

$$M = \begin{pmatrix} \lambda - 2u(u-v)^{-1} & -2(u-v)^{-1} \\ 2uv(u-v)^{-1} & \lambda + 2v(u-v)^{-1} \end{pmatrix},$$

$$U = \lambda^{-1}(u-v_{-1})^{-1} \begin{pmatrix} u+v_{-1} & 2 \\ -2uv_{-1} & -(u+v_{-1}) \end{pmatrix}.$$

The recursion operator $\mathcal{R}$ has a weakly nonlocal inverse:

$$\mathcal{R}^{-1} = \mathcal{H}\mathcal{J}' = \begin{pmatrix} \dfrac{(u-v)^2}{(u_{-1}-v)^2}\mathcal{S}^{-1} & \dfrac{(u-u_{-1})^2}{(u_{-1}-v)^2} \\ -\dfrac{(v-v_1)^2}{(u-v_1)^2} & \dfrac{(u-v)^2}{(u-v_1)^2}\mathcal{S} - \dfrac{2(u-u_{-1})(v-v_1)}{(u-v_1)(u_{-1}-v)} \end{pmatrix} -$$

$$- 2 \begin{pmatrix} \dfrac{(u-v)(u_{-1}-u)}{u_{-1}-v} \\ \dfrac{(u-v)(v-v_1)}{u-v_1} \end{pmatrix} (\mathcal{S}-1)^{-1} \left(\dfrac{v-v_1}{(u-v)(u-v_1)} \quad \dfrac{u-u_{-1}}{(u-v)(u_{-1}-v)} \right),$$

where the symplectic operator $\mathcal{J}'$ is given by

$$\mathcal{J}' = \begin{pmatrix} 0 & -(u-v_1)^{-2}\mathcal{S} + \dfrac{(u-u_{-1})(v-v_1)}{(u-v)^2(u-v_1)(u_{-1}-v)} \\ (u_{-1}-v)^{-2}\mathcal{S}^{-1} - \dfrac{(u-u_{-1})(v-v_1)}{(u-v)^2(u-v_1)(u_{-1}-v)} & 0 \end{pmatrix} +$$

$$+ \begin{pmatrix} \dfrac{v-v_1}{(u-v)(u-v_1)} \\ \dfrac{u-u_{-1}}{(u-v)(u_{-1}-v)} \end{pmatrix} (\mathcal{S}+1)(\mathcal{S}-1)^{-1} \begin{pmatrix} \dfrac{v-v_1}{(u-v)(u-v_1)} & \dfrac{u-u_{-1}}{(u-v)(u_{-1}-v)} \end{pmatrix}.$$

The Heisenberg ferromagnet lattice is a special case (if we set $a=1$, $b=0$, and $h(u,v)=(u-v)^2/2$) of the Landau–Lifshitz (Sklyanin) chain [14], [75], [76]:

$$\begin{aligned} u_t &= a\left(\frac{2h}{u_1-v}+h_v\right)+b\left(\frac{2h}{u_{-1}-v}+h_v\right), \\ v_t &= a\left(\frac{2h}{u-v_{-1}}-h_u\right)+b\left(\frac{2h}{u-v_1}-h_u\right), \end{aligned} \tag{87}$$

where

$$h(u,v)=\frac{i}{4}(K_1(1-uv)^2-K_2(1+uv)^2+K_3(u+v)^2), \qquad K_1K_2K_3\neq 0, \quad K_n\in\mathbb{R}.$$

We note that the coefficients of a and b in the Landau–Lifshitz (Sklyanin) chain (87) commute.

The Lax representation for Eq. (87) was obtained in [75] in the form

$$U = i\sum_{k=1}^{3} aN_k^+(\lambda)S_k(u,v_{-1})\sigma_k + bN_k^-(\lambda)S_k(v,u_{-1})\sigma_k,$$

$$M = \frac{1}{\sqrt{\langle \mathbf{SKS}\rangle}}\left(I-\sum_{k=1}^{3} M_k(\lambda)S_k(u,v)\sigma_k\right).$$

Here, σ_k are the Pauli matrices

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}, \qquad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}, \qquad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

the vector function $S(p,q)$ is defined as

$$(S_1(p,q),S_2(p,q),S_3(p,q))=\left(\frac{1-pq}{p-q},\frac{i+ipq}{p-q},\frac{p+q}{p-q}\right),$$

satisfying $S_1^2+S_2^2+S_3^2=1$,

$$\langle \mathbf{SKS}\rangle=\sum_{l=1}^{3} K_l S_l^2(u,v),$$

and $M_l(\lambda), N_l^{\pm}(\lambda)$ can be expressed in terms of Jacobi elliptic functions of the spectral parameter λ:

$$M_1(\lambda)=\sqrt{1-K_1K_2^{-1}}\,\mathrm{sn}(\lambda,\kappa), \qquad N_1^{\pm}(\lambda)=\frac{\sqrt{K_2K_3}}{2}M_1(\lambda\pm\mu),$$

$$M_2(\lambda)=\sqrt{1-K_2K_1^{-1}}\,\mathrm{cn}(\lambda,\kappa), \qquad N_2^{\pm}(\lambda)=\frac{\sqrt{K_1K_3}}{2\kappa}M_2(\lambda\pm\mu),$$

$$M_3(\lambda)=\sqrt{1-K_3K_1^{-1}}\,\mathrm{dn}(\lambda,\kappa), \qquad N_3^{\pm}(\lambda)=\frac{\sqrt{K_1K_2}}{2}M_3(\lambda\pm\mu),$$

where κ and μ are defined by the equations

$$\kappa = \sqrt{\frac{K_3(K_1 - K_2)}{K_2(K_1 - K_3)}}, \qquad \mathrm{cn}(\mu, \kappa) = \frac{K_1}{K_3}.$$

Instead of explicit uniformization of the elliptic curve, we can use the identities

$$\frac{M_l^2 - 1}{K_l} = \frac{M_j^2 - 1}{K_j}, \qquad N_l^{\pm} = \frac{K_1}{2(M_1^2 - 1)}(M_l \pm M_j M_k),$$
$$l, j, k \in \{1, 2, 3\}, \qquad l \neq j \neq k \neq l.$$

One local Hamiltonian operator for Eq. (87) has the form [75]

$$\mathcal{H} = h(u, v) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad f = a \log \frac{h(u, v)}{(u_1 - v)^2} + b \log \frac{h(u, v)}{(u_{-1} - v)^2}.$$

Knowing the Lax representation, we can, in principle, compute its recursion operators (as done in Sec. 3.2) with the multipliers

$$\mu_{\pm}(\lambda) = (N_1^{\pm}(\lambda))^2, \qquad \nu_{\pm}(\lambda) = N_1^{\pm}(\lambda) N_2^{\pm}(\lambda) N_3^{\pm}(\lambda).$$

But the calculations involved are rather large and we have not obtained a compact representation of the operators.

4.18. The Belov–Chaltikian lattice.

- Equation [77]:

$$u_t = u(v_2 - v_{-1}),$$
$$v_t = u_{-1} - u + v(v_1 - v_{-1}).$$

- Hamiltonian structure [77]:

$$\mathcal{H}_1 = \begin{pmatrix} u(\mathcal{S} - \mathcal{S}^{-1})(\mathcal{S} + 1 + \mathcal{S}^{-1})u & u(\mathcal{S} - 1)(\mathcal{S} + 1 + \mathcal{S}^{-1})v \\ v(1 - \mathcal{S}^{-1})(\mathcal{S} + 1 + \mathcal{S}^{-1})u & v(\mathcal{S} - \mathcal{S}^{-1})v + \mathcal{S}^{-1}u - u\mathcal{S} \end{pmatrix}, \qquad f_1 = v,$$

$$\mathcal{H}_2 = \begin{pmatrix} 0 & u(1 + \mathcal{S} + \mathcal{S}^2)(u\mathcal{S} - \mathcal{S}^{-2}u) \\ (u\mathcal{S}^2 - \mathcal{S}^{-1}u)(1 + \mathcal{S}^{-1} + \mathcal{S}^{-2})u & v(1 + \mathcal{S})(u\mathcal{S} - \mathcal{S}^{-2}u) + (u\mathcal{S}^2 - \mathcal{S}^{-1}u)(1 + \mathcal{S}^{-1})v \end{pmatrix} +$$

$$+ \begin{pmatrix} u(1 + \mathcal{S} + \mathcal{S}^2)(\mathcal{S}^{-1}v - v\mathcal{S})(1 + \mathcal{S}^{-1} + \mathcal{S}^{-2})u & u(1 + \mathcal{S} + \mathcal{S}^2)(\mathcal{S}^{-1}v - v\mathcal{S})(1 + \mathcal{S}^{-1})v \\ v(1 + \mathcal{S})(\mathcal{S}^{-1}v - v\mathcal{S})(1 + \mathcal{S}^{-1} + \mathcal{S}^{-2})u & v(1 + \mathcal{S})(\mathcal{S}^{-1}v - v\mathcal{S})(1 + \mathcal{S}^{-1})v \end{pmatrix},$$

$$f_2 = -\frac{1}{3} \log u.$$

- Nontrivial symmetry:

$$\mathcal{H}_2\delta f_1 = \begin{pmatrix} uv_{-1}(v+v_{-1}+v_{-2}) - uv_2(v_1+v_2+v_3) + u(u_1+u_2-u_{-1}-u_{-2}) \\ (u-vv_1)(v+v_1+v_2) + (vv_{-1}-u_{-1})(v+v_{-1}+v_{-2}) - v(u_{-2}-u_1) \end{pmatrix}.$$

- Master symmetry [78]:

$$\begin{pmatrix} nu_t + uv_1 + 4uv_{-1} + uv \\ nv_t + u - vv_1 - 4u_{-1} + 4vv_{-1} + v^2 \end{pmatrix}.$$

- Lax representation [79]:

$$M = \begin{pmatrix} \lambda & \lambda v & \lambda u_{-1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} v-\lambda & -\lambda v & -\lambda u_{-1} \\ -1 & v_{-1} & 0 \\ 0 & -1 & v_{-2} \end{pmatrix}.$$

The Belov–Chaltikian lattice is the Boussinesq lattice related to the lattice W_3-algebra [79].

The Boussinesq lattice related to the lattice W_m-algebra for the dependent variables $u^1, u^2, \ldots, u^m$ and independent variables n and t was recently written [80] as

$$\begin{aligned} u_t^1 &= -u^1(u_m^2 - u_{-1}^2), \\ u_t^i &= u^{i+1} - u_{-1}^{i+1} - u^i(u_{i-1}^2 - u_{-1}^2), \quad i = 2,3,\ldots,m-1, \\ u_t^m &= u^1 - u_{-1}^1 - u^m(u_{m-1}^2 - u_{-1}^2). \end{aligned}$$

The vector $\boldsymbol{\tau} = (\tau^1, \ldots, \tau^n)^{\mathrm{T}}$ defined by

$$\begin{aligned} \tau^1 &= nu_t^1 - u^1\left((m+1)u_m^2 + \sum_{l=0}^{m} u_l^2\right), \\ \tau^i &= nu_t^i - u^i\left(iu_{i-1}^2 + \sum_{l=0}^{i-1} u_l^2\right) + (i+1)u^{i+1}, \quad i = 2,3,\ldots,m-1, \\ \tau^m &= nu_t^n - u^m\left(mu_{m-1}^2 + \sum_{l=0}^{m-1} u_l^2\right) + (m+1)u^1, \end{aligned}$$

is a master symmetry. Its Hamiltonian structures were also studied in [80].

4.19. The Blaszak–Marciniak lattice.

- Equation [81]:

$$\begin{aligned} u_t &= w_1 - w_{-1}, \\ v_t &= u_{-1}w_{-1} - uw, \\ w_t &= w(v - v_1). \end{aligned}$$

- Hamiltonian structure [81]:

$$\mathcal{H}_1 = \begin{pmatrix} \mathcal{S}-\mathcal{S}^{-1} & 0 & 0 \\ 0 & 0 & (\mathcal{S}^{-1}-1)w \\ 0 & -w(\mathcal{S}-1) & 0 \end{pmatrix}, \qquad f_1 = uw + \frac{1}{2}v^2,$$

$$\mathcal{H}_2 = \begin{pmatrix} \mathcal{S}v - v\mathcal{S}^{-1} - \\ \quad - u(\mathcal{S}+1)^{-1}(1-\mathcal{S})u & \mathcal{S}w\mathcal{S}-\mathcal{S}^{-1}w & u(\mathcal{S}+1)^{-1}(1-\mathcal{S})w \\ w\mathcal{S}-\mathcal{S}^{-1}w\mathcal{S}^{-1} & \mathcal{S}^{-1}uw - uw\mathcal{S} & v(\mathcal{S}^{-1}-1)w \\ w(\mathcal{S}+1)^{-1}(1-\mathcal{S})u & -w(\mathcal{S}-1)v & w(\mathcal{S}^{-1}-\mathcal{S})w - \\ & & \quad - w(\mathcal{S}+1)^{-1}(1-\mathcal{S})w \end{pmatrix}, \qquad f_2 = v.$$

- Recursion operator:

$$\mathcal{R} = \mathcal{H}_2\mathcal{H}_1^{-1},$$

where

$$\mathcal{H}_1^{-1} = \begin{pmatrix} \frac{1}{2}(\mathcal{S}-1)^{-1} + \frac{1}{2}(\mathcal{S}+1)^{-1} & 0 & 0 \\ 0 & 0 & -(\mathcal{S}-1)^{-1}\frac{1}{w} \\ 0 & -\frac{1}{w}\mathcal{S}(\mathcal{S}-1)^{-1} & 0 \end{pmatrix}.$$

- Nontrivial symmetry:

$$\mathcal{H}_2\delta f_1 = \mathcal{H}_2 \begin{pmatrix} w \\ v \\ u \end{pmatrix} = \begin{pmatrix} w_1(v_1+v_2) - w_{-1}(v+v_{-1}) \\ u_{-1}w_{-1}(v+v_{-1}) - uw(v+v_1) - w_{-1}w_{-2} + ww_1 \\ w(v^2-v_1^2) + w(w_{-1}u_{-1} - w_1u_1) \end{pmatrix}.$$

- Master symmetry:

$$\mathcal{R}\begin{pmatrix} \frac{u}{2} \\ v \\ \frac{3w}{2} \end{pmatrix}.$$

- Lax representation [81]:

$$L = \mathcal{S}^2 + u_1\mathcal{S} - v_1 + w\mathcal{S}^{-1}, \qquad A = \mathcal{S}^2 + u_1\mathcal{S} - v_1.$$

We do not explicitly write the recursion operator, which is no longer weakly nonlocal although both the operators $\mathcal{H}_2$ and $\mathcal{H}_1^{-1}$ are weakly nonlocal. The statement that such a recursion operator generates local symmetries can be proved in the same way as in [31] for weakly nonlocal differential recursion operators. We can compute the next Hamiltonian equal to $uvw + uv_1w + v^3/3 - ww_1$. Its master symmetry is highly nonlocal [78].

Another three-component lattice was given in [29] as

$$p_t = q_1 - q,$$

$$q_t = q(p_{-1} - p) + r - r_{-1},$$

$$r_t = r(p_{-1} - p_1).$$

It has the Lax representation

$$M = \begin{pmatrix} 0 & 1 & 0 \\ q & u+\lambda & 1 \\ r & 0 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} -p_{-1} & 1 & 0 \\ q & \lambda & 1 \\ r_{-1} & 0 & -p \end{pmatrix}.$$

This lattice is related to the Blaszak–Marciniak lattice by the Miura transformation

$$p = v_1, \qquad q = -uw, \qquad r = -ww_1.$$

5. Conclusion

We have reviewed two close concepts directly related to the Lax representations for integrable systems: Darboux transformations and recursion operators. We used the well-known NLS equation, whose Lax representation is polynomial in the spectral parameter, and a deformation of the derivative NLS equation, whose Lax representation is invariant under the dihedral reduction group $\mathbb{D}_2$, as two typical examples. We then presented a list of integrable differential–difference equations containing the equations themselves, Hamiltonian structures, recursion operators, nontrivial generalized symmetries, and Lax representations. For most equations, we also added notes on their relations to other known equations and the weakly nonlocal inverse recursion operators if they exist.

The theory of integrable partial difference (or discrete) equations is a relatively recent but very active area of research. Thanks to the work of Adler, Bobenko and Suris, affine-linear quadrilateral equations were classified [82], [83], based on the condition of three-dimensional consistency. Levi and Yamilov then proposed using the existence of a generalized symmetry as a criterion for integrability [84] to classify integrable partial difference equations. They extended the Adler–Bobenko–Suris list. The symmetry flows for all integrable partial difference equations can be regarded as integrable differential–difference equations. The concept of a recursion operator was extended to difference equations, and it was shown that it generates an infinite sequence of symmetries and canonical conservation laws for a partial difference equation [35], [11], [53]. It can be proved that the difference equation shares the same recursion operator for its symmetry flows. Therefore, studying the symmetry structure is in fact the same for integrable difference equations as for integrable differential–difference equations. This is one of our motivations for producing the list presented here. Moreover, this list can serve as a benchmark for developing computer software packages for the symbolic computation of recursion operators, Lax representations, symmetries, and conservation laws for nonlinear differential–difference equations [72].

We did not present a rigorous proof here that the operators computed from Darboux transformations are in fact Nijenhuis recursion operators for the equations obtained from the corresponding Lax representations. For Lax representations that are polynomial in the spectral parameter, under certain technical conditions, a sketch of the proof was given in [29]. We believe that the main statement in [29] can be significantly generalized and simplified and also that there is a neat rigorous algebraic proof that the operators obtained from Darboux transformations invariant under reduction groups are indeed Nijenhuis recursion operators.

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