Theoretical and Mathematical Physics, **174**(1): 134–153 (2013)

INTEGRABLE DEFORMATIONS IN THE ALGEBRA OF PSEUDODIFFERENTIAL OPERATORS FROM A LIE ALGEBRAIC PERSPECTIVE

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We split the algebra of pseudodifferential operators in two different ways into the direct sum of two Lie subalgebras and deform the set of commuting elements in one subalgebra in the direction of the other component. The evolution of these deformed elements leads to two compatible systems of Lax equations that both have a minimal realization. We show that this Lax form is equivalent to a set of zero-curvature relations. We conclude by presenting linearizations of these systems, which form the key framework for constructing the solutions.

Keywords: integrable deformation, pseudodifferential operator, Lax equation, Kadomtsev–Petviashvili hierarchy, zero-curvature relation, linearization

1. Introduction

In this paper, we study the algebraic aspects of two compatible systems of Lax equations for pseudodifferential operators. They are both linked to splitting this Lie algebra as a vector space into the direct sum of two Lie subalgebras and deforming a basic set of commuting elements in one component of this decomposition in the direction of the other component. One decomposition leads to the Kadomtsev–Petviashvili (KP) hierarchy and the other leads to a strict KP hierarchy. Both systems appear naturally in the two-dimensional Toda hierarchy as formulated in [1]. We present a minimal realization of both systems of Lax equations and show in Theorem 2 below that this Lax form in both cases is equivalent to a system of zero-curvature equations. This last fact indicates the existence of a system of compatible linear equations. We conclude by describing these linearizations, which form the algebraic setting for constructing actual solutions from infinite-dimensional varieties (see [2], [3]).

The order of exposition is as follows. In Sec. 2, we present the Lie algebraic aspects of compatible systems of Lax equations in spaces of finite-dimensional matrices. Next, we present the algebra of pseudodifferential operators and its relevant properties and decompositions. Section 4 is devoted to describing the systems of Lax equations in this algebra. In Sec. 5, we show that there exists a realization of both systems with a minimum number of relations between the coefficients of the deformed operators. In Sec. 6, we prove that the Lax form of the equations is equivalent to the zero-curvature form. We describe the linearizations in the concluding section.

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Prepared from an English manuscript submitted by the authors; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 174, No. 1, pp. 154–176, January, 2013.

 0040-5779/13/1741-0134

2. Compatible Lax equations for matrices

We consider a differentiable map $t \mapsto g(t)$, $t \in \mathbb{R}$, to the invertible $n\times n$ matrices. For each $n\times n$ matrix V, we can deform V by conjugating with this map, which results in the flow

$$t \mapsto L(t) := g(t)^{-1} V g(t). \tag{1}$$

We set $M(t) := (dg(t)/dt)g(t)^{-1} = \dot{g}(t)g(t)^{-1}$. A direct calculation then shows that the evolution of the deformation L of V is given by

$$\dot{L} := \frac{dL}{dt} = LM - ML = [L, M]. \tag{2}$$

Equation (2) is an example of a so-called Lax equation. Many systems in mechanics have such a Lax form (see, e.g., [4]), a prototype being the finite open Toda lattice (cf. [5]). Their link to representation theory was studied in detail in [6].

The next step is to consider systems of compatible Lax equations such as (2). Ordinary derivatives are then replaced with partial derivatives. We briefly recall what, from a Lie algebraic standpoint, the ingredients are for obtaining compatible systems of Lax equations (first for $n\times n$ matrices). This serves as a starting point for considerations in the infinite-dimensional case.

We start with a real or complex Lie algebra $\mathfrak{g}$ of $n\times n$ matrices and a matrix Lie group G associated with it. Our first assumption is that the Lie algebra $\mathfrak{g}$ as a vector space is the direct sum

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \tag{3}$$

of Lie subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of $\mathfrak{g}$. We write $\pi_i \colon \mathfrak{g} \to \mathfrak{g}_i$ for the corresponding projections. The next necessary condition is that each $\mathfrak{g}_i$, $i = 1, 2$, is the Lie algebra of a Lie subgroup G_i of the group G. This does not present a difficulty in the finite-dimensional case, but satisfying the condition in the infinite-dimensional case can turn out to be problematic.

Further, we must assume that the direct sum decomposition has its counterpart on the group level, i.e., the map $(g_1, g_2) \mapsto g_1 g_2$ is a diffeomorphism from $G_1 \times G_2$ to G. We may not have this property in both the finite- and the infinite-dimensional cases.

The systems of Lax equations we discuss are linked to commuting flows in G_2. We consider a family $\{F_i \mid F_i \in \mathfrak{g}_2,\ 1 \le i \le m\}$ of matrices that, first, are linearly independent and, second, mutually commute:

$$[F_{i_1}, F_{i_2}] = 0 \tag{4}$$

for all i_1 and $i_2 \in \{1, \ldots, m\}$. The corresponding commuting flows are

$$\gamma(t) = \gamma(t_1, \ldots, t_m) = e^{\sum_{i=1}^{m} t_i F_i}. \tag{5}$$

The idea is now to deform the generators of the commuting flows in $\mathfrak{g}_2$ in the direction of G_1 and to consider the evolution of these deformations. For this, we take any $g \in G$ and the commuting flows γ as in (5). Because of the unique decomposition $G = G_1 G_2$, we know that there are elements $g_1 \in G_1$ and $g_2 \in G_2$ such that

$$\gamma(t) g \gamma(t)^{-1} = g_1(t)^{-1} g_2(t). \tag{6}$$

We now use the G_1-component to deform the directions in $\mathfrak{g}_2$, i.e., we define multidimensional flows $\mathcal{F}_i$ in $\mathfrak{g}$ by

$$\mathcal{F}_i := g_1 F_i g_1^{-1}, \quad 1 \le i \le m. \tag{7}$$

This deformation preserves commutativity:

$$[\mathcal{F}_{i_1}, \mathcal{F}_{i_2}] = 0. \tag{8}$$

The evolution of the $\{\mathcal{F}_i\}$ with respect to the parameters of the commuting flows is given by Lax equations. More precisely, we have the following theorem.

Theorem 1. *In the notation used, the deformations $\{\mathcal{F}_i\}$ of the original commuting directions satisfy the equalities*

$$\frac{\partial}{\partial t_{i_1}}(\mathcal{F}_{i_2}) = [\mathcal{F}_{i_2}, \pi_1(\mathcal{F}_{i_1})] = [\pi_2(\mathcal{F}_{i_1}), \mathcal{F}_{i_2}]. \tag{9}$$

Proof. We note that the second equality in (9) follows because the deformations $\{\mathcal{F}_i\}$ commute and the identity

$$\pi_1(\mathcal{F}_{i_1}) = \mathcal{F}_{i_1} - \pi_2(\mathcal{F}_{i_1})$$

holds. Because all the $\{\mathcal{F}_i\}$ are obtained by conjugating a constant matrix with a matrix depending on the parameters $\{t_i\}$, we directly verify that

$$\frac{\partial}{\partial t_{i_1}}(\mathcal{F}_{i_2}) = \left[\frac{\partial}{\partial t_{i_1}}(g_1)g_1^{-1}, \mathcal{F}_{i_2}\right]. \tag{10}$$

We therefore merely need to show that the commutators in the right-hand side of (10) are equal to those stated in the theorem. For this, we differentiate the expressions $g_1\gamma g$ and $g_2\gamma$ with respect to t_{i_1}. We obtain

$$\frac{\partial}{\partial t_{i_1}}(g_1)\gamma g + g_1 F_{i_1}\gamma g = \left(\frac{\partial}{\partial t_{i_1}}(g_1)g_1^{-1} + \mathcal{F}_{i_1}\right)g_1\gamma g,$$

$$\frac{\partial}{\partial t_{i_1}}(g_2) + g_2 F_{i_1}\gamma = \left(\frac{\partial}{\partial t_{i_1}}(g_2)g_2^{-1} + g_2 F_{i_1} g_2^{-1}\right)g_2\gamma,$$

and by the equality $g_1\gamma g = g_2\gamma$, we have

$$\frac{\partial}{\partial t_{i_1}}(g_1)g_1^{-1} + \mathcal{F}_{i_1} = \frac{\partial}{\partial t_{i_1}}(g_2)g_2^{-1} + g_2 F_{i_1} g_2^{-1}. \tag{11}$$

Because $F_{i_1} \in \mathfrak{g}_2$, we have $g_2 F_{i_1} g_2^{-1} \in \mathfrak{g}_2$. The factor $(\partial g_2/\partial t_{i_1})g_2^{-1}$, as the tangent vector of a flow in G_2 shifted back to the origin, belongs to $\mathfrak{g}_2$, and all the right-hand side of (11) therefore belongs to $\mathfrak{g}_2$. A similar argument shows that the factor $(\partial g_1/\partial t_{i_1})g_1^{-1}$ belongs to $\mathfrak{g}_1$, and the component $\mathcal{F}_{i_1}$ of identity (11), as the G-conjugate of a matrix in $\mathfrak{g}$, belongs to $\mathfrak{g}$. We have just seen that the $\mathfrak{g}_1$ component of the left-hand side is equal to zero. This yields

$$\frac{\partial}{\partial t_{i_1}}(g_1)g_1^{-1} = -\pi_1(\mathcal{F}_{i_1}),$$

which results in Lax equations (9). The theorem is proved.

We now assume that instead of the real or complex $n\times n$ matrices, we have an infinite-dimensional Lie algebra consisting of integral and differential operators. The compatible systems of Lax equations that we describe for this algebra are generalizations of the Lax form of the Korteweg–de Vries (KdV) equation. We recall that this equation describes the time evolution of the height $u(x,t)$ of waves propagating on the water surface in a narrow channel:

$$u_t := \frac{\partial u}{\partial t} = \frac{1}{4}\frac{\partial^3 u}{\partial x^3} + \frac{3}{2}u\frac{\partial u}{\partial x} = \frac{1}{4}u_{xxx} + \frac{3}{2}uu_x. \tag{12}$$

The KdV equation can be written as a Lax equation for differential operators in $\partial = \partial/\partial x$ with coefficients in a ring R of differentiable functions in x and t.

For any $a \in R$ and $m \in \mathbb{N}$, let $a\partial^m$ represents the endomorphism of R of differentiating m times with respect to x and then multiplying with $a \in R$. Finite sums of such operators are called differential operators in ∂ with coefficients in R. They form an algebra, denoted here by $R[\partial]$.

The KdV equation can now be written as an equality between operators from $R[\partial]$. To see this, with any function $v \in R$, we associate the differential operators

$$\mathcal{L}_2 = \partial^2 + v, \qquad P_3 = \partial^3 + \frac{3}{2}v\partial + \frac{3}{4}\partial(v). \tag{13}$$

Because P_3 and $\mathcal{L}_2$ are endomorphisms of the ring R, we can speak of the commutator $[P_3, \mathcal{L}_2]$ of P_3 and $\mathcal{L}_2$. A direct calculation shows that

$$[P_3, \mathcal{L}_2] = \frac{1}{4}v_{xxx} + \frac{3}{2}v\partial(v).$$

In other words, the operator $[P_3, \mathcal{L}_2]$ is a zeroth-order operator in ∂. Further, we view $t \mapsto \partial^2 + v(x,t)$ as a flow of Schrödinger operators and $\partial_t := \partial/\partial t$ as an operator acting coefficientwise on elements of $R[\partial]$, one of which is $\mathcal{L}_2$. We note that the statement "v is a solution of the KdV equation" is equivalent to the Lax equation

$$\frac{\partial}{\partial t}(\mathcal{L}_2) = 0 \cdot \frac{\partial^2}{\partial x^2} + 2\frac{\partial v}{\partial t} = [P_3, \mathcal{L}_2]. \tag{14}$$

Lax first wrote the KdV equation in this form [7], which explains the name *Lax form* of the KdV equation for (14). To understand how Eq. (14) arose, we must pass to a proper extension of $R[\partial]$, the algebra of pseudodifferential operators with respect to ∂. It can be shown that in this algebra, there exists a square root $L = (\mathcal{L}_2)^{1/2}$ of $\mathcal{L}_2$ of the form

$$L = \partial + \sum_{j=1}^{\infty} \ell_{j+1}\partial^{-j}.$$

Then P_3 is the component in $R[\partial]$ of the third power of L, and the equation

$$[P_3, \mathcal{L}_2] = [\mathcal{L}_2, L^3 - P_3]$$

explains why $[P_3, \mathcal{L}_2]$ is a zeroth-order operator in ∂. Such an algebra of pseudodifferential operators is also an appropriate context for studying compatible systems of Lax equations of form (14) and is the topic of the next section.

3. The ring $R[\xi, \xi^{-1})$ of pseudodifferential operators

Our aim in this section is to describe the extension of the differential operators that explains the Lax form of equations like the KdV equation and its generalizations. To stress the algebraic character of our considerations and to handle real and complex solutions on an equal footing, we start abstractly with an algebra R over a field k of characteristic zero that serves as the source of solutions of the nonlinear equations under consideration. Further, as a substitute for the operators $\partial/\partial x$ in the KdV setting, we assume that the algebra R has a privileged k-linear derivation $\partial\colon R \mapsto R$. Because ∂ is k-linear, the field k is in the ring of constants

$$R_{\text{const}} = \{r \in R \mid \partial(r) = 0\}$$

inside R. We present examples that are prototypes of this starting point.

Example 1. Let R be the ring $k[t_i]$ of polynomials in the variables $\{t_i \mid i \in I\}$ with coefficients in k or the ring of formal power series $k[[t_i]]$ in these variables, and let ∂ be the partial derivative with respect to one of them, for example, t_{i_0}. The ring of constants R_{const} in these cases is respectively equal to $k[t_i, i \neq i_0]$ or $k[[t_i, i \neq i_0]]$.

If R and ∂ are given, then we can form differential operators in ∂ with coefficients in R. They comprise the family $R[\partial]$ of k-linear endomorphisms of R of the form $\sum_{i=0}^{n} a_i \partial^i$, $a_i \in R$, i.e., the maps

$$r \to \sum_{i=0}^{n} a_i \partial^i(r),$$

acting from R to R.

The rule

$$\partial^m \circ r_1 = \sum_{i=0}^{m} \binom{m}{i} \partial^i(r_1) \partial^{m-i}$$

determines the composition of two such endomorphisms $\sum_i a_i \partial^i$ and $\sum_j b_j \partial^j$. The result is

$$\sum_{i,j} \sum_{k=0}^{i} \binom{i}{k} a_i \partial^k(b_j) \partial^{j+i-k}, \tag{15}$$

i.e., again an element of $R[\partial]$. It might turn out that the powers of ∂ are not R-linear independent. To avoid this, we introduce an algebra $R[\xi]$ whose multiplication is basically the same as in $R[\partial]$, but the corresponding relations in it are decoupled. The elements of $R[\xi]$ are formal expressions

$$\sum_{i=0}^{n} a_i \xi^i, \quad a_i \in R, \quad i \geq 0.$$

Their addition, product structure, and scalar multiplication are given by the rules

$$\sum_i a_i \xi^i + \sum_i b_i \xi^i = \sum_i (a_i + b_i) \xi^i, \tag{16}$$

$$\left(\sum_{i=0}^{n} a_i \xi^i\right)\left(\sum_{j=0}^{m} b_j \xi^i\right) = \sum_{\substack{0 \leq i \leq n \\ 0 \leq j \leq m}} \sum_{0 \leq k \leq i} \binom{i}{k} a_i \partial^k(b_j) \xi^{i+j-k}, \tag{17}$$

$$\lambda.a = \lambda.\left(\sum_j a_j \xi^j\right) := \sum_j \lambda a_j \xi^j. \tag{18}$$

It can be shown that this multiplicative structure is associative, and $R[\xi]$ therefore becomes a k-algebra and also a free R-module with the basis $\{\xi^i \mid i \geq 0\}$. The map

$$a = \sum_{i=0}^{n} a_i \xi^i \mapsto \phi(a) := \sum_{i=0}^{n} a_i \partial^i$$

is a surjective k-algebra homomorphism ϕ from $R[\xi]$ onto $R[\partial]$. In the degenerate case, i.e., when $\partial = 0$, the ring $R[\xi]$ becomes the commutative algebra

$$R[\lambda] = \left\{ \sum_{i=0}^{n} a_i \lambda^i \;\middle|\; a_i \in R \right\}$$

of polynomials in λ with coefficients in R. We can view $R[\xi]$ as a deformation of the commutative algebra $R[\lambda]$. In several works (see, e.g., [8]), it was proposed to work directly with $R[\xi]$ instead of $R[\partial]$. Elements of $R[\xi]$ are called formal differential operators with respect to ∂.

In $R[\xi]$, it makes sense to introduce the notion of degree.

Definition 1. The *degree* $\deg(a)$ of a nonzero element $a = \sum_{i=0}^{N} a_i\xi^i$ in $R[\xi]$ is the number N if $a_N \neq 0$. We set the degree of the zero element to be $-\infty$.

The elements ξ^m, $m \geq 1$, are not invertible in $R[\xi]$, and we therefore construct an extension of the algebra $R[\xi]$ that contains all formal inverses $\{\xi^{-m} \mid m \geq 1\}$. Viewing the elements of $R[\xi]$ as "differential operators," we can realize this process by adding "integral operators" to $R[\xi]$ but doing this purely algebraically, as is shown below. In this extended algebra, first of all, we should have

$$\xi^n\xi^m = \xi^{n+m}$$

for all m and $n \in \mathbb{Z}$. To clarify how ξ^{-1} should act on an element of the form $b\xi^m$, $b \in R$, we successively apply the rule $\xi^{-1}c = c\xi^{-1} - \xi^{-1}\partial(c)\xi^{-1}$, which leads to the supposition that $\xi^{-1}(b\xi^m)$ has the form of an infinite series in negative powers of ξ:

$$\xi^{-1}(b\xi^m) = \sum_{s=0}^{\infty}(-1)^s\partial^s(b)\xi^{m-1-s}. \tag{19}$$

Formula (19) also determines the action of the other negative powers ξ^n, $n < 0$, and setting

$$\binom{n}{k} := \frac{n(n-1)\cdots(n-k+1)}{k!} \tag{20}$$

$n \in \mathbb{Z}$, we can show by induction on $|n|$ that the action of ξ^n is given by

$$\xi^n b\xi^m = \sum_{s=0}^{\infty}\binom{n}{s}\partial^s(b)\xi^{m+n-s}. \tag{21}$$

This formula allows introducing a multiplicative structure on the set $R[\xi,\xi^{-1})$ of all formal series

$$p = \sum_{j=-\infty}^{N} p_j\xi^j, \quad p_j \in R.$$

The product of two such series $a = \sum_j a_j\xi^j$ and $b = \sum_i b_i\xi^i$ is defined as

$$a.b := \sum_j\sum_i\sum_{s=0}^{\infty}\binom{j}{s}a_j\,\partial^s(b_i)\xi^{i+j-s}, \tag{22}$$

which is an obvious extension of (17). Further, if we respectively define addition and multiplication by scalars from k on $R[\xi,\xi^{-1})$ by (16) and (18), then $R[\xi,\xi^{-1})$ becomes a k-algebra. This algebra, being a mixture of differential operators and integral operators, has a special name.

Definition 2. The elements of $R[\xi,\xi^{-1})$ are called *pseudodifferential operators with respect to* ∂ or, briefly, *pseudodifferential operators* if the differentiation operation on which they are based does not require a special mention. We also use the brief notation Psd for the k-algebra $R[\xi,\xi^{-1})$.

Further, the notion of degree can also be extended to pseudodifferential operators.

Definition 3. The *degree* $\deg(a)$ of a nonzero element a of $R[\xi, \xi^{-1})$ is the integer n such that

$$a = a_n \xi^n + \sum_{j<n} a_j \xi^j, \quad a_n \neq 0.$$

Adding the inverses of the $\{\xi^m \mid m \geq 1\}$ increases the number of invertible elements in $R[\xi, \xi^{-1})$ because we have the following lemma.

Lemma 1. *Every pseudodifferential operator* $P = \sum_{j \leq m} p_j \xi^j$*, where* $p_m \in R^*$*, has an inverse* P^{-1} *of the form* $\sum_{i \leq -m} q_i \xi^i$*, where* $q_{-m} = p_m^{-1}$.

Proof. The product of the elements $\sum_{j \leq m} p_j \xi^j$ and $\sum_{i \leq -m} q_i \xi^i$ is by definition equal to

$$\sum_{j \leq m} p_j \sum_{i \leq -m} \sum_{s=0}^{\infty} \binom{j}{s} p_j \, \partial^s(q_i) \xi^{j+i-s}.$$

This is an operator of nonpositive degree, and if it is equal to 1, then the leading coefficient q_{-m} must be the inverse of p_m, and for all $k \geq 1$, we must have the equality

$$\sum_{\substack{i,j,s,\\ i+j-s=-k}} \binom{j}{s} p_j \, \partial^s(q_i) = p_m q_{-m-k} + \sum_{\substack{i,j,s,\\ i+j-s=-k,\\ i>-m-k}} \binom{j}{s} p_j \, \partial^s(q_i) = 0.$$

Because p_m is invertible, we can therefore find q_{-m-k} assuming that all the q_i, $i > -m-k$, are known. This proves the existence of an operator inverse to P. The lemma is proved.

Because the algebra $R[\xi, \xi^{-1})$ has a wide collection of invertible elements, it has a large potential for the dressing procedure.

Definition 4. An element $P \in R[\xi, \xi^{-1})$ is said to be obtained by *dressing* an element $Q \in R[\xi, \xi^{-1})$ if there is an invertible element $K \in R[\xi, \xi^{-1})$ such that $P = KQK^{-1}$. The operator K in this case is called the *dressing operator*.

In what follows, we encounter two subgroups of invertible elements of $R[\xi, \xi^{-1})$ that deserve special attention: the group

$$D(0) = \left\{ p_0 + \sum_{j<0} p_j \xi^j \;\middle|\; p_0 \in R^* \right\}$$

and its normal subgroup $D(0)_1$ of all elements of the form $1 + \sum_{j<0} p_j \xi^j$.

Like any associative k-algebra, $R[\xi, \xi^{-1})$ is a Lie algebra over the field k with respect to the commutator. Inside the Lie algebra $R[\xi, \xi^{-1})$, we can use different decompositions. Any $P = \sum_j p_j \xi^j \in R[\xi, \xi^{-1})$ can be split into the sum of two of its components: the differential operator $P_{\geq 0}$ and the strictly integral operator $P_{<0}$ where

$$P_{\geq 0} = \sum_{j \geq 0} p_j \xi^j, \qquad P_{<0} = \sum_{j<0} p_j \xi^j. \tag{23}$$

Similarly, P can also be represented as the sum of its purely differential part $P_{>0}$ and integral part $P_{\leq 0}$

$$P_{>0} = \sum_{j>0} p_j \xi^j, \qquad P_{\leq 0} = \sum_{j \leq 0} p_j \xi^j. \tag{24}$$

It follows from the multiplication rules given in $R[\xi,\xi^{-1})$ that these two decompositions yield two ways to split the Lie algebra $R[\xi,\xi^{-1})$ into the direct sum of two Lie subalgebras. The first way is

$$\mathrm{Psd} = \{P \in \mathrm{Psd},\ P = P_{<0}\} \oplus \{P \in \mathrm{Psd},\ P = P_{\geq 0}\} := \mathrm{Psd}_{<0} \oplus \mathrm{Psd}_{\geq 0}. \tag{25}$$

The second is

$$\mathrm{Psd} = \{P \in \mathrm{Psd},\ P = P_{\leq 0}\} \oplus \{P \in \mathrm{Psd},\ P = P_{>0}\} := \mathrm{Psd}_{\leq 0} \oplus \mathrm{Psd}_{>0}. \tag{26}$$

Neither the supposition $\mathfrak{g}_2 = \mathrm{Psd}_{\geq 0}$ nor the supposition $\mathfrak{g}_2 = \mathrm{Psd}_{>0}$ allow successfully choosing the group G_2. Nevertheless, if $\mathfrak{g}_1 = \mathrm{Psd}_{<0}$, then an appropriate variant for the group G_1 appears. Namely, for each element $P = \sum_{j<0} p_j \xi^j \in R[\xi,\xi^{-1})_{<0}$ and any $m \geq 1$, the element P^m has a degree less than or equal to $-m$. Hence, the formula

$$e^P = \sum_{m=0}^{\infty} \frac{P^m}{m!}$$

determines a well-defined element of $D(0)_1$. Conversely, the same argument shows that for each element $P = \sum_{j<0} p_j \xi^j \in R[\xi,\xi^{-1})_{<0}$, the formula

$$\log(1+P) := \sum_{m=1}^{\infty} (-1)^{m+1} \frac{P^m}{m}$$

determines a well-defined element in $R[\xi,\xi^{-1})_{<0}$. Obviously, it satisfies the equality $1 + P = e^{\log(1+P)}$. Therefore, $D(0)_1$ can be viewed as the group G_1 corresponding to the Lie algebra $\mathfrak{g}_1 = \mathrm{Psd}_{<0}$.

For $\mathfrak{g}_1 = \mathrm{Psd}_{\leq 0}$, we could, as above, consider the exponential map for the elements $P = \sum_{j\leq 0} p_j \xi^j \in R[\xi,\xi^{-1})_{\leq 0}$, but that requires taking convergence into account, as can be seen from the zeroth-order term. It is necessary that $e^{p_0} \in R$ for every $p_0 \in R$. Hence, if this exponential map yields a well-defined element of $\mathrm{Psd}_{\leq 0}$, then it determines an element of $D(0)$. Therefore, $D(0)$ can be viewed as the group corresponding to $\mathrm{Psd}_{\leq 0}$.

4. Lax equations for pseudodifferential operators

In this section, we describe compatible systems of Lax equations in $R[\xi,\xi^{-1})$ that are analogues of the finite-dimensional systems considered in Theorem 1 and are based on the decompositions presented in Sec. 3. Other k-linear derivations of R commuting with ∂, like the operator $\partial/\partial t$ in Eq. (14), appear here. They also act naturally on elements of Psd. We introduce this action and use it hereafter without further mention.

Let Δ be another k-linear derivation of R that commutes with ∂. We can then extend Δ to $R[\xi,\xi^{-1})$ by setting

$$\Delta(P) = \sum_j \Delta(p_j)\xi^j$$

for each $P = \sum_j p_j \xi^j \in R[\xi,\xi^{-1})$. Because Δ and ∂ commute, a direct verification shows that the following lemma holds.

Lemma 2. *The extension of Δ to $R[\xi,\xi^{-1})$ is a k-linear derivation of this algebra.*

An obvious choice for the respective commuting elements in $\mathrm{Psd}_{>0}$ and $\mathrm{Psd}_{\geq 0}$ is $\{\xi^m \mid m \geq 1\}$ and $\{\xi^m \mid m \geq 0\}$. Because the deformations that we have in mind for these directions consist of conjugating with suitable elements from the group G_1, we can neglect the element ξ^0 and restrict ourself to deforming $\{\xi^m \mid m \geq 1\}$ in both cases.

We first consider the deformations with the group $D(0)_1$. Conjugating with an element of $D(0)_1$ gives the set $\{L^m,\ m \geq 1\}$, where $L = K\xi K^{-1}$, $K \in D(0)_1$, has the form

$$L = \xi + \sum_{j=1}^{\infty} \ell_{j+1}\xi^{-j}. \tag{27}$$

Under a mild condition, any L of form (27) can be obtained by dressing ξ with elements of $D(0)_1$.

Lemma 3. *If ∂ is surjective, then any $P = \xi + \sum_{i=1}^{\infty} p_{i+1}\xi^{-i}$ can be obtained by dressing the operator ξ by an element of $D(0)_1$.*

Proof. The proof consists in solving the equation $PK = K\xi$ with $K \in D(0)_1$ step by step. If $K = 1 + \sum_{j>0} k_j\xi^{-j}$, then the right-hand side is $\xi + \sum_{j>0} k_j\xi^{1-j}$, and the left-hand side is equal to

$$\begin{aligned} PK = \left(\xi + \sum_{i=1}^{\infty} p_{i+1}\xi^{-i}\right)\left(1 + \sum_{j>0} k_j\xi^{-j}\right) = \xi + \sum_{i>0} k_i\xi^{1-i} + \\ + \sum_{i>0} \partial(k_i)\xi^{-i} + \sum_{i=1}^{\infty} p_{i+1}\xi^{-i} + \sum_{i\geq 1}\sum_{j\geq 1}\sum_{l\geq 0} p_{i+1}\binom{-i}{l}\partial^l(k_j)\xi^{-i-j-l}. \end{aligned} \tag{28}$$

It hence follows that we must choose K such that

$$\sum_{j\geq 1} \partial(k_j)\xi^{-j} + \sum_{i=1}^{\infty} p_{i+1}\xi^{-i} + \sum_{i\geq 1}\sum_{j\geq 1}\sum_{l\geq 0} p_{i+1}\binom{-i}{l}\partial^l(k_j)\xi^{-i-j-l} = 0.$$

The coefficient of ξ^{-1} in the expression in the left-hand side is equal to $\partial(k_1) + p_2$, and thanks to the condition imposed on ∂, we can find a k_1 such that this coefficient is zero. Assuming that we have found $\{k_1, \ldots, k_m\}$, $m \geq 1$, such that the coefficients of all the ξ^{-l}, $l \leq m$, are zero, we then find that the next coefficient has the form

$$\partial(k_{m+1}) + p_{m+2} + (\text{polynomial expression in } \partial^l(k_i) \text{ and } p_{j+1}), \quad i \leq m, \quad j \leq m,$$

and we can choose k_{m+1} such that this equals zero. The coefficients of K can thus be found inductively. The lemma is proved.

The deformations with $D(0)$ are slightly more general. Conjugating the basic directions with an element of $D(0)$ yields the set $\{M^m, m \geq 1\}$, where $M = D\xi D^{-1}$, $D \in D(0)$, has the form

$$M = \xi + \sum_{j=0}^{\infty} m_{j+1}\xi^{-j}. \tag{29}$$

For any M of form (29) to be obtainable by dressing ξ with elements of $D(0)$, we need another property in addition to the surjectivity of ∂ (cf. Lemma 3).

Lemma 4. *Let R and ∂ be such that ∂ is surjective and such that the element*

$$e^r = \sum_{i=0}^{\infty} \frac{r^i}{i!}$$

for each $r \in R$ is a well-defined element of R^* satisfying $\partial(e^r) = \partial(r)e^r$. Then every element

$$P = \xi + \sum_{i=0}^{\infty} p_{i+1}\xi^{-i}, \quad p_0 \neq 0,$$

can be obtained by dressing the operator ξ by an element of $D(0)$.

Proof. The goal is to find a zeroth-order operator D with an invertible leading coefficient such that $PD = D\xi$. We consider $D = k_0 K$, where k_0 is invertible and K is an operator of the form $K = 1 + K_- = 1 + \sum_{i \geq 1} k_i \xi^{-i}$. If we similarly represent the operator $P = \xi + p_1 + P_-$, where $P_- = \sum_{i=1}^{\infty} p_{i+1}\xi^{-i}$, then by Lemma 3, to solve the equation $Pk_0K = k_0K\xi$, it suffices to find $k_0 \in R^*$ such that $k_0^{-1}Pk_0$ has no constant term. A direct calculation shows that the coefficient of ξ^0 in $k_0^{-1}Pk_0$ is equal to $k_0^{-1}\partial(k_0) + p_1$. Therefore, we must solve the equation $\partial(k_0) + p_1k_0 = 0$. If l_1 is an antiderivative of p_1 with respect to ∂, i.e., $\partial(l_1) = p_1$, then we can choose $k_0 = e^{-l_1}$ in R as the solution of this equation. The lemma is proved..

Example 2. A concrete example of a k-algebra R and a k-linear derivation ∂ of R that satisfies the conditions of Lemma 3 is the choice $k = \mathbb{R}$ or $\mathbb{C}$, $R = k[[t_i, i \geq 1]]$, and $\partial := \partial/\partial t_1$.

We now turn to the analogue of the Lax equations in Theorem 1 for both decompositions. We first consider $\mathfrak{g}_1 = \mathrm{Psd}_{<0}$. We then have $\pi_1(P) = P_{<0}$. To each basic direction ξ^i, $i \geq 1$, in $\mathfrak{g}_2$, there should correspond an infinitesimal generator of a flow, i.e., a k-linear derivation ∂_i of R that commutes with ∂. We need deformations $\{L^m, m \geq 1\}$ with L of form (27) that satisfy the equations

$$\partial_i(L^m) = [L^m, \pi_1(L^i)] = [L^i_{\geq 0}, L^m] = [B_i, L^m], \quad i \geq 1, \quad m \geq 1, \tag{30}$$

where B_i is a brief notation for $L^i_{\geq 0}$. Because ∂_i and the operation of taking the commutator with B_i are both derivations of Psd, it suffices to find L satisfying

$$\partial_i(L) = [B_i, L] = [L, \pi_1(L^i)] \tag{31}$$

for all $i, i \geq 1$. We note that the last equality shows that $[B_i, L]$ is an operator of an order ≤ -1 like $\partial_i(L)$. As is seen in what follows, it follows from these equations that the differential operators $\{B_i\}$ in $R[\xi, \xi^{-1})$ satisfy

$$\partial_{i_1}(B_{i_2}) - \partial_{i_2}(B_{i_1}) - [B_{i_1}, B_{i_2}] = 0.$$

Let the derivations ∂, ∂_2, and ∂_3 be

$$\partial = \frac{\partial}{\partial x}, \qquad \partial_2 = \frac{\partial}{\partial t_2}, \qquad \partial_3 = \frac{\partial}{\partial t_3}.$$

Then this equation for $i_1 = 3$ and $i_2 = 2$ reduces to the equation

$$3(\ell_2)_{t_2t_2} = \left(2(\ell_2)_{t_3} - \frac{1}{2}(\ell_2)_{xxx} - 6\ell_2(\ell_2)_x\right)_x \tag{32}$$

for ℓ_2, which up to a scaling factor is the KP equation, and this explains the following terminology.

Definition 5. Let a k-algebra R equipped with a privileged k-linear derivation ∂ and a set $\{\partial_i \mid i \geq 1\}$ of k-linear derivations commuting with ∂ be given. Equations (31) for an operator L in $R[\xi, \xi^{-1})$ of form (27) are called the *Lax equations of the KP hierarchy*. We call L a *solution* of the hierarchy and the set $(R, \partial, \{\partial_i \mid i \geq 1\})$ a *realization* of this nonlinear system.

Remark 1. We note that any realization of the KP hierarchy admits the trivial solution $L = \xi$.

Further, we take $\mathfrak{g}_1 = \mathrm{Psd}_{\leq 0}$. Then $\pi_1(P) = P_{\leq 0}$. Again, for each $r \geq 1$, we need an infinitesimal generator of a flow corresponding to each basic direction ξ^r. To distinguish this case from the foregoing case, we let $\underline{\partial}_r$ denote this k-linear derivation that commutes with ∂. In this case, we seek deformations $\{M^m, m \geq 1\}$ with M of form (29) that satisfy the equations

$$\underline{\partial}_r(M^m) = [M^m, \pi_1(M^r)] = [M^r_{>0}, L^m] = [C_r, M^m], \quad r \geq 1, \quad m \geq 1, \tag{33}$$

where C_r is a brief notation for $M^r_{>0}$. Because $\underline{\partial}_r$ and the operation of taking the commutator with C_r are both derivations of Psd, it suffices to find M satisfying

$$\underline{\partial}_r(M) = [C_r, M] = [M, \pi_1(M^r)] \tag{34}$$

for all $r \geq 1$. We note that the last equality shows that $[C_r, M]$ is an operator of an order ≤ 0 like $\underline{\partial}_r(M)$. Because the KP hierarchy is based on the decomposition where $\pi_2(P)$ is the full differential operator part of a pseudodifferential operator P and the present decomposition is a strict version of it, we use the following terminology.

Definition 6. Let a k-algebra R equipped with a privileged k-linear derivation ∂ and a set $\{\underline{\partial}_r \mid r \geq 1\}$ of k-linear derivations commuting with ∂ be given. Equations (34) for an operator M in $R[\xi, \xi^{-1})$ of form (29) are called the *Lax equations of the strict KP hierarchy*. We call M a *solution* of the hierarchy and the set $(R, \partial, \{\underline{\partial}_r \mid r \geq 1\})$ a *realization* of this nonlinear system.

Remark 2. We note that any realization of the strict KP hierarchy also admits the trivial solution $M = \xi$.

As previously noted, each $[C_r, M]$ is an operator of an order ≤ 0. Similarly, each $[B_i, L]$ is an operator of an order ≤ -1. We assume that $Q \in R[\xi]$ has a degree $k \geq 1$ without a constant term and the degree of $[Q, M]$ does not exceed zero. Also, let P be an element in $R[\xi]$ whose commutator with L is of an order ≤ -1. It then makes sense to consider a k-linear derivation $\underline{\partial}_Q$ of R such that

$$\underline{\partial}_Q(M) = [Q, M]$$

and a k-linear derivation ∂_P of R such that

$$\partial_P(L) = [P, L].$$

The following proposition shows that this does not add anything new to the equations considered both in the case of the KP hierarchy and in the case of the strict KP hierarchy.

Proposition 1. *The following statements hold*:

1. *Let $Q \in R[\xi]$ be an element of a degree $k \geq 1$ without a constant term such that the degree of the commutator $[Q, M]$ does not exceed zero. Then there exists a unique set of elements $c_r \in R$, $\partial(c_r) = 0$, $1 \leq r \leq k$, such that $Q = \sum_{r=1}^{k} c_r C_r$.*

2. *Let $P \in R[\xi]$ be an element of a degree k such that the commutator $[P, L]$ has a negative degree. Then there exists a unique set of elements $p_i \in R$, $0 \leq i \leq k$, $\partial(p_i) = 0$, such that $P = \sum_{i=0}^{k} p_i B_i$.*

Proof. We use induction on the degree of Q for the proof. If the degree of Q is one, then $Q = c_1\xi$, where the degree of $[c_1\xi, M]$ does not exceed zero. In particular, this means that the only term of degree one produced by the commutator $[Q, M]$, namely, $[c_1\xi, \xi] = -\partial(c_1)\xi$, must be zero. Then $Q = c_1 C_1$ with $\partial(c_1) = 0$. Now let $Q = c_{k+1}\xi^{k+1} + R_k$, where R_k has a degree $\leq k$. We then obtain $[c_{k+1}\xi^{k+1}, \xi] = -\partial(c_{k+1})\xi^{k+1}$, and this is the only term in $[Q, L]$ of degree $k+1$. Hence, $\partial(c_{k+1})$ must be zero. The operator $Q - c_{k+1}C_{k+1}$ has a degree $\leq k$, no constant term, and the same property as Q. Therefore, it has the form

$$Q - c_{k+1}C_{k+1} = \sum_{i=1}^{k} c_i C_i, \qquad \partial(c_i) = 0.$$

The first statement is thus proved. The second is proved similarly.

The next section is devoted to a minimal realization of the Lax equations for both hierarchies.

5. Minimal realization of the Lax equations

In this section, we want to realize Eqs. (31) and (34) with a minimum number of relations between the coefficients of the respective potential solutions M and L and their derivatives with respect to ∂. We formalize this as follows: as the algebra R, we consider the algebra

$$\underline{\widetilde{R}} := k[\widetilde{m}_{j+1}^{(s)} \mid j \geq 0,\ s \geq 0]$$

of all polynomials in the unknowns $\{\widetilde{m}_{j+1}^{(s)} \mid j \geq 0,\ s \geq 0\}$ with coefficients in k in the case of the strict KP hierarchy and the algebra

$$\widetilde{R} := k[\tilde{\ell}_{j+1}^{(s)} \mid j \geq 1,\ s \geq 0]$$

of all polynomials in the unknowns $\{\tilde{\ell}_{j+1}^{(s)} \mid j \geq 1,\ s \geq 0\}$ with coefficients in k in the KP case. We recall that a k-linear derivation Δ of a polynomial ring $k[x_s]$ in any number of variables is uniquely determined by prescribing the images $\Delta(x_s)$ of all the $\{x_s\}$ by virtue of the derivation property

$$\Delta(fg) = \Delta(f)g + f\Delta(g)$$

for any f and $g \in k[x_s]$. Moreover, $\Delta(x_s)$ can be chosen arbitrarily. Taking this into account, we define the privileged derivation $\underline{\tilde{\partial}}$ in the first case as

$$\underline{\tilde{\partial}}(\widetilde{m}_{j+1}^{(s)}) = \widetilde{m}_{j+1}^{(s+1)}, \quad j \geq 0, \quad s \geq 0.$$

Similarly, we define the basic derivation $\tilde{\partial}$ on $\widetilde{R}$ as

$$\tilde{\partial}(\tilde{\ell}_{i+1}^{(s)}) = \tilde{\ell}_{i+1}^{(s+1)}, \quad i \geq 1, \quad s \geq 0.$$

Starting from the respective pseudodifferential operators $\widetilde{M}$ and $\widetilde{L}$ defined as

$$\widetilde{M} := \xi + \sum_{j=0}^{\infty} \widetilde{m}_{j+1}^{(0)} \xi^{-j}, \qquad \widetilde{L} := \xi + \sum_{j=1}^{\infty} \tilde{\ell}_{j+1}^{(0)} \xi^{-j},$$

we hence see that the superscript measures how many times $\underline{\tilde{\partial}}$ and $\tilde{\partial}$ have been applied to the coefficients of $\widetilde{M}$ and $\widetilde{L}$. Obviously, there are no relations between the coefficients of $\widetilde{M}$ and $\widetilde{L}$ and their respective derivatives with respect to $\underline{\tilde{\partial}}$ and $\tilde{\partial}$.

Further, we want to define k-derivations $\{\underline{\tilde{\partial}}_r \mid r \geq 1\}$ of $\underline{\widetilde{R}}$ that commute with $\underline{\tilde{\partial}}$. This last property leads to

$$\underline{\tilde{\partial}}_r(\widetilde{m}_{j+1}^{(s)}) = \underline{\tilde{\partial}}^s \underline{\tilde{\partial}}_r(\widetilde{m}_{j+1}^{(0)}),$$

and it therefore suffices to define the action of $\underline{\tilde{\partial}}_r$ on the coefficients $\widetilde{m}_{j+1}^{(0)}$ of $\widetilde{M}$. This can be done optimally in terms of a relation that $\widetilde{M}$ must satisfy. In accordance with the preceding section, we let $\widetilde{C}_r$ denote the operator $\widetilde{M}_{>0}^r$ for each $r \geq 1$. Further, we define the derivation $\underline{\tilde{\partial}}_r$ of $\underline{\widetilde{R}}$ by the identity

$$\underline{\tilde{\partial}}_r(\widetilde{M}) = \sum_{j=0}^{\infty} \underline{\tilde{\partial}}_r(\widetilde{m}_{j+1}^{(0)})\xi^{-j} := [\widetilde{C}_r, \widetilde{M}] = [\widetilde{M}, \widetilde{M}_{\leq 0}^r] \tag{35}$$

in $\underline{\widetilde{R}}[\xi, \xi^{-1})$. Similarly, we can define a set of k-derivations $\{\tilde{\partial}_i \mid i \geq 1\}$ of $\widetilde{R}$ that commute with $\tilde{\partial}$ by prescribing their action on the coefficients of $\widetilde{L}$ in terms of a set of relations for $\widetilde{L}$. As before, we simply write $\widetilde{B}_i$ for the pseudodifferential operator $\widetilde{L}_{\geq 0}^i$ for each $i \geq 1$. The equations that $\widetilde{L}$ must satisfy are then

$$\tilde{\partial}_i(\widetilde{L}) = \sum_{j=1}^{\infty} \tilde{\partial}_i(\tilde{\ell}_{j+1}^{(0)})\xi^{-j} := [\widetilde{B}_i, \widetilde{L}] = [\widetilde{L}, \widetilde{L}_{<0}^i]. \tag{36}$$

The Lax equations in both cases hold by definition. Nevertheless, we can derive several consequences from them. The first concerns a series of nonlinear equations for the corresponding sets of differential operators $\{\widetilde{C}_r\}$ and $\{\widetilde{B}_i\}$.

Proposition 2. 1. *The differential operators $\{\widetilde{C}_r\}$ in $\underline{\widetilde{R}}[\xi, \xi^{-1})$ satisfy the equations*

$$\underline{\tilde{\partial}}_{r_1}(\widetilde{C}_{r_2}) - \underline{\tilde{\partial}}_{r_2}(\widetilde{C}_{r_1}) - [\widetilde{C}_{r_1}, \widetilde{C}_{r_2}] = 0, \tag{37}$$

called the zero-curvature relations of the solution $\widetilde{M}$ of the strict KP hierarchy.

2. *The differential operators $\{\widetilde{B}_i\}$ in $\widetilde{R}[\xi, \xi^{-1})$ satisfy the equations*

$$\tilde{\partial}_{i_1}(\widetilde{B}_{i_2}) - \tilde{\partial}_{i_2}(\widetilde{B}_{i_1}) - [\widetilde{B}_{i_1}, \widetilde{B}_{i_2}] = 0, \tag{38}$$

called the zero-curvature relations of the solution $\widetilde{L}$ of the KP hierarchy.

Proof. The first statement holds if we can prove that the left-hand side of (37) belongs to both $\underline{\widetilde{R}}[\xi, \xi^{-1})_{>0}$ and $\underline{\widetilde{R}}[\xi, \xi^{-1})_{\leq 0}$. The first is obvious because all the $\widetilde{C}_r$ belong to $\underline{\widetilde{R}}[\xi, \xi^{-1})_{>0}$ and the $\underline{\tilde{\partial}}_r$ act coefficientwise. As previously mentioned, the Lax equations for $\widetilde{M}$ imply those for its powers

$$\underline{\tilde{\partial}}_{r_1}(\widetilde{M}^{r_2}) = [\widetilde{C}_{r_1}, \widetilde{M}^{r_2}] = -[\widetilde{M}_{<0}^{r_1}, \widetilde{M}^{r_2}]. \tag{39}$$

Further, we substitute the identity

$$\widetilde{C}_{r_k} = \widetilde{M}^{r_k} - \widetilde{M}_{\leq 0}^{r_k}$$

for $k = 1, 2$ in the left-hand side of (37) and use relations (39). We obtain

$$\begin{aligned} \underline{\tilde{\partial}}_{r_1}(\widetilde{C}_{r_2}) - \underline{\tilde{\partial}}_{r_2}(\widetilde{C}_{r_1}) - [\widetilde{C}_{r_1}, \widetilde{C}_{r_2}] &= [\widetilde{C}_{r_1}, \widetilde{M}^{r_2}] - \underline{\tilde{\partial}}_{r_1}(\widetilde{M}_{\leq 0}^{r_2}) - [\widetilde{C}_{r_2}, \widetilde{M}^{r_1}] + \\ &\quad + \underline{\tilde{\partial}}_{r_2}(\widetilde{M}_{\leq 0}^{r_1}) + [\widetilde{M}_{\leq 0}^{r_1}, \widetilde{M}^{r_2}] - [\widetilde{M}^{r_1}, \widetilde{M}_{\leq 0}^{r_2}] - [\widetilde{M}_{<0}^{r_1}, \widetilde{M}_{\leq 0}^{r_2}] = \\ &= \underline{\tilde{\partial}}_{r_2}(\widetilde{M}_{\leq 0}^{r_1}) - \underline{\tilde{\partial}}_{r_1}(\widetilde{M}_{\leq 0}^{r_2}) - [\widetilde{M}_{\leq 0}^{r_1}, \widetilde{M}_{\leq 0}^{r_2}]. \end{aligned}$$

Here, the last expression obviously belongs to $\underline{\widetilde{R}}[\xi, \xi^{-1})_{\leq 0}$. Arguing similarly, we can prove the second statement.

A consequence of Proposition 2 is a property that unites the flows belonging to one hierarchy.

Corollary 1. 1. *The derivations* $\{\tilde{\underline{\partial}}_r\}$ *of* $\widetilde{\underline{R}}$ *commute not only with* $\tilde{\underline{\partial}}$ *but also among themselves.*
2. *The derivations* $\{\tilde{\partial}_i\}$ *of* $\widetilde{R}$ *commute not only with* $\tilde{\partial}$ *but also among themselves.*

Proof. We must show that for all indices,

$$\tilde{\underline{\partial}}_{r_1} \circ \tilde{\underline{\partial}}_{r_2} = \tilde{\underline{\partial}}_{r_2} \circ \tilde{\underline{\partial}}_{r_1}, \qquad \tilde{\partial}_{i_1} \circ \tilde{\partial}_{i_2} = \tilde{\partial}_{i_2} \circ \tilde{\partial}_{i_1}.$$

Because $\{\tilde{\underline{\partial}}_r\}$ and $\{\tilde{\partial}_i\}$ respectively commute with $\tilde{\underline{\partial}}$ and $\tilde{\partial}$, the same holds for their products. Hence, if their differences are zero on the respective coefficients of $\widetilde{M}$ and $\widetilde{L}$, then they are identically zero on the whole of $\widetilde{\underline{R}}$ and $\widetilde{R}$. It hence suffices to show that

$$\tilde{\underline{\partial}}_{r_1} \circ \tilde{\underline{\partial}}_{r_2}(\widetilde{M}) = \tilde{\underline{\partial}}_{r_2} \circ \tilde{\underline{\partial}}_{r_1}(\widetilde{M}), \qquad \tilde{\partial}_{i_1} \circ \tilde{\partial}_{i_2}(\widetilde{L}) = \tilde{\partial}_{i_2} \circ \tilde{\partial}_{i_1}(\widetilde{L}).$$

For the first relation, we use the Lax equations for $\widetilde{M}$ and the fact that each $\tilde{\underline{\partial}}_{r_k}$ is a derivation of $\widetilde{\underline{R}}[\xi, \xi^{-1})$ to obtain

$$\tilde{\underline{\partial}}_{r_1} \circ \tilde{\underline{\partial}}_{r_2} - \tilde{\underline{\partial}}_{r_2} \circ \tilde{\underline{\partial}}_{r_1}(\widetilde{M}) = [\tilde{\underline{\partial}}_{r_1}(\widetilde{C}_{r_2}), \widetilde{M}] + [\widetilde{C}_{r_2}, [\widetilde{C}_{r_1}, \widetilde{M}]] - [\tilde{\underline{\partial}}_{r_2}(\widetilde{C}_{r_1}), \widetilde{M}] - [\widetilde{C}_{r_1}, [\widetilde{C}_{r_2}, \widetilde{M}]].$$

Because the commutator satisfies

$$[\widetilde{C}_{r_2}, [\widetilde{C}_{r_1}, \widetilde{M}]] - [\widetilde{C}_{r_1}, [\widetilde{C}_{r_2}, \widetilde{M}]] = [[\widetilde{C}_{r_2}, \widetilde{C}_{r_1}], \widetilde{M}],$$

and by virtue of statement 1 in Proposition 2, we have

$$[\tilde{\underline{\partial}}_{r_1}(\widetilde{C}_{r_2}) - \tilde{\underline{\partial}}_{r_2}(\widetilde{C}_{r_1}) - [\widetilde{C}_{r_1}, \widetilde{C}_{r_2}], \widetilde{M}] = [0, \widetilde{M}] = 0.$$

This concludes the proof for $\widetilde{M}$; the proof for $\widetilde{L}$ is similar.

6. Zero-curvature relations

The goal in this section is to describe other realizations of solutions of both hierarchies algebraically. The starting point in both cases is the operators $\widetilde{M}$ and $\widetilde{L}$ introduced in Sec. 5. Let $\underline{R}$ and R be other k-algebras equipped with the respective privileged k-linear derivations $\underline{\partial}$ and ∂. Further, we consider the potential solutions

$$M = \xi + \sum_{j=0}^{\infty} m_{j+1} \xi^{-j}, \qquad L = \xi + \sum_{j=1}^{\infty} \ell_{j+1} \xi^{-j}$$

in $\underline{R}[\xi, \xi^{-1})$ and $R[\xi, \xi^{-1})$ and the corresponding cutoffs

$$C_r := M^r_{>0}, \quad r \geq 1, \qquad B_i := L^i_{\geq 0}, \quad i \geq 1.$$

Then M uniquely determines a k-algebra morphism $i_M \colon \widetilde{\underline{R}} \to \underline{R}$ by the prescription

$$i_M(\widetilde{m}^{(s)}_{j+1}) = \underline{\partial}^s(m_{j+1}), \tag{40}$$

and this k-algebra morphism by definition satisfies

$$i_M \circ \tilde{\underline{\partial}} = \underline{\partial} \circ i_M. \tag{41}$$

Similarly, the operator L determines a k-algebra morphism $i_L \colon \widetilde{R} \to R$ by

$$i_L(\tilde{\ell}_{j+1}^{(s)}) = \partial^s(\ell_{j+1}), \tag{42}$$

and this k-algebra morphism by definition satisfies

$$i_L \circ \tilde{\partial} = \partial \circ i_L. \tag{43}$$

The maps i_M and i_L can be extended to k-algebra morphisms from the respective pseudodifferential operators $\underline{\widetilde{R}}[\xi, \xi^{-1})$ and $\widetilde{R}[\xi, \xi^{-1})$, introduced in Sec. 5, to $\underline{R}[\xi, \xi^{-1})$ and $R[\xi, \xi^{-1})$ such that

$$i_M(\widetilde{M}) = M, \qquad i_L(\widetilde{L}) = L.$$

In order to speak about solutions of the hierarchies, there should exist counterparts inside $\underline{R}$ and R of the respective collections of derivations $\{\underline{\tilde{\partial}}_r\}$ and $\{\tilde{\partial}_i\}$. This means that we need sets of k-linear derivations $\{\underline{\partial}_r\}$ and $\{\partial_i\}$ of $\underline{R}$ and R that commute with $\underline{\partial}$ and ∂. Assuming that these maps exist, we have introduced all the key ingredients, and it now makes sense to see if M and L satisfy the corresponding Lax equations with respect to these sets of derivations. Therefore, if M is a solution of the Lax equations of the strict KP hierarchy, then for all $r \geq 1$, we have

$$\underline{\partial}_r(M) = \underline{\partial}_r \circ i_M(\widetilde{M}) = [C_r, M] = [i_M(\widetilde{C}_r), i_M(\widetilde{M})] = i_M([\widetilde{C}_r, \widetilde{M}]) = i_M \circ \underline{\tilde{\partial}}_r(\widetilde{M}).$$

Hence, the k-linear maps $\underline{\partial}_r \circ i_M$ and $i_M \circ \underline{\tilde{\partial}}_r$ are equal on the coefficients of $\widetilde{M}$, but because of relation (41) and because the derivations $\{\underline{\partial}_r\}$ commute with ∂, we obtain the compatibilities

$$\underline{\partial}_r \circ i_M = i_M \circ \underline{\tilde{\partial}}_r, \quad r \geq 1, \tag{44}$$

on $\underline{\widetilde{R}}[\xi, \xi^{-1})$. On the other hand, if compatibilities (44) hold, then we apply these identities to $\widetilde{M}$, and because i_M is a k-algebra morphism, we obtain the Lax equations for M. Hence, relations (44) are equivalent to M being a solution of the strict KP hierarchy with respect to the $\{\underline{\partial}_r\}$.

There is also a similar reformulation of the statement that L is a solution of the KP hierarchy. Indeed, if L is a solution of the Lax equations of the KP hierarchy, then for all $i \geq 1$, we have

$$\partial_i(L) = \partial_i \circ i_L(\widetilde{L}) = [B_i, L] = [i_L(\widetilde{B}_i), i_L(\widetilde{L})] = i_L([\widetilde{B}_i, \widetilde{L}]) = i_L \circ \tilde{\partial}_i(\widetilde{L}).$$

Hence, the k-linear maps $\partial_i \circ i_L$ and $i_L \circ \tilde{\partial}_i$ are equal on the coefficients of $\widetilde{L}$, but by similar arguments as for M, we obtain the relations

$$\partial_i \circ i_L = i_L \circ \tilde{\partial}_i, \quad i \geq 1, \tag{45}$$

on $\widetilde{R}[\xi, \xi^{-1})$. Conversely, if compatibilities (45) hold, then we apply these identities to $\widetilde{L}$, and because i_L is a k-algebra morphism, we obtain the Lax equations for L. Therefore, relations (45) are equivalent to L being a solution of the KP hierarchy with respect to the $\{\partial_i\}$.

Further, we consider an analogue of zero-curvature relations (37) and (38) for the respective cutoffs $\{C_r\}$ and $\{B_i\}$ of M and L. If we couple identities (44) and (45) with the result in Proposition 2, then we see, first, that the strict differential operators $\{C_r\}$ in $\underline{R}[\xi, \xi^{-1})$ corresponding to a solution M of the strict KP hierarchy satisfy

$$\underline{\partial}_{r_1}(C_{r_2}) - \underline{\partial}_{r_2}(C_{r_1}) - [C_{r_1}, C_{r_2}] = 0. \tag{46}$$

We use the same terminology as for the minimal realization and call Eqs. (46) the *zero-curvature relations* for the strict cutoffs $\{C_r\}$ of the solution M of the strict KP hierarchy. Second, for a solution L of the KP hierarchy, it follows that the differential operators $\{B_i\}$ in $R[\xi, \xi^{-1})$ satisfy

$$\partial_{i_1}(B_{i_2}) - \partial_{i_2}(B_{i_1}) - [B_{i_1}, B_{i_2}] = 0. \tag{47}$$

Here, we use the same terminology as in the minimal case and call Eqs. (47) the *zero-curvature relations* for the differential operators $\{B_i\}$ corresponding to the solution L of the KP hierarchy.

The zero-curvature relations in both cases also suffice for obtaining the Lax equations for M and for L, namely, we have the following theorem.

Theorem 2. *Let $\underline{R}$ and R be k-algebras equipped with the respective privileged k-linear derivations $\underline{\partial}$ and ∂, and let M and L respectively be elements in $\underline{R}[\xi, \xi^{-1})$ and $R[\xi, \xi^{-1})$ of forms* (29) *and* (27).

1. *If $\underline{R}$ has a set of k-linear derivations $\{\underline{\partial}_r,\ r \geq 1\}$ that all commute with $\underline{\partial}$, then M satisfies the Lax equations of the strict KP hierarchy if and only if zero-curvature relations* (46) *hold for $\{C_r,\ r \geq 1\}$.*
2. *If R has a set of k-linear derivations $\{\partial_i,\ i \geq 1\}$ that commute with ∂, then L satisfies the Lax equations of the KP hierarchy with respect to $\{\partial_i\}$ if and only if zero-curvature relations* (47) *hold for $\{B_i,\ i \geq 1\}$.*

Proof. In both cases, we need only show the sufficiency. We present only the proof of the first statement because the proof of the second is absolutely analogous. We consider an operator M for which

$$\underline{\partial}_{r_1}(C_{r_2}) - \underline{\partial}_{r_2}(C_{r_1}) - [C_{r_1}, C_{r_2}] = 0$$

for all r_1 and r_2. If, for simplicity, we write that $M^{r_2} = C_{r_2} + D_{r_2}$ for all $r_2 \geq 1$, then this relation has an interesting consequence

$$\begin{aligned}\underline{\partial}_{r_1}(M^{r_2}) - [C_{r_1}, M^{r_2}] &= \underline{\partial}_{r_1}(C_{r_2} + D_{r_2}) - [C_{r_1}, C_{r_2}] - [C_{r_1}, D_{r_2}] = \\ &= \underline{\partial}_{r_2}(C_{r_1}) + \underline{\partial}_{r_1}(D_{r_2}) - [C_{r_1}, D_{r_2}].\end{aligned}$$

Because D_{r_2} has negative degrees in ξ, the last expression has a degree in ξ not exceeding $r_1 - 2$ for all $r_2 \geq 1$. We suppose that for some r_1, the Lax equation with respect to this derivation does not hold for M, i.e.,

$$\underline{\partial}_{r_1}(M) - [C_{r_1}, M] = \beta\xi_m + (\text{lower order in } \xi), \quad \beta \neq 0.$$

Because $\underline{\partial}_{r_1}$ and the operation of taking the commutator with C_{r_1} are both k-linear derivations, we have

$$\begin{aligned}\underline{\partial}_{r_1}(M^{r_2}) - [C_{r_1}, M^{r_2}] &= \sum_{r=0}^{r_2-1} M^r(\underline{\partial}_{r_1}(M) - [C_{r_1}, L])M^{r_2-1-r} = \\ &= r_2\beta\xi_{m+r_2-1} + (\text{lower order in } \xi)\end{aligned}$$

for all $r_2 \geq 1$. In particular, the degree in ξ of the operator $\underline{\partial}_{r_1}(M^{r_2}) - [C_{r_1}, M^{r_2}]$ is not bounded for r_2 tending to infinity. This contradicts the prior result that this degree is bounded by $r_1 - 2$. Hence, the supposition is false, and all the Lax equations for M must hold. This completes the proof of the theorem.

Remark 3. Zero-curvature relations (46) and (47) indicate the existence of linear systems from which the compatibility conditions are formed. We present such systems in the next section, and they give the key to constructing solutions of the hierarchies.

7. The linearization

Our starting point is a realization $(\underline{R}, \underline{\partial}, \{\underline{\partial}_r\})$ of the strict KP hierarchy and a realization $(R, \partial, \{\partial_i\})$ of the KP hierarchy. Let M be a potential solution M of the first hierarchy in $\underline{R}[\xi, \xi^{-1})$ and L be a potential solution of the second hierarchy in $R[\xi, \xi^{-1})$. The goal in this section is, on one hand, to describe a linear system in an appropriate $\underline{R}[\xi, \xi^{-1})$-module that leads to the Lax equations of the strict KP hierarchy and, on the other hand, to describe a linear system in an appropriate $R[\xi, \xi^{-1})$-module that leads to the Lax equations of the KP hierarchy. For the potential solution M of the strict KP hierarchy, the equalities

$$M\phi = z\phi, \qquad \underline{\partial}_r(\phi) = C_r(\phi), \quad r \geq 1, \tag{48}$$

must be satisfied. This system is called the *linearization of the strict KP hierarchy*. In the case of the KP hierarchy, the potential solution L must satisfy the so-called *linearization of the KP hierarchy*, given by the equations

$$L\psi = z\psi, \qquad \partial_i(\psi) = B_i(\psi), \quad i \geq 1. \tag{49}$$

Before specifying ψ and ϕ, we first show which manipulations are needed to obtain the Lax equations for M and L, and we then describe the context in which they hold.

We apply the derivation $\underline{\partial}_r$ to the first equation in (48). Assuming the Leibnitz rule for the action of $\underline{\partial}_r$ on $M\phi$ and that z is a scalar with respect to both the $\underline{R}[\xi]$-action and the $\underline{\partial}_r$-action, we substitute the second equation and obtain

$$\begin{aligned}\underline{\partial}_r(M\phi - z\phi) &= \underline{\partial}_r(M)\phi + M\underline{\partial}_r(\phi) - z\underline{\partial}_r(\phi) = \\ &= \underline{\partial}_r(M)\phi + MC_r\phi - C_r(z\phi) = (\underline{\partial}_r(M) - [C_r, M])\phi = 0.\end{aligned} \tag{50}$$

If we can omit the function ϕ, then M satisfies the Lax equations of the strict KP hierarchy.

Similarly, applying the derivation ∂_i to the first equation in (49), we obtain

$$\begin{aligned}\partial_i(L\psi - z\psi) &= \partial_i(L)\psi + L\partial_i(\psi) - z\partial_i(\psi) = \\ &= \partial_i(L)\psi + LB_i\psi - B_i(z\psi) = (\partial_i(L) - [B_i, L])\psi = 0.\end{aligned} \tag{51}$$

Hence, if we can omit ψ from the last equation, we obtain the Lax equations for L.

To make sense of Eqs. (48), we need a left action of operators like M and all the $\{C_r\}$. Therefore, we build an appropriate $\underline{R}[\xi, \xi^{-1})$-module, and the form of the elements in this module is guided by the solution of (48) for the trivial solution $M = \xi$. In this case, we have $C_r = \xi^r = M^r$ for all $r \geq 1$, and Eqs. (48) become

$$M\phi = z\phi \qquad \text{and} \qquad \underline{\partial}_r(\phi) = M^r(\phi) = z^r\phi \quad \text{for all } r \geq 1.$$

In particular, we can see that the first-order approximation of the flow corresponding to $\underline{\partial}_r$ is multiplication by z^r. Let s_r be the parameter for the flow corresponding to $\underline{\partial}_r$. Then $\underline{\partial}_r$ acts by taking the partial derivative $\partial/\partial s_r$. The equations of the linearization can then be integrated formally. We therefore consider the formal series

$$\phi_0 = e^{\sum_{r=1}^{\infty} s_r z^r}. \tag{52}$$

With the introduced operation $\underline{\partial}_r$, it satisfies the linearization equations for the trivial solution of the strict KP hierarchy. The space $\mathcal{O}$ of so-called *oscillating functions* is a space for which we can make sense of Eqs. (48), and it can be seen as a collection of perturbations of the trivial solution ϕ_0. It is defined as

$$\mathcal{O} = \left\{ \left(\sum_{j=-\infty}^{N} a_j z^j \right) e^{\sum_{r=1}^{\infty} s_r z^r} = \left(\sum_{j=-\infty}^{N} a_j z^j \right) \phi_0 \;\middle|\; a_j \in R \text{ for all } j \right\}.$$

We note that the product $(\sum_{j=-\infty}^{N} a_j z^j)\psi_0$ of the elements of $\mathcal{O}$ is formal. If we express ψ_0 in terms of the elementary Schur functions, i.e.,

$$e^{\sum_{i=1}^{\infty} s_r z^r} = \sum_{n=0}^{\infty} p_n(s) z^n,$$

then this product as a series in z and z^{-1} is formally equal to

$$\left(\sum_{j=-\infty}^{N} a_j z^j\right) e^{\sum_{r=1}^{\infty} s_r z^r} = \sum_{\ell\in\mathbb{Z}} \left(\sum_{k=0}^{\infty} a_{\ell-k} p_k\right) z^\ell.$$

Therefore, we need suitable convergence conditions to speak of the coefficients $\sum_{k=0}^{\infty} a_{\ell-k}p_k$ for all $\ell \in \mathbb{Z}$. This can be done by considering the product in a suitable class of boundary values (see [3]). A natural embedding of $k[s_r]$ as a k-subalgebra of R and also the fact that the $\underline{\partial}_r$ are suitable extensions of the derivation $\partial/\partial s_r$ on $k[s_r]$ help to place these coefficients in R. The space $\mathcal{O}$ becomes an $\underline{R}[\xi, \xi^{-1})$-module in accordance with the natural extension of the actions

$$b.\left(\sum_{j=-\infty}^{N} a_j z^j\right)\phi_0 = \left(\sum_{j=-\infty}^{N} b a_j z^j\right)\phi_0, \quad b \in R,$$

$$\xi.\left(\sum_{j=-\infty}^{N} a_j z^j\right)\phi_0 = \left(\sum_{j=-\infty}^{N} \partial(a_j) z^j + \sum_{j=-\infty}^{N} a_j z^{j+1}\right)\phi_0.$$

We also assume that each $\underline{\partial}_r$ acts on $\mathcal{O}$ according to the Leibnitz rule

$$\underline{\partial}_r\left(\left(\sum_{j=-\infty}^{N} a_j z^j\right)\phi_0\right) = \left(\sum_{j=-\infty}^{N} \underline{\partial}_r(a_j) z^j + \sum_{j=-\infty}^{N} a_j z^{j+r}\right)\phi_0.$$

This defines all the operators in (48). We note that $\mathcal{O}$ is a free $\underline{R}[\xi, \xi^{-1})$-module with the generator ϕ_0 because the equality

$$\left(\sum_j p_j \xi^j\right)\phi_0 = \left(\sum_j p_j z^j\right)\phi_0$$

is satisfied. Hence, if we have relations $Q\phi = 0$, where $Q \in \underline{R}[\xi, \xi^{-1})$, and $\phi = P\phi_0$, where $P \in \underline{R}[\xi, \xi^{-1})$, in $\mathcal{O}$, then we can conclude that $QP = 0$ and, moreover, $Q = 0$ if P is invertible. This is just the case for the class of oscillating functions in which an element $\phi \in \mathcal{O}$ is called an *oscillating function of type* $\alpha_\ell z^\ell$, where α_ℓ is invertible in $\underline{R}$, if it has the form

$$\phi = \phi(t, z) = \left\{\alpha_\ell z^\ell + \sum_{k<\ell} \alpha_k z^k\right\}\phi_0 = K.\phi_0, \qquad K = \sum_{k\le\ell} \alpha_k \xi^k. \tag{53}$$

An oscillating function ϕ of type $\alpha_\ell z^\ell$ is called a wave function of the strict KP hierarchy if there is an operator M such that Eqs. (48) hold for M and ϕ. We note that in this case, M is a solution of the strict KP hierachy because all manipulations described in this section are allowed. The operator M is then totally defined by ϕ. If $\phi = K.\phi_0$ as in (53), then the first equation in (48) can be written as

$$M\phi = MK.\phi_0 = z\phi = zK.\phi_0 = K\xi.\phi_0,$$

and M is hence obtained by dressing ξ with K, i.e., $M = K\xi K^{-1}$. We note that to obtain Eqs. (48), it suffices to prove a weaker result.

Proposition 3. *Let ϕ be an oscillating function of type $\alpha_\ell z^\ell$ and M be a first-order operator in $\underline{R}[\xi,\xi^{-1})$ of form (29). We assume that for all $r \geq 1$, there exists a differential operator $Q_r \in \underline{R}[\xi]$ without a constant term such that*

$$M\phi = z\phi, \qquad \underline{\partial}_r(\phi) = Q_r\phi.$$

Then $Q_r = C_r$ for all $r \geq 1$. In particular, M is a solution of the strict KP hierarchy.

Proof. Let K be such that $\phi = K\phi_0$. As was shown, the first equation ensures that $M = K\xi K^{-1}$. By the definition of the action of $\underline{\partial}_r$ on $\mathcal{O}$, on one hand, we have

$$\underline{\partial}_r(\phi) = \{\underline{\partial}_r(K) + K\xi^r\}.\phi_0,$$

and, on the other hand, this must equal $Q_r K.\phi_0$. This leads to the identity

$$\underline{\partial}_r(K)K^{-1} + K\xi^r K^{-1} = \underline{\partial}_r(K)K^{-1} + M^r = Q_r.$$

Because $\underline{\partial}_r(K)K^{-1}$ has only terms of a degree in ξ not exceeding zero, taking the strict differential operator part of both sides results in $M^r_{>0} = (Q_r)_{>0} = Q_r$. The proposition is proved.

We assume that we have two wave functions $\phi_1 = K_1.\phi_0$ and $\phi_2 = K_2.\phi_0$ of the strict KP hierarchy corresponding to the same solution M of the strict KP hierarchy, i.e.,

$$M\phi_k = z\phi_k \qquad \text{and} \qquad \underline{\partial}_r(\phi_k) = C_r\psi_k, \quad k = 1, 2.$$

Then, first, we know that $K := K_1^{-1}K_2$ commutes with ξ. Hence, the coefficients of K are constants for the derivation ∂, i.e., $\partial(K) = 0$. Moreover, it follows from the proof of Proposition 3 that for all $r \geq 1$,

$$\underline{\partial}_r(K_1) = (C_r - M^r)K_1, \qquad \underline{\partial}_r(K_2) = (C_r - M^r)K_2.$$

This means that the coefficients of K are constants for all derivations $\underline{\partial}_r$ because

$$\begin{aligned}\underline{\partial}_r(K) &= -K_1^{-1}\underline{\partial}_r(K_1)K_1^{-1}K_2 + K_1^{-1}\underline{\partial}_r(K_2) = \\ &= -K_1^{-1}(C_r - M^r)K_1K_1^{-1}K_2 + K_1^{-1}\underline{\partial}_r(K_2) = \\ &= K_1^{-1}(-(C_r - M^r)K_2 + \underline{\partial}_r(K_2)) = 0.\end{aligned}$$

The following statement has thus been proved.

Corollary 2. *If $\phi_1 = K_1.\phi_0$ and $\phi_2 = K_2.\phi_0$ are two wave functions of the strict KP hierarchy corresponding to the same solution M, then the element $K := K_1^{-1}K_2$ in $D(0)$ is constant for all relevant flows, i.e., $\underline{\partial}_r(K) = \underline{\partial}(K) = 0$ for all $r \geq 1$.*

In the KP case, we build a similar $R[\xi,\xi^{-1})$-module. Again, we should start with the linearization for the trivial solution $L_0 = \xi$ of the hierarchy:

$$L_0\psi = z\psi, \qquad \partial_i(\psi) = L_0^i(\psi) = z^i\psi, \quad i \geq 1.$$

Let t_i be the parameter for the flow corresponding to ∂_i, i.e., ∂_i acts as taking the partial derivative $\partial/\partial t_i$ with respect to t_i. Then the linearization has the solution

$$\psi_0 = e^{\sum_{i=1}^{\infty} t_i z^i}. \tag{54}$$

The appropriate $R[\xi,\xi^{-1})$-module is again a collection $\mathcal{M}$ of perturbations of the trivial solution ψ_0. Concretely, it is given by

$$\mathcal{M} = \left\{ \left(\sum_{j=-\infty}^{N} a_j z^j \right) e^{\sum_{i=1}^{\infty} t_i z^i} = \left(\sum_{j=-\infty}^{N} a_j z^j \right) \psi_0 \;\middle|\; a_j \in R \text{ for all } j \right\},$$

and its elements $\mathcal{M}$ are also called *oscillating functions.* The space $\mathcal{M}$ can be made into an $R[\xi,\xi^{-1})$-module analogously to obtaining the $\underline{R}[\xi,\xi^{-1})$-module structure on $\mathcal{O}$. In particular, it is a free $R[\xi,\xi^{-1})$-module with the generator ψ_0.

We again distinguish a special class of elements of $\mathcal{M}$: an element $\psi \in \mathcal{M}$ is called an *oscillating function of type* z^ℓ if it has the form

$$\psi = \psi(t,z) = \left\{ z^\ell + \sum_{k<\ell} \alpha_k z^k \right\} \psi_0 = K.\psi_0, \qquad K = \xi^\ell + \sum_{k<\ell} \alpha_k \xi^k. \tag{55}$$

An oscillating function ψ of type z^ℓ is called a wave function of the KP hierarchy if there is an operator L of form (27) such that Eqs. (49) hold for L and ψ. We note that in this case, L is a solution of the KP hierarchy because all manipulations described in this section are allowed, and if $\psi = K.\psi_0$, then L is obtained by dressing ξ with K, i.e., $L = K\xi K^{-1}$. Analogues of Proposition 3 and Corollary 2 hold in the KP setting. We formulate them for completeness of the exposition; the proofs repeat those presented above.

Proposition 4. 1. *Let ψ be an oscillating function of type z^ℓ and L be a first-order operator in $R[\xi,\xi^{-1})$ that is a potential solution of the KP hierarchy. We assume that for all $i \geq 1$, there exists a differential operator $P_i \in R[\xi]$ such that*

$$L\psi = z\psi, \qquad \partial_i(\psi) = P_i\psi.$$

Then $P_i = B_i$ for all $i \geq 1$. In particular, L is a solution of the KP hierarchy.

2. *Let $\psi_1 = K_1.\psi_0$ and $\psi_2 = K_2.\psi_0$ be two wave functions of the KP hierarchy corresponding to the same solution L. Then the dressing operators K_1 and K_2 differ by an element K in $D(0)_1$ that is constant for all relevant derivations, i.e., for all $i \geq 1$,*

$$\partial_i(K) = \partial(K) = 0.$$

Remark 4. In [3], we will present a geometric setting that allows constructing wave functions of the strict KP hierarchy, where the product of the perturbation and the exponential factor converges.

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