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EXTENDED RESOLVENT OF THE HEAT OPERATOR WITH A MULTISOLITON POTENTIAL

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We consider the heat operator with a general multisoliton potential and derive its extended resolvent depending on a parameter $q \in \mathbb{R}^2$. We show that it is bounded in all variables and find its singularities in q. We introduce the Green's functions and study their properties in detail.

Keywords: Kadomtsev–Petviashvili equation, heat operator, extended resolvent, soliton

1. Introduction

We consider the Kadomtsev–Petviashvili II (KP-II) equation [1]

$$(u_t - 6uu_{x_1} + u_{x_1x_1x_1})_{x_1} = -3u_{x_2x_2},$$

where $u = u(x,t)$, $x = (x_1, x_2)$, and the subscripts x_1, x_2, and t denote partial derivatives. From the early 1970s, it has been known that this equation is integrable [2], [3] and can be considered a prototypical (2+1)-dimensional equation, being a generalization of the celebrated Korteweg–de Vries (KdV) equation. The KP-II equation is integrable by virtue of its association with the operator

$$\mathcal{L}(x, \partial_x) = -\partial_{x_2} + \partial_{x_1}^2 - u(x), \tag{1.1}$$

which yields the well-known heat conduction equation or heat equation, for short. The spectral theory of operator (1.1) was developed in [4]–[7] for a real potential $u(x)$ rapidly decaying at spatial infinity. But this is not the most interesting case, because the KP-II equation was in fact proposed in [1] to describe a weak two-dimensional transverse perturbation of the one-soliton solution of the KdV equation. The main difficulty in studying this problem arises because the soliton solutions of the KP equations do not decay at spatial infinity and have a ray behavior on the x plane (see, e.g., [8]–[11]). Correspondingly, the integral equations defining the Jost solutions in this case are just meaningless because their kernels do not exist.

A theory of the KP-II equation that also includes solitons still awaits its construction, similarly to what was successfully done for the KP-I equation in [12]. Following the approach in that paper, we generalize the standard inverse scattering transform (IST) method and consider the so-called scattering on a nontrivial background, i.e., we consider a potential

$$\widetilde{u}(x) = u(x) + u'(x), \tag{1.2}$$

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where $u(x)$ is some multisoliton potential and $u'(x)$ is a smooth function decaying sufficiently rapidly in both its variables, which we can consider a perturbation of the soliton potential. The Jost solution corresponding to the operator $\widetilde{\mathcal{L}}$ with the potential $\widetilde{u}$ can be defined as the solution of the integral equation

$$\widetilde{\Phi}(x,\mathbf{k}) = \Phi(x,\mathbf{k}) + \int dy\, \mathcal{G}(x,y,\mathbf{k})u'(y)\widetilde{\Phi}(y,\mathbf{k}), \tag{1.3}$$

where $\mathbf{k} \in \mathbb{C}$ is a spectral parameter, $\Phi(x,\mathbf{k})$ is the Jost solution corresponding to the operator $\mathcal{L}$ with the multisoliton potential u, and $\mathcal{G}(x,y,\mathbf{k})$ is its total Green's function,

$$\bigl(-\partial_{x_2} + \partial_{x_1}^2 - u(x)\bigr)\mathcal{G}(x,x',\mathbf{k}) = \delta(x-x').$$

Let the perturbation $u'(x)$ decrease at infinity faster than $1/(x_1^2+x_2^2)$. Then if the Green's function satisfies the *boundedness condition*, i.e., if the function

$$G(x,x',\mathbf{k}) = e^{i\mathbf{k}(x_1-x_1')+\mathbf{k}^2(x_2-x_2')}\mathcal{G}(x,x',\mathbf{k}) \tag{1.4}$$

is bounded in $x, x' \in \mathbb{R}^2$ and $\mathbf{k} \in \mathbb{C}$ and has finite limits at infinity, then the kernel of (1.3) is well defined, and we can in fact use the standard technique [4]–[7] to prove that the solution $\widetilde{\Phi}(x,\mathbf{k})$ exists. We were thus able to develop the IST for a solution describing a perturbation of the one-soliton solution [13]. But the case of an arbitrary number of solitons remains open for investigation. For this case in [14], we derived a total Green's function, which is a natural generalization of the Green's function for a decaying potential. As shown in [12] and [13], to describe the singularities of the Jost solutions, we need some additional Green's functions. Therefore, to be able to work with a heat operator with a generic multisoliton potential, following [12], we introduce its extended resolvent (this was done for some multisoliton solutions in [15]), which is a more general object than the Green's function. Namely, we introduce a two-dimensional real parameter $q = (q_1, q_2)$ and consider the *extended* Lax operator

$$\mathcal{L}(x, \partial_x + q) = -\partial_{x_2} - q_2 + (\partial_{x_1} + q_1)^2 - u(x). \tag{1.5}$$

Then the extended resolvent of heat operator (1.1) is defined as the tempered distribution $M(x,x';q)$ with respect to all its six variables x, x', q satisfying the differential equation

$$\mathcal{L}(x,\partial_x+q)M(x,x';q) = \mathcal{L}^{\mathrm{d}}(x',\partial_{x'}+q)M(x,x';q) = \delta(x-x'), \tag{1.6}$$

where $\mathcal{L}^{\mathrm{d}}(x,\partial_x)$ is the operator dual to $\mathcal{L}(x,\partial_x)$.

The extended resolvent can be considered the generating functional of the different Green's functions of the operator $\mathcal{L}$ given by (1.1). Indeed, we introduce the "hatted" operator

$$\widehat{M}(x,x';q) = e^{q(x-x')}M(x,x';q), \qquad qx = q_1x_1 + q_2x_2, \tag{1.7}$$

which, of course, is not necessarily a tempered distribution. Nevertheless, it is easy to see that

$$\mathcal{L}(x,\partial_x)\widehat{M}(x,x';q) = \mathcal{L}^{\mathrm{d}}(x',\partial_{x'})\widehat{M}(x,x';q) = \delta(x-x'). \tag{1.8}$$

In particular, the total Green's function is obtained via the reduction

$$\mathcal{G}(x,x',\mathbf{k}) = \widehat{M}(x,x';q)\Big|_{\substack{q_1=\operatorname{Im}\mathbf{k},\\ q_2=\operatorname{Im}^2\mathbf{k}-\operatorname{Re}^2\mathbf{k}}}. \tag{1.9}$$

Constructing the distribution $M(x,x';q)$ is the subject of this paper. It was shown in [16] that heat operator (1.1) with a multisoliton potential $u(x)$ can have left and right annihilators in some polygonal regions of the q plane, and the resolvent consequently cannot exist in those regions. We provide an explicit expression for the extended resolvent and prove that it exists as a tempered distribution and satisfies (1.6) outside some special polygonal regions (the same as shown in [16]). We show that in this region, reduction (1.9) is always possible for an arbitrary $\mathbf{k}\in\mathbb{C}$ and, as a result, we obtain the total Green's function derived in [14]. Under this condition, the fact that $M(x,x';q)$ is a tempered distribution means that the boundedness condition is satisfied for function (1.4). In accordance with the procedure followed in the case of the nonstationary Schrödinger operator in [12], describing the singularities of the total Green's function on the complex plane of the spectral parameter requires some special reductions of the extended resolvent, i.e., auxiliary Green's functions. We conclude this paper with a detailed study of these singularities.

2. Heat operator with a multisoliton potential and its Jost solutions

Soliton potentials [8]–[11], [15]–[17] are characterized by two natural numbers (topological charges) $N_a\geq 1$ and $N_b\geq 1$. Let $\mathcal{N}=N_a+N_b$; hence, $\mathcal{N}\geq 2$. We introduce $\mathcal{N}$ real parameters

$$\kappa_1<\kappa_2<\cdots<\kappa_{\mathcal{N}}$$

and functions

$$K_n(x)=\kappa_n x_1+\kappa_n^2 x_2,\quad n=1,\dots,\mathcal{N}. \tag{2.1}$$

Let $e^{K(x)}=\operatorname{diag}\{e^{K_n(x)}\}_{n=1}^{\mathcal{N}}$ be a diagonal $\mathcal{N}\times\mathcal{N}$ matrix, $\mathcal{D}$ be an $\mathcal{N}\times N_b$ constant matrix, and $\mathcal{V}$ be an "incomplete Vandermonde matrix," i.e., the $N_b\times\mathcal{N}$ matrix

$$\mathcal{V}=\begin{pmatrix}1 & 1 & \cdots & 1\\ \kappa_1 & \kappa_2 & \cdots & \kappa_{\mathcal{N}}\\ \vdots & \vdots & \ddots & \vdots\\ \kappa_1^{N_b-1} & \kappa_2^{N_b-1} & \cdots & \kappa_{\mathcal{N}}^{N_b-1}\end{pmatrix}.$$

Then the soliton potential is given by

$$u(x)=-2\partial_{x_1}^2\log\tau(x),$$

where the τ-function can be expressed as

$$\tau(x)=\det\bigl(\mathcal{V}e^{K(x)}\mathcal{D}\bigr). \tag{2.2}$$

For the Jost and dual Jost solutions (the respective solutions of heat operator (1.1) and its dual), we have

$$\begin{aligned}\Phi(x,\mathbf{k})&=e^{-i\mathbf{k}x_1-\mathbf{k}^2x_2}\chi(x,\mathbf{k}), &\qquad \chi(x,\mathbf{k})&=\frac{\tau_\Phi(x,\mathbf{k})}{\tau(x)},\\ \Psi(x,\mathbf{k})&=e^{i\mathbf{k}x_1+\mathbf{k}^2x_2}\xi(x,\mathbf{k}), &\qquad \xi(x,\mathbf{k})&=\frac{\tau_\Psi(x,\mathbf{k})}{\tau(x)},\end{aligned} \tag{2.3}$$

and the equality (the Miwa shift)

$$\tau_\Phi(x,\mathbf{k})=\det\bigl(\mathcal{V}e^{K(x)}(\kappa+i\mathbf{k})\mathcal{D}\bigr),\qquad \tau_\Psi(x,\mathbf{k})=\det\biggl(\mathcal{V}\frac{e^{K(x)}}{\kappa+i\mathbf{k}}\mathcal{D}\biggr),$$

where $\kappa + i\mathbf{k} = \mathrm{diag}\{\kappa_n + i\mathbf{k}\}_{n=1}^{\mathcal{N}}$.

To study the properties of the potential and the Jost solutions, it is convenient to use the representation for the τ-functions that follows from the Binet–Cauchy formula for the determinant of a product of matrices:

$$\tau(x) = \frac{1}{N_b!} \sum_{\{n_i\}=1}^{\mathcal{N}} \mathcal{D}(\{n_i\}) V(\{n_i\}) \prod_{l=1}^{N_b} e^{K_{n_l}(x)}, \tag{2.4}$$

$$\chi(x,\mathbf{k}) = \frac{1}{N_b!\,\tau(x)} \sum_{\{m_i\}=1}^{\mathcal{N}} \mathcal{D}(\{m_i\})\, V(\{m_i\}) \prod_{l=1}^{N_b} (\kappa_{m_l} + i\mathbf{k}) e^{K_{m_l}(x)}, \tag{2.5}$$

$$\xi(x,\mathbf{k}) = \frac{1}{N_b!\,\tau(x)} \sum_{\{n_i\}=1}^{\mathcal{N}} \mathcal{D}(\{n_i\})\, V(\{n_i\}) \prod_{l=1}^{N_b} \frac{e^{K_{n_l}(x)}}{\kappa_{n_l} + i\mathbf{k}}, \tag{2.6}$$

where we use the notation

$$\begin{gathered} V(\{n_i\}) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \kappa_{n_1} & \kappa_{n_2} & \cdots & \kappa_{n_{N_b}} \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_{n_1}^{N_b-1} & \kappa_{n_2}^{N_b-1} & \cdots & \kappa_{n_{N_b}}^{N_b-1} \end{pmatrix} \equiv \prod_{1\le i<j\le N_b} (\kappa_{n_j} - \kappa_{n_i}), \\ \mathcal{D}(\{n_i\}) = \det \begin{pmatrix} \mathcal{D}_{n_1,1} & \cdots & \mathcal{D}_{n_1,N_b} \\ \vdots & \ddots & \vdots \\ \mathcal{D}_{n_{N_b},1} & \cdots & \mathcal{D}_{n_{N_b},N_b} \end{pmatrix} \end{gathered} \tag{2.7}$$

for the maximal minors of the matrices $\mathcal{V}$ and $\mathcal{D}$ and $\{m_i\} = \{m_1, \dots, m_{N_b}\}$ and $\{n_i\} = \{n_1, \dots, n_{N_b}\}$ denote unordered sets of N_b indices from the interval $\{1, \dots, \mathcal{N}\}$. We recall that the maximal minors of a matrix satisfy the Plücker relation: for any subsets $\{m_1, \dots, m_{N_b}\}$ and $\{n_1, \dots, n_{N_b}\}$ of indices ranging from 1 to $\mathcal{N}$ and arbitrary $j \in \{1, \dots, N_b\}$,

$$\begin{aligned} \mathcal{D}(\{m_i\})\mathcal{D}(\{n_i\}) = \sum_{s=1}^{N_b} &\mathcal{D}(m_1, \dots, m_{s-1}, n_j, m_{s+1}, \dots, m_{N_b}) \times \\ &\times \mathcal{D}(n_1, \dots, n_{j-1}, m_s, n_{j+1}, \dots, n_{N_b}). \end{aligned} \tag{2.8}$$

We note that the only x-dependent terms in (2.4)–(2.6) are exponentials of sums of linear functions (2.1). Consequently, the asymptotic behavior of the function $\tau(x)$ and of the potential has a sectorial structure on the x plane. To specify this structure as $x \to \infty$, we introduce the directions (rays) r_n such that along r_n, the function $x_1 + (\kappa_n + \kappa_{n+N_b})x_2$ is bounded and $(\kappa_{n+N_b} - \kappa_n)x_2 \to -\infty$, $n = 1, \dots, \mathcal{N}$. Here, we assume that the indices are defined modulo $\mathcal{N}$, and because $\mathcal{N} = N_a + N_b$, we hence say that $n + N_b = n - N_a$ for $n > N_a$. As a result, we have N_a rays in the direction $x_2 \to -\infty$ and N_b rays in the direction $x_2 \to +\infty$. The sector σ_n is swept by the ray r_n rotating counterclockwise up to the ray r_{n+1}. These sectors are nonintersecting and cover the entire x plane with the exception of the rays. In [9], we proved that the leading exponentials in $\tau(x)$ as $x \to \infty$ are the exponentials $\exp\bigl(\sum_{l=n}^{n+N_b-1} K_l(x)\bigr)$, each leading in the corresponding sector σ_n of the x plane. More precisely, if the coefficients $z_n = V(n, \dots, n+N_b-1)\mathcal{D}(n, \dots, n+N_b-1)$

are nonzero for all $n = 1, \dots, \mathcal{N}$ (again under the condition that the indices are defined modulo $\mathcal{N}$), then the function $\tau(x)$ has the following asymptotic behavior along rays and inside sectors:

$$
\begin{aligned}
&x \xrightarrow{r_n} \infty: \quad \tau(x) = \bigl(z_n + z_{n+1} e^{K_{N_b+n}(x) - K_n(x)} + o(1)\bigr) \exp\biggl(\sum_{l=n}^{n+N_b-1} K_l(x)\biggr), \\
&x \xrightarrow{\sigma_n} \infty: \quad \tau(x) = \bigl(z_n + o(1)\bigr) \exp\biggl(\sum_{l=n}^{n+N_b-1} K_l(x)\biggr).
\end{aligned}
\tag{2.9}
$$

The regularity of the potential $u(x)$ on the x plane is equivalent to the absence of zeros of $\tau(x)$. It is clear that it suffices to impose the condition that the matrix $\mathcal{D}$ is totally nonnegative (TNN), i.e., that $\mathcal{D}(n_1, \dots, n_{N_b}) \geq 0$ for all $1 \leq n_1 < \dots < n_{N_b} \leq \mathcal{N}$. On the other hand, it follows directly from (2.9) that it suffices to require $z_n > 0$ for the nonsingularity of the asymptotic potential. In the case of a TNN matrix, this condition is equivalent [18] to the condition that all maximal minors of the matrix $\mathcal{D}$ are positive, i.e., a totally positive (TP) matrix.

We also mention that the asymptotic forms of the functions $\chi(x, \mathbf{k})$ and $\xi(x, \mathbf{k})$ are bounded on the x plane because the x-dependent exponentials enter the denominators and numerators of expressions (2.5) and (2.6) with coefficients proportional to $\mathcal{D}(\{n_i\})$. This means that the leading asymptotic behavior of the denominators of $\chi(x, \mathbf{k})$ and $\xi(x, \mathbf{k})$ on the x plane is not weaker than the behavior of their numerators (for more details, see [8], [9], [16], [17], where the same notation is used).

In what follows, we need the values $\chi(x, i\kappa_n)$ of $\chi(x, \mathbf{k})$ and the residues $\xi_n(x)$ of $\xi(x, \mathbf{k})$ at $\mathbf{k} = i\kappa_n$. Taking (2.3) and (2.5), (2.6) into account, we obtain

$$
\chi(x, i\kappa_n) = \frac{(-1)^{N_b}}{N_b!\,\tau(x)} \sum_{\{m_i\}=1}^{\mathcal{N}} \mathcal{D}(\{m_i\})\, V(\{m_i\}, n) \prod_{l=1}^{N_b} e^{K_{m_l}(x)}, \tag{2.10}
$$

$$
\xi_n(x) = \frac{1}{iN_b!\,\tau(x)} \sum_{\{n_i\}=1}^{\mathcal{N}} \mathcal{D}(\{n_i\}) \sum_{j=1}^{N_b} \delta_{n_j n} (-1)^{j-1} V(n_1, \dots, \widehat{n_j}, \dots, n_{N_b}) \prod_{l=1}^{N_b} e^{K_{n_l}(x)}, \tag{2.11}
$$

where $\{\{m_i\}, n\} = \{m_1, \dots, m_{N_b}, n\}$, the hat over $\widehat{n_j}$ indicates that this index is omitted, and the Kronecker symbol $\delta_{n_j n}$ in the right-hand side of (2.11) appears because the residues of the terms in the sum are nonzero only when $n_j = n$ for some j.

Using the analytic properties of $\chi(x, \mathbf{k})$ and $\xi(x, \mathbf{k})$ (see (2.5), (2.6)), we can write their product in terms of the values $\chi(x, i\kappa_n)$ and $\xi_n(x)$ as

$$
\chi(x, \mathbf{k})\xi(x', \mathbf{k}) = 1 + \sum_{n=1}^{\mathcal{N}} \frac{\chi(x, i\kappa_n)\xi_n(x')}{\mathbf{k} - i\kappa_n}, \tag{2.12}
$$

which is also useful in what follows. In [16], we demonstrated that the Jost solutions satisfy the Hirota bilinear identity

$$
\sum_{n=1}^{\mathcal{N}} \Phi(x, i\kappa_n)\Psi_n(x') = 0, \tag{2.13}
$$

where $\Psi_n(x)$ analogously to (2.11) denotes the residue of $\Psi(x, \mathbf{k})$ at $\mathbf{k} = i\kappa_n$.

3. Extended resolvent $M(x, x'; q)$

We prove that the resolvent $M(x,x';q)$, i.e., a tempered distribution in its six real variables satisfying (1.6), can be written as a sum of continuous (in some sense) and discrete parts whose definitions and properties are given below:

$$M(x,x';q) = M_{\mathrm{c}}(x,x';q) + M_{\mathrm{d}}(x,x';q). \tag{3.1}$$

We also identify the infinite region on the q plane where $M(x,x';q)$ is a tempered distribution and prove that $M(x,x';q)$ is in fact bounded for q in this region. For both summands, we use the hat notation introduced in (1.7):

$$M_{\mathrm{c}}(x,x';q) = e^{-q(x-x')}\widehat{M}_{\mathrm{c}}(x,x';q), \qquad M_{\mathrm{d}}(x,x';q) = e^{-q(x-x')}\widehat{M}_{\mathrm{d}}(x,x';q). \tag{3.2}$$

Hence, we first define

$$\begin{aligned}\widehat{M}_{\mathrm{c}}(x,x';q) = &-\frac{\operatorname{sgn}(x_2-x_2')}{2\pi}\int d\alpha\,\theta((q_2+\alpha^2-q_1^2)(x_2-x_2'))\times\\ &\times\Phi(x,\alpha+iq_1)\Psi(x',\alpha+iq_1),\end{aligned} \tag{3.3}$$

where the Jost and dual Jost solutions $\Phi(x,\mathbf{k})$ and $\Psi(x,\mathbf{k})$ are defined in (2.3), the integration is along the real axis, and θ is the Heaviside function. The properties of M_{c} are given in the following lemma.

Lemma 1. *The integral in the right-hand side of* (3.3) *exists, and the function* M_{c} *given in* (3.2) *is a bounded function of its arguments for all* $x, x', q \in \mathbb{R}^2$ *and has finite limits at infinity.*

Proof. Because of (2.3) and (3.2), we can write

$$\begin{aligned}M_{\mathrm{c}}(x,x';q) = M_0(x,x';q) - \sum_{n=1}^{\mathcal{N}}\chi(x,i\kappa_n)\xi_n(x')\frac{\operatorname{sgn}(x_2-x_2')}{2\pi}\int d\alpha\,\frac{\theta((q_2+\alpha^2-q_1^2)(x_2-x_2'))}{\alpha+i(q_1-\kappa_n)}\times\\ \times e^{-i\alpha(x_1-x_1'+2q_1(x_2-x_2'))-(q_2+\alpha^2-q_1^2)(x_2-x_2')},\end{aligned} \tag{3.4}$$

where we use (2.12) for the product $\chi(x,\mathbf{k})\xi(x',\mathbf{k})$ and $M_0(x,x';q)$ is the extended resolvent of operator (1.5) in the case of a zero potential:

$$\begin{aligned}M_0(x,x';q) = &-\frac{\operatorname{sgn}(x_2-x_2')}{2\pi}\int d\alpha\,\theta((q_2+\alpha^2-q_1^2)(x_2-x_2'))\times\\ &\times e^{-i\alpha(x_1-x_1'+2q_1(x_2-x_2'))-(q_2+\alpha^2-q_1^2)(x_2-x_2')}.\end{aligned} \tag{3.5}$$

Then the statement of the lemma follows directly because the factors in (3.4) and (3.5) decrease exponentially.

Applying heat operator (1.1) to $\widehat{M}_{\mathrm{c}}(x,x';q)$ in (3.3), we obtain

$$\mathcal{L}(x,\partial_x)\widehat{M}_{\mathrm{c}}(x,x';q) = \frac{\delta(x_2-x_2')}{2\pi}\int ds\,\Phi(x,s+iq_1)\Psi(x',s+iq_1).$$

The integral in the right-hand side can be computed explicitly by virtue of (2.3): substituting (2.8) in it, we obtain

$$\frac{\delta(x_2-x_2')}{2\pi}\int ds\,\Phi(x,s+iq_1)\Psi(x',s+iq_1)=\delta(x-x')-\delta(x_2-x_2')p(x,x',q_1)+$$
$$+\,i\delta(x_2-x_2')\theta(x_1'-x_1)\sum_{n=1}^{\mathcal{N}}\Phi(x,i\kappa_n)\Psi_n(x'), \tag{3.6}$$

where the last term cancels because of (2.13). In (3.6), we set

$$p(x,x',q_1)=i\sum_{n=1}^{\mathcal{N}}\theta(q_1-\kappa_n)\Phi(x,i\kappa_n)\Psi_n(x'). \tag{3.7}$$

We note that $p(x,x',q_1)$ does not belong to the space of Schwartz distributions, because it can increase exponentially in some directions in the x plane [16]. It follows from (2.13) that

$$p(x,x',q_1)=0\quad\text{for any } q_1\notin[\kappa_1,\kappa_{\mathcal{N}}], \tag{3.8}$$

and this function can be rewritten in either of the forms

$$p(x,x',q_1)=\frac{i}{2}\sum_{n=1}^{\mathcal{N}}\operatorname{sgn}(q_1-\kappa_n)\Phi(x,i\kappa_n)\Psi_n(x')\equiv -i\sum_{n=1}^{\mathcal{N}}\theta(\kappa_n-q_1)\Phi(x,i\kappa_n)\Psi_n(x'). \tag{3.9}$$

We also note that by definition,

$$\mathcal{L}(x,\partial_x)p(x,x',q_1)=0\quad\text{for any } q\in\mathbb{R}^2. \tag{3.10}$$

We define the second term in (3.1) as (also see (3.2))

$$\widehat{M}_{\mathrm{d}}(x,x';q)=\mp\theta(\pm(x_2-x_2'))p(x,x',q_1), \tag{3.11}$$

and by (3.10), we have

$$\mathcal{L}(x,\partial_x)\widehat{M}_{\mathrm{d}}(x,x';q)=\delta(x_2-x_2')p(x,x',q_1)$$

for any choice of sign in the right-hand side. With (3.6) taken into account, this proves that the function

$$\widehat{M}(x,x';q)=\widehat{M}_{\mathrm{c}}(x,x';q)+\widehat{M}_{\mathrm{d}}(x,x';q) \tag{3.12}$$

satisfies the first equality in (1.8) or $M(x,x';q)$ given by (3.1) satisfies the first equality in (1.6) by virtue of (1.7). The second equalities in (1.6) and (1.8) are proved analogously. We note that by (3.8),

$$M_{\mathrm{d}}(x,x';q)=0\quad\text{for all } q_1\notin[\kappa_1,\kappa_{\mathcal{N}}]. \tag{3.13}$$

Therefore, to prove that $M(x,x';q)$ is the extended resolvent, we must prove that $M_{\mathrm{d}}(x,x';q)$ belongs to the class of tempered distributions and specify the choice of signs in (3.11). For this, we must find the dependence of $M_{\mathrm{d}}(x,x';q)$ on its variables explicitly.

We first consider the function $p(x,x',q_1)$ defined in (3.7). Substituting equalities (2.10) and (2.11), by virtue of the antisymmetry of the minors of the matrices $\mathcal{D}$ and $\mathcal{V}$ (see (2.7)), after summing over n, we obtain

$$p(x,x',q_1)=\frac{-1}{N_b!(N_b-1)!\,\tau(x)\tau(x')}\sum_{\{m_i\}=1}^{\mathcal{N}}\sum_{\{n_i\}=1}^{\mathcal{N}}\mathcal{D}(\{m_i\})\mathcal{D}(\{n_i\})\theta(q_1-\kappa_{n_{N_b}})\times$$
$$\times V(\{m_i\},n_{N_b})V(n_1,\dots,n_{n_{N_b-1}})\times$$
$$\times\exp\biggl(\sum_{l=1}^{N_b}K_{m_l}(x)+K_{n_{N_b}}(x)+\sum_{l=1}^{N_b-1}K_{n_l}(x')\biggr). \tag{3.14}$$

Further, we use formula (2.8) for the product of the minors of the matrix $\mathcal{D}$ with $j=N_b$ and exchange the summation indices: $m_s\leftrightarrow n_{N_b}$, $s=1,\dots,N_b$. We note that under this exchange, the first Vandermonde determinant changes sign, while the second Vandermonde determinant and also the exponential are unchanged. Therefore, we have

$$p(x,x',q_1)=\frac{1}{N_b!(N_b-1)!\,\tau(x)\tau(x')}\sum_{s=1}^{N_b}\sum_{\{m_i\}=1}^{\mathcal{N}}\sum_{\{n_i\}=1}^{\mathcal{N}}\mathcal{D}(\{m_i\})\mathcal{D}(\{n_i\})\times$$
$$\times\theta(q_1-\kappa_{m_s})V(\{m_i\},n_{N_b})V(n_1,\dots,n_{N_b-1})\times$$
$$\times\exp\biggl(\sum_{l=1}^{N_b}K_{m_l}(x)+K_{n_{N_b}}(x)+\sum_{l=1}^{N_b-1}K_{n_l}(x')\biggr). \tag{3.15}$$

Now exchanging m_s and m_{N_b} and summing over s, we obtain N_b equal terms. Finally, we multiply (3.14) by N_b, add to (3.15), and divide this sum by N_b+1, which gives

$$p(x,x',q_1)=\frac{1}{((N_b-1)!)^2(N_b+1)\tau(x)\tau(x')}\sum_{\{m_i\}=1}^{\mathcal{N}}\sum_{\{n_i\}=1}^{\mathcal{N}}\mathcal{D}(\{m_i\})\mathcal{D}(\{n_i\})\times$$
$$\times\bigl(\theta(q_1-\kappa_{m_{N_b}})-\theta(q_1-\kappa_{n_{N_b}})\bigr)V(\{m_i\},n_{N_b})V(n_1,\dots,n_{N_b-1})\times$$
$$\times\exp\biggl(\sum_{l=1}^{N_b}K_{m_l}(x)+K_{n_{N_b}}(x)+\sum_{l=1}^{N_b-1}K_{n_l}(x')\biggr),$$

and hence for all $n=1,\dots,\mathcal{N}-1$ for $\kappa_n\le q_1\le\kappa_{n+1}$,

$$p(x,x',q_1)=\frac{1}{((N_b-1)!)^2(N_b+1)\tau(x)\tau(x')}\sum_{\{m_i\},\{n_i\}}\mathcal{D}(\{m_i\})\mathcal{D}(\{n_i\})\times$$
$$\times\bigl(\theta(q_1-\kappa_{m_{N_b}})-\theta(q_1-\kappa_{n_{N_b}})\bigr)V(\{m_i\},n_{N_b})V(n_1,\dots,n_{N_b-1})\times$$
$$\times\exp\biggl(\sum_{l=1}^{N_b}K_{m_l}(x)+K_{n_{N_b}}(x)+\sum_{l=1}^{N_b-1}K_{n_l}(x')\biggr).$$

Here and hereafter, the sum over $\{m_i\},\{n_i\}$ assumes summation over all selections $m_1,\dots,m_{N_b}$ and n_1, $\dots,n_{N_b}$ from the set $\{1,\dots,\mathcal{N}\}$ such that the interval $[\kappa_n,\kappa_{n+1}]$ belongs to the interval $[\kappa_{m_{N_b}},\kappa_{n_{N_b}}]$. We note that this sum does not contain terms with $m_{N_b}=n_{N_b}$ because of the factor $V(\{m_i\},n_{N_b})$.

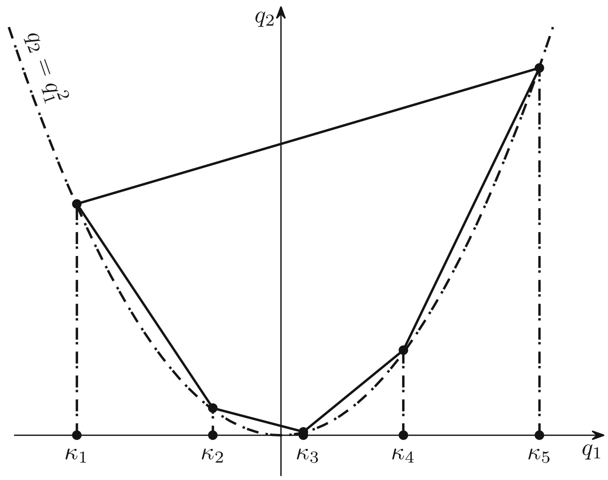

Fig. 1. The polygon $\mathcal{P}$ in the case $\mathcal{N}=5$.

By (3.2) and (3.11), we find that for all $n=1,\dots,\mathcal{N}-1$ for $\kappa_n \le q_1 \le \kappa_{n+1}$,

$$M_{\mathrm{d}}(x,x';q) = \frac{\mp\theta(\pm(x_2-x_2'))e^{-q(x-x')}}{((N_b-1)!)^2(N_b+1)\tau(x)\tau(x')} \sum_{\{m_i\},\{n_i\}} \mathcal{D}(\{m_i\})\mathcal{D}(\{n_i\}) \times$$
$$\times \bigl(\theta(q_1-\kappa_{m_{N_b}})-\theta(q_1-\kappa_{n_{N_b}})\bigr)V(\{m_i\},n_{N_b})V(n_1,\dots,n_{N_b-1}) \times$$
$$\times \exp\biggl(\sum_{l=1}^{N_b} K_{m_l}(x) + K_{n_{N_b}}(x) + \sum_{l=1}^{N_b-1} K_{n_l}(x')\biggr) \tag{3.16}$$

under the same condition on the summation over $\{m_i\},\{n_i\}$. Representation (3.16) for $M_{\mathrm{d}}(x,x';q)$ gives one more proof of relation (3.13).

In [16], we demonstrated that extended operator (1.5) can have annihilators when q belongs to some polygons on the q plane. Therefore, the operator inverse to (1.5) (the extended resolvent) cannot exist for any value of q. We introduce the polygon $\mathcal{P}$ inscribed in the parabola $q_2=q_1^2$ in the q plane (see Fig. 1) with vertices at the points (κ_n,κ_n^2) for $n=1,\dots,\mathcal{N}$ and the characteristic function

$$\epsilon(q) = \sum_{m=1}^{\mathcal{N}-1} \bigl(\theta(q_1-\kappa_{n+1})-\theta(q_1-\kappa_n)\bigr)\bigl(\theta(q_{n,n+1})-\theta(q_{1\mathcal{N}})\bigr), \tag{3.17}$$

where

$$q_{mn} = q_2 - (\kappa_m+\kappa_n)q_1 + \kappa_m\kappa_n. \tag{3.18}$$

Obviously, this polygon divides the strip $\kappa_1 < q_1 < \kappa_{\mathcal{N}}$ in the q plane into two disconnected parts. Moreover, this polygon consists of substrips given by subsequent κ values as

$$\kappa_n < q_1 < \kappa_{n+1}, \qquad q_{n,n+1} > 0, \quad n=1,\dots,\mathcal{N}-1, \qquad q_{1,\mathcal{N}} < 0.$$

Taking (3.13) into account, we can now prove the following result.

Lemma 2. *Let the $\{N_a,N_b\}$-soliton potential $u(x)$ be such that its τ-function* (2.2), (2.4) *satisfies the condition that on any subset $\{n_1<\dots<n_{N_b}\}\subset\{1,\dots,\mathcal{N}\}$, the ratio*

$$\frac{1}{\tau(x)}\prod_{l=1}^{N_b} e^{K_{n_l}(x)} \tag{3.19}$$

is bounded for all x and has finite limits at spatial infinity. Then $M_{\rm d}(x,x';q)$ for all q in the strip $\kappa_1 \le q_1 \le \kappa_{\mathcal{N}}$ and outside the polygon $\mathcal{P}$ is a bounded function of all its arguments including values at infinities if the upper sign in (3.11) is chosen for q above the polygon $\mathcal{P}$ and the bottom sign is chosen for q below the polygon $\mathcal{P}$.

Proof. Because of (3.13), we consider only q belonging to the strip $\kappa_1 \le q_1 \le \kappa_{\mathcal{N}}$. We set $z_{mn} = x_1 + (\kappa_m + \kappa_n)x_2$. Using the identity

$$\theta(q_1-\kappa_m)-\theta(q_1-\kappa_n)=\operatorname{sgn}(z_{mn}-z'_{mn})\big[\theta((q_1-\kappa_m)(z_{mn}-z'_{mn}))-\theta((q_1-\kappa_n)(z_{mn}-z'_{mn}))\big],$$

we then rewrite (3.16) in the form

$$\begin{aligned}M_{\rm d}(x,x';q) = {} & \frac{\mp\theta(\pm(x_2-x'_2))}{((N_b-1)!)^2(N_b+1)\tau(x)\tau(x')}\times\\ & \times\sum_{\{m_i\},\{n_i\}}\mathcal{D}(\{m_i\})\mathcal{D}(\{n_i\})\operatorname{sgn}(z_{m_{N_b}n_{N_b}}-z'_{m_{N_b}n_{N_b}})V(\{m_i\},n_{N_b})V(n_1,\dots,n_{N_b-1})\times\\ & \times\big[\theta((q_1-\kappa_{m_{N_b}})(z_{m_{N_b}n_{N_b}}-z'_{m_{N_b}n_{N_b}}))-\theta((q_1-\kappa_{n_{N_b}})(z_{m_{N_b}n_{N_b}}-z'_{m_{N_b}n_{N_b}}))\big]\times\\ & \times\exp\Bigg(\sum_{l=1}^{N_b}K_{m_l}(x)+K_{n_{N_b}}(x)+\sum_{l=1}^{N_b-1}K_{n_l}(x')-q(x-x')\Bigg)\end{aligned}$$

(we recall that the summation over $\{m_i\},\{n_i\}$ ranges all $m_1,\dots,m_{N_b}$ and $n_1,\dots,n_{N_b}$ in the set $\{1,\dots,\mathcal{N}\}$ such that $[\kappa_n,\kappa_{n+1}]\subseteq[\kappa_{m_{N_b}},\kappa_{n_{N_b}}]$).

We decompose this expression into a sum of two terms in correspondence with the two terms in the square brackets, replace $q(x-x')$ in the exponential with the identity $q(x-x') = K_m(x)-K_m(x')+q_{mn}(x_2-x'_2)+(q_1-\kappa_m)(z_{mn}-z'_{mn})$, where $K_m(x)$ and q_{mn} are defined in (2.1) and (3.18), and finally set $m=m_{N_b}$ and $n=n_{N_b}$ in the first term and $n=m_{N_b}$ and $m=n_{N_b}$ in the second term. We thus obtain

$$M_{\rm d}(x,x';q)=M^{(1)}(x,x';q)+M^{(2)}(x,x';q), \tag{3.20}$$

where in each substrip $\kappa_n \le q_1 \le \kappa_{n+1}$, $n=1,\dots,\mathcal{N}$,

$$\begin{aligned}M^{(1)}(x,x';q) = {} & \frac{\mp\theta(\pm(x_2-x'_2))}{((N_b-1)!)^2(N_b+1)}\sum_{\{m_i\},\{n_i\}}\operatorname{sgn}\big(z_{m_{N_b}n_{N_b}}-z'_{m_{N_b}n_{N_b}}\big)\times\\ & \times e^{-q_{m_{N_b}n_{N_b}}(x_2-x'_2)}V(\{m_i\},n_{N_b})V(n_1,\dots,n_{N_b-1})\times\\ & \times\theta((q_1-\kappa_{m_{N_b}})(z_{m_{N_b}n_{N_b}}-z'_{m_{N_b}n_{N_b}}))e^{-(q_1-\kappa_{m_{N_b}})(z_{m_{N_b}n_{N_b}}-z'_{m_{N_b}n_{N_b}})}\times\\ & \times\frac{\mathcal{D}(\{m_i\})\exp\big(\sum_{l=1}^{N_b-1}K_{m_l}(x)+K_{n_{N_b}}(x)\big)}{\tau(x)}\times\\ & \times\frac{\mathcal{D}(\{n_i\})\exp\big(\sum_{l=1}^{N_b-1}K_{n_l}(x')+K_{m_{N_b}}(x')\big)}{\tau(x')}\end{aligned} \tag{3.21}$$

and

$$M^{(2)}(x,x';q) = \frac{\pm\theta(\pm(x_2-x_2'))}{((N_b-1)!)^2(N_b+1)} \sum_{\{m_i\},\{n_i\}} \operatorname{sgn}\big(z_{m_{N_b}n_{N_b}} - z'_{m_{N_b}n_{N_b}}\big) \times$$
$$\times\, e^{-q_{m_{N_b}n_{N_b}}(x_2-x_2')} V(\{m_i\}, n_{N_b}) V(n_1,\ldots,n_{N_b-1}) \times$$
$$\times\, \theta((q_1-\kappa_{n_{N_b}})(z_{m_{N_b}n_{N_b}} - z'_{m_{N_b}n_{N_b}}))e^{-(q_1-\kappa_{n_{N_b}})(z_{m_{N_b}n_{N_b}}-z'_{m_{N_b}n_{N_b}})} \times$$
$$\times\, \frac{\mathcal{D}(\{m_i\})\exp\big(\sum_{l=1}^{N_b} K_{m_l}(x)\big)}{\tau(x)} \frac{\mathcal{D}(\{n_i\})\exp\big(\sum_{l=1}^{N_b} K_{n_l}(x')\big)}{\tau(x')}. \tag{3.22}$$

The right-hand sides of these relations depend on x and x' via the exponential factors and the τ-functions in the denominators. Because the extended resolvent must belong to the space of tempered distributions, it suffices to show that the right-hand sides of equalities (3.21) and (3.22) cannot increase at the spatial infinity. We consider the behavior with respect to x of these two expressions in detail.

First, recalling definition (3.18) of q_{mn}, we note that for q belonging to the nth substrip $\kappa_n \le q_1 \le \kappa_{n+1}$ such that $q_{n,n+1} \le 0$, i.e., for q in this substrip below the polygon $\mathcal{P}$ or on its bottom border, all other $q_{m_{N_b},n_{N_b}}$ in the sum are nonpositive. Similarly, if $q_{1,\mathcal{N}} \ge 0$, i.e., q belongs to the substrip $\kappa_n \le q_1 \le \kappa_{n+1}$ above the polygon $\mathcal{P}$ or on its upper border, then all other $q_{m_{N_b},n_{N_b}}$ in the summation are nonnegative. Therefore, if we choose the signs in the right-hand side of (3.21) and (3.22) as indicated in the lemma condition, then the exponentials in the second lines decrease or are bounded when x or q tends to infinity. The exponentials in the third lines of (3.21) and (3.22) decrease or at least do not increase by virtue of the factors in front of them.

Hence, we should verify the behavior with respect to x and x' of the last two fractions in (3.21) and (3.22). The situation is trivial in the case of relation (3.22). The last two exponentials in the numerators of those fractions have the same coefficient (minor of $\mathcal{D}$) as in $\tau(x)$ and $\tau(x')$ in the denominators, and these fractions are therefore bounded as x and x' increase. The situation with (3.21) is more complicated. We consider the next-to-last factor in that expression. If the minor $\mathcal{D}(\{m_i\})$ in the numerator is nonzero and $\mathcal{D}(m_1,\ldots,m_{N_b-1},n_{N_b}) \neq 0$, then the same exponential is present in both the numerator and $\tau(x)$ in the denominator, and the fraction is therefore bounded. But if $\mathcal{D}(m_1,\ldots,m_{N_b-1},n_{N_b}) = 0$, then such an exponential is not contained in $\tau(x)$, and in the direction where that exponential dominates (if such a direction exists), the fraction increases at large distances. The same also holds for the last factor in the right-hand side of equality (3.21). The boundedness of these fractions and consequently of the entire expression (3.21) is guaranteed by the condition imposed on (3.19). The lemma is proved.

Remark 1. The lemma condition imposed on expression (3.19) is sufficient for the lemma to hold, but it is not necessary, and the boundedness of $M_{\mathrm{d}}(x,x';q)$ requires additional study. Nevertheless, it is clear that this condition is satisfied for a TP matrix $\mathcal{D}$. If we impose the condition $z_n > 0$ on the matrix $\mathcal{D}$ instead of the TP condition, then all leading exponentials are present in $\tau(x)$, as noted when discussing (2.9). Therefore, the lemma condition is satisfied if $\tau(x)$ has no zeros in the finite domain. To avoid these singularities, it suffices to additionally require that the matrix $\mathcal{D}$ be TNN.

Remark 2. The boundedness of $M(x,x';q)$ in q for $q \in \mathbb{R}^2\backslash\mathcal{P}$, on the boundaries of $\mathcal{P}$, and in the limit as $q_{1,2} \to \infty$ follows from the boundedness of $M_{\mathrm{c}}(x,x';q)$ in (3.4) and also $M^{(1)}(x,x';q)$ and $M^{(2)}(x,x';q)$ in (3.21) and (3.22). In the next section, we consider the behavior of $M(x,x';q)$ with respect to q in detail.

Remark 3. We have proved that under the conditions in Lemma 2, the function $M(x,x';q)$ is a bounded function of its arguments and has a finite asymptotic behavior. This means that this function belongs to the class of tempered distributions, i.e., it is the extended resolvent of the heat operator $\mathcal{L}(x,x';q)$ for q outside the polygon $\mathcal{P}$.

4. Properties of the resolvent and Green's functions

4.1. Extended resolvent inside and outside the parabola $q_2 = q_1^2$. The local properties of the extended resolvent are easier to study in terms of kernel (1.7). Under special reduction (1.9), this kernel is just the total Green's function $\mathcal{G}(x,x',\mathbf{k})$, where $\mathbf{k}\in\mathbb{C}$ is the spectral parameter. Indeed, in this case, the difference $q_2 - q_1^2 = -\operatorname{Re}^2\mathbf{k}$ is nonpositive, and $q_{1\mathcal{N}}$ (see (3.18)) is less than or equal to zero in the interval $\kappa_1 \le q_1 \le \kappa_{\mathcal{N}}$. Therefore, this reduction maps the exterior region of the parabola $q_2 = q_1^2$ to the complex-$\mathbf{k}$ plane (more precisely, it is a one-to-two map because the reduction depends on $|\operatorname{Re}\mathbf{k}|$ and not on $\operatorname{Re}\mathbf{k}$). Taking condition (3.13) into account, we see that the part of the strip $\kappa_1 \le q_1 \le \kappa_{\mathcal{N}}$ outside the parabola is located below the polygon $\mathcal{P}$, and by Lemma 2, we should therefore choose the lower sign in (3.11). Hence, by (3.1), (3.3), and (3.7), we obtain

$$\begin{aligned}\mathcal{G}(x,x',\mathbf{k}) = {}& -\frac{\operatorname{sgn}(x_2-x_2')}{2\pi}\int d\alpha\,\theta((\alpha^2-\operatorname{Re}^2\mathbf{k})(x_2-x_2'))\times\\ &\times\Phi(x,\alpha+i\operatorname{Im}\mathbf{k})\Psi(x',\alpha+i\operatorname{Im}\mathbf{k})+\\ &+i\theta(x_2'-x_2)\sum_{n=1}^{\mathcal{N}}\theta(\operatorname{Im}\mathbf{k}-\kappa_n)\Phi(x,i\kappa_n)\Psi_n(x'),\end{aligned}\tag{4.1}$$

which coincides with the Green's function derived in [14]. The boundedness (see (1.4)) proved in that paper now follows from the boundedness of the resolvent at infinity.

The interior part of the parabola, as noted when discussing (3.17), is divided by the polygon $\mathcal{P}$ into the part above the polygon ($q_{1,\mathcal{N}} \ge 0$) and $\mathcal{N}-1$ lenses bounded by the parabola and its chords connecting the points (κ_n,κ_n^2) and $(\kappa_{n+1},\kappa_{n+1}^2)$, $n=1,\dots,\mathcal{N}-1$ (see Fig. 1). All these lenses are below the polygon $\mathcal{P}$. Taking into account that both $\widehat{M}_{\mathrm{c}}(x,x';q)$ and $\widehat{M}_{\mathrm{d}}(x,x';q)$ are independent of q_2 for $q_2 \ge q_1^2$ and by analogy with (1.9), we introduce the auxiliary Green's functions

$$\begin{aligned}&\mathcal{G}^{+}(x,x',\operatorname{Im}\mathbf{k}) = \widehat{M}(x,x';q)\big|_{q_1=\operatorname{Im}\mathbf{k}},\quad q_2\ge q_1^2,\quad q_2\ge(\kappa_1+\kappa_{\mathcal{N}})q_1-\kappa_1\kappa_{\mathcal{N}},\\ &\mathcal{G}^{-}(x,x',\operatorname{Im}\mathbf{k}) = \widehat{M}(x,x';q)\big|_{q_1=\operatorname{Im}\mathbf{k}},\quad q_2\ge q_1^2,\quad q_2\le(\kappa_n+\kappa_{n+1})q_1-\kappa_n\kappa_{n+1}\end{aligned}\tag{4.2}$$

for $n=1,\dots,\mathcal{N}-1$. Again, by (3.1), (3.3), (3.7), and (3.11) (with the appropriate sign chosen according to Lemma 2), we obtain

$$\begin{aligned}\mathcal{G}^{+}(x,x',\operatorname{Im}\mathbf{k}) = {}& -\frac{\theta(x_2-x_2')}{2\pi}\int d\alpha\,\Phi(x,\alpha+i\operatorname{Im}\mathbf{k})\Psi(x',\alpha+i\operatorname{Im}\mathbf{k})-\\ &-i\theta(x_2-x_2')\sum_{n=1}^{\mathcal{N}}\theta(\operatorname{Im}\mathbf{k}-\kappa_n)\Phi(x,i\kappa_n)\Psi_n(x'),\end{aligned}\tag{4.3}$$

$$\begin{aligned}\mathcal{G}^{-}(x,x',\operatorname{Im}\mathbf{k}) = {}& -\frac{\theta(x_2-x_2')}{2\pi}\int d\alpha\,\Phi(x,\alpha+i\operatorname{Im}\mathbf{k})\Psi(x',\alpha+i\operatorname{Im}\mathbf{k})+\\ &+i\theta(x_2'-x_2)\sum_{n=1}^{\mathcal{N}}\theta(\operatorname{Im}\mathbf{k}-\kappa_n)\Phi(x,i\kappa_n)\Psi_n(x').\end{aligned}\tag{4.4}$$

The boundedness of these Green's functions follows from the boundedness of the extended resolvent according to Lemma 2. Namely, by (1.7) and (4.2), the function $e^{\operatorname{Im}\mathbf{k}(x_1'-x_1)+s(x_2'-x_2)}\mathcal{G}^+(x,x',\operatorname{Im}\mathbf{k})$ is bounded for any real s such that

$$s \ge \operatorname{Im}^2\mathbf{k} + \max\{0, (\kappa_{\mathcal{N}} - \operatorname{Im}\mathbf{k})(\operatorname{Im}\mathbf{k} - \kappa_1)\},$$

and $e^{\operatorname{Im}\mathbf{k}(x_1'-x_1)+s(x_2'-x_2)}\mathcal{G}^-(x,x',\operatorname{Im}\mathbf{k})$ is bounded for any real s such that

$$\operatorname{Im}^2\mathbf{k} \le s \le \operatorname{Im}^2\mathbf{k} + (\kappa_{n+1} - \operatorname{Im}\mathbf{k})(\operatorname{Im}\mathbf{k} - \kappa_n), \quad n = 1,\dots,\mathcal{N}-1.$$

4.2. Singularities of the Green's functions. To describe the singularities of the considered reductions of the resolvent, it is convenient to use representation (3.4) for it, and by (1.7), (2.3), (3.12), and (3.7), (3.11), we can hence write

$$\widehat{M}(x,x';q) = \widehat{M}_0(x,x';q) - \sum_{n=1}^{\mathcal{N}} \Phi(x,i\kappa_n)\Psi_n(x')\biggl\{\frac{\operatorname{sgn}(x_2-x_2')}{2\pi}\int d\alpha\,\frac{\theta((q_2+\alpha^2-q_1^2)(x_2-x_2'))}{\alpha+i(q_1-\kappa_n)}\times$$
$$\times e^{(q_1-\kappa_n-i\alpha)(x_1-x_1')-((\alpha+iq_1)^2+\kappa_n^2)(x_2-x_2')} \pm i\theta(\pm(x_2-x_2'))\theta(q_1-\kappa_n)\biggr\}, \tag{4.5}$$

where

$$\widehat{M}_0(x,x';q) = -\frac{\operatorname{sgn}(x_2-x_2')}{2\pi}\int d\alpha\,\theta((q_2+\alpha^2-q_1^2)(x_2-x_2'))e^{(q_1-i\alpha)(x_1-x_1')+(q_1-i\alpha)^2(x_2-x_2')}$$

as a result of (1.7) and (3.5) and the sign in the right-hand side of (4.5) must be chosen according to Lemma 2. It is easy to see that the kernel $\widehat{M}(x,x';q)$ outside the parabola $q_2 = q_1^2$ is a continuous function of q for $q \neq (\kappa_n,\kappa_n^2)$, $n = 1,\dots,\mathcal{N}$.

From (4.5), using reduction (1.9), we obtain the following representation of the total Green's function given by (4.1):

$$\begin{aligned}\mathcal{G}(x,x',\mathbf{k}) = {} & -\frac{\operatorname{sgn}(x_2-x_2')}{2\pi}\int_{\operatorname{Im}\mathbf{k}'=\operatorname{Im}\mathbf{k}} d\operatorname{Re}\mathbf{k}'\,\theta\bigl((\operatorname{Re}^2\mathbf{k}'-\operatorname{Re}^2\mathbf{k})(x_2-x_2')\bigr)\times\\ & \times e^{-i\mathbf{k}'(x_1-x_1')-\mathbf{k}'^2(x_2-x_2')} - \sum_{n=1}^{\mathcal{N}}\Phi(x,i\kappa_n)\Psi_n(x')\times\\ & \times\biggl\{\frac{\operatorname{sgn}(x_2-x_2')}{2\pi}\int_{\operatorname{Im}\mathbf{k}'=\operatorname{Im}\mathbf{k}} d\operatorname{Re}\mathbf{k}'\,\frac{\theta\bigl((\operatorname{Re}^2\mathbf{k}'-\operatorname{Re}^2\mathbf{k})(x_2-x_2')\bigr)}{\mathbf{k}'-i\kappa_n}\times\\ & \times e^{-(\kappa_n+i\mathbf{k}')(x_1-x_1')-(\kappa_n^2+\mathbf{k}'^2)(x_2-x_2')} - \frac{i}{2}\theta(x_2'-x_2)\operatorname{sgn}(\operatorname{Im}\mathbf{k}-\kappa_n)\biggr\},\end{aligned} \tag{4.6}$$

where we use expression (3.9) for $p(x,x',q_1)$. By the preceding discussion, formula (4.6) defines a continuous function of $\mathbf{k}\in\mathbb{C}$ for all $\mathbf{k}\neq i\kappa_n$.

To study the behavior of the Green's functions near the points $\mathbf{k} = i\kappa_n$, we first consider the Green's function $\mathcal{G}^+$ given by (4.3). By relation (4.5), we have

$$\begin{aligned}\mathcal{G}^+(x,x',\operatorname{Im}\mathbf{k}) = {} & -\frac{\theta(x_2-x_2')}{2\pi}\biggl\{\int_{\operatorname{Im}\mathbf{k}'=\operatorname{Im}\mathbf{k}} d\operatorname{Re}\mathbf{k}'\,e^{-i\mathbf{k}'(x_1-x_1')-\mathbf{k}'^2(x_2-x_2')} +\\ & + \sum_{n=1}^{\mathcal{N}}\Phi(x,i\kappa_n)\Psi_n(x')\biggl(\int_{\operatorname{Im}\mathbf{k}'=\operatorname{Im}\mathbf{k}} d\operatorname{Re}\mathbf{k}'\,\frac{e^{-(\kappa_n+i\mathbf{k}')(x_1-x_1')-(\kappa_n^2+\mathbf{k}'^2)(x_2-x_2')}}{\mathbf{k}'-i\kappa_n}+\\ & + i\pi\operatorname{sgn}(\operatorname{Im}\mathbf{k}-\kappa_n)\biggr)\biggr\}.\end{aligned} \tag{4.7}$$

It is easy to see that this Green's function is continuous in $\operatorname{Im}\mathbf{k}$ and, moreover, $\mathcal{G}^+(x,x',\operatorname{Im}\mathbf{k})$ is independent of $\operatorname{Im}\mathbf{k}$. Indeed, differentiating with respect to $\operatorname{Im}\mathbf{k}$, we use the fact that the exponentials are analytic in $\mathbf{k}' = \operatorname{Re}\mathbf{k}' + i\operatorname{Im}\mathbf{k}$ and decay rapidly as $\operatorname{Re}\mathbf{k}' \to \infty$ and also that $\overline{\partial}_{\mathbf{k}'}(\mathbf{k}' - i\kappa_n)^{-1} = \pi\delta(\operatorname{Re}\mathbf{k}')\delta(\operatorname{Im}\mathbf{k}' - \kappa_n)$. Hence, the derivative of the first term and of each term in the summation are independently equal to zero, and we can therefore set

$$\mathcal{G}^+(x,x') = -\frac{\theta(x_2 - x_2')}{2\pi}\Bigg\{\int d\alpha\, e^{-i\alpha(x_1-x_1')-\alpha^2(x_2-x_2')} + \\ + \sum_{n=1}^{\mathcal{N}} \Phi(x,i\kappa_n)\Psi_n(x') \int \frac{d\alpha}{\alpha} e^{-i\alpha(x_1-x_1'+2\kappa_n(x_2-x_2'))-\alpha^2(x_2-x_2')}\Bigg\}, \tag{4.8}$$

where the integral is understood in the principal value sense. The result can also be calculated explicitly in terms of hypergeometric functions.

On the other hand, using (4.3) and (4.4), we directly obtain

$$\mathcal{G}^-(x,x',\operatorname{Im}\mathbf{k}) = \mathcal{G}^+(x,x') + \frac{i}{2}\sum_{n=1}^{\mathcal{N}} \operatorname{sgn}(\operatorname{Im}\mathbf{k} - \kappa_n)\Phi(x,i\kappa_n)\Psi_n(x'), \tag{4.9}$$

and $\mathcal{G}^-(x,x',\operatorname{Im}\mathbf{k})$ is discontinuous for all $\operatorname{Im}\mathbf{k} = \kappa_n$, more precisely,

$$\mathcal{G}^-(x,x',\kappa_n + 0) - \mathcal{G}^-(x,x',\kappa_n - 0) = i\Phi(x,i\kappa_n)\Psi_n(x'), \quad n = 2,\ldots,\mathcal{N}-1. \tag{4.10}$$

Therefore, considering the difference between the expressions in (4.6) and (4.7), we deduce that the total Green's function can be written in the form

$$\mathcal{G}(x,x',\mathbf{k}) = \mathcal{G}_{\mathrm{reg}}(x,x',\mathbf{k}) + \mathcal{G}_{\Delta}(x,x',\mathbf{k}), \tag{4.11}$$

where

$$\mathcal{G}_{\mathrm{reg}}(x,x',\mathbf{k}) = \mathcal{G}^+(x,x') + \frac{1}{2\pi}\int_{\substack{|\operatorname{Re}\mathbf{k}'|\le|\operatorname{Re}\mathbf{k}|,\\ \operatorname{Im}\mathbf{k}'=\operatorname{Im}\mathbf{k}}} d\operatorname{Re}\mathbf{k}'\, e^{-i\mathbf{k}'(x_1-x_1')-\mathbf{k}'^2(x_2-x_2')} + \\ + \frac{1}{2\pi}\sum_{n=1}^{\mathcal{N}} \Phi(x,i\kappa_n)\Psi_n(x') \times \\ \times \int_{\substack{|\operatorname{Re}\mathbf{k}'|\le|\operatorname{Re}\mathbf{k}|,\\ \operatorname{Im}\mathbf{k}'=\operatorname{Im}\mathbf{k}}} d\operatorname{Re}\mathbf{k}'\, \frac{e^{-(\kappa_n+i\mathbf{k}')(x_1-x_1')-(\kappa_n^2+\mathbf{k}'^2)(x_2-x_2')} - 1}{\mathbf{k}' - i\kappa_n}. \tag{4.12}$$

The last integral term is regularized by subtracting 1 in the numerator, which compensates the zero in the denominator at $\mathbf{k}' = i\kappa_n$. Correspondingly,

$$\mathcal{G}_{\Delta}(x,x',\mathbf{k}) = \frac{1}{i\pi}\sum_{n=1}^{\mathcal{N}} \Phi(x,i\kappa_n)\Psi_n(x') \operatorname{arccot}\frac{\operatorname{Im}\mathbf{k} - \kappa_n}{|\operatorname{Re}\mathbf{k}|}, \tag{4.13}$$

where a term proportional to the left-hand side of (2.13) is omitted. This expression, obviously, is discontinuous at $\mathbf{k} = i\kappa_n$, $n = 1,\ldots,\mathcal{N}$, and only at those points. Namely,

$$\mathcal{G}_{\Delta}(x,x',\mathbf{k}) = \frac{1}{i\pi}\Phi(x,i\kappa_n)\Psi_n(x') \operatorname{arccot}\frac{\operatorname{Im}\mathbf{k} - \kappa_n}{|\operatorname{Re}\mathbf{k}|} - \\ - i\sum_{m=n+1}^{\mathcal{N}} \Phi(x,i\kappa_m)\Psi_m(x') + o(1), \quad \mathbf{k} \sim i\kappa_n, \quad n = 1,\ldots,\mathcal{N}. \tag{4.14}$$

We also note that because of (4.11)–(4.13),

$$\mathcal{G}(x,x',\mathbf{k})\big|_{\operatorname{Re}\mathbf{k}=0}=\mathcal{G}^{+}(x,x')-\frac{i}{2}\sum_{n=1}^{\mathcal{N}}\operatorname{sgn}(\kappa_n-\operatorname{Im}\mathbf{k})\Phi(x,i\kappa_n)\Psi_n(x'),\tag{4.15}$$

where we assume that $\operatorname{Im}\mathbf{k}\neq\kappa_n$, $n=1,\dots,\mathcal{N}$. As a result of (2.13) and (4.9), this gives

$$\mathcal{G}(x,x',\mathbf{k})\big|_{\operatorname{Re}\mathbf{k}=0}=\mathcal{G}^{-}(x,x',\operatorname{Im}\mathbf{k}).\tag{4.16}$$

On the other hand, in the vicinity of the points $i\kappa_n$, we have

$$\begin{aligned}\mathcal{G}(x,x',\mathbf{k})=\mathcal{G}^{+}(x,x')+\frac{1}{i\pi}\Phi(x,i\kappa_n)\Psi_n(x')\operatorname{arccot}\frac{\operatorname{Im}\mathbf{k}-\kappa_n}{|\operatorname{Re}\mathbf{k}|}-{}\\ -i\sum_{m=n+1}^{\mathcal{N}}\Phi(x,i\kappa_m)\Psi_m(x')+o(1),\quad \mathbf{k}\sim i\kappa_n,\quad n=1,\dots,\mathcal{N}.\end{aligned}\tag{4.17}$$

In a subsequent publication, we will show that these properties of the Green's functions allow solving the problem formulated in the introduction, i.e., allow generalizing the IST method to the case of perturbed multisoliton potentials (1.2) of heat operator (1.1).

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