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NEW INTEGRABLE SYSTEMS AS A LIMIT OF THE ELLIPTIC $SL(N,\mathbb{C})$ TOP

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We consider the scaling limit of an elliptic top. This limit is a combination of a scaling of the elliptic top variables, an infinite shift of the spectral parameter, and the trigonometric limit. We give general necessary constraints on the scaling of the variables and examples of such a degeneracy. A certain subclass of limit systems is integrable in the Liouville sense, which can also be shown directly.

Keywords: integrable system, Inozemtsev limit, integrability test, elliptic top

1. Introduction

The limit that we consider here is a generalization of the Inozemtsev limit applied to the elliptic $SL(N,\mathbb{C})$ top. The Inozemtsev limit is a procedure for the degeneration of the elliptic Calogero–Moser system to the Toda chain [1]. On the other hand, a correspondence between the smallest-dimension orbit of the elliptic $SL(N,\mathbb{C})$ top and the phase space of the Calogero–Moser elliptic system was shown in [2], which suggests a transfer to the elliptic top of the technique for the degeneration of the Calogero–Moser elliptic system to Toda chain. But the degeneration of the elliptic top may lead to a wider class of systems if we do not restrict it to the smallest-dimension orbit. Certain possibilities for the degeneration of the elliptic top lead to systems whose integrability was proved directly [3]. Here, we describe these systems more elegantly in the language of the root system of the initial algebra $\mathfrak{sl}(N,\mathbb{C})$ and show that particular cases of this degeneration corresponding to limit transitions to the periodic and nonperiodic Toda chains are naturally related to the Inozemtsev limit of the Calogero–Moser system for the root system of the algebra $\mathfrak{sl}(N,\mathbb{C})$.

The limit under consideration leads to contraction of the initial Lie algebra $\mathfrak{sl}(N,\mathbb{C})$ with respect to the Poisson brackets. In the examples given in Sec. 4, the limit algebra turns out to be solvable. To show the integrability of the limit systems, we must find the phase space dimension. One way to do this is to estimate the degeneracy degree of the Poisson tensor from above and to indicate the necessary set of independent Casimir functions. For a certain subclass of subsystems considered in Sec. 4, we manage to determine the phase space dimension directly and to produce a sufficient number of independent Hamiltonians, thus showing that the limit systems are integrable in the Liouville sense.

2. Elliptic top

The elliptic $SL(N,\mathbb{C})$ top is an example of the Euler–Arnold top [4]. It is defined on a coadjoint orbit of the group $(N,\mathbb{C})$

$$\mathcal{R}^{\rm rot} = \{\mathbf{S}\in\mathfrak{sl}(N,\mathbb{C}),\ \mathbf{S} = g^{-1}\mathbf{S}^{(0)}g\},$$

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where $\mathbf{S}^{(0)} \in \mathfrak{sl}(N,\mathbb{C})$ is the element fixing the orbit and $g \in SL(N,\mathbb{C})$ is defined up to a left multiplication by the element $\mathbf{S}^{(0)}$ of the stationary subgroup G_0. A Kirillov–Kostant symplectic form

$$\omega^{\mathrm{rot}} = \mathrm{Tr}\{\mathbf{S}^{(0)}(dg)g^{-1} \wedge (dg)g^{-1}\}$$

is given on the phase space $\mathcal{R}^{\mathrm{rot}}$. The Hamiltonian is defined as the quadratic form of the operator J, the inverse of the inertia tensor:

$$H^{\mathrm{rot}} = -\frac{1}{2}\,\mathrm{Tr}\,\mathbf{S}J(\mathbf{S}).$$

In what follows, we consider a special operator J that ensures the integrability of the system. In the sine algebra basis in $\mathfrak{sl}(N,\mathbb{C}) \ni \mathbf{S}$ (see Appendix A), it becomes diagonal:

$$J(\mathbf{S}) = \sum_{m,n} J_{mn} s_{mn} T_{mn}, \qquad J_{mn} = E_2\biggl(\frac{m+n\tau}{N},\tau\biggr),$$

where $m,n \in \{0,\dots,N-1\}$, $m^2+n^2 \neq 0$. Here, $E_2(z,\tau)$ is the second Eisenstein function [5] defined on the complex torus $T^2\colon \mathbb{C}/(2\omega_1\mathbb{Z}+2\omega_2\mathbb{Z})$ with the half-periods $\omega_1 = 1/2$ and $\tau = \omega_2/\omega_1$, and s_{mn} are the coordinates $\mathbf{S}$ in the sine algebra basis $\{T_{mn}\}$.

The Hamilton equations of motion for the elliptic top can be represented in the Lax form [6]:

$$\frac{dL^{\mathrm{rot}}}{dt} = \{H^{\mathrm{rot}}, L^{\mathrm{rot}}\} = N[L^{\mathrm{rot}}, M^{\mathrm{rot}}], \tag{1}$$

where

$$L^{\mathrm{rot}} = \sum_{m,n} s_{mn}\varphi\begin{bmatrix} m \\ n \end{bmatrix}(z)T_{mn}, \qquad \varphi\begin{bmatrix} m \\ n \end{bmatrix}(z) = \mathrm{e}\biggl(-\frac{nz}{N}\biggr)\phi\biggl(-\frac{m+n\tau}{N},z\biggr),$$

$$M^{\mathrm{rot}} = \sum_{m,n} s_{mn} f\begin{bmatrix} m \\ n \end{bmatrix}(z)T_{mn}, \qquad f\begin{bmatrix} m \\ n \end{bmatrix}(z) = \mathrm{e}\biggl(-\frac{nz}{N}\biggr)\partial_u\phi(u,z)\big|_{u=-(m+n)/N},$$

$\mathrm{e}(z) \equiv e^{2\pi i z}$, and ϕ is a combination of theta functions (see formula (B.1) in Appendix B). The factor N is taken out of the second Lax matrix M^{rot} for convenience in comparing with papers where a different normalization of the sine algebra basis elements T_{mn} is used. The Lax matrix is quasiperiodic in the spectral parameter:

$$L^{\mathrm{rot}}(z+1) = T_{10}L^{\mathrm{rot}}(z)T_{10}^{-1}, \qquad L^{\mathrm{rot}}(z+\tau) = T_{01}L^{\mathrm{rot}}(z)T_{01}^{-1}. \tag{2}$$

Hence, $\mathrm{Tr}(L^{\mathrm{rot}}(z))^k$ are doubly periodic functions with poles of at most the kth order, and they can consequently be expanded in the basis of doubly periodic functions on the torus, for example, composed of the second Eisenstein function and its derivatives:

$$\mathrm{Tr}(L^{\mathrm{rot}}(z))^k = H_{k,0} + E_2(z)H_{k,2} + E_2'(z)H_{k,3} + \dots + E_2^{(k-2)}(z)H_{k,k}. \tag{3}$$

The quadratic Hamiltonian

$$H^{\mathrm{rot}} = \frac{1}{2}H_{2,0} = \frac{1}{2}\,\mathrm{Tr}(L^{\mathrm{rot}})^2 - \frac{1}{2}\,\mathrm{Tr}\,S^2E_2(z,\tau)$$

is also present among the coefficients of the expansion. The Poisson bracket in the sine algebra basis has the form

$$\{s_{ab}, s_{cd}\} = 2i\sin\biggl(\frac{\pi}{N}(bc-ad)\biggr)s_{a+c,b+d}. \tag{4}$$

Using transition formulas (A.2) (see Appendix A), we can obtain the Poisson brackets for the components S_{ij} in the matrix form $\mathbf{S}$ from relation (4):

$$\{S_{ij}, S_{kl}\} = N(S_{kj}\delta_{il} - S_{il}\delta_{kj}). \tag{5}$$

From the components S_{ij}, $i, j \in \{1, \ldots, N\}$, we can choose coordinates in $\mathfrak{sl}(N, \mathbb{C})$, excluding any of the diagonal elements using the relation $\sum_{i=1}^{N} S_{ii} = 0$. In what follows, we call S_{ij} the coordinates in the standard basis in $\mathfrak{sl}(N, \mathbb{C})$, understanding that one of the diagonal elements is not an independent coordinate but simply a notation, for example, $S_{NN} = -\sum_{i=1}^{N-1} S_{ii}$.

Linear brackets (4) and (5) can be written in terms of the Belavin–Drinfeld elliptic r-matrix $r(z)$ [7]–[9]:

$$\{L_1^{\mathrm{rot}}(z_1), L_2^{\mathrm{rot}}(z_2)\} = [r(z_1 - z_2), L_1^{\mathrm{rot}}(z_1) + L_2^{\mathrm{rot}}(z_2)],$$

where $L_1(z) = L(z) \otimes \mathrm{Id}$ and $L_2(z) = \mathrm{Id} \otimes L(z)$. The classical r-matrix itself is defined as

$$r(z) = -\sum_{m,n} \varphi \begin{bmatrix} m \\ n \end{bmatrix} (z) T_{mn} \otimes T_{-m,-n}.$$

3. Scaling limit of elliptic top

The limit transition consists in scaling the elliptic top variables, infinitely shifting the spectral parameter $z = \tilde{z} + \tau/2$, and passing to the trigonometric limit $\operatorname{Im}\tau \to +\infty$. Different scaling variants are possible for the elliptic top coordinates in the standard basis:

$$\widetilde{S}_{ij} = S_{ij} q^{\mathfrak{g}(i,j)}, \qquad q = e^{2\pi i\tau}. \tag{6}$$

We consider the constraints imposed on $\mathfrak{g}(i,j)$ by the requirement that the Lax matrix and the structure constants of the Poisson bracket Lie algebra converge.

3.1. Convergence of Lax matrices. Using Lax matrix definition (1) and expansions (B.3) (see Appendix B), we find that the condition $s_{mn} = O(q^{-\min\{n \bmod N, (-n) \bmod N\}/2N})$ must be satisfied for the Lax matrices to converge. We use formulas (A.2) in Appendix A to pass from coordinates in the sine algebra basis to coordinates in the standard basis.

Avoiding cumbersome formulas, we can interpret the coordinates in the sine algebra basis very simply. Let E_{ij} be the matrix with unity in the ith row and jth column and all other elements zero. We fix a basis in the Cartan subalgebra $\mathfrak{sl}(N, \mathbb{C})$ as usual:

$$\mathfrak{h} = \mathcal{L}(\{h_i = E_{ii} - E_{i+1,i+1},\ 1 \le i < N\}).$$

Here and hereafter, we understand $\mathcal{L}(X)$ as the linear span of the set X. As simple roots, we choose the natural set

$$\Delta_+ = \{e_i = E_{i,i+1}, 1 \le i < N\}. \tag{7}$$

Based on formulas (A.2), we conclude that the element s_{ab} is a linear combination of coordinates corresponding to the roots $\alpha \in \Phi$, $\operatorname{ht}\alpha \equiv b \bmod N$, where Φ is the $\mathfrak{sl}(N, \mathbb{C})$ root lattice. Further, taking (A.1) into account, we see that the index a in s_{ab} corresponds to the number of the harmonic in the Fourier expansion of the indicated elements. We thus obtain the necessary condition for the existence of the limit of the Lax matrices L^{rot} and M^{rot}:

$$\mathfrak{g}(i,j) \le \frac{1}{2N} \min\{(i-j) \bmod N, (j-i) \bmod N\}, \tag{8}$$

where the constraint on $\mathfrak{g}(i,j)$ depends only on the difference $i - j$.

3.2. Convergence of structure constants. We consider the transformation of Poisson bracket (5) of Lie algebra structure constants under substitution (6),

$$S_{ij} = \widetilde{S}_{ij} q^{-\mathfrak{g}(i,j)},$$

whence we obtain the brackets of the new variables:

$$\{\widetilde{S}_{ij}, \widetilde{S}_{kl}\} = N q^{\mathfrak{g}(i,j)+\mathfrak{g}(k,i)} (\delta_{il} \widetilde{S}_{kj} q^{-\mathfrak{g}(k,j)} - \delta_{kj} \widetilde{S}_{il} q^{-\mathfrak{g}(i,l)}). \tag{9}$$

The structure constants converge as $\operatorname{Im}\tau \to +\infty$ or, equivalently, as $q \to 0$ if for all i, j, and k except $i = j = k$,

$$\mathfrak{g}(i,j) + \mathfrak{g}(k,i) - \mathfrak{g}(k,j) \geq 0. \tag{10}$$

Below, we show that under the condition for convergence of the Lax matrix, inequality (10) also holds for $i = j = k$. It is easy to see that the case of the equality in (10) corresponds to the Lie algebra structure constant $C^{(k,j)}_{(i,j),(k,i)}$ that does not vanish in the limit; all other constants tend to zero in the limit $\operatorname{Im}\tau \to +\infty$. We can immediately note that scalings preserving all structure constants of the form described above are just diagonal q-dependent gauges. We assume that the equality holds in (10). Then

1. $\mathfrak{g}(i,i) = 0$ if $N > 1$, and
2. the equalities $\mathfrak{g}(i,j) + \mathfrak{g}(j,i) = 0$ and $\mathfrak{g}(i,j) = \operatorname{sgn}(j-i) \sum_{k=i}^{j-1} \mathfrak{g}(k,k+1)$ hold.

It follows directly from the last relation that the new variables $\widetilde{S}_{ij}$ are the result of gauging S_{ij} using the matrix

$$Q = \operatorname{diag}\{q^{\sum_{k=1}^{i} \mathfrak{g}(k,k+1)},\ 1 \leq i < N\}$$

except in the trivial case $N = 1$. In the cases where not all constants are preserved, the described procedure is called an algebra contraction.

3.3. Analysis of degeneracy possibilities. Imposing both convergence conditions, we consider some properties of the obtained limit systems. For this, we introduce the notation

$$\begin{aligned} \mathcal{S}_1 &= \left\{\widetilde{S}_{ij}\colon\ \mathfrak{g}(i,j) < \frac{1}{2N} \min\{(i-j) \bmod N, (j-i) \bmod N\}\right\}, \\ \mathcal{S}_2 &= \left\{\widetilde{S}_{ij}\colon\ \mathfrak{g}(i,j) = \min\{(i-j) \bmod N, (j-i) \bmod N\}\right\}. \end{aligned} \tag{11}$$

The variables $\mathcal{S}_2$ enter the Lax matrices; hence, their limit equations of motion can also be represented in the Lax form.

From (10), we find that $\mathfrak{g}(i,i) \geq 0$ if $N > 1$. The inequality $\mathfrak{g}(i,i) \leq 0$ follows from (8). Combining the two preceding statements, we conclude that $N > 1$ necessarily implies $\mathfrak{g}(i,i) = 0$.

Statement 1. *After taking the limit $q \to 0$, we find that $\mathcal{L}(\mathcal{S}_2)$ is a subalgebra with respect to the Poisson bracket.*

This statement is sufficient for the subsystem depending on variables from the set $\mathcal{S}_2$ to be closed. But the statement can be strengthened because $\mathcal{L}(\mathcal{S}_2)$ has two subalgebras $\mathcal{L}(\mathcal{S}_2^+)$ and $\mathcal{L}(\mathcal{S}_2^-)$ such that $\mathcal{L}(\mathcal{S}_2^+ \cup \mathcal{S}_2^-) = \mathcal{L}(\mathcal{S}_2)$, where

$$\mathcal{S}_2^+ = \left\{\widetilde{S}_{ij}\colon\ \mathfrak{g}(i,j) = \frac{(j-i) \bmod N}{2N}\right\}, \qquad \mathcal{S}_2^- = \left\{\widetilde{S}_{ij}\colon\ \mathfrak{g}(i,j) = \frac{(i-j) \bmod N}{2N}\right\}.$$

Statement 2. *In the limit $q \to 0$, we find that $L(\mathcal{S}_2^{\pm})$ is a subalgebra with respect to the Poisson bracket.*

Proof. It suffices to prove the statement only for $\mathcal{L}(\mathcal{S}_2^{+})$; the argument for $\mathcal{L}(\mathcal{S}_2^{-})$ is analogous. Let $\widetilde{S}_{ij}, \widetilde{S}_{jk} \in \mathcal{S}_2^{+}$. We note that in the case where variables have no common indices, their bracket is certainly zero, and we therefore do not restrict the generality of the argument. Further, if

$$(j-i) \bmod N \le \frac{N}{2}, \qquad (k-i) \bmod N \le \frac{N}{2},$$

then $0 \le (j-i) \bmod N + (k-j) \bmod N \le N$. If

$$(j-i) \bmod N + (k-j) \bmod N = N,$$

then $(k-i) \equiv 0 \bmod N$, and consequently $k = i$ because $k, i \in \{1, \ldots, N\}$. Therefore, $\mathfrak{g}(i,j) + \mathfrak{g}(j,i) > 0 = \mathfrak{g}(i,i) = \mathfrak{g}(j,j)$, and hence $\{\widetilde{S}_{ij}, \widetilde{S}_{jk}\} = 0 \in \mathcal{L}(\mathcal{S}_2^{+})$.

If

$$(j-i) \bmod N + (k-j) \bmod N = 0,$$

then $j \equiv i \equiv k \bmod N$, and consequently $i = j = k$ because $i, j, k \in \{1, \ldots, N\}$. Therefore, $\widetilde{S}_{ij}, \widetilde{S}_{jk} \in \mathfrak{h}$, and hence $\{\widetilde{S}_{ij}, \widetilde{S}_{jk}\} = 0 \in \mathcal{L}(\mathcal{S}_2^{+})$.

If

$$0 < (j-i) \bmod N + (k-j) \bmod N < N,$$

then $(j-i) \bmod N + (k-j) \bmod N = (k-i) \bmod N$. It follows from inequality (8) that $\mathfrak{g}(i,k) \le ((k-i) \bmod N)/2N$. We consider two cases. If

$$\mathfrak{g}(i,k) < \frac{(k-i) \bmod N}{2N} = \mathfrak{g}(i,j) + \mathfrak{g}(j,k),$$

then $\{\widetilde{S}_{ij}, \widetilde{S}_{jk}\} = 0 \in \mathcal{L}(\mathcal{S}_2^{+})$, and if

$$\mathfrak{g}(i,k) = \frac{(k-i) \bmod N}{2N} = \mathfrak{g}(i,j) + \mathfrak{g}(j,k),$$

then $\{\widetilde{S}_{ij}, \widetilde{S}_{jk}\} = -N\widetilde{S}_{ik} \in \mathcal{L}(\mathcal{S}_2^{+})$. The statement is proved.

Statement 3. *The linear span $\mathcal{L}(\mathcal{S}_2^{+})/\mathfrak{h}$ commutes with $\mathcal{L}(\mathcal{S}_2^{-})/\mathfrak{h}$.*

Proof. Let $\widetilde{S}_{ij} \in \mathcal{L}(\mathcal{S}_2^{+})/\mathfrak{h}$ and $S_{jk} \in \mathcal{S}_2^{-}/\mathfrak{h}$. Then

$$\mathfrak{g}(i,k) = \frac{1}{2N} \min\{(i-k) \bmod N, (k-i) \bmod N\} <$$
$$< \frac{1}{2N} \max\{(j-i) \bmod N, (j-k) \bmod N\} < \mathfrak{g}(i,j) + \mathfrak{g}(j,k).$$

Statement 4. *The subalgebra $\mathcal{L}(\mathcal{S}_2)$ is solvable.*

Proof. We let $G^{(i)}$ denote the elements of the derived series of the subalgebra $\mathcal{L}(S_2)$. Then $G^{(0)} = \mathcal{L}(\mathcal{S}_2)$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$. It follows from Statement 3 that $G^{(1)} \subset \mathcal{L}(\mathcal{S}_2)/\mathfrak{h}$. We now show by induction that

$$G^{(i)} = \mathcal{L}\big(\{\widetilde{S}_{ij} \colon \widetilde{S}_{ij} \in \mathcal{S}_2,\ \big((j-i) \bmod N\big) \in \{2^{i-1}, \dots, N-2^{i-1}\}\}\big). \tag{12}$$

For $G^{(1)}$, the induction base is obvious. To show the induction step, we recall that $\mathcal{S}_2 = \mathcal{S}_2^+ \cup \mathcal{S}_2^-$. Taking Statement 3 into account, we find that for $i \ge 1$,

$$G^{(i)} = \big(G^{(i)} \cap \mathcal{L}(\mathcal{S}_2^+)\big) + \big(G^{(i)} \cap \mathcal{L}(\mathcal{S}_2^-)\big)$$

(this sum is always direct for $i \ge 2$). It therefore suffices to show that the statement is true for $\mathcal{L}(\mathcal{S}_2^\pm)$. We present the argument only for $\mathcal{S}_2^+$. We set

$$G_+^{(i)} \equiv \mathcal{L}\big(\{\widetilde{S}_{ij} \colon\ \widetilde{S}_{ij} \in \mathcal{S}_2, \big((j-i) \bmod N\big) \in \{2^{i-1}, \dots, \lfloor N/2 \rfloor\}\}\big). \tag{13}$$

Here and hereafter, $\lfloor x \rfloor$ denotes the integer part of x. It follows from (9) that all nonzero structure constants of the Poisson bracket Lie algebra that survive the limit $\operatorname{Im}\tau \to +\infty$ must be nonzero before passing to the limit. Taking into account that $\mathcal{L}(\mathcal{S}_2^+)$ is a subalgebra, we obtain

$$[G_+^{(i)}, G_+^{(i)}] \subset G_+^{(i+1)},$$

where $G_+^{(i+1)}$ is defined by formula (13). The argument for $\mathcal{S}_2^-$ is analogous. Taking the relation $G^{(i)} = G_+^{(i)} + G_-^{(i)}$ into account, we obtain the sought statement (12).

4. Examples of degeneration

We now present some examples of degeneration of the elliptic $SL(N,\mathbb{C})$ top. For this, we consider scalings of variables depending only on the root height in our chosen basis (7):

$$\widetilde{S}_{ij} = q^{\mathfrak{g}(j-i)} S_{ij}. \tag{14}$$

This approach was proposed in [3]. Here, we show the main results as an illustration of Statements 1–4. Passing to the sine algebra basis does not mix roots of different heights; relation (14) therefore allows writing the scalings in the sine algebra basis

$$s_{mn} = \tilde{s}_{mn} q^{-g(n)}, \quad m,n \in \{0,\dots,N-1\}, \quad m^2+n^2 \ne 0.$$

We consider the following possibilities for $g(n)$, parameterized by an integer p, $1 \le p < N/2$:

$$g(i) = \begin{cases} \dfrac{k}{2N}, & 0 \le k \le p < N/2, \\[2mm] \dfrac{p}{2N}, & p < k < N-p, \\[2mm] \dfrac{N-k}{2N}, & N-p \le k < N, \end{cases} \tag{15}$$

where $k \equiv i \bmod N$, $0 \le k \le N$. Inequality (10) then becomes $g(k) + g(n) - g(k+n) \ge 0$, whose validity for any k and n can be easily verified directly.

4.1. Limit of the Lax matrix. The function $\mathfrak{g}(i,j) = g(j-i)$ given in (15) satisfies inequality (8) for all p. In the limit $\operatorname{Im}\tau \to +\infty$, the elements of the Lax matrix L^{rot} corresponding to the coordinates

$$\mathcal{S}_2 = \bigl\{\widetilde{S}_{ij} \colon \bigl((j-i) \bmod N\bigr) \equiv \{p+1,\dots,N-p-1\}\bigr\}$$

vanish if the indicated set is nonempty. In contrast, the elements corresponding to the coordinates

$$\mathcal{S}_1 = \bigl\{\widetilde{S}_{ij} \colon \bigl((j-i) \bmod N\bigr) \equiv \{0,\dots,p,N-p,\dots,N-1\}\bigr\},$$

are preserved, and the matrix $\widetilde{L}^{\mathrm{rot}} = \lim_{q\to 0} L^{\mathrm{rot}}$ therefore has $2p{+}1$ diagonals.

4.2. Poisson bracket algebra of the limit system. In passing to the limit $\operatorname{Im}\tau \to +\infty$, only the following brackets remain nonzero:

$$\{\widetilde{S}_{ii}, \widetilde{S}_{jk}\} = N(\widetilde{S}_{ji}\delta_{ik} - \widetilde{S}_{ik}\delta_{ij}), \tag{16a}$$

$$\{\widetilde{S}_{ij}, \widetilde{S}_{kl}\} = N(\widetilde{S}_{kj}\delta_{il} - \widetilde{S}_{il}\delta_{kj}) \quad \text{for } 0 < (j-i) \bmod N \le p, \quad 0 < (l-k) \bmod N \le p, \quad 0 < (j+l-i-k) \bmod N \le p, \tag{16b}$$

$$\{\widetilde{S}_{ij}, \widetilde{S}_{kl}\} = N(\widetilde{S}_{kj}\delta_{il} - \widetilde{S}_{il}\delta_{kj}) \quad \text{for } N-p \le (j-i) \bmod N < N, \quad N-p \le (l-k) \bmod N < N, \quad N-p \le (j+l-i-k) \bmod N < N. \tag{16c}$$

We can use notation (11) to clarify the meaning of these rather cumbersome expressions. Expression (16a) corresponds to the preserved brackets of elements of $\mathfrak{h}$ and $\mathcal{L}(\mathcal{S}_1 \cup \mathcal{S}_2)$ in the form that they had before the limit transition. It follows from (16b) that $\widetilde{S}_{ij} \in \mathcal{S}_2^+$ and $\widetilde{S}_{kl} \in \mathcal{S}_2^+$; if $i = l$, then $\widetilde{S}_{kj} \in \mathcal{S}_2^+$, and if $k = j$, then $\widetilde{S}_{il} \in \mathcal{S}_2^+$. The same statements, up to replacing $\mathcal{S}_2^+$ with $\mathcal{S}_2^-$, hold for (16c). We can thus directly see that $\mathcal{L}(\mathcal{S}_2^\pm)$ are subalgebras of the Poisson bracket Lie algebra of the limit system. Indeed, an even stronger statement following from an explicit form of (16) holds for such a scaling.

Statement 5. *Subalgebras $\mathcal{L}(\mathcal{S}_2^+)$ and $\mathcal{L}(\mathcal{S}_2^-)$ are ideals of Poisson-bracket Lie algebra* (16).

We also note that the brackets in question can be written using the r-matrix that results from passing to the limit from the elliptic r-matrix in terms of variables $\widetilde{S}_{ij}$:

$$\{\widetilde{L}_1^{\mathrm{rot}}(\tilde z_1), \widetilde{L}_2^{\mathrm{rot}}(\tilde z_2)\} = [\tilde r(\tilde z_1 - \tilde z_2), \widetilde{L}_1^{\mathrm{rot}}(\tilde z_1) + \widetilde{L}_1^{\mathrm{rot}}(\tilde z_1)].$$

4.3. Integrability. Under certain restrictions on p, we can directly prove the integrability of the systems obtained in the limit, indicating a sufficient number of independent integrals in involution. Here, we formulate the main statements without cumbersome proofs, which can be found in [3]. To construct a family of Hamiltonians in involution, we note that before the limit transition, the Lax matrix $\widetilde{L}^{\mathrm{rot}}(\tilde z)$ as a function of the spectral parameter has two quasiperiods, 1 and τ (see formulas (2)); we consequently seek the Hamiltonians as coefficients of expansion (3) in doubly periodic functions on the torus. After passing to the limit $\operatorname{Im}\tau \to +\infty$, only one quasiperiod, equal to 1, remains, and the complex torus transforms into the complex cylinder. It is therefore natural to consider expansion in the Fourier basis $\{\mathrm{e}(j\tilde z) \equiv w^j,\ j \in \mathbb{Z}\}$.

Statement 6. *In the Fourier expansion of the trace of the kth power of the Lax matrix, only a finite number of terms are nonzero, namely,*

$$\operatorname{Tr}\bigl(\widetilde{L}^{\mathrm{rot}}(\tilde z)\bigr)^k = \sum_{j=-M}^{M} H_{kj} w^j, \qquad M = \left\lfloor \frac{kp}{N} \right\rfloor, \quad w \equiv \mathrm{e}(\tilde z). \tag{17}$$

Statement 7. *The coefficients H_{kj} are in involution:*

$$\{H_{k_1 j_1}, H_{k_2 j_2}\} = 0. \tag{18}$$

Statement 8. *If N and p are coprime, then the coefficients H_{kj}, where $k > 0$, $\lfloor jN/p \rfloor \le k \le N$, and $|j| < p$, together with the Casimir functions $\widetilde{S}_{i,i+p}\widetilde{S}_{i+p,i}$, $1 \le i \le N$, and H_{Np} are independent at a generic point.*

Combining all the above statements, we have the following conclusion.

Statement 9. *If N and p are coprime, then the systems described above are integrable in the Liouville sense at a generic point.*

5. Conclusion

We have shown that in the passage to a scaling limit of the elliptic $SL(N,\mathbb{C})$ top, systems arise that have a closed subsystem related to a solvable subalgebra (Statement 4). Moreover, the equations of motion of these systems can be represented in the Lax form. In all the presented examples, the variables not in the Lax matrix can be expressed algebraically on a given symplectic leaf in terms of variables in the Lax matrix. The dimension of the whole limit system phase space therefore coincides with the dimension of the subsystem. Unfortunately, without imposing additional conditions on the scaling of variables, we cannot say the same in the general case. As a counterexample, we can even consider a trivial scaling $\widetilde{S}_{ij} = S_{ij}$ for $N \ge 3$. We show that the dimension of the Lie algebra $\mathcal{L}(\mathcal{S}_1 \cup \mathcal{S}_2)$ symplectic leaf at a generic point does not coincide with the dimension of the Lie subalgebra $\mathcal{L}(S_2)$ symplectic leaf. In the considered case of a trivial scaling, we have

$$\mathcal{L}(S_2) = \mathfrak{h}, \qquad \mathcal{L}(S_1) = \mathcal{L}\big(\{\widetilde{S}_{ij} \colon\ i \ne j\}\big).$$

Moreover, the structure constants of the Lie algebra $\mathcal{L}(\mathcal{S}_1 \cup \mathcal{S}_2)$ with respect to the Poisson bracket do not change in the limit $q \to 0$, and the dimension of the Lie algebra $\mathcal{L}(\mathcal{S}_1 \cup \mathcal{S}_2)$ symplectic leaf is consequently also preserved and differs from zero at a generic point. On the other hand, the subalgebra $\mathcal{L}(\mathcal{S}_2)$ is Abelian in this case, and the dimension of the Lie algebra $\mathcal{L}(\mathcal{S}_2)$ symplectic leaf is therefore identically zero.

The technique for passing to the limit described in Sec. 3 with slight modifications can be applied to the degeneration of the Calogero–Moser system with spin for analyzing the possible degenerations of spin variables. Undoubtedly, by passing to the Inozemtsev limit in the Calogero–Moser system with spin [10]–[13], we can obtain a wider class of systems than by degeneration of the Calogero–Moser system without spin. This idea, together with an example of the elliptic Calogero–Moser $SL(3,\mathbb{C})$ system degeneration, was first proposed in [14]. For comparison, we note that the possible degenerations of the Calogero–Moser system without spin were analyzed in [15] for an arbitrary root system; only the elements corresponding to simple roots and to the maximal root can be preserved in the limit.

Among the systems described in Sec. 4, the system corresponding to the case $p = 1$ is a Toda chain. In this case, the coordinates corresponding to the basis elements of the Cartan subalgebra of the initial Poisson bracket Lie algebra $\mathfrak{sl}(N,\mathbb{C})$ and also corresponding to the roots with the height ± 1 and $\pm(N-1)$ remain in $\mathcal{S}_1$. Because in the case of a spinless Calogero–Moser system, we can separately scale only the coupling constants for long and short roots (and in the case of $\mathfrak{sl}(N,\mathbb{C})$, there is only one constant), it does not seem possible to preserve the terms with interaction corresponding to roots other than the basis ones and the maximal one. The situation is different in the Calogero–Moser system with spin. Using different scalings for different scaling variables, we can also keep terms with interactions corresponding to roots with heights different from ± 1.

The Lax matrix for the Calogero–Moser system with spin in the elliptic case has the form [16]–[18]

$$L^{\mathrm{SCM}}(z) = p + \zeta(z)\sum_{i=1}^{N}\xi_i h_i + \sum_{\alpha\in\Phi}\phi(\alpha\cdot u, z)\xi_\alpha e_\alpha,$$

where Φ is the root lattice and the function $\phi(u,z)$ is defined in (B.1). In such a case, the Lax matrix convergence condition follows not from expansions (B.3) but from expansion of the function $\phi(u,z)$.

Appendix A: Sine algebra

To simplify formulas, we introduce the notation

$$\tilde{\delta}(n) = \begin{cases} 1, & n \equiv 0 \bmod N, \\ 0, & n \not\equiv 0 \bmod N, \end{cases} \qquad \mathrm{e}(z) = e^{2\pi i z}.$$

The elements T_{mn} of the sine algebra basis in $\mathfrak{sl}(N,\mathbb{C})$ are defined as

$$(T_{mn})_{ij} = \mathrm{e}\biggl(\frac{mn}{2N}\biggr)\,\mathrm{e}\biggl(\frac{im}{N}\biggr)\tilde{\delta}(j-i-n), \quad m,n\in\{0,\dots,N-1\}, \quad m^2+n^2\neq 0. \tag{A.1}$$

For elements with indices $m,n\in\mathbb{Z}$ ($m\neq 0 \bmod N$ or $n\neq 0 \bmod N$), we can introduce the quasiperiodicity condition

$$T_{mn} = \mathrm{e}\biggl(\frac{mn-(m \bmod N)(n \bmod N)}{2N}\biggr)T_{m \bmod N,\, n \bmod N},$$

$$s_{mn} = \mathrm{e}\biggl(\frac{(m \bmod N)(n \bmod N)-mn}{2N}\biggr)s_{m \bmod N,\, n \bmod N},$$

where

$$\mathrm{e}\biggl(\frac{mn-(m \bmod N)(n \bmod N)}{2N}\biggr) = \pm 1.$$

The commutation relations in the sine algebra basis then become

$$[T_{mn}, T_{kl}] = 2i\sin\biggl(\frac{\pi}{N}(kn-ml)\biggr)T_{m+k,\,n+l}.$$

The coordinates $\{s_{mn}\}$ in the sine algebra basis are related to the coordinates $\{S_{ij}\}$ in the expansion in the standard basis in $SL(N,\mathbb{C})$ as

$$S_{ij} = \sum_{m,n}s_{mn}(T_{mn})_{ij}, \qquad s_{mn} = \frac{1}{N}\sum_{i,j}S_{ij}(T_{-m,n})_{ij}. \tag{A.2}$$

Appendix B: Expansion of elliptic functions

Our definitions and notation are mainly borrowed from [5] and [19]. The basic object in our consideration is the Riemann theta function

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z,\tau) = \sum_{j\in\mathbb{Z}} q^{(j+a)^2/2}\,\mathrm{e}((j+a)(z+b)).$$

To write the Lax matrices, we need the functions

$$\vartheta(z) = \theta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z, \tau), \qquad \phi(u, z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}, \tag{B.1}$$

and also

$$\begin{aligned} \varphi \begin{bmatrix} m \\ n \end{bmatrix} (z) &= \mathrm{e}\Big(-\frac{nz}{N}\Big)\phi\Big(-\frac{m+n\tau}{N}, z\Big), \\ f \begin{bmatrix} m \\ n \end{bmatrix} (z) &= \mathrm{e}\Big(-\frac{nz}{N}\Big)\partial_u \phi(u, z)\big|_{u-(m+n\tau)/N}. \end{aligned} \tag{B.2}$$

We consider the expansion of functions (B.2) as $z = \tilde{z} + \tau/2$, $\operatorname{Im}\tau \to +\infty$. The following expansions were derived in detail in [3]:

$$\begin{aligned} \varphi \begin{bmatrix} m \\ n \end{bmatrix} \Big(\tilde{z} + \frac{\tau}{2}\Big) &= \begin{cases} -\dfrac{\pi\,\mathrm{e}(m/2N)}{\sin(\pi m/N)} + o(1), & n = 0, \\ 2\pi i q^{n/2N}\,\mathrm{e}\Big(\dfrac{m}{N} - \dfrac{n\tilde{z}}{N}\Big) + o(q^{n/2N}), & 0 < n < \dfrac{N}{2}, \\ 4\pi q^{1/4}\,\mathrm{e}\Big(\dfrac{m}{2N}\Big)\sin\Big(\pi\Big(\tilde{z} - \dfrac{m}{N}\Big)\Big) + o(q^{1/4}), & n = \dfrac{N}{2}, \\ -2\pi i q^{(N-n)/2N}\,\mathrm{e}\Big(\dfrac{N-n}{N}\tilde{z}\Big) + o(q^{(N-n)/2N}), & \dfrac{N}{2} < n < N, \end{cases} \\ f \begin{bmatrix} m \\ n \end{bmatrix} \Big(\tilde{z} + \frac{\tau}{2}\Big) &= \begin{cases} -\dfrac{\pi^2}{\sin^2(\pi m/N)} + o(1), & n = 0, \\ 4\pi^2\,\mathrm{e}\Big(\dfrac{m}{N}\Big)\,\mathrm{e}\Big(-\dfrac{n\tilde{z}}{N}\Big)q^{n/2N} + o(q^{n/2N}), & 0 < n < \dfrac{3N}{4}, \\ 4\pi^2\Big[\mathrm{e}\Big(\dfrac{m}{N}\Big) - \mathrm{e}\Big(-\dfrac{n}{N} + \tilde{z}\Big)\Big]\,\mathrm{e}\Big(-\dfrac{3\tilde{z}}{4}\Big)q^{3/8} + o(q^{3/8}), & n = \dfrac{3N}{4}, \\ -4\pi^2\,\mathrm{e}\Big(-\dfrac{m}{N} + \tilde{z}\Big)\,\mathrm{e}\Big(-\dfrac{n\tilde{z}}{N}\Big)q^{3(1-n/N)/2} + o(q^{3(1-n/N)/2}), & \dfrac{3N}{4} < n < N. \end{cases} \end{aligned} \tag{B.3}$$

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