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DEGENERATING THE ELLIPTIC SCHLESINGER SYSTEM

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We study various ways of degenerating the Schlesinger system on the elliptic curve with R marked points. We construct a limit procedure based on an infinite shift of the elliptic curve parameter and on shifts of the marked points. We show that using this procedure allows obtaining a nonautonomous Hamiltonian system describing the Toda chain with additional spin $\mathfrak{sl}(N,\mathbb{C})$ degrees of freedom.

Keywords: integrable system, isomonodromic deformation, Schlesinger system, Toda chain, Inozemtsev limit

1. Introduction

This paper is devoted to using limit procedures to study various degenerations of the elliptic generalization [1]–[4] of the Schlesinger system [5] and of the nonautonomous elliptic $SL(N,\mathbb{C})$ top. The methods considered for constructing these limit transitions can be generalized to the case of the Painlevé equations related to the elliptic Schlesinger system and to integrable systems arising in various domains of theoretical physics. In particular, the Painlevé and Schlesinger equations have important applications in matrix model theory [6]–[8] and conformal field theory [9], [10]. Some integrable systems are related to the low-energy effective action in supersymmetric gauge theories [11]–[13].

Relations between different integrable systems were studied in [14]–[17] and other papers. Inozemtsev [14] proposed a procedure establishing a limit relation between the Toda chains and the elliptic Calogero–Moser model. The Inozemtsev limit method was later generalized and used to establish relations between other integrable systems [15]. A singular symplectic transformation of the Calogero–Moser system into the elliptic $SL(N,\mathbb{C})$ top was constructed in [16]. Using this transformation, Smirnov obtained integrable systems of tops on the $\mathfrak{sl}(N,\mathbb{C})$ algebra equivalent to the N-particle trigonometric and rational Calogero–Moser systems [17].

The Schlesinger system [5] is the following system of first-order differential equations for R matrices $\mathbf{S}^i$, $i=1,\dots,R$, located at R marked points $x_j\in\mathbb{CP}^1$, $j=1,\dots,R$:

$$\begin{aligned}\frac{\partial\mathbf{S}^i}{\partial x_j}&=\frac{[\mathbf{S}^i,\mathbf{S}^j]}{x_i-x_j},\quad i\neq j,\\ \frac{\partial\mathbf{S}^i}{\partial x_i}&=-\sum_{j\neq i}\frac{[\mathbf{S}^i,\mathbf{S}^j]}{x_i-x_j}.\end{aligned}\tag{1.1}$$

Equations (1.1) describe the preservation conditions for the monodromies of the linear system on $\mathbb{CP}^1$ of the form

$$\frac{d\Psi}{dz}=\sum_{i=1}^{R}\frac{\mathbf{S}^i}{z-x_i}\Psi.$$

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It was shown in [18] that in the case of 2×2 matrices and four marked points, the Schlesinger system is equivalent to the Painlevé VI equation [19].

The generalization in [1]–[4] of the Schlesinger system describing isomonodromic deformations on an elliptic curve is called the elliptic Schlesinger system. It is a nonautonomous Hamiltonian system defined on the space of R copies of the Lie coalgebra $\mathfrak{sl}(N,\mathbb{C})^*$, and the role of time in this system is played by the parameter τ of the elliptic curve $\Sigma_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and the positions x_i, $i = 1,\dots,R$, of the marked points on Σ_τ. The phase space of this Hamiltonian system is the direct product of R copies of coadjoint orbits of the group $SL(N,\mathbb{C})$, and each orbit is related to one of the marked points x_i. We can therefore regard the elliptic Schlesinger system as a system of interacting nonautonomous elliptic $SL(N,\mathbb{C})$ tops. An autonomous version of the elliptic Schlesinger system is the integrable Gaudin system [20]–[22], which is a Hitchin system on the elliptic curve Σ_τ with R marked points.

From the elliptic $SL(N,\mathbb{C})$ top, we previously obtained the Toda chains for $N \geq 2$ in the autonomous case [23], [24] and for $N = 2$ in the nonautonomous case [25]. Here, we generalize the limit procedure in [23]–[25] to the case of the elliptic Schlesinger system and obtain a nonautonomous Hamiltonian system describing the Toda chain with additional spin $\mathfrak{sl}(N,\mathbb{C})$ degrees of freedom.

The proposed procedure is based on the generalizations [23]–[25] of the Inozemtsev limit [14], i.e., the limit transition from the elliptic Calogero–Moser system to the Toda chain in the autonomous case. In the Inozemtsev limit, we perform infinite shifts of coordinates of the elliptic Calogero–Moser system, renormalize the coupling constant, and take the trigonometric limit in which the imaginary part of the elliptic curve parameter tends to infinity. In the case of the nonautonomous elliptic Schlesinger system, the elliptic curve parameter τ plays the role of time and cannot be simply taken to infinity. We therefore introduce an infinite shift of the parameter τ using the change of variable

$$\tau = \tau_1 + \tau_2, \tag{1.2}$$

where the first summand τ_1 becomes the time in the limit system and we use the second, constant term to take the trigonometric limit $\operatorname{Im}\tau_2 \to +\infty$. From the elliptic curve standpoint, in the trigonometric limit, we obtain the complex cylinder $\mathbb{C}/\mathbb{Z}$ from the complex torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$.

In Sec. 2, we obtain a nonautonomous Toda chain from the system of the elliptic $SL(N,\mathbb{C})$ top with the spectral parameter z for $N > 2$. For this, in addition to replacement (1.2), we perform the time-dependent shift of the spectral parameter $z = \tilde{z} + \tau/2$ and time-independent scaling transformations of coordinates analogous to those previously proposed in [23].

In Sec. 3, we consider the reduction of the Schlesinger elliptic system in which the Lax operator $L(z)$ is the sum of R copies of the Lax operator of the elliptic $SL(N,\mathbb{C})$ top,

$$L(z) = \sum_{i=1}^{R} L^i(z - x_i).$$

Introducing various shifts of the marked points x_i, we can realize its own type of reduction on each copy of the elliptic $SL(N,\mathbb{C})$ top. We construct the limit procedure that describes two possible variants of reductions of copies of the elliptic top. We take the limit procedure described in Sec. 2 as the first reduction variant and the trigonometric limit as the second. To simplify expressions, we apply the first reduction variant to the first r copies of the elliptic top and the second reduction variant to the remaining $p = R - r$ copies. For this, in addition to replacement (1.2), we perform shifts of the marked points $x_i = \tilde{x}_i - \tau/2$, $i = 1,\dots,r$, together with scaling transformations of the coordinates of the corresponding copies of the Lie coalgebra $\mathfrak{sl}(N,\mathbb{C})^*$. In Sec. 3.2.1, we show that the limit systems obtained at fixed p in the case $r > 1$ are equivalent to limit systems obtained in the case $r = 1$ in which only one copy of the elliptic top transforms into the Toda chain. Using the proposed reduction scheme for the elliptic Schlesinger system, we thus obtain the nonautonomous system describing the Toda chain with additional spin $\mathfrak{sl}(N,\mathbb{C})$ degrees of freedom.

We present the necessary facts about the systems under consideration.

1.1. The elliptic Schlesinger system. The elliptic Schlesinger system is defined on the space of R copies of the Lie coalgebra $\mathfrak{g}^* \sim \mathfrak{sl}(N,\mathbb{C})^*$:

$$\mathcal{P}_{R,N} = \bigoplus_{i=1}^{R} \mathfrak{g}_i^*, \qquad \mathfrak{g}_i^* = \left\{ \mathbf{S}^i = \sum_{m,n} s^i_{mn} T_{mn} \right\},$$

where the basis $\{T_{mn}\}$, $m,n = 0,\dots,N-1$, $m^2+n^2 \neq 0$, is described in Appendix A. On the space $\mathcal{P}_{R,N}$, we have the linear Poisson structure

$$\{s^i_{mn}, s^j_{kl}\} = 2\mathrm{i}\delta_{ij} \sin\left(\frac{\pi(nk-ml)}{N}\right) s^i_{m+k,n+l}, \tag{1.3}$$

where $\mathrm{i} = \sqrt{-1}$. This Poisson system is degenerate, and its symplectic leaves are R copies of the coadjoint orbits $\mathcal{O}_i$, $i = 1,\dots,R$, of $SL(N,\mathbb{C})$. The phase space of the elliptic Schlesinger system has the form

$$\mathcal{R}_{R,N} \sim \mathcal{P}_{R,N} / \{c^\mu(i) = c^\mu(i)_0\} \sim \prod_i \mathcal{O}_i,$$

where $c^\mu(i)$ are Casimir functions of the orders $\mu = 2,\dots,N$ corresponding to the orbits $\mathcal{O}_i$.

The role of system times is played by the parameter τ of the elliptic curve $\Sigma_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and the positions x_i, $i = 1,\dots,R$, of the marked points on Σ_τ. For the singular point positions x_i and the elliptic curve parameter τ to have the sense of local coordinates in an open cell in the moduli space of elliptic curves with R marked points, we can impose the condition $\sum_{i=1}^R x_i \in (\mathbb{Z}+\tau\mathbb{Z})$ on x_i.

The elliptic Schlesinger system is a nonautonomous Hamiltonian system with the equations of motion

$$\begin{aligned} &\partial_\tau \mathbf{S}^i = \{H_0, \mathbf{S}^i\}, \\ &\partial_{x_k} \mathbf{S}^i = \{H_k, \mathbf{S}^i\}, \quad i,k = 1,\dots,R, \end{aligned} \tag{1.4}$$

where

$$H_0 = \frac{1}{4\pi\mathrm{i}} \left(\sum_{j\neq i} \sum_{m,n} s^i_{mn}\, s^j_{-m,-n}\, f\begin{bmatrix} m \\ n \end{bmatrix}(x_j - x_i) - \right.$$

$$\left. - \sum_{i=1}^{R} \sum_{m,n} s^i_{mn}\, s^i_{-m,-n}\, E_2\left(\frac{m+n\tau}{N}\right) \right), \tag{1.5}$$

$$H_k = -\sum_{i\neq k} \sum_{m,n} s^k_{mn}\, s^i_{-m,-n}\, \varphi\begin{bmatrix} m \\ n \end{bmatrix}(x_i - x_k) \tag{1.6}$$

and

$$\varphi\begin{bmatrix} m \\ n \end{bmatrix}(z) = \mathrm{e}\left(-\frac{nz}{N}\right) \phi\left(-\frac{m+n\tau}{N}, z\right), \tag{1.7}$$

$$f\begin{bmatrix} m \\ n \end{bmatrix}(z) = \mathrm{e}\left(-\frac{nz}{N}\right) \partial_u \phi(u,z)\big|_{u=-(m+n\tau)/N}, \tag{1.8}$$

$\mathrm{e}(z) \equiv e^{2\pi \mathrm{i} z}$, and the functions $\phi(u,z)$ and $E_k(z)$ are defined in Appendix B (see the respective formulas (B.3) and (B.1)).

We can represent Hamiltonian equations of motion (1.4) as the zero-curvature conditions

$$
\begin{aligned}
&\partial_\tau L - \partial_z M^0 = [L, M^0],\\
&\partial_{x_k} L - \partial_z M^k = [L, M^k], \quad k = 1, \dots, R,
\end{aligned}
\tag{1.9}
$$

where

$$
L(z) = \sum_{i=1}^{R} \sum_{m,n} s^i_{mn}\, \varphi\begin{bmatrix} m \\ n \end{bmatrix}(z - x_i) T_{mn}, \tag{1.10}
$$

$$
M^0(z) = \frac{1}{2\pi \mathrm{i}} \sum_{i=1}^{R} \sum_{m,n} s^i_{mn}\, f\begin{bmatrix} m \\ n \end{bmatrix}(z - x_i) T_{mn}, \tag{1.11}
$$

$$
M^k(z) = -\sum_{m,n} s^k_{mn}\, \varphi\begin{bmatrix} m \\ n \end{bmatrix}(z - x_k) T_{mn}. \tag{1.12}
$$

Hamiltonians (1.5) and (1.6) are related to the expansion of $\operatorname{Tr} L^2(z)$ over the elliptic function basis,

$$
\frac{1}{2N} \operatorname{Tr} L^2(z) = 2\pi \mathrm{i} H_0 + \sum_{k=1}^{R} \big(H_{2,k} E_2(z - x_k) + H_k E_1(z - x_k) \big), \tag{1.13}
$$

where $H_{2,k}$ are the quadratic Casimir functions corresponding to the orbits $\mathcal{O}_k$,

$$
H_{2,k} = \frac{1}{2} \sum_{m,n} s^k_{mn}\, s^k_{-m,-n}. \tag{1.14}
$$

We can write Poisson brackets (1.3) using the classical Belavin–Drinfeld r-matrix $r(z)$ [26]–[28]:

$$
\{L_1(z), L_2(w)\} = [r(z - w), L_1(z) + L_2(w)],
$$

where

$$
r(z) = -\sum_{m,n} \varphi\begin{bmatrix} m \\ n \end{bmatrix}(z) T_{mn} \otimes T_{-m,-n}, \tag{1.15}
$$

$L_1(z) = L(z) \otimes \mathrm{Id}$, and $L_2(w) = \mathrm{Id} \otimes L(w)$.

1.2. The elliptic $SL(N,\mathbb{C})$ top. The elliptic $SL(N,\mathbb{C})$ top is an example of the Euler–Arnold top [29], whose phase space is given by a coadjoint orbit of $SL(N,\mathbb{C})$,

$$
\mathcal{R}^{\mathrm{rot}} = \{\mathbf{S} \in \mathfrak{sl}(N,\mathbb{C}),\ \mathbf{S} = g^{-1} \mathbf{S}^{(0)} g\},
$$

where $g \in SL(N,\mathbb{C})$ is defined modulo left shifts G_0 commuting with $\mathbf{S}^{(0)}$. In the phase space $\mathcal{R}^{\mathrm{rot}}$ of the top, we have the nondegenerate symplectic Kirillov–Kostant form $\omega^{\mathrm{rot}} = \operatorname{Tr}\big(\mathbf{S}^{(0)} (dg) g^{-1} \wedge (dg) g^{-1}\big)$.

The system dynamics is governed by the Hamiltonian

$$
H^{\mathrm{rot}} = -\frac{1}{2} \operatorname{Tr} \mathbf{S} J(\mathbf{S}), \tag{1.16}
$$

where $J(\mathbf{S})=\sum_{m,n} J_{mn}\, s_{mn}T_{mn}$, s_{mn} are the coordinates in the basis $\{T_{mn}\}$ of the algebra $\mathfrak{sl}(N,\mathbb{C})$, which constitute the sine-algebra (see Appendix A),

$$J_{mn}=E_2\biggl(\frac{m+n\tau}{N},\tau\biggr),\quad m,n=0,\dots,N-1,\quad m^2+n^2\neq 0,$$

and $E_2(z,\tau)$ is the second Eisenstein function [30] defined on the complex torus $\mathbb{C}/(2\omega_1\mathbb{Z}+2\omega_2\mathbb{Z})$ with $\omega_1=1/2$ and $\tau=\omega_2/\omega_1$.

We can write the equations of motion in the Lax form [31],

$$\frac{dL^{\mathrm{rot}}}{dt}=N[L^{\mathrm{rot}},M^{\mathrm{rot}}].$$

The multiplier N is related to the definition of the Lax matrices in the sine-algebra basis (see Appendix A):

$$L^{\mathrm{rot}}=\sum_{m,n} s_{mn}\,\varphi\begin{bmatrix}m\\ n\end{bmatrix}(z)T_{mn},\qquad M^{\mathrm{rot}}=\sum_{m,n} s_{mn}\,f\begin{bmatrix}m\\ n\end{bmatrix}(z)T_{mn}.$$

Hamiltonian (1.16) is related to the Lax matrix:

$$H^{\mathrm{rot}}=\frac{1}{2}\operatorname{Tr}(L^{\mathrm{rot}})^2-\frac{1}{2}\operatorname{Tr}S^2E_2(z,\tau). \tag{1.17}$$

We obtain the Poisson brackets for the s_{mn} from the commutator $[T_{ab},T_{cd}]$ of the basis elements T_{ab} and T_{cd} (see formula (A.1) in Appendix A),

$$\{s_{ab},s_{cd}\}=2\mathrm{i}\sin\biggl(\frac{\pi(bc-ad)}{N}\biggr)s_{a+c,b+d}. \tag{1.18}$$

Passing to standard basis (A.2), we obtain $\{S_{ij},S_{kl}\}=N(S_{kj}\delta_{il}-S_{il}\delta_{kj})$.

1.3. The Toda chains. Periodic and nonperiodic Toda chains composed from N interacting particles are defined in the center-of-mass frame on the phase space

$$\mathcal{R}^{\mathrm{T}}=\biggl\{(\mathbf{u},\mathbf{v})\colon\ \sum_{i=1}^{N}u_i=0,\ \sum_{i=1}^{N}v_i=0\biggr\}$$

with the canonical symplectic form $\omega^{\mathrm{T}}=(d\mathbf{v}\wedge d\mathbf{u})$. The Hamiltonian of the nonperiodic Toda chain is

$$H^{\mathrm{AT}}=\frac{1}{2}\sum_{i=1}^{N}v_i^2+4\pi^2M^2\sum_{i=1}^{N-1}\mathrm{e}(u_{i+1}-u_i),$$

while we must add the term corresponding to interaction between the first and last particles for the periodic chain,

$$H^{\mathrm{PT}}=\frac{1}{2}\sum_{i=1}^{N}v_i^2+4\pi^2M^2\sum_{i=1}^{N}\mathrm{e}(u_{i+1}-u_i),\qquad u_{N+1}=u_1.$$

The equations of motion of both the periodic and the nonperiodic Toda chains admit the Lax form [32]–[34]:

$$\frac{dL^{\mathrm{AT}}}{dt}=[L^{\mathrm{AT}},M^{\mathrm{AT}}],\qquad \frac{dL^{\mathrm{PT}}}{dt}=[L^{\mathrm{PT}},M^{\mathrm{PT}}].$$

2. The nonautonomous $SL(N,\mathbb{C})$ top and the Toda chain

We consider the nonautonomous elliptic $SL(N,\mathbb{C})$ top in which the elliptic curve parameter τ plays the role of time. The Lax pair of the top satisfies the isomonodromy deformation equations:

$$\partial_\tau L^{\rm rot} - \frac{1}{2\pi \rm i}\partial_z M^{\rm rot} = N[L^{\rm rot}, M^{\rm rot}], \tag{2.1}$$

which are equivalent to the Hamiltonian equations of motion

$$\frac{ds_{mn}}{d\tau} = \{H^{\rm rot}, s_{mn}\}.$$

We previously applied various limit procedures to the elliptic $SL(N,\mathbb{C})$ top to obtain Toda chains in the autonomous case for $N \geq 2$ and in the nonautonomous case for $N = 2$ [23]–[25]. In this section, we construct a nonautonomous Toda chain from the elliptic $SL(N,\mathbb{C})$ top for $N > 2$. The Hamiltonian equations of motion of this nonautonomous system are

$$\partial_{\tau_1} u_j = N^2 v_j, \qquad \partial_{\tau_1} v_j = 8\pi^3 {\rm i} M^2 N^2 q_1^{1/N}\big({\rm e}(u_{j+1}-u_j) - {\rm e}(u_j - u_{j-1})\big), \tag{2.2}$$

where the factor $q_1^{1/N}$ in the right-hand side of the second equation depends on time explicitly, $q_1 \equiv {\rm e}(\tau_1)$.

As mentioned above, the limit procedure is based on an infinite shift of τ, which we describe by the variable change $\tau = \tau_1 + \tau_2$, where the first term τ_1 becomes the time in the limit system and we use the second, constant term to take the trigonometric limit $\operatorname{Im}\tau_2 \to +\infty$. After such a replacement, the Hamiltonian of the nonautonomous elliptic top becomes $H^{\rm rot} = H^{\rm rot}(\mathbf{s}, \tau_1 + \tau_2)$. We now consider the system dynamics with respect to τ_1, interpreting the constant parameter τ_2 as the time reference point. The Hamiltonian equations of motion with respect to τ_1,

$$\frac{ds_{mn}}{d\tau_1} = \{H^{\rm rot}, s_{mn}\},$$

admit the Lax representation equivalent to (2.1)

$$\partial_{\tau_1} L^{\rm rot} - \frac{1}{2\pi \rm i}\partial_z M^{\rm rot} = N[L^{\rm rot}, M^{\rm rot}], \tag{2.3}$$

where $L^{\rm rot} = L^{\rm rot}(\mathbf{s}, z, \tau_1 + \tau_2)$ and $M^{\rm rot} = M^{\rm rot}(\mathbf{s}, z, \tau_1 + \tau_2)$. We next perform a spectral parameter shift depending on τ_1,

$$z = \tilde z + \frac{\tau}{2}, \tag{2.4}$$

and a scaling transformation of coordinates independent of τ_1,

$$s_{mn} = \tilde s_{mn} q_2^{-g(n)}, \qquad g(n) = \frac{1-\tilde\delta(n)}{2N}, \tag{2.5}$$

where we introduce the notation

$$q_2 \equiv {\rm e}(\tau_2), \qquad \tilde\delta(n) = \begin{cases} 1, & n \equiv 0 \bmod N, \\ 0, & n \not\equiv 0 \bmod N. \end{cases}$$

The trigonometric limit as $\operatorname{Im}\tau_2 \to +\infty$ then corresponds to the limit as $q_2 \to 0$.

After replacement (2.5), Poisson bracket (1.18) becomes

$$\{\tilde{s}_{ab}, \tilde{s}_{cd}\} = 2\mathrm{i}\sin\left(\frac{\pi(bc-ad)}{N}\right) q_2^{g(b)+g(d)-g(b+d)} \tilde{s}_{a+c,b+d}.$$

The coordinates $\tilde{s}_{mn}$ hence constitute a Lie algebra with respect to the Poisson bracket in the limit as $q_2 \to 0$ under the condition

$$g(k) + g(n) - g(k+n) \geq 0 \tag{2.6}$$

for all k and n. The function $g(n)$ defined in (2.5) satisfies condition (2.6), and all the nonzero Poisson brackets of the limit system variables are

$$\{\tilde{s}_{a0}, \tilde{s}_{cd}\} = -2\mathrm{i}\sin\left(\frac{\pi ad}{N}\right) \tilde{s}_{a+c,d}.$$

Hence, we find that the limit algebra is solvable and is obtained by the contraction from $\mathfrak{sl}(N, \mathbb{C})$.

In what follows, we conveniently use the standard basis in which replacement (2.5) becomes

$$S_{ij} = \widetilde{S}_{ij} q_2^{-\mathfrak{g}(i,j)}, \qquad \mathfrak{g}(i,j) = \frac{1-\delta_{ij}}{2N}.$$

In the variables $\widetilde{S}_{ij}$, the limit algebra becomes

$$\{\widetilde{S}_{ii}, \widetilde{S}_{jk}\} = N\widetilde{S}_{jk}(\delta_{ik} - \delta_{ij}). \tag{2.7}$$

Because spectral parameter shift (2.4) is time dependent in our procedure, Eq. (2.3) becomes

$$\partial_{\tau_1} L^{\mathrm{rot}} - \partial_{\tilde{z}}\left(\frac{M^{\mathrm{rot}}}{2\pi\mathrm{i}} + \frac{L^{\mathrm{rot}}}{2}\right) = N[L^{\mathrm{rot}}, M^{\mathrm{rot}}],$$

where $L^{\mathrm{rot}} = L^{\mathrm{rot}}(\mathbf{s}, \tilde{z} + \tau/2, \tau)$ and $M^{\mathrm{rot}} = M^{\mathrm{rot}}(\mathbf{s}, \tilde{z} + \tau/2, \tau)$. In the limit under consideration, the isomonodromy deformation equation hence transforms into the equation

$$\partial_{\tau_1} \widetilde{L}^{\mathrm{rot}} - \partial_{\tilde{z}} \widetilde{M}^{\mathrm{rot}} = [\widetilde{L}^{\mathrm{rot}}, \widetilde{M}^{\mathrm{rot}}],$$

where

$$\widetilde{L}^{\mathrm{rot}} = \lim_{q_2 \to 0} 2\pi\mathrm{i} N L^{\mathrm{rot}}, \qquad \widetilde{M}^{\mathrm{rot}} = \lim_{q_2 \to 0} (N M^{\mathrm{rot}} + \pi\mathrm{i} N L^{\mathrm{rot}}).$$

We then represent the limit Lax matrices in the standard basis:

$$\begin{aligned}
\widetilde{L}^{\mathrm{rot}}_{ij} = &-4\pi^2 \sum_{m=1}^{N} \sum_{k=1}^{N-1} \widetilde{S}_{mm} \frac{\mathrm{e}(k(i-m)/N)}{\mathrm{e}(-k/N) - 1} \delta_{ij} - \\
&- 4\pi^2 N q_1^{1/2N} \widetilde{S}_{i+1,i+2}\, \mathrm{e}\left(-\frac{\tilde{z}}{N}\right) \delta_{i+1,j} + 4\pi^2 N q_1^{1/2N} \widetilde{S}_{i,i-1}\, \mathrm{e}\left(\frac{\tilde{z}}{N}\right) \delta_{i,j+1}, \\
\widetilde{M}^{\mathrm{rot}}_{ij} = &-\pi^2 \sum_{m=1}^{N} \sum_{k=1}^{N-1} \cot\left(\frac{\pi k}{N}\right) \sin^{-1}\left(\frac{\pi k}{N}\right) \mathrm{e}\left(\frac{i-m+1/2}{N}\right) \widetilde{S}_{mm} \delta_{ij} + \\
&+ 2\pi^2 N q_1^{1/2N} \left(\widetilde{S}_{i+1,i+2}\, \mathrm{e}\left(-\frac{\tilde{z}}{N}\right) \delta_{i+1,j} + \widetilde{S}_{i,i-1}\, \mathrm{e}\left(\frac{\tilde{z}}{N}\right) \delta_{i,j+1}\right),
\end{aligned}$$

where again $q_1 \equiv \mathrm{e}(\tau_1)$.

As was shown in [23] in the autonomous case, algebra (2.7) admits the bosonization

$$\begin{gathered}\widetilde{S}_{ii} = \frac{N}{2\pi\mathrm{i}}(v_{i-1} - v_i), \qquad \widetilde{S}_{i,i+1} = MN\,\mathrm{e}(u_i), \qquad \widetilde{S}_{i+1,i} = MN\,\mathrm{e}(-u_i), \\ \widetilde{S}_{i,i+k} = c_{i,i+k}\,\mathrm{e}\biggl(\sum_{n=i}^{i+k-1} u_n\biggr), \quad k = 2,\dots,N-2, \quad c_{i,i+k} = \mathrm{const},\end{gathered} \tag{2.8}$$

where $\mathbf{u}$ and $\mathbf{v}$ are the canonical coordinates of the nonautonomous periodic Toda chain in the center-of-mass frame,

$$\{v_i, u_j\} = \delta_{ij}, \quad i,j = 1,\dots,N, \qquad \sum_{i=1}^{N} u_i = 0, \qquad \sum_{i=1}^{N} v_i = 0. \tag{2.9}$$

Passing to the coordinates $\mathbf{u}, \mathbf{v}$, we obtain the Lax pair for the nonautonomous Toda chain,

$$\widetilde{L}^{\mathrm{rot}} = -4\pi^2 MN^2 \begin{pmatrix} \bar v_1 & \mathrm{e}_2^+ & 0 & \cdots & 0 & \mathrm{e}_N^- \\ \mathrm{e}_1^- & \bar v_2 & \mathrm{e}_3^+ & \ddots & \vdots & 0 \\ 0 & \mathrm{e}_2^- & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & \mathrm{e}_N^+ \\ \mathrm{e}_1^+ & 0 & \cdots & 0 & \mathrm{e}_{N-1}^- & \bar v_N \end{pmatrix}, \tag{2.10}$$

where we introduce the notation

$$\bar v_k = \frac{v_k}{2\pi\mathrm{i}M}, \qquad \mathrm{e}_k^{\pm} = \pm\,\mathrm{e}\biggl(\pm u_k \mp \frac{\tilde z}{N}\biggr) q^{1/2N}, \quad k = 1,\dots,N,$$

for brevity. The elements of the matrix M^{rot} are

$$\begin{gathered}\widetilde{M}_{ij}^{\mathrm{rot}} = \pi N \sum_{m=1}^{N}\sum_{k=1}^{N-1} \cot\biggl(\frac{\pi k}{N}\biggr)\,\mathrm{e}\biggl(\frac{k(i-m)}{N}\biggr) v_m \delta_{ij} + \\ + 2\pi^2 MN^2 q_1^{1/2N}\biggl(\mathrm{e}\biggl(u_{i+1} - \frac{\tilde z}{N}\biggr)\delta_{i+1,j} + \mathrm{e}\biggl(\frac{\tilde z}{N} - u_{i-1}\biggr)\delta_{i,j+1}\biggr).\end{gathered}$$

We write the Hamiltonian in the form

$$\widetilde{H}^{\mathrm{rot}} = N^2 \sum_{i=1}^{N} \frac{v_i^2}{2} + 4\pi^2 M^2 N^2 q_1^{1/N} \sum_{i=1}^{N} \mathrm{e}(u_{i+1} - u_i).$$

We note that the explicit time dependence is contained in $q_1^{1/N} \equiv \mathrm{e}(\tau_1/N)$.

The Hamiltonian equations of motion in $\mathbf{u}$ and $\mathbf{v}$ are (2.2).

3. Reducing the elliptic Schlesinger system

The equations of motion of the elliptic Schlesinger system admit Lax form (1.9), where the Lax matrices L and M^0 are sums of R copies of Lax matrices (1.10) and (1.11) of the elliptic $SL(N,\mathbb{C})$ top, the matrices M^k have form (1.12), and the elliptic curve parameter τ and the marked points x_k, $k=1,\ldots,R$, are the system times.

Taking the marked points x_k on the elliptic curve as the limit procedure parameters, we can obtain a separate reduction type for each copy of the elliptic $SL(N,\mathbb{C})$ top. In this paper, we construct a limit procedure allowing two types of reductions for these copies of the elliptic top. The first reduction type is analogous to the limit procedure described in Sec. 2. It produces a nonautonomous Toda chain from the nonautonomous elliptic $SL(N,\mathbb{C})$ top. In the second reduction type, the Lax matrix of the elliptic top transforms into the free system Lax matrix with preservation of the algebra $\mathfrak{sl}(N,\mathbb{C})$. To simplify expressions, we apply the first reduction type to the first r copies of the elliptic top and the second reduction type to the remaining $R-r$ copies. For this, we perform the infinite shift of the elliptic curve parameter τ using coordinate change (1.2) together with the following shifts of the marked points and scaling transformations of the coordinates:

$$x_i=\tilde{x}_i-\frac{\tau}{2},\qquad s^i_{mn}=\tilde{s}^i_{mn}q_2^{-g(n)},\quad i=1,\ldots,r. \tag{3.1}$$

3.1. The Toda chain and the $\mathfrak{sl}(N,\mathbb{C})$-spin degrees of freedom for $N>2$. Before describing the limit system equations of motion and their Lax representation, we define the new set of commuting Hamiltonians and the corresponding times. We here consider only equations of motion generated by quadratic Hamiltonians.

3.1.1. The quadratic Hamiltonians of the limit system. The coordinate change in (3.1) results in contracting the first r copies of $\mathfrak{sl}(N,\mathbb{C})$ in the limit as $\operatorname{Im}\tau_2\to+\infty$. The algebra of the remaining $R-r$ copies of the elliptic top is then preserved. All the nonzero Poisson brackets of the limit system are therefore

$$\begin{aligned}&\{\tilde{s}^i_{m0},\tilde{s}^i_{nk}\}=-2\mathrm{i}\sin\left(\frac{\pi mk}{N}\right)\tilde{s}^i_{m+n,k},\quad i=1,\ldots,r,\\&\{s^j_{mn},s^j_{kl}\}=2\mathrm{i}\sin\left(\frac{\pi(nk-ml)}{N}\right)s^j_{m+k,n+l},\quad j=r+1,\ldots,R,\end{aligned} \tag{3.2}$$

or, in the standard basis,

$$\begin{aligned}&\{\widetilde{S}^i_{mm},\widetilde{S}^i_{nk}\}=N\widetilde{S}^i_{nk}(\delta_{mk}-\tilde{\delta}_{nm}),\quad i=1,\ldots,r,\\&\{S^j_{mn},S^j_{kl}\}=N(S^j_{kn}\delta_{ml}-S^j_{ml}\delta_{kn}),\quad j=r+1,\ldots,R.\end{aligned} \tag{3.3}$$

Because the algebra of the first r copies of the elliptic top is not preserved in the limit under consideration, the initial Hamiltonians (1.5) and (1.6) of the Schlesinger elliptic system do not transform directly into the limit system Hamiltonians. To find the quadratic Hamiltonians of the limit system, we turn to elliptic function basis expansion (1.13) of the trace of the squared Lax matrix (1.10) of the Schlesinger system. In the limit procedure under consideration, matrix (1.10) transforms into the limit system Lax matrix:

$$\widetilde{L}(z)=\lim_{\operatorname{Im}\tau_2\to+\infty}L(z)=\sum_{i=1}^{r}\widetilde{L}^i(z-\tilde{x}_i)+\sum_{j=r+1}^{R}\widetilde{L}^j(z-x_j), \tag{3.4}$$

where

$$\widetilde{L}^i(z-\tilde{x}_i)=\sum_{m,n}\tilde{s}^i_{mn}\,\varphi^{\mathrm{T}}_{mn}(z-\tilde{x}_i)T_{mn},\quad i=1,\dots,r,$$

$$\widetilde{L}^j(z-x_j)=\sum_{m,n}s^j_{mn}\,\varphi^{\mathrm{S}}_{mn}(z-x_j)T_{mn},\quad j=r+1,\dots,R,$$

and we define the functions $\varphi^{\mathrm{T}}_{mn}(z)$ and $\varphi^{\mathrm{S}}_{mn}(z)$ in (3.6) below. We find the expansion of the squared Lax matrix of the limit system by taking the limit in expansion (1.13):

$$\frac{1}{2N}\operatorname{Tr}\widetilde{L}^2(z)=2\pi\mathrm{i}\biggl(\widetilde{H}_0-\frac{\widetilde{H}_r}{2}\biggr)+\pi\sum_{i=r+1}^{R}\bigl(\widetilde{H}_i\cot(\pi(z-x_i))+\pi H_{2,i}\sin^{-2}(\pi(z-x_i))\bigr),\tag{3.5}$$

where

$$\begin{aligned}\widetilde{H}_0=\lim_{\operatorname{Im}\tau_2\to+\infty}H_0={}&\frac{1}{2\pi\mathrm{i}}\sum_{i=1}^{r}\sum_{m,n}\biggl(\sum_{j=i+1}^{r}\tilde{s}^i_{mn}\,\tilde{s}^j_{-m,-n}\,f^{\mathrm{S}}_{mn}(\tilde{x}_j-\tilde{x}_i)+\\&+\sum_{j=r+1}^{R}\tilde{s}^i_{mn}\,s^j_{-m,-n}\,f^{\mathrm{T}}_{mn}(x_j-\tilde{x}_i)\biggr)+\frac{1}{2\pi\mathrm{i}}\sum_{i=1}^{r}H^{\mathrm{T},i}-\\&-\frac{\pi}{4\mathrm{i}}\sum_{i=r+1}^{R}\sum_{m=1}^{N-1}\sin^{-2}\biggl(\frac{\pi m}{N}\biggr)\biggl(s^i_{m0}\,s^i_{-m,0}+2\sum_{j=i+1}^{R}s^i_{m0}\,s^j_{-m,0}\biggr),\\ \widetilde{H}_r=\lim_{\operatorname{Im}\tau_2\to+\infty}\sum_{i=1}^{r}H_i={}&-\sum_{i=1}^{r}\sum_{j=r+1}^{R}\sum_{m,n}\tilde{s}^i_{mn}\,s^j_{-m,-n}\varphi^{\mathrm{T}}_{mn}(x_j-\tilde{x}_i),\\ \widetilde{H}_i=\lim_{\operatorname{Im}\tau_2\to+\infty}H_i={}&\sum_{j=1}^{r}\sum_{m,n}\tilde{s}^j_{mn}\,s^i_{-m,-n}\,\varphi^{\mathrm{T}}_{mn}(x_i-\tilde{x}_j)-\\&-\sum_{\substack{r\le j\le R,\\ j\ne i}}\sum_{m,n}s^i_{mn}\,s^j_{-m,-n}\,\varphi^{\mathrm{S}}_{mn}(x_j-x_i),\quad i=r+1,\dots,R.\end{aligned}$$

In the formula for $\widetilde{H}_0$, $H^{\mathrm{T},i}$ denotes the terms equivalent to the Hamiltonians of the nonautonomous Toda chain whose explicit form is given in (3.11) below. In the latter expressions, we also introduce the new functions $f^{\mathrm{T}}_{mn}(z)$, $f^{\mathrm{S}}_{mn}(z)$, $\varphi^{\mathrm{T}}_{mn}(z)$, and $\varphi^{\mathrm{S}}_{mn}(z)$, which are obtained from the expansions of $f\bigl[{m\atop n}\bigr](z)$ and $\varphi\bigl[{m\atop n}\bigr](z)$ in the parameter q_2 at the origin. Some of these functions manifest an explicit dependence on the time τ_1. For example, $f^{\mathrm{T}}_{mn}(z)=f^{\mathrm{T}}_{mn}(z,\tau_1)$, $f^{\mathrm{S}}_{mn}(z)=f^{\mathrm{S}}_{mn}(z,\tau_1)$, and $\varphi^{\mathrm{T}}_{mn}(z)=\varphi^{\mathrm{T}}_{mn}(z,\tau_1)$. Using formulas in Appendix B, we find that for $N>2$,

$$f^{\mathrm{T}}_{mn}(z)=\begin{cases}-\pi^2\sin^{-2}\biggl(\dfrac{\pi m}{N}\biggr), & n=0,\\ 4\pi^2q_1^{1/2N}\,\mathrm{e}\biggl(\dfrac{m-z}{N}\biggr), & n=1,\\ 0, & 1<n<N,\end{cases}$$

$$
f^{\mathrm{S}}_{mn}(z) = \begin{cases} -\pi^2 \sin^{-2}\left(\dfrac{\pi m}{N}\right), & n = 0, \\ 4\pi^2 q_1^{1/N}\,\mathrm{e}\left(\dfrac{m-z}{N}\right), & n = 1, \\ 0, & 1 < n < N-1, \\ 4\pi^2 q_1^{1/N}\,\mathrm{e}\left(\dfrac{z-m}{N}\right), & n = N-1, \end{cases}
$$

$$
\varphi^{\mathrm{T}}_{mn}(z) = \begin{cases} -\pi\,\mathrm{e}\left(\dfrac{m}{2N}\right)\sin^{-1}\left(\dfrac{\pi m}{N}\right), & n = 0, \\ 2\pi \mathrm{i} q_1^{1/2N}\,\mathrm{e}\left(\dfrac{m-z}{N}\right), & n = 1, \\ 0, & 1 < n < N-1, \\ -2\pi \mathrm{i} q_1^{1/2N}\,\mathrm{e}\left(\dfrac{z}{N}\right), & n = N-1, \end{cases} \tag{3.6}
$$

$$
\varphi^{\mathrm{S}}_{mn}(z) = \begin{cases} \pi\left(\cot(\pi z) - \cot\left(\dfrac{\pi m}{N}\right)\right), & n = 0, \\ \pi\,\mathrm{e}\left(\dfrac{z}{2} - \dfrac{zn}{N}\right)\sin^{-1}(\pi z), & 0 < n < N. \end{cases}
$$

The obtained coefficients in expansion (3.5) determine the set of quadratic Hamiltonians and Casimir functions of the limit system. The coefficients $H_{2,i}$, $i = r+1, \ldots, R$, are the limit system Casimir functions because the algebra of coordinates of the corresponding copies of the elliptic top is preserved. The functions $\widetilde{H} = \widetilde{H}_0 - \widetilde{H}_r/2$ and $\widetilde{H}_i$, $i = r+1, \ldots, R$, comprise the set of quadratic Hamiltonians. We now prove that these functions are indeed the limit system Hamiltonians.

Proposition 1. *The coefficients $\widetilde{H}$ and $\widetilde{H}_i$, $i = r+1, \ldots, R$, of expansion* (3.5) *are in involution,*

$$
\{\widetilde{H}, \widetilde{H}_i\} = 0, \quad i = r+1, \ldots, R,
$$

$$
\{\widetilde{H}_{i_1}, \widetilde{H}_{i_2}\} = 0, \quad i_1, i_2 = r+1, \ldots, R.
$$

Proof. We write Poisson brackets (3.2) and (3.3) in the r-matrix form,

$$
\{\widetilde{L}_1(z), \widetilde{L}_2(w)\} = [\tilde{r}(z-w), \widetilde{L}_1(z) + \widetilde{L}_2(w)],
$$

where $\widetilde{L}_1(z) = \widetilde{L}(z) \otimes \mathrm{Id}$, $\widetilde{L}_2(w) = \mathrm{Id} \otimes \widetilde{L}(w)$, and the matrix $\tilde{r}(z)$ is the limit of classical elliptic r-matrix (1.15) as $\operatorname{Im}\tau_2 \to +\infty$,

$$
\tilde{r}(z) = \lim_{\operatorname{Im}\tau_2 \to +\infty} r(z) = -\sum_{m,n} \varphi^{\mathrm{S}}_{m,n}(z) T_{m,n} \otimes T_{-m,-n}.
$$

We then obtain

$$
\{\operatorname{Tr}\widetilde{L}^2(z), \operatorname{Tr}\widetilde{L}^2(w)\} = \operatorname{Tr}\{\widetilde{L}^2(z), \widetilde{L}^2(w)\} = 0 \tag{3.7}
$$

for the Poisson bracket between $\operatorname{Tr}\widetilde{L}^2(z)$ and $\operatorname{Tr}\widetilde{L}^2(w)$. Because the Hamitonians $\widetilde{H}_i$, $i = r+1, \ldots, R$, are the residues of $\operatorname{Tr}\widetilde{L}^2(z)/2N$ at the points $z = x_i$, expression (3.7) implies that they are in involution: $\{\widetilde{H}_{i_1}, \widetilde{H}_{i_2}\} = 0$ for all $i_1, i_2 = r+1, \ldots, R$. Because the Hamiltonians $H_{2,i}$, $i = r+1, \ldots, R$, are Casimir functions of the limit system, it follows from expansion (3.5) that for the free term $\widetilde{H} = \widetilde{H}_0 - \widetilde{H}_r/2$, we have $\{\widetilde{H}, \operatorname{Tr}\widetilde{L}^2(z)\} = 0$ or $\{\widetilde{H}, \widetilde{H}_i\} = 0$ for all $i = r+1, \ldots, R$.

3.1.2. The equations of motion. We have obtained quadratic Hamiltonians and can now determine the time set of the limit system. As stated above, instead of τ in the limit system, we have τ_1 corresponding to the dynamics governed by the Hamiltonian $\widetilde{H}$. The remaining Hamiltonians $\widetilde{H}_i$ determine the dynamics with respect to the times x_i, $i = r+1, \ldots, R$. The limit system is therefore a nonautonomous Hamiltonian system with the equations of motion

$$\partial_{\tau_1} \tilde{\mathbf{s}}^i = \{\widetilde{H}, \tilde{\mathbf{s}}^i\}, \qquad \partial_{x_k} \tilde{\mathbf{s}}^i = \{\widetilde{H}_k, \tilde{\mathbf{s}}^i\}, \quad i = 1, \ldots, r, \quad k = r, \ldots, R, \tag{3.8}$$

$$\partial_{\tau_1} \mathbf{s}^i = \{\widetilde{H}, \mathbf{s}^i\}, \qquad \partial_{x_k} \mathbf{s}^i = \{\widetilde{H}_k, \mathbf{s}^i\}, \quad i = r, \ldots, R, \quad k = r, \ldots, R. \tag{3.9}$$

We must make an important comment here. By the definition of the function $\varphi^{\mathrm{T}}_{mn}(z)$ (see formulas (3.6)), the dependence of Lax matrix (3.4) on $\tilde{\mathbf{s}}^i$, $i = 1, \ldots, r$, reduces to the dependence on sums of the form $\sum_{k=1}^{r} \mathrm{e}(c_n \tilde{x}_k) \tilde{s}^k_{mn}$, where $c_n = 0, \pm 1/N$ depending on n. The same holds for the dependence of the limit system Hamiltonians, defined in expansion (3.5), on $\tilde{\mathbf{s}}^i$. This results in equations of motion (3.8) becoming equivalent at different $i = 1, \ldots, r$. We can say that the marked points x_i, $i = 1, \ldots, r$, merge at infinity after shifts (3.1). Hence, in the proposed limit procedure, it makes sense to consider only the case $r = 1$, in which one copy of the elliptic top transforms into the Toda chain. We can then write the equations of motion of the limit system as

$$\begin{aligned} &\partial_{\tau_1} \widetilde{\mathbf{S}}^1 = \{\widetilde{H}, \widetilde{\mathbf{S}}^1\}, \qquad \partial_{x_k} \widetilde{\mathbf{S}}^1 = \{\widetilde{H}_k, \widetilde{\mathbf{S}}^1\}, \quad k = 2, \ldots, R, \\ &\partial_{\tau_1} \mathbf{s}^i = \{\widetilde{H}, \mathbf{s}^i\}, \qquad \partial_{x_k} \mathbf{s}^i = \{\widetilde{H}_k, \mathbf{s}^i\}, \quad i, k = 2, \ldots, R. \end{aligned} \tag{3.10}$$

Before writing the equations of motion in an explicit form, we pass from the coordinates of the first copy of the elliptic top to the coordinates representing the generalized degrees of freedom of the Toda chain. As shown in Sec. 2, the limit algebra of the variables of $\widetilde{\mathbf{S}}^1$ admits such a transition, and the bosonization formulas are then analogous to (2.8):

$$\widetilde{S}^1_{ii} = \frac{N}{2\pi \mathrm{i}}(v_{i-1} - v_i), \qquad \widetilde{S}^1_{i,i+1} = MN\,\mathrm{e}(u_i), \qquad \widetilde{S}^1_{i+1,i} = MN\,\mathrm{e}(-u_i),$$

$$\widetilde{S}^1_{i,i+k} = c^i_{i,i+k}\,\mathrm{e}\biggl(\sum_{n=i}^{i+k-1} u_n\biggr), \quad k = 2, \ldots, N-2, \quad c_{j,j+k} = \mathrm{const},$$

where $\mathbf{u}$ and $\mathbf{v}$ are the canonical coordinates in the center-of-mass frame and satisfy relations (2.9).

After passing to the coordinates $\mathbf{u}, \mathbf{v}$, we must add the following Lax matrix of the nonautonomous Toda chain (analogous to matrix (2.10)) to the Lax matrix (3.4) of limit system:

$$\widetilde{L}^{\mathrm{rot}} = 2\pi \mathrm{i} MN \begin{pmatrix} \bar{v}_1 & \mathrm{e}^+_2 & 0 & \cdots & 0 & \mathrm{e}^-_N \\ \mathrm{e}^-_1 & \bar{v}_2 & \mathrm{e}^+_3 & \ddots & \vdots & 0 \\ 0 & \mathrm{e}^-_4 & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & \ddots & \mathrm{e}^+_{N-1} \\ \mathrm{e}^+_1 & 0 & \cdots & 0 & \mathrm{e}^-_{N-1} & \bar{v}_N \end{pmatrix},$$

where now

$$\bar{v}_k = \frac{v_k}{2\pi \mathrm{i} M}, \qquad \mathrm{e}^\pm_k = \pm\,\mathrm{e}\biggl(\pm u_k \mp \frac{z}{N}\biggr) q^{1/2N}, \quad k = 1, \ldots, N.$$

We must also add the Hamiltonian of the nonautonomous Toda chain

$$H^{\mathrm{T},1} = N\sum_{j=1}^{N}\frac{v_j^2}{2} + 4\pi^2 M^2 N q_1^{1/N}\sum_{j=1}^{N}\mathrm{e}(u_{j+1}-u_j) \tag{3.11}$$

to $\widetilde{H}$.

We can now write the explicit form of the Hamiltonian equations of motion for the Toda-chain generalized degrees of freedom $\mathbf{u}$ and $\mathbf{v}$ and for the $\mathfrak{sl}(N,\mathbb{C})$ degrees of freedom $\mathbf{s}^j$, $j=2,\dots,R$. We first describe the dynamics of these degrees of freedom with respect to the time τ_1. The equations of motion for $\mathbf{u}$ and $\mathbf{v}$ are determined by the nonautonomous Toda chain Hamiltonian $H^{\mathrm{T},1}$ and by the terms describing the interaction between the Toda chain degrees of freedom and the $\mathfrak{sl}(N,\mathbb{C})$ degrees of freedom from $\widetilde{H}_0$ and $\widetilde{H}_1$:

$$\begin{aligned}
\partial_{\tau_1} u_j &= \frac{N}{2\pi\mathrm{i}}v_j - \frac{1}{2\mathrm{i}}\sum_{k=2}^{R}\sum_{m=1}^{N-1} s^k_{m0}\,\mathrm{e}\Big(\frac{mj}{N}\Big)\cot\Big(\frac{\pi m}{N}\Big),\\
\partial_{\tau_1} v_j &= 4\pi^2 M^2 N q_1^{1/N}\big(\mathrm{e}(u_{j+1}-u_j)-\mathrm{e}(u_j-u_{j-1})\big) -\\
&\quad - 2\pi^2 M q_1^{1/2N}\sum_{k=2}^{R}\bigg(\mathrm{e}\Big(u_j+\frac{\tilde{x}_1-x_k}{N}\Big)S^k_{j,j-1} - \mathrm{e}\Big(-u_j+\frac{x_k-\tilde{x}_1}{N}\Big)S^k_{j,j+1}\bigg).
\end{aligned}$$

The equations of motion for $\mathbf{s}^j$, $j=2,\dots,R$, are determined by terms with interactions between different copies of the $\mathfrak{sl}(N,\mathbb{C})$ degrees of freedom from $\widetilde{H}_0$ and by terms with interactions between the Toda chain degrees of freedom and the $\mathfrak{sl}(N,\mathbb{C})$ degrees of freedom from $\widetilde{H}_0$ and $\widetilde{H}_1$:

$$\begin{aligned}
\partial_{\tau_1} s^i_{mn} &= 2\pi M q_1^{1/2N}\sum_{j=1}^{N}\sum_{l=0}^{N-1}\mathrm{e}\Big(\frac{l}{2N}-\frac{jl}{N}\Big)\bigg(\sin\Big(\frac{\pi(nl-m)}{N}\Big)\mathrm{e}\Big(u_j+\frac{\tilde{x}_1-x_i}{N}\Big)s^i_{m-l,n-1} +\\
&\quad + \sin\Big(\frac{\pi(nl+m)}{N}\Big)\mathrm{e}\Big(-u_{j-1}+\frac{x_i-\tilde{x}_1}{N}\Big)s^i_{m-l,n+1}\bigg) +\\
&\quad + \sum_{j=1}^{N}\sum_{l=1}^{N-1} v_j s^i_{m+l,n}\,\mathrm{e}\Big(\frac{jl}{N}\Big)\sin\Big(\frac{\pi nl}{N}\Big)\cot\Big(\frac{\pi l}{N}\Big) -\\
&\quad - \pi\sum_{k=2}^{R}\sum_{l=1}^{N-1} s^k_{l0}\,s^i_{m-l,n}\sin\Big(\frac{\pi nl}{N}\Big)\sin^{-2}\Big(\frac{\pi l}{N}\Big).
\end{aligned}$$

To describe the dynamics with respect to the times x_k, $k=2,\dots,R$, we write the corresponding Hamiltonians $\widetilde{H}_k$ explicitly:

$$\begin{aligned}
\widetilde{H}_k &= \sum_{j=1}^{N} v_j S^k_{jj} + 2\pi\mathrm{i} M q_1^{1/2N}\sum_{j=1}^{N}\bigg(\mathrm{e}\Big(u_j+\frac{\tilde{x}_1-x_k}{N}\Big)S^k_{j,j-1} -\\
&\quad - \mathrm{e}\Big(-u_j+\frac{x_k-\tilde{x}_1}{N}\Big)S^k_{j,j+1}\bigg) - \sum_{\substack{2\le i\le R,\\ i\ne k}}\sum_{m,n} s^k_{mn}\,s^i_{-m,-n}\,\varphi^{\mathrm{S}}_{mn}(x_i-x_k).
\end{aligned}$$

These Hamiltonians contain terms of two types. The terms of the first type describe interactions between the Toda-chain degrees of freedom and the $\mathfrak{sl}(N,\mathbb{C})$ degrees of freedom; these terms determine the equations of motion for $\mathbf{u}$ and $\mathbf{v}$,

$$\partial_{x_k} u_j = S^k_{jj},$$

$$\partial_{x_k} v_j = 4\pi^2 M q_1^{1/2N}\left(\mathrm{e}\left(u_j + \frac{\tilde{x}_1 - x_k}{N}\right)S^k_{j,j-1} + \mathrm{e}\left(-u_j + \frac{x_k - \tilde{x}_1}{N}\right)S^k_{j,j+1}\right).$$

On the other hand, the terms of the second type describe the interaction between different copies of the $\mathfrak{sl}(N,\mathbb{C})$ degrees of freedom; these terms determine the equations of motion for $\mathbf{s}^i$, $i = 1,\ldots,R$, $i \neq k$:

$$\partial_{x_k} s^i_{mn} = 2\mathrm{i}\sum_{l,j} s^k_{lj}\, s^i_{m-l,n-j} \sin\left(\frac{\pi(mj-nl)}{N}\right)\varphi^{\mathrm{S}}_{lj}(x_i - x_k).$$

The equations of motion for the variables $\mathbf{s}^k$ are determined by the terms of both types and are

$$\begin{aligned}\partial_{x_k} s^k_{mn} = 4\pi M q_1^{1/2N} \sum_{j=1}^{N}\sum_{l=0}^{N-1} \mathrm{e}\left(\frac{l}{2N} - \frac{jl}{N}\right)\Bigg(\sin\left(\frac{\pi(nl+m)}{N}\right)\mathrm{e}\left(-u_{j-1} + \frac{x_k - \tilde{x}_1}{N}\right)s^k_{m-l,n+1} - \\ - \sin\left(\frac{\pi(nl-m)}{N}\right)\mathrm{e}\left(u_j + \frac{\tilde{x}_1 - x_k}{N}\right)s^k_{m-l,n-1}\Bigg) - \\ - 2\mathrm{i}\sum_{\substack{2\le i\le R,\\ i\neq k}}\sum_{l,j} s^k_{m+l,n+j}\, s^i_{-l,-j}\sin\left(\frac{\pi(mj-nl)}{N}\right)\varphi^{\mathrm{S}}_{lj}(x_i - x_k) - \\ - 2\mathrm{i}\sum_{j=1}^{N}\sum_{l=1}^{N-1} v_j\, s^k_{m+l,n}\,\mathrm{e}\left(\frac{jl}{N}\right)\sin\left(\frac{\pi nl}{N}\right).\end{aligned}$$

3.1.3. The Lax representation for the limit equations of motion. We can represent equations of motion (3.10) in the Lax form:

$$\partial_{\tau_1}\widetilde{L} - \partial_z \widetilde{M}^0 = [\widetilde{L}, \widetilde{M}^0], \qquad (3.12)$$

$$\partial_{x_k}\widetilde{L} - \partial_z \widetilde{M}^k = [\widetilde{L}, \widetilde{M}^k], \quad k = 2,\ldots,R, \qquad (3.13)$$

where the Lax matrix $\widetilde{L}$ is defined in (3.4) and the matrices $\widetilde{M}^k$, $k = 2,\ldots,R$, are the limits of the matrices M^k as $\operatorname{Im}\tau_2 \to +\infty$,

$$\widetilde{M}^k(z) = \lim_{\operatorname{Im}\tau_2\to+\infty} M^k(z) = -\sum_{m,n} s^k_{mn}\,\varphi^{\mathrm{S}}_{mn}(z - x_k)T_{mn}, \quad k = 2,\ldots,R.$$

We find the Lax matrix $\widetilde{M}^0$ by considering the equations of motion of the limit system with respect to the time τ_1:

$$\partial_{\tau_1}\widetilde{\mathbf{S}}^1 = \{\widetilde{H}, \widetilde{\mathbf{S}}^1\}, \qquad \partial_{\tau_1}\mathbf{s}^j = \{\widetilde{H}, \mathbf{s}^j\}, \quad j = 2, \ldots, R.$$

The Hamiltonian $\widetilde{H}$ is related to the Hamiltonians of the elliptic Schlesinger system:

$$\widetilde{H} = \lim_{\operatorname{Im}\tau_2 \to +\infty}\left(H_0 - \frac{H_1}{2}\right).$$

We now construct the Lax representation for the Hamiltonian equations of motion of the elliptic Schlesinger system with the Hamiltonian $H_0 - H_1/2$. We can write these equations of motion as

$$\partial_{\tau_1}\mathbf{s}^i = \left\{H_0 - \frac{H_1}{2}, \mathbf{s}^i\right\} = \partial_\tau \mathbf{s}^i - \frac{1}{2}\partial_{x_1}\mathbf{s}^i, \quad i = 1, \ldots, R.$$

Rewriting Eqs. (1.9) in the form

$$\sum_{i=1}^{R}\sum_{m,n}\varphi\begin{bmatrix} m \\ n \end{bmatrix}(z - x_i)\, T_{mn}\, \partial_\tau s^i_{mn} = [L, M^0],$$

$$\sum_{i=1}^{R}\sum_{m,n}\varphi\begin{bmatrix} m \\ n \end{bmatrix}(z - x_i)\, T_{mn}\, \partial_{x_k} s^i_{mn} = [L, M^k], \quad k = 1, \ldots, R,$$

we obtain the expression for $\partial_{\tau_1}\mathbf{s}^i$:

$$\sum_{i=1}^{R}\sum_{m,n}\varphi\begin{bmatrix} m \\ n \end{bmatrix}(z - x_i)\, T_{mn}\, \partial_{\tau_1} s^i_{mn} = \left[L, M^0 - \frac{M^1}{2}\right].$$

Using the replacement $x_1 = \tilde{x}_1 - \tau/2$, we transform the obtained expression into the sought Lax equation

$$\partial_{\tau_1} L - \partial_z\left(M^0 - \frac{M^1}{2}\right) = \left[L, M^0 - \frac{M^1}{2}\right]. \tag{3.14}$$

In the limit procedure under consideration, Eq. (3.14) transforms into (3.12). For the Lax matrix $\widetilde{M}^0$ of the limit system, we then obtain

$$\widetilde{M}^0 = \lim_{\operatorname{Im}\tau_2 \to +\infty}\left(M^0 - \frac{M^1}{2}\right).$$

Using the functions $f^{\mathrm{T}}_{mn}(z)$, $f^{\mathrm{S}}_{mn}(z)$, and $\varphi^{\mathrm{T}}_{mn}(z)$, we reduce the matrix $\widetilde{M}^0$ to the form

$$\widetilde{M}^0(z) = \frac{1}{2\pi\mathrm{i}}\sum_{m,n}\tilde{s}^1_{mn}\big(f^{\mathrm{T}}_{mn}(z - \tilde{x}_1) + \pi\mathrm{i}\varphi^{\mathrm{T}}_{mn}(z - \tilde{x}_1)\big)T_{mn} +$$

$$+ \frac{1}{2\pi\mathrm{i}}\sum_{i=2}^{R}\sum_{m=1}^{N-1} s^i_{m0}\, f^{\mathrm{S}}_{m0}(z - x_i)\, T_{m0}.$$

4. Conclusion

We have described several ways to reduce the elliptic Schlesinger system using a limit procedure based on infinite shifts of the elliptic curve parameter τ and of r marked points x_i. We showed that for a fixed $p = R - r$, the case $r > 1$ is equivalent to the case $r = 1$ because the shifted marked points merge at infinity. As a result, we obtained a nonautonomous Hamiltonian system describing the Toda chain with additional spin $\mathfrak{sl}(N, \mathbb{C})$ degrees of freedom and represented the equations of motion of the obtained system in the form of zero-curvature equations (3.12) and (3.13).

The described limit procedure can be adapted to the case of the elliptic Gaudin system, which is an integrable version of the elliptic Schlesinger system. In the limit, we then obtain an autonomous version of the system constructed in Sec. 3. It can be expected that this autonomous system will be integrable because its Lax operator is analogous to (3.4) except that it is explicitly time dependent.

Appendix A: The sine-algebra

The basis elements T_{mn} of the sine-algebra in $\mathfrak{sl}(N,\mathbb{C})$ are defined as

$$(T_{mn})_{ij} = \mathrm{e}\biggl(\frac{mn}{2N}\biggr)\,\mathrm{e}\biggl(\frac{im}{N}\biggr)\tilde{\delta}(j-i-n), \quad m,n=0,\dots,N-1, \quad m^2+n^2\neq 0.$$

For elements with the indices $m,n\in\mathbb{Z}$ such that $m\not\equiv 0 \bmod N$ or $n\not\equiv 0 \bmod N$, we can introduce the quasiperiodicity condition

$$T_{mn} = \mathrm{e}\biggl(\frac{mn-(m \bmod N)(n \bmod N)}{2N}\biggr)T_{m \bmod N,\, n \bmod N},$$

$$s_{mn} = \mathrm{e}\biggl(\frac{(m \bmod N)(n \bmod N)-mn}{2N}\biggr)s_{m \bmod N,\, n \bmod N},$$

where

$$\mathrm{e}\biggl(\frac{mn-(m \bmod N)(n \bmod N)}{2N}\biggr) = \pm 1.$$

The commutation relations in the sine-algebra basis then become

$$[T_{mn},T_{kl}] = 2\mathrm{i}\sin\biggl(\frac{\pi(kn-ml)}{N}\biggr)T_{m+k,n+l}. \tag{A.1}$$

The coordinates $\{s_{mn}\}$ in the sine-algebra basis are related to the coordinates $\{S_{ij}\}$ in the standard basis expansion of the $SL(N,\mathbb{C})$ algebra as

$$S_{ij} = \sum_{m,n} s_{mn}(T_{mn})_{ij}, \qquad s_{mn} = \frac{1}{N}\sum_{i,j} S_{ij}(T_{-m,n})_{ij}. \tag{A.2}$$

Appendix B: Elliptic functions

We borrow the definitions and properties of the elliptic functions from [30] and [35]. The principal object is the theta function with characteristics,

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(z,\tau) = \sum_{j\in\mathbb{Z}} q^{(j+a)^2/2}\,\mathrm{e}((j+a)(z+b)),$$

where $q=\mathrm{e}(\tau)\equiv e^{2\pi\mathrm{i}\tau}$. We also need the Eisenstein functions

$$\begin{aligned}\varepsilon_k(z) &= \lim_{M\to+\infty}\sum_{n=-M}^{M}(z+n)^{-k},\\ E_k(z) &= \lim_{M\to+\infty}\sum_{n=-M}^{M}\varepsilon_k(z+n\tau), \quad k\in\mathbb{N}.\end{aligned} \tag{B.1}$$

For taking limits of the Lax matrices, we use the series expansion

$$\vartheta(z)=\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z,\tau)=\sum_{j\in\mathbb{Z}}q^{(j+1/2)^2/2}\,\mathrm{e}((j+1/2)(z+1/2)). \tag{B.2}$$

We set

$$\phi(u,z)=\frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}. \tag{B.3}$$

This function satisfies the equalities

$$\begin{aligned}&\phi(u,z)\phi(-u,z)=E_2(z)-E_2(u),\\&\partial_u\phi(u,z)=\phi(u,z)\big(E_1(u+z)-E_1(u)\big).\end{aligned} \tag{B.4}$$

We also have the parity properties

$$E_k(-z)=(-1)^kE_k(z),\qquad \vartheta(-z)=-\vartheta(z),\qquad \phi(u,z)=\phi(z,u)=-\phi(-u,-z)$$

and the quasiperiodicity properties

$$\begin{aligned}&E_1(z+1)=E_1(z), && E_1(z+\tau)=E_1(z)-2\pi\mathrm{i},\\&E_2(z+1)=E_2(z), && E_2(z+\tau)=E_2(z),\\&\vartheta(z+1)=-\vartheta(z), && \vartheta(z+\tau)=-q^{-1/2}\,\mathrm{e}(-z)\vartheta(z),\\&\phi(u+1,z)=\phi(u,z), && \phi(u+\tau,z)=\mathrm{e}(-z)\phi(u,z).\end{aligned} \tag{B.5}$$

We set $z=\tilde z+\tau/2$ and investigate the reductions of the elliptic functions $\varphi\left[\begin{smallmatrix}m\\n\end{smallmatrix}\right]$ in (1.7) and $f\left[\begin{smallmatrix}m\\n\end{smallmatrix}\right]$ in (1.8) in the limit as $\operatorname{Im}\tau\to+\infty$. By (B.3), the series expansion of $\varphi\left[\begin{smallmatrix}m\\n\end{smallmatrix}\right](z)$ reduces to the series expansion of the theta function. Taking the leading nonzero term of the expansion, we obtain

$$\vartheta\left(-\frac{m}{N}-\frac{n\tau}{N}\right)=\begin{cases}2q^{1/8}\sin\left(\dfrac{\pi m}{N}\right)+o(q^{1/8}), & n=0,\\ \mathrm{i}q^{1/8-n/2N}\,\mathrm{e}\left(-\dfrac{m}{2N}\right)+o(q^{1/8-n/2N}), & 0<n<N,\end{cases}$$

$$\vartheta\left(\tilde z+\frac{\tau}{2}-\frac{m}{N}-\frac{n}{N}\tau\right)=\begin{cases}-\mathrm{i}q^{n/2N-1/8}\,\mathrm{e}\left(\dfrac{m/N-\tilde z}{2}\right)+o(q^{n/2N-1/8}), & 0\le n<\dfrac{N}{2},\\ -2q^{1/8}\sin\left(\pi\left(\tilde z-\dfrac{m}{N}\right)\right)+o(q^{1/8}), & n=\dfrac{N}{2},\\ \mathrm{i}q^{3/8-n/2N}\,\mathrm{e}\left(\dfrac{\tilde z-m/N}{2}\right)+o(q^{3/8-n/2N}), & \dfrac{N}{2}<n<N,\end{cases}$$

which gives

$$\phi\left(-\frac{m+n\tau}{N};\tilde z+\frac{\tau}{2}\right)=\begin{cases}-\pi\,\mathrm{e}\left(\dfrac{m}{2N}\right)\sin^{-1}\left(\dfrac{\pi m}{N}\right)+o(1), & n=0,\\ 2\pi\mathrm{i}q^{n/N}\,\mathrm{e}\left(\dfrac{m}{N}\right)+o(q^{n/N}), & 0<n<\dfrac{N}{2},\\ 4\pi q^{1/2}\sin\left(\pi\left(\tilde z-\dfrac{m}{N}\right)\right)\mathrm{e}\left(\dfrac{m/N+\tilde z}{2}\right)+o(q^{1/2}), & n=\dfrac{N}{2},\\ -2\pi\mathrm{i}q^{1/2}\,\mathrm{e}(\tilde z)+o(q^{1/2}), & \dfrac{N}{2}<n<N,\end{cases}$$

and

$$\varphi\begin{bmatrix} m \\ n \end{bmatrix}\left(\tilde{z}+\frac{\tau}{2}\right)=\begin{cases} -\pi\,\mathrm{e}\left(\dfrac{m}{2N}\right)\sin^{-1}\left(\dfrac{\pi m}{N}\right)+o(1), & n=0,\\ 2\pi\mathrm{i}q^{n/2N}\,\mathrm{e}\left(\dfrac{m-n\tilde{z}}{N}\right)+o(q^{n/2N}), & 0<n<\dfrac{N}{2},\\ 4\pi q^{1/4}\,\mathrm{e}\left(\dfrac{m}{2N}\right)\sin\left(\pi\left(\tilde{z}-\dfrac{m}{N}\right)\right)+o(q^{1/4}), & n=\dfrac{N}{2},\\ -2\pi\mathrm{i}q^{(N-n)/2N}\,\mathrm{e}\left(\dfrac{N-n}{N}\tilde{z}\right)+o(q^{(N-n)/2N}), & \dfrac{N}{2}<n<N. \end{cases}$$

To find the limit of $f\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right]$, we expand $E_1(\tilde{x}-\sigma\tau)$ in a series in $q=\mathrm{e}(\tau)$. Taking (B.1) into account, we obtain

$$E_1(\tilde{x}-\sigma\tau)=\lim_{M\to+\infty}\sum_{n=-M}^{M}\varepsilon_1(\tilde{x}+(n-\sigma)\tau)=$$

$$=\varepsilon_1(\tilde{x}-\sigma\tau)+\lim_{M\to+\infty}\sum_{n=1}^{M}\big(\varepsilon_1(\tilde{x}+(n-\sigma)\tau)+\varepsilon_1(\tilde{x}-(n+\sigma)\tau)\big).$$

Using the explicit expression for $\varepsilon_1(x)$ [30],

$$\varepsilon_1(x)=\pi\cot(\pi x)=\pi\mathrm{i}\frac{\mathrm{e}(x)+1}{\mathrm{e}(x)-1}=\pi\mathrm{i}\begin{cases} -1-2\,\mathrm{e}(x)+o(\mathrm{e}(x)), & \operatorname{Im}x\to+\infty,\\ 1+2\,\mathrm{e}(x)+o(\mathrm{e}(x)), & \operatorname{Im}x\to-\infty, \end{cases}$$

we obtain the result for the leading term in the expansion of $E_1(\tilde{x}-\sigma\tau)$:

$$E_1(\tilde{x}-\sigma\tau)=\begin{cases} \pi\cot(\pi\tilde{x})+o(1), & \sigma=0,\\ \pi\mathrm{i}+2\pi\mathrm{i}q^{\sigma}\,\mathrm{e}(-\tilde{x})+o(q^{\sigma}), & 0<\sigma<\dfrac{1}{2},\\ \pi\mathrm{i}+2\pi\mathrm{i}q^{1/2}\big(\mathrm{e}(-\tilde{x})-\mathrm{e}(\tilde{x})\big)+o(q^{1/2}), & \sigma=\dfrac{1}{2},\\ \pi\mathrm{i}-2\pi\mathrm{i}q^{1-\sigma}\,\mathrm{e}(\tilde{x})+o(q^{1-\sigma}), & \dfrac{1}{2}<\sigma<1. \end{cases}$$

Using (B.5), we generalize this formula to the case $\sigma\in\mathbb{R}$:

$$E_1(\tilde{x}-\sigma\tau)=\begin{cases} 2\pi\mathrm{i}\lfloor\sigma\rfloor+\pi\cot(\pi\tilde{x})+o(1), & \{\sigma\}=0,\\ 2\pi\mathrm{i}\lfloor\sigma\rfloor+\pi\mathrm{i}+2\pi\mathrm{i}q^{\{\sigma\}}\,\mathrm{e}(-\tilde{x})+o(q^{\{\sigma\}}), & 0<\{\sigma\}<\dfrac{1}{2},\\ 2\pi\mathrm{i}\lfloor\sigma\rfloor+\pi\mathrm{i}+2\pi\mathrm{i}q^{\{\sigma\}}\,\mathrm{e}(-\tilde{x})+o(q^{\{\sigma\}}), & \{\sigma\}=\dfrac{1}{2},\\ 2\pi\mathrm{i}\lfloor\sigma\rfloor+\pi\mathrm{i}-2\pi\mathrm{i}q^{1-\{\sigma\}}\big(\mathrm{e}(-\tilde{x})-\mathrm{e}(\tilde{x})\big)+o(q^{1/2}), & \dfrac{1}{2}<\{\sigma\}<1, \end{cases}$$

where we let $\{\sigma\}$ denote the fractional part of σ.

We now consider the expansion of the function $\partial_u\phi(u,z)\big|_{u=\tilde u-\sigma\tau}$ in the limit as $\operatorname{Im}\tau\to+\infty$ using formula (B.4). Setting $z=\tilde z+\tau/2$ and taking all possible values of σ into account, we obtain

$$\partial_u\phi(u,z)\big|_{u=\tilde u-\sigma\tau}=\begin{cases}-\pi^2\sin^{-2}\pi\tilde u+o(1), & \sigma=0,\\ 4\pi^2 q\,\mathrm{e}(-\tilde u)+o(q), & 0<\sigma<\dfrac34,\\ 4\pi^2q^{3/4}\big(\mathrm{e}(-\tilde u)-\mathrm{e}(\tilde u+\tilde z)\big)+o(q^{3/4}), & \sigma=\dfrac34,\\ -4\pi^2q^{3/2-\sigma}\,\mathrm{e}(\tilde u+\tilde z)+o(q^{3/2-\sigma}), & \dfrac34<\sigma<1.\end{cases}$$

Taking definition (1.8) into account, we finally obtain the expression

$$f\begin{bmatrix}m\\n\end{bmatrix}\left(\tilde z+\frac{\tau}{2}\right)=\begin{cases}-\pi^2\sin^{-2}\left(\dfrac{\pi m}{N}\right)+o(1), & n=0,\\ 4\pi^2\,\mathrm{e}\left(\dfrac{m}{N}\right)\mathrm{e}\left(-\dfrac{n\tilde z}{N}\right)q^{n/2N}+o(q^{n/2N}), & 0<n<\dfrac{3N}{4},\\ 4\pi^2\left(\mathrm{e}\left(\dfrac{m}{N}\right)-\mathrm{e}\left(-\dfrac{n}{N}+\tilde z\right)\right)\mathrm{e}\left(-\dfrac{3\tilde z}{4}\right)q^{3/8}+o(q^{3/8}), & n=\dfrac{3N}{4},\\ -4\pi^2\,\mathrm{e}\left(-\dfrac{m}{N}+\tilde z\right)\mathrm{e}\left(-\dfrac{n\tilde z}{N}\right)q^{3(1-n/N)/2}+o(q^{3(1-n/N)/2}), & \dfrac{3N}{4}<n<N.\end{cases}$$

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