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LIMIT RELATION BETWEEN TODA CHAINS AND THE ELLIPTIC $SL(N,\mathbb{C})$ TOP

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We study a limit relation between the elliptic $SL(N,\mathbb{C})$ top and Toda chains. We show that in the case of the nonautonomous $SL(2,\mathbb{C})$ top, whose equations of motion are related to the Painlevé VI equation, it turns out to be possible to modify the previously proposed procedure and in the limit obtain the nonautonomous Toda chain, whose equations of motion are equivalent to a particular case of the Painlevé III equation. We obtain the limit of the Lax pair for the elliptic $SL(2,\mathbb{C})$ top, which allows representing the equations of motion of the nonautonomous Toda chain as the equation for isomonodromic deformations.

Keywords: integrable system, nonautonomous system, Painlevé equation, Inozemtsev limit, topological structure of phase space

1. Introduction

This paper is a continuation of [1], where the case of the autonomous $SL(N,\mathbb{C})$ top was considered. Here, we consider four integrable systems whose equations of motion have a Lax representation with the spectral parameter [2]–[4]: the periodic and nonperiodic Toda chains, the elliptic Calogero–Moser model, and the elliptic $SL(N,\mathbb{C})$ top. These systems are related to each other, which was previously established in [5]–[8]. Inozemtsev proposed a procedure giving a limit relation between the Toda chains and the elliptic Calogero–Moser system [5]. Levin, Olshanetsky, and Zotov constructed a symplectic transformation from the Calogero–Moser system to the elliptic $SL(N,\mathbb{C})$ top [6].

To obtain the limit systems of the elliptic top, which are equivalent to Toda chains, we use a procedure similar to the Inozemtsev limit. The Inozemtsev limit is a combination of the trigonometric limit, infinite shifts of particle coordinates, and a rescaling of the coupling constant. To obtain the limit systems equivalent to the Toda chains, we must combine the Inozemtsev limit and an infinite shift of the spectral parameter. Because the spectral parameter of the elliptic $SL(N,\mathbb{C})$ top is defined on a complex torus T^2 with a modulus τ, under the trigonometric limit $\operatorname{Im}\tau \to +\infty$, we obtain systems with the spectral parameter on an infinite complex cylinder $\mathbb{C}/\mathbb{Z}$.

In the case of the elliptic $SL(2,\mathbb{C})$ top, it is convenient to use an explicit form of a symplectic map from the phase space of the elliptic Calogero–Moser system to the phase space of the top (see Eq. (7) below). The coordinate shifts of the elliptic Calogero–Moser system used in the Inozemtsev limit then induce a rescaling of the elliptic $SL(2,\mathbb{C})$ top coordinates (see Sec. 2).

In the case of the elliptic $SL(N{>}2,\mathbb{C})$ top, it is much more complicated to derive the explicit form of the symplectic map between the phase spaces of the elliptic Calogero–Moser system and elliptic top. We therefore rescale the coordinates in accordance with the limit behavior of the Lax matrix, thus generalizing the method developed for $N=2$ to the case $N>2$ (see Sec. 3).

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The limit procedure described above can be further generalized to the nonautonomous case. In Sec. 4, we consider the system of the nonautonomous elliptic $SL(2,\mathbb{C})$ top, where the role of time is played by the elliptic curve parameter τ [9], [10]. The equations of motion of the nonautonomous $SL(2,\mathbb{C})$ top can be written as an isomonodromic deformation equation [11]–[13]:

$$\partial_\tau L^{\mathrm{rot}} - \frac{1}{2\pi \mathrm{i}}\,\partial_z M^{\mathrm{rot}} = 2[L^{\mathrm{rot}}, M^{\mathrm{rot}}],$$

where L^{rot} and M^{rot} are Lax matrices of the elliptic $SL(2,\mathbb{C})$ top. A generalization of the limit procedure is to represent the elliptic curve parameter as a sum of two terms $\tau = \tau_1 + \tau_2$. The imaginary part of the second term τ_2 tends to infinity in the limit, while the first term plays the role of time in the limit system. Thus, the nonautonomous Toda chain and the representation of its equations of motion as an equation for isomonodromic deformations are obtained simultaneously.

Because the equations of motion of the nonautonomous elliptic $SL(2,\mathbb{C})$ top are equivalent to a particular case of the Painlevé VI equation [14], [15], the equations of motion of the obtained nonautonomous Toda chain are equivalent to a particular case of the Painlevé III equation [16], [17].

We review the necessary information concerning the integrable systems that we consider here.

1.1. Elliptic $SL(N,\mathbb{C})$ top. The elliptic $SL(N,\mathbb{C})$ top is an example of the Euler–Arnold top [18], whose phase space is defined by a coadjoint action orbit of the group $SL(N,\mathbb{C})$:

$$\mathcal{R}^{\mathrm{rot}} = \{\mathbf{S} \in \mathfrak{sl}(N,\mathbb{C}),\ \mathbf{S} = g^{-1}\mathbf{S}^{(0)}g\},$$

where $g \in SL(N,\mathbb{C})$ is defined modulo left shifts G_0 commuting with $\mathbf{S}^{(0)}$. There is the nondegenerate Kirillov–Kostant symplectic form

$$\omega^{\mathrm{rot}} = \mathrm{Tr}(\mathbf{S}^{(0)} dgg^{-1} \wedge dgg^{-1})$$

on the phase space of the top $\mathcal{R}^{\mathrm{rot}}$.

The dynamics of the system is governed by the Hamiltonian

$$H^{\mathrm{rot}} = -\frac{1}{2}\,\mathrm{Tr}\,\mathbf{S}J(\mathbf{S}), \tag{1}$$

where

$$J(\mathbf{S}) = \sum_{mn} J_{mn} s_{mn} T_{mn}, \qquad J_{mn} = E_2\biggl(\frac{m+n\tau}{N},\tau\biggr),$$

$$m,n \in \{0,\dots,N-1\}, \quad m^2+n^2 \neq 0,$$

$E_2(z,\tau)$ is the second Eisenstein function [19] defined on the complex torus $T^2\colon \mathbb{C}/(2\omega_1\mathbb{Z}+2\omega_2\mathbb{Z})$ with $\omega_1 = 1/2$, $\tau = \omega_2/\omega_1$, while s_{mn} are the coordinates in a special basis $\{T_{mn}\}$ in the algebra $\mathfrak{sl}(N,\mathbb{C})$, which forms the sine algebra (see Appendix A).

The equations of motion can be written in the Lax form [20]

$$\frac{dL^{\mathrm{rot}}}{dt} = N[L^{\mathrm{rot}}, M^{\mathrm{rot}}]. \tag{2}$$

The factor N in Eq. (2) is related to the definition of the Lax matrices in the sine algebra basis (see Appendix A):

$$L^{\mathrm{rot}} = \sum_{m,n} s_{mn}\varphi\begin{bmatrix} m\\ n\end{bmatrix}(z)T_{mn}, \qquad \varphi\begin{bmatrix} m\\ n\end{bmatrix}(z) = \mathrm{e}\biggl(-\frac{nz}{N}\biggr)\phi\biggl(-\frac{m+n\tau}{N},z\biggr),$$

$$M^{\mathrm{rot}} = \sum_{m,n} s_{mn} f\begin{bmatrix} m\\ n\end{bmatrix}(z)T_{mn}, \qquad f\begin{bmatrix} m\\ n\end{bmatrix}(z) = \mathrm{e}\biggl(-\frac{nz}{N}\biggr)\partial_u\phi(u,z)|_{u=-(m+n\tau)/N},$$

where $\mathrm{e}(z) \equiv e^{2\pi \mathrm{i} z}$, $\mathrm{i} \equiv \sqrt{-1}$, and ϕ is a combination of theta functions (see Appendix B). Hamiltonian (1) is related to the Lax matrix as

$$H^{\mathrm{rot}} = \frac{1}{2}\operatorname{Tr}(L^{\mathrm{rot}})^2 - \frac{1}{2}\operatorname{Tr} S^2 E_2(z,\tau). \tag{3}$$

The Poisson brackets for the variables s_{mn} follow from the commutator $[T_{ab}, T_{cd}]$ (see (A.1) in Appendix A) of the basis elements T_{ab} and T_{cd}:

$$\{s_{ab}, s_{cd}\} = 2\mathrm{i}\sin\left[\frac{\pi}{N}(bc-ad)\right] s_{a+c,b+d}. \tag{4}$$

Passing to standard basis (A.2) yields

$$\{S_{ij}, S_{kl}\} = N(S_{kj}\delta_{il} - S_{il}\delta_{kj}). \tag{5}$$

1.2. The elliptic Calogero–Moser model. The elliptic Calogero–Moser system was first described in the quantum case [21], [22]. The phase space represents the space of momenta and coordinates in the center-of-mass frame,

$$\mathcal{R}^{\mathrm{CM}} = \left\{(\mathbf{u},\mathbf{v}),\ \sum_{i=1}^{N} u_i = 0,\ \sum_{i=1}^{N} v_i = 0\right\},$$

with the canonical symplectic form $\omega^{\mathrm{CM}} = (d\mathbf{v} \wedge d\mathbf{u})$. The Hamiltonian is quadratic in the momenta $\mathbf{v}$,

$$H^{\mathrm{CM}} = \sum_{i=1}^{N} \frac{v_i^2}{2} + m^2 \sum_{i>j} E_2(u_i - u_j, \tau).$$

The equations of motion produced by this Hamiltonian can be represented in the Lax form:

$$\frac{dL^{\mathrm{CM}}}{dt} = [L^{\mathrm{CM}}, M^{\mathrm{CM}}],$$

where

$$L_{ij}^{\mathrm{CM}} = \delta_{ij} v_i + m(1-\delta_{ij})\phi(u_i - u_j, z),$$

$$M_{ij}^{\mathrm{CM}} = -\delta_{ij}\sum_{k\neq j} E_2(u_j - u_k) + \left.\frac{\partial \phi(u,z)}{\partial u}\right|_{u=u_i-u_j}.$$

1.3. Toda chains. The periodic and nonperiodic Toda chains are systems with N interacting particles on the line. The phase space is a space of momenta and coordinates of particles in the center-of-mass frame,

$$\mathcal{R}^{\mathrm{T}} = \left\{(\mathbf{u},\mathbf{v}),\ \sum_{i=1}^{N} u_i = 0,\ \sum_{i=1}^{N} v_i = 0\right\},$$

with the canonical symplectic form $\omega^{\mathrm{T}} = (d\mathbf{v} \wedge d\mathbf{u})$. The Hamiltonian of the nonperiodic system is

$$H^{\mathrm{aT}} = \frac{1}{2}\sum_{i=1}^{N} v_i^2 + 4\pi^2 M^2 \sum_{i=1}^{N-1} \mathrm{e}(u_{i+1} - u_i),$$

and we have

$$H^{\mathrm{pT}} = \frac{1}{2}\sum_{i=1}^{N} v_i^2 + 4\pi^2 M^2 \sum_{i=1}^{N} \mathrm{e}(u_{i+1} - u_i), \qquad u_{N+1} = u_1,$$

in the periodic case.

The equations of motion of the periodic and the nonperiodic Toda chains are equivalent to the corresponding Lax equations [23]–[25]

$$\frac{d}{dt}L^{\mathrm{aT}} = [L^{\mathrm{aT}}, M^{\mathrm{aT}}], \qquad \frac{d}{dt}L^{\mathrm{pT}} = [L^{\mathrm{pT}}, M^{\mathrm{pT}}].$$

2. Elliptic $SL(2,\mathbb{C})$ top and Toda chains

In [6], Levin, Olshanetsky, and Zotov established the relation between the Calogero–Moser system and the elliptic $SL(N,\mathbb{C})$ top in the form of the singular gauge transformation

$$L^{\mathrm{rot}}(z) = \Xi(z) L^{\mathrm{CM}}(z) \Xi^{-1}(z).$$

This transformation leads to the symplectic map between the phase spaces

$$\mathcal{R}^{\mathrm{CM}} \to \mathcal{R}^{\mathrm{rot}}, \qquad (\mathbf{u},\mathbf{v}) \mapsto \mathbf{S}, \tag{6}$$

which for $N=2$ takes the form

$$\begin{aligned} s_{01} &= -v\frac{\theta_{01}(0)\theta_{01}(2u)}{\vartheta'(0)\vartheta(2u)} - m\frac{\theta_{01}^2(0)}{\theta_{00}(0)\theta_{10}(0)}\frac{\theta_{00}(2u)\theta_{10}(2u)}{\vartheta^2(2u)}, \\ s_{10} &= v\frac{\theta_{10}(0)\theta_{10}(2u)}{\vartheta'(0)\vartheta(2u)} + m\frac{\theta_{10}^2(0)}{\theta_{00}(0)\theta_{01}(0)}\frac{\theta_{00}(2u)\theta_{01}(2u)}{\vartheta^2(2u)}, \\ s_{11} &= -\mathrm{i}v\frac{\theta_{00}(0)\theta_{00}(2u)}{\vartheta'(0)\vartheta(2u)} - \mathrm{i}m\frac{\theta_{00}^2(0)}{\theta_{10}(0)\theta_{01}(0)}\frac{\theta_{10}(2u)\theta_{01}(2u)}{\vartheta^2(2u)}. \end{aligned} \tag{7}$$

The basic idea of the procedure used is to regard the coordinates of the elliptic $SL(2,\mathbb{C})$ top as functions of the coordinates of the Calogero–Moser system $(\mathbf{u},\mathbf{v})$ and then shift the coordinates $(\mathbf{u},\mathbf{v})$ as in passing to the Inozemtsev limit.

2.1. The limit system equivalent to a periodic Toda chain. To obtain a periodic Toda chain in the limit, we combine the coordinate shift $u = U + \tau/4$, the coupling-constant scaling $M = mq^{1/4}$ ($q \equiv \mathrm{e}(\tau)$), the spectral parameter shift $z = \tilde{z} + \tau/2$, and the trigonometric limit $q \to 0$. From symplectic map (7) between the phase spaces, we then obtain an expansion of the functions $s_{ij}(u,v)$ in powers of q. We take the renormalized coordinates of the $SL(2,\mathbb{C})$ top as the coordinates of the limit top:

$$\begin{aligned} \tilde{s}_{10} &= \lim_{q\to 0} s_{10} = -\frac{\mathrm{i}v}{\pi}, \\ \tilde{s}_{01} &= \lim_{q\to 0} s_{01} q^{1/4} = M\cos(2\pi U), \\ \tilde{s}_{11} &= \lim_{q\to 0} s_{11} q^{1/4} = -M\sin(2\pi U). \end{aligned} \tag{8}$$

Renormalized coordinates (8) form the Lie algebra obtained by contracting the algebra $\mathfrak{sl}(2,\mathbb{C})$:

$$\{\tilde{s}_{10},\tilde{s}_{11}\} = 2\mathrm{i}\tilde{s}_{01}, \qquad \{\tilde{s}_{11},\tilde{s}_{01}\} = 0, \qquad \{\tilde{s}_{01},\tilde{s}_{10}\} = 2\mathrm{i}\tilde{s}_{11}. \tag{9}$$

A symplectic leaf is defined by the condition

$$\tilde{s}_{01}^2 + \tilde{s}_{11}^2 = \mathrm{const} = M^2,$$

which is a limit of the Casimir function of the elliptic $SL(2,\mathbb{C})$ top,

$$s_{01}^2 + s_{10}^2 + s_{11}^2 = \mathrm{const} = m^2.$$

Formulas (8) define a symplectic map of the canonical coordinates (U, v) to the coordinates on the symplectic leaf of the limit top and are hereafter called the bosonization formulas.

Taking the behavior of the functions $\varphi\big[\begin{smallmatrix} m \\ n \end{smallmatrix}\big](z)$ and $f\big[\begin{smallmatrix} m \\ n \end{smallmatrix}\big](z)$ (see (A.10) and (A.11) in Appendix B) in the considered limit into account, we obtain the Lax pair and the Hamiltonian of the limit top [1]:

$$\tilde{L}^{\mathrm{rot}} = \lim_{q\to 0} L^{\mathrm{rot}} = 4\pi \begin{pmatrix} \dfrac{\mathrm{i}}{4}\tilde{s}_{10} & \tilde{s}_{01}\sin(\pi\tilde{z}) - \tilde{s}_{11}\cos(\pi\tilde{z}) \\ \tilde{s}_{01}\sin(\pi\tilde{z}) + \tilde{s}_{11}\cos(\pi\tilde{z}) & -\dfrac{\mathrm{i}}{4}\tilde{s}_{10} \end{pmatrix},$$

$$\widetilde{M}^{\mathrm{rot}} = \lim_{q\to 0} M^{\mathrm{rot}} = \pi^2 \begin{pmatrix} \tilde{s}_{10} & 4(\tilde{s}_{01} + \mathrm{i}\tilde{s}_{11})\,\mathrm{e}\biggl(-\dfrac{\tilde{z}}{2}\biggr) \\ 4(\tilde{s}_{01} - \mathrm{i}\tilde{s}_{11})\,\mathrm{e}\biggl(-\dfrac{\tilde{z}}{2}\biggr) & -\tilde{s}_{10} \end{pmatrix},$$

$$\widetilde{H}^{\mathrm{rot}} = \lim_{q\to 0} H^{\mathrm{rot}} = -\pi^2\tilde{s}_{10}^2 + 8\pi^2\tilde{s}_{01}^2 - 8\pi^2\tilde{s}_{11}^2.$$

By direct verification, we can confirm that the equations of motion preserve the Lax form in the limit:

$$\frac{d\tilde{L}^{\mathrm{rot}}}{dt} = \{\widetilde{H}^{\mathrm{rot}}, \tilde{L}^{\mathrm{rot}}\} = 2[\tilde{L}^{\mathrm{rot}}, \widetilde{M}^{\mathrm{rot}}].$$

Using bosonization formulas (8), we can obtain a periodic Toda chain [1].

2.2. The limit system equivalent to a nonperiodic Toda chain. In the case of a nonperiodic Toda chain, we use the parameters of the coordinate shift $u = U + \tau/8$ and of the coupling constant renormalization $M = mq^{1/8}$. The shift of the spectral parameter remains the same: $z = \tilde{z} + \tau/2$. The trigonometric limit $q \to 0$ yields the coordinates of the limit top:

$$\begin{aligned} \tilde{s}_{10} &= \lim_{q\to 0} s_{10} = -\frac{\mathrm{i}v}{\pi}, \\ \tilde{s}_{01} &= \lim_{q\to 0} s_{01}q^{1/4} = \frac{1}{2}M\,\mathrm{e}(U), \\ \tilde{s}_{11} &= \lim_{q\to 0} s_{11}q^{1/4} = \frac{\mathrm{i}}{2}M\,\mathrm{e}(U). \end{aligned} \tag{10}$$

Renormalized coordinates (10) define the same algebra (9) as in the periodic case, but the symplectic leaf is defined by a different expression:

$$\tilde{s}_{01}^2 + \tilde{s}_{11}^2 = 0.$$

The Lax pair and the Hamiltonian of the limit top take the form [1]

$$\tilde{L}^{\mathrm{rot}} = 4\pi \begin{pmatrix} \dfrac{\mathrm{i}}{4}\tilde{s}_{10} & \tilde{s}_{01}\sin(\pi\tilde{z}) - \tilde{s}_{11}\cos(\pi\tilde{z}) \\ \tilde{s}_{01}\sin(\pi\tilde{z}) + \tilde{s}_{11}\cos(\pi\tilde{z}) & -\dfrac{\mathrm{i}}{4}\tilde{s}_{10} \end{pmatrix},$$

$$\widetilde{M}^{\mathrm{rot}} = \pi^2 \begin{pmatrix} \tilde{s}_{10} & 4(\tilde{s}_{01} + \mathrm{i}\tilde{s}_{11})\,\mathrm{e}\biggl(-\dfrac{\tilde{z}}{2}\biggr) \\ 4(\tilde{s}_{01} - \mathrm{i}\tilde{s}_{11})\,\mathrm{e}\biggl(-\dfrac{\tilde{z}}{2}\biggr) & -\tilde{s}_{10} \end{pmatrix},$$

$$\widetilde{H}^{\mathrm{rot}} = -\pi^2\tilde{s}_{10}^2 + 8\pi^2\tilde{s}_{01}^2 - 8\pi^2\tilde{s}_{11}^2.$$

Bosonization formulas (10) transform the limit top into a nonperiodic Toda chain [1].

3. Elliptic $SL(N>2,\mathbb{C})$ top and Toda chains

We consider a limit that is a combination of the spectral parameter shift $z = \tilde{z} + \tau/2$, the coordinate scaling, and the trigonometric limit $q \to 0$. In this case, the coordinate scaling is defined not by symplectic map (6) between phase spaces as in the case $N = 2$ but by the behavior of the Lax matrix of the elliptic top system in the described limit.

3.1. The limit system equivalent to a periodic Toda chain. In the described limit, the behavior of the Lax matrix of the elliptic top is defined by the expansion of the function $\varphi\left[\begin{smallmatrix} m \\ n \end{smallmatrix}\right](z)$ in a series in q (see Appendix B). Taking this into account, we use the coordinate renormalization

$$s_{mn} = \tilde{s}_{mn} q^{-g(n)}, \qquad g(n) = \frac{1-\delta_{n0}}{2N}, \quad m, n \in \{0, \dots, N-1\}, \quad m^2 + n^2 \neq 0,$$

which yields the three-diagonal Lax matrix of the limit system. In the limit, the renormalized coordinates form a Lie algebra with respect to the Poisson brackets [1]. The limit Lax matrices and the Hamiltonian

$$\tilde{L}^{\mathrm{rot}} = \lim_{q\to 0} L^{\mathrm{rot}}, \qquad \widetilde{M}^{\mathrm{rot}} = \lim_{q\to 0} M^{\mathrm{rot}}, \qquad \widetilde{H}^{\mathrm{rot}} = \lim_{q\to 0} H^{\mathrm{rot}}$$

satisfy the Lax equation

$$\frac{d\tilde{L}^{\mathrm{rot}}}{dt} = \{\widetilde{H}^{\mathrm{rot}}, \tilde{L}^{\mathrm{rot}}\} = N[\tilde{L}^{\mathrm{rot}}, \widetilde{M}^{\mathrm{rot}}] \tag{11}$$

and are independent of the variables $\tilde{s}_{mn}$, $1 < n < N-1$, $0 \le m \le N-1$. Therefore, the Hamiltonian equations of motion in these variables are not described by Eq. (11). But these variables on the generic symplectic leaf are functions of the variables included in the Lax matrix [1].

We can introduce the bosonization formulas for the coordinates of the limit system. For this, it is convenient to pass to the standard basis using formula (A.2). In the standard basis, the bosonization formulas for the renormalized coordinates $\widetilde{S}_{ij}$ have the form [1]

$$\begin{aligned}
&\widetilde{S}_{ii} = \frac{N}{2\pi\mathrm{i}}(v_{i-1} - v_i), \\
&\widetilde{S}_{i,i+1} = MN\,\mathrm{e}(u_i), \\
&\widetilde{S}_{i+1,i} = MN\,\mathrm{e}(-u_i), \\
&\widetilde{S}_{i,i+k} = c_{i,i+k}\,\mathrm{e}\biggl(\sum_{n=i}^{i+k-1} u_n\biggr), \quad 2 \le k \le N-2, \quad c_{i,i+k} = \mathrm{const},
\end{aligned} \tag{12}$$

where $\mathbf{u}$ and $\mathbf{v}$ are canonical coordinates,

$$\{v_i, u_j\} = \delta_{ij}, \quad i, j \in \{1, \dots, N\},$$

and

$$\sum_{i=1}^{N} u_i = 0, \qquad \sum_{i=1}^{N} v_i = 0.$$

The variables $\mathbf{u}$ and $\mathbf{v}$ have the dynamics of a periodic Toda chain in the center-of-mass frame. Using (12) and the gauge transformation

$$\begin{aligned}
&\tilde{L}^{\mathrm{rot}} \to g^{-1}\tilde{L}^{\mathrm{rot}} g, \qquad \widetilde{M}^{\mathrm{rot}} \to g^{-1}\widetilde{M}^{\mathrm{rot}} g + \frac{1}{N} g^{-1}\dot{g}, \\
&g_{ij} = \delta_{ij}\,\mathrm{e}\biggl(\frac{i\tilde{z}}{N}\biggr) \prod_{k=1}^{i-1} \mathrm{e}(-u_k),
\end{aligned} \tag{13}$$

we transform the limit Lax matrices and the Hamiltonian into the known Lax matrices and the Hamiltonian of a periodic Toda chain:

$$\tilde{L}^{\mathrm{rot}} = 2\pi \mathrm{i} MN \begin{pmatrix} \dfrac{v_1}{2\pi\mathrm{i}M} & \mathrm{e}(u_2 - u_1) & 0 & \cdots & 0 & -\mathrm{e}(\tilde{z}) \\ -1 & \dfrac{v_2}{2\pi\mathrm{i}M} & \mathrm{e}(u_3 - u_2) & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \mathrm{e}(u_N - u_{N-1}) \\ \mathrm{e}(u_1 - u_N - \tilde{z}) & 0 & \cdots & 0 & -1 & \dfrac{v_n}{2\pi\mathrm{i}M} \end{pmatrix},$$

$$\widetilde{M}^{\mathrm{rot}} = 4\pi^2 MN \begin{pmatrix} 0 & \mathrm{e}(u_2 - u_1) & 0 & \cdots & 0 \\ \vdots & 0 & \mathrm{e}(u_3 - u_2) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \mathrm{e}(u_N - u_{N-1}) \\ \mathrm{e}(u_1 - u_N - \tilde{z}) & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

$$\widetilde{H}^{\mathrm{rot}} = N^2 \sum_{i=1}^{N} \frac{v_i^2}{2} + 4\pi^2 M^2 N^2 \sum_{i=1}^{N} \mathrm{e}(u_{i+1} - u_i) = N^2 H^{\mathrm{pT}}.$$

3.2. The limit system equivalent to a nonperiodic Toda chain. To obtain the Lax matrix for a nonperiodic Toda chain, we must consider a renormalization of the standard basis coordinates that differs from the preceding case:

$$S_{ij} = \widetilde{S}_{ij} q^{-\mathfrak{g}(i,j)}, \qquad \mathfrak{g}(i,j) = \frac{1 - \delta_{ij} - \delta_{i1}\delta_{jN}/2}{2N}, \quad i, j \in \{1, \ldots, N\}.$$

The algebra of coordinates of the limit system in the standard basis has the same limit as in the periodic case, and the limit Lax matrices and the Hamiltonian satisfy the same Lax matrix equation (11). The variables

$$\bigl\{\widetilde{S}_{ij},\ 1 < (j-i) \bmod N < N-1,\ 1 \le i \le N,\ 1 \le j \le N\bigr\} \cup \{\widetilde{S}_{1N}\}$$

are not included in the limit Lax matrices and Hamiltonian. The Hamiltonian equations of motion for these variables can be integrated after solving Eq. (11) [1].

We can introduce bosonization formulas for the variables of the limit system [1]:

$$\begin{aligned} &\widetilde{S}_{ii} = \frac{N}{2\pi\mathrm{i}}(v_{i-1} - v_i), && i \in \{1, \ldots, N\}, \\ &\widetilde{S}_{i,i+1} = MN\,\mathrm{e}(u_i), && i \in \{1, \ldots, N\}, \\ &\widetilde{S}_{i+1,i} = MN\,\mathrm{e}(-u_i), && i \in \{1, \ldots, N-1\}, \\ &\widetilde{S}_{1,N} = \mathrm{const} \cdot \mathrm{e}(-u_N), && \\ &\widetilde{S}_{i,i+k} = c_{i,i+k}\,\mathrm{e}\biggl(\sum_{n=i}^{i+k-1} u_n\biggr), \quad 2 \le k \le N-2, \quad c_{i,i+k} = \mathrm{const}. && \end{aligned} \tag{14}$$

The canonical coordinates $\mathbf{u}$ and $\mathbf{v}$ have the dynamics of a nonperiodic Toda chain in the center-of-mass frame. After substitution (14) and gauge transformation (13), the limit Lax matrices and Hamiltonian take the known form of the Lax matrices and Hamiltonian of a nonperiodic Toda chain:

$$\tilde{L}^{\mathrm{rot}} = 2\pi\mathrm{i}MN \begin{pmatrix} \dfrac{v_1}{2\pi\mathrm{i}M} & \mathrm{e}(u_2-u_1) & 0 & \cdots & \cdots & 0 \\ -1 & \dfrac{v_2}{2\pi\mathrm{i}M} & \mathrm{e}(u_3-u_2) & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \mathrm{e}(u_N-u_{N-1}) \\ \mathrm{e}(u_1-u_N-\tilde{z}) & 0 & \cdots & 0 & -1 & \dfrac{v_n}{2\pi\mathrm{i}M} \end{pmatrix},$$

$$\widetilde{M}^{\mathrm{rot}} = 4\pi^2 MN \begin{pmatrix} 0 & \mathrm{e}(u_2-u_1) & 0 & \cdots & 0 \\ \vdots & 0 & \mathrm{e}(u_3-u_2) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \mathrm{e}(u_N-u_{N-1}) \\ \mathrm{e}(u_1-u_N-\tilde{z}) & 0 & \cdots & \cdots & 0 \end{pmatrix},$$

$$\widetilde{H}^{\mathrm{rot}} = N^2 \sum_{i=1}^{N} \frac{v_i^2}{2} + 4\pi^2 M^2 N^2 \sum_{i=1}^{N-1} \mathrm{e}(u_{i+1}-u_i) = N^2 H^{\mathrm{aT}}.$$

4. Nonautonomous $SL(2,\mathbb{C})$ top and Toda chain

We consider the nonautonomous $SL(2,\mathbb{C})$ top, where the role of time is played by the elliptic curve parameter τ. The Lax pair for this top satisfies the isomonodromic deformation equation

$$\partial_\tau L^{\mathrm{rot}} - \frac{1}{2\pi\mathrm{i}}\,\partial_z M^{\mathrm{rot}} = 2[L^{\mathrm{rot}}, M^{\mathrm{rot}}], \tag{15}$$

which is equivalent to the equations of motion

$$\frac{ds_{mn}}{d\tau} = \{H^{\mathrm{rot}}, s_{mn}\}.$$

We obtained the Toda chain from the elliptic $SL(2,\mathbb{C})$ top in the autonomous case above. Our goal now is to obtain the nonautonomous Toda chain whose equations of motion (in the case $N = 2$) are equivalent to a particular case of the Painlevé III equation. For this, we modify the limit procedure as follows.

We represent the elliptic curve parameter τ as $\tau = \tau_1 + \tau_2$ and consider the trigonometric limit $\operatorname{Im}\tau_2 \to +\infty$, keeping τ_1 as the time. Thus,

$$\frac{ds_{mn}}{d\tau} = \frac{ds_{mn}}{d\tau_1}.$$

Just as in the autonomous case, we shift the spectral parameter as $z = \tilde{z} + \tau/2$ and scale the coordinates as

$$s_{mn} = \tilde{s}_{mn} q_2^{-g(n)}$$

or

$$s_{10} = \tilde{s}_{10}, \qquad s_{01} = \tilde{s}_{01} q_2^{-1/4}, \qquad s_{11} = \tilde{s}_{11} q_2^{-1/4},$$

where $q_2 \equiv \mathrm{e}(\tau_2)$.

Because the spectral parameter shift is time dependent in the considered limit procedure, Eq. (15) is transformed as

$$\partial_{\tau_1} L^{\mathrm{rot}} - \partial_{\tilde{z}} \left(\frac{M^{\mathrm{rot}}}{2\pi \mathrm{i}} + \frac{1}{2} L^{\mathrm{rot}} \right) = 2[L^{\mathrm{rot}}, M^{\mathrm{rot}}],$$

where $L^{\mathrm{rot}} = L^{\mathrm{rot}}(\mathbf{s}, \tilde{z} + \tau/2, \tau)$ and $M^{\mathrm{rot}} = M^{\mathrm{rot}}(\mathbf{s}, \tilde{z} + \tau/2, \tau)$. The Hamiltonian of the elliptic $SL(2, \mathbb{C})$ top is written as (1)

$$H^{\mathrm{rot}} = -J_{01} s_{01}^2 - J_{10} s_{10}^2 - J_{11} s_{11}^2.$$

In the limit $q_2 \to 0$,

$$J_{10} = \pi^2 + o(1), \qquad J_{01} = -8\pi^2 q^{1/2} + o(q_2^{1/2}), \qquad J_{11} = 8\pi^2 q^{1/2} + o(q_2^{1/2}),$$

where $q \equiv \mathrm{e}(\tau)$. For the Hamiltonian, we then have

$$\widetilde{H}^{\mathrm{rot}} = \lim_{q_2 \to 0} H^{\mathrm{rot}} = -\pi^2 s_{10}^2 - 8\pi^2 q_1^{1/2} (\tilde{s}_{11}^2 - \tilde{s}_{01}^2),$$

where $q_1 \equiv \mathrm{e}(\tau_1)$.

The limit algebra is obtained by contraction of the algebra $\mathfrak{sl}(2, \mathbb{C})$:

$$\{\tilde{s}_{10}, \tilde{s}_{11}\} = 2\mathrm{i}\tilde{s}_{01}, \qquad \{\tilde{s}_{11}, \tilde{s}_{01}\} = 0, \qquad \{\tilde{s}_{01}, \tilde{s}_{10}\} = 2\mathrm{i}\tilde{s}_{11}.$$

The equations of motion preserve their form in this limit,

$$\frac{d\tilde{s}_{mn}}{d\tau_1} = \{\widetilde{H}^{\mathrm{rot}}, \tilde{s}_{mn}\},$$

and are equivalent to the isomonodromic deformation equation

$$\partial_{\tau_1} \tilde{L}^{\mathrm{rot}} - \partial_{\tilde{z}} \widetilde{M}^{\mathrm{rot}} = [\tilde{L}^{\mathrm{rot}}, \widetilde{M}^{\mathrm{rot}}], \tag{16}$$

where

$$\tilde{L}^{\mathrm{rot}} = \lim_{q_2 \to 0} 4\pi \mathrm{i} L^{\mathrm{rot}}, \qquad \widetilde{M}^{\mathrm{rot}} = \lim_{q_2 \to 0} (2M^{\mathrm{rot}} + 2\pi \mathrm{i} L^{\mathrm{rot}}).$$

Just as in the autonomous case, the algebra admits the bosonization

$$\tilde{s}_{10} = -\frac{\mathrm{i}v}{\pi}, \qquad \tilde{s}_{01} = M \cos(2\pi U), \qquad \tilde{s}_{11} = -M \sin(2\pi U).$$

The Lax pair then becomes

$$\tilde{L}^{\mathrm{rot}} = \begin{pmatrix} 4\mathrm{i}\pi v & 16\mathrm{i} M \pi^2 \sin(\pi(2U + \tilde{z})) q_1^{1/4} \\ -16\mathrm{i} M \pi^2 \sin(\pi(2U - \tilde{z})) q_1^{1/4} & -4\mathrm{i}\pi v \end{pmatrix},$$

$$\widetilde{M}^{\mathrm{rot}} = \begin{pmatrix} 0 & 8M\pi^2 \cos(\pi(2U + \tilde{z})) q_1^{1/4} \\ 8M\pi^2 \cos(\pi(2U - \tilde{z})) q_1^{1/4} & 0 \end{pmatrix}.$$

We write the Hamiltonian

$$\widetilde{H}^{\mathrm{rot}} = v^2 + 8M^2\pi^2\,\mathrm{e}\biggl(\frac{\tau_1}{2}\biggr)\cos(4\pi U)$$

and the equation of motion of the limit system in the variables U and v,

$$\begin{aligned} \frac{dv}{d\tau_1} &= 32M^2\pi^3\,\mathrm{e}\biggl(\frac{\tau_1}{2}\biggr)\sin(4\pi U), \\ \frac{dU}{d\tau_1} &= 2v. \end{aligned} \tag{17}$$

We show that Eqs. (17) are equivalent to a particular case of the Painlevé III equation. For this, we change the variable as $32\mathrm{i}M\pi\,\mathrm{e}(\tau_1/4) = t$ and represent system (17) by the single second-order differential equation

$$\frac{1}{t}\frac{d}{dt}t\frac{d(4\pi U)}{dt} = \sin(4\pi U).$$

5. Conclusion

In our previous paper [1], we obtained systems equivalent to Toda chains from the elliptic $SL(N,\mathbb{C})$ top using a limit procedure close to the Inozemtsev limit. Here, we have given a generalization of the previously proposed procedure to the nonautonomous $SL(2,\mathbb{C})$ top and Toda chain. It was previously known that the nonautonomous $SL(2,\mathbb{C})$ top and Toda chain are related by some Inozemtsev limit [16]. But we here showed that the nonautonomous Toda chain together with the linear problem can be obtained from the nonautonomous $SL(2,\mathbb{C})$ top; namely, the proposed limit of the Lax matrices for the elliptic $SL(2,\mathbb{C})$ top allows writing the equations of motion for the nonautonomous Toda chain in the form of an isomonodromic deformation equation. An analogous result can also be obtained for the nonautonomous $SL(N{>}2,\mathbb{C})$ top, but the equations of motion for such nonautonomous systems are not directly related to Painlevé equations.

In this paper, we have considered the limit relation between the linear problems for particular cases of the Painlevé VI and Painlevé III equations. A natural continuation of this problem is to describe the limit relation between the linear problems for the general forms of the Painlevé VI and Painlevé III equations.

Appendix A: Sine algebra

To simplify the formulas, we use the notation $\vee$ for the logical "or" and also the notation

$$\tilde{\delta}(n) = \begin{cases} 1, & n \equiv 0 \bmod N, \\ 0, & n \not\equiv 0 \bmod N. \end{cases}$$

The elements T_{mn} of the basis in $\mathfrak{sl}(N,\mathbb{C})$, which generates the sine algebra, can be defined as

$$(T_{mn})_{ij} = \mathrm{e}\biggl(\frac{mn}{2N}\biggr)\mathrm{e}\biggl(\frac{im}{N}\biggr)\tilde{\delta}(j-i-n), \quad (m\neq 0)\vee(n\neq 0), \quad m,n\in\{0,\dots,N-1\}.$$

For $m,n\in\mathbb{Z}$ $(m\not\equiv 0 \bmod N)\vee(n\not\equiv 0 \bmod N)$, the quasiperiodicity properties hold for the basis elements of the sine algebra and for the coordinates in this basis:

$$T_{mn} = \mathrm{e}\biggl(\frac{mn-(m \bmod N)(n \bmod N)}{2N}\biggr)T_{m \bmod N,\, n \bmod N},$$

$$s_{mn} = \mathrm{e}\biggl(\frac{(m \bmod N)(n \bmod N)-mn}{2N}\biggr)s_{m \bmod N,\, n \bmod N},$$

where $\mathrm{e}\big((mn-(m \bmod N)(n \bmod N))/(2N)\big)=\pm 1$.

The commutation relations have the form

$$[T_{mn},T_{kl}]=2\mathrm{i}\sin\left[\frac{\pi}{N}(kn-ml)\right]T_{m+k,n+l}. \tag{A.1}$$

The relations

$$S_{ij}=\sum_{m,n}s_{mn}(T_{mn})_{ij},\qquad s_{mn}=\frac{1}{N}\sum_{i,j}S_{ij}(T_{-m,n})_{ij} \tag{A.2}$$

hold between the coordinates $\{S_{ij}\}$ in the standard basis and the coordinates $\{s_{mn}\}$ in the sine algebra basis.

Appendix B: Degenerate elliptic functions

The definitions and properties of elliptic functions are essentially taken from [19] and [26]. The basic object is the theta function with the characteristics

$$\theta\begin{bmatrix}a\\b\end{bmatrix}(z,\tau)=\sum_{j\in\mathbb{Z}}q^{(j+a)^2/2}\,\mathrm{e}((j+a)(z+b)).$$

We also need the Eisenstein functions

$$\begin{aligned}\varepsilon_k(z)&=\lim_{M\to+\infty}\sum_{n=-M}^{M}(z+n)^{-k},\quad k\in\mathbb{N},\\ E_k(z)&=\lim_{M\to+\infty}\sum_{n=-M}^{M}\varepsilon_k(z+n\tau).\end{aligned} \tag{A.3}$$

To determine the limits of the Lax matrices, we use the series expansions of the functions

$$\vartheta(z)=\theta\begin{bmatrix}1/2\\1/2\end{bmatrix}(z,\tau)=\sum_{j\in\mathbb{Z}}q^{(j+1/2)^2/2}\,\mathrm{e}\left(\left(j+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)\right), \tag{A.4}$$

$$\phi(u,z)=\frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}, \tag{A.5}$$

$$\varphi\begin{bmatrix}m\\n\end{bmatrix}(z)=\mathrm{e}\left(-\frac{nz}{N}\right)\phi\left(-\frac{m+n\tau}{N},z\right), \tag{A.6}$$

$$f\begin{bmatrix}m\\n\end{bmatrix}(z)=\mathrm{e}\left(-\frac{nz}{N}\right)\partial_u\phi(u,z)|_{u-(m+n\tau)/N}. \tag{A.7}$$

These functions satisfy the relations

$$\begin{aligned}&\phi(u,z)\phi(-u,z)=E_2(z)-E_2(u),\\ &\partial_u\phi(u,z)=\phi(u,z)(E_1(u+z)-E_1(u))\end{aligned} \tag{A.8}$$

and have the parities

$$E_k(-z)=(-1)^kE_k(z),\qquad \vartheta(-z)=-\vartheta(z),\qquad \phi(u,z)=\phi(z,u)=-\phi(-u,-z)$$

and quasiperiodicity properties

$$\begin{aligned}
&E_1(z+1)=E_1(z), && E_1(z+\tau)=E_1(z)-2\pi\mathrm{i},\\
&E_2(z+1)=E_2(z), && E_2(z+\tau)=E_2(z),\\
&\vartheta(z+1)=-\vartheta(z), && \vartheta(z+\tau)=-q^{-1/2}\,\mathrm{e}(-z)\vartheta(z),\\
&\phi(u+1,z)=\phi(u,z), && \phi(u+\tau,z)=\mathrm{e}(-z)\phi(u,z).
\end{aligned} \tag{A.9}$$

We set $z=\tilde z+\tau/2$ and examine the degeneration of elliptic functions (A.6) and (A.7) in the limit $\operatorname{Im}\tau\to+\infty$. In view of (A.5), the series expansion of the function $\varphi\left[\begin{smallmatrix} m\\ n\end{smallmatrix}\right](z)$ reduces to the series expansion of the theta function. Considering the leading nonzero term in the expansions, we obtain

$$\vartheta\left(-\frac{m}{N}-\frac{n}{N}\tau\right)=\begin{cases} 2q^{1/8}\sin\left(\pi\dfrac{m}{N}\right)+o(q^{1/8}), & n=0,\\ iq^{1/8-n/(2N)}\,\mathrm{e}\left(-\dfrac{m}{2N}\right)+o(q^{1/8-n/(2N)}), & 0<n<N,\end{cases}$$

$$\vartheta\left(\tilde z+\frac{\tau}{2}-\frac{m}{N}-\frac{n}{N}\tau\right)=\begin{cases} -\mathrm{i}q^{n/(2N)-1/8}\,\mathrm{e}\left(\dfrac12\left(\dfrac{m}{N}-\tilde z\right)\right)+o(q^{n/(2N)-1/8}), & 0\le n<\dfrac{N}{2},\\ -2q^{1/8}\sin\left(\pi\left(\tilde z-\dfrac{m}{N}\right)\right)+o(q^{1/8}), & n=\dfrac{N}{2},\\ \mathrm{i}q^{3/8-n/(2N)}\,\mathrm{e}\left(\dfrac12\left(\tilde z-\dfrac{m}{N}\right)\right)+o(q^{3/8-n/(2N)}), & \dfrac{N}{2}<n<N,\end{cases}$$

which yields

$$\phi\left(-\frac{m+n\tau}{N},\tilde z+\frac{\tau}{2}\right)=\begin{cases} -\pi\,\mathrm{e}\left(\dfrac{m}{2N}\right)\sin^{-1}\left(\pi\dfrac{m}{N}\right)+o(1), & n=0,\\ 2\pi\mathrm{i}q^{n/N}\,\mathrm{e}\left(\dfrac{m}{N}\right)+o(q^{n/N}), & 0<n<\dfrac{N}{2},\\ 4\pi q^{1/2}\sin\left(\pi\left(\tilde z-\dfrac{m}{N}\right)\mathrm{e}\left(\dfrac12\left(\dfrac{m}{N}+\tilde z\right)\right)\right)+o(q^{1/2}), & n=\dfrac{N}{2},\\ -2\pi\mathrm{i}q^{1/2}\,\mathrm{e}(\tilde z)+o(q^{1/2}), & \dfrac{N}{2}<n<N,\end{cases},$$

$$\varphi\begin{bmatrix} m\\ n\end{bmatrix}\left(\tilde z+\frac{\tau}{2}\right)=\begin{cases} -\pi\,\mathrm{e}\left(\dfrac{m}{2N}\right)\sin^{-1}\left(\pi\dfrac{m}{N}\right)+o(1), & n=0,\\ 2\pi\mathrm{i}q^{n/(2N)}\,\mathrm{e}\left(\dfrac{m}{N}-\dfrac{n\tilde z}{N}\right)+o(q^{n/(2N)}), & 0<n<\dfrac{N}{2},\\ 4\pi q^{1/4}\,\mathrm{e}\left(\dfrac{m}{2N}\right)\sin\left(\pi\left(\tilde z-\dfrac{m}{N}\right)\right)+o(q^{1/4}), & n=\dfrac{N}{2},\\ -2\pi\mathrm{i}q^{(N-n)/(2N)}\,\mathrm{e}\left(\dfrac{N-n}{N}\tilde z\right)+o(q^{(N-n)/(2N)}), & \dfrac{N}{2}<n<N.\end{cases} \tag{A.10}$$

To find the limit of the function $f\left[\begin{smallmatrix} m\\ n\end{smallmatrix}\right]$, we expand $E_1(\tilde x-\sigma\tau)$ in a series in q. In view of (A.3), we obtain

$$E_1(\tilde x-\sigma\tau)=\lim_{M\to+\infty}\sum_{n=-M}^{M}\varepsilon_1(\tilde x+(n-\sigma)\tau)=\varepsilon_1(\tilde x-\sigma\tau)+$$

$$+\lim_{M\to+\infty}\sum_{n=1}^{M}(\varepsilon_1(\tilde x+(n-\sigma)\tau)+\varepsilon_1(\tilde x-(n+\sigma)\tau)).$$

Using the explicit expression for the function $\varepsilon_1(x)$ [19]

$$\varepsilon_1(x) = \pi\cot(\pi x) = \pi\mathrm{i}\frac{\mathrm{e}(x)+1}{\mathrm{e}(x)-1} = \pi\mathrm{i}\times\begin{cases} -1-2\,\mathrm{e}(x)+o(\mathrm{e}(x)), & \operatorname{Im}x\to+\infty,\\ 1+2\,\mathrm{e}(x)+o(\mathrm{e}(x)), & \operatorname{Im}x\to-\infty,\end{cases}$$

we can obtain the result for the leading term in the expansion of the function $E_1(\tilde{x}-\sigma\tau)$:

$$E_1(\tilde{x}-\sigma\tau) = \begin{cases} \pi\cot(\pi\tilde{x})+o(1), & \sigma=0,\\ \pi\mathrm{i}+2\pi\mathrm{i}q^{\sigma}\,\mathrm{e}(-\tilde{x})+o(q^{\sigma}), & 0<\sigma<\dfrac{1}{2},\\ \pi\mathrm{i}+2\pi\mathrm{i}q^{1/2}(\mathrm{e}(-\tilde{x})-\mathrm{e}(\tilde{x}))+o(q^{1/2}), & \sigma=\dfrac{1}{2},\\ \pi\mathrm{i}-2\pi\mathrm{i}q^{1-\sigma}\,\mathrm{e}(\tilde{x})+o(q^{1-\sigma}), & \dfrac{1}{2}<\sigma<1,\end{cases}$$

and, using (A.9), the generalization of the last formula in the case $\sigma\in\mathbb{R}$:

$$E_1(\tilde{x}-\sigma\tau) = \begin{cases} 2\pi\mathrm{i}\lfloor\sigma\rfloor+\pi\cot(\pi\tilde{x})+o(1), & \{\sigma\}=0,\\ 2\pi\mathrm{i}\lfloor\sigma\rfloor+\pi\mathrm{i}+2\pi\mathrm{i}q^{\{\sigma\}}\,\mathrm{e}(-\tilde{x})+o(q^{\{\sigma\}}), & 0<\{\sigma\}<\dfrac{1}{2},\\ 2\pi\mathrm{i}\lfloor\sigma\rfloor+\pi\mathrm{i}+2\pi\mathrm{i}q^{1/2}(\mathrm{e}(-\tilde{x})-\mathrm{e}(\tilde{x}))+o(q^{1/2}), & \{\sigma\}=\dfrac{1}{2},\\ 2\pi\mathrm{i}\lfloor\sigma\rfloor+\pi\mathrm{i}-2\pi\mathrm{i}q^{1-\{\sigma\}}\,\mathrm{e}(\tilde{x})+o(q^{1-\{\sigma\}}), & \dfrac{1}{2}<\{\sigma\}<1,\end{cases}$$

where $\{\sigma\}$ is the fractional part of the number σ.

We now consider the expansion of the function $\partial_u\phi(u,z)|_{u=\tilde{u}-\sigma\tau}$ in the limit $\operatorname{Im}\tau\to+\infty$ taking (A.8) into account. Assuming that $z=\tilde{z}+\tau/2$ and considering all possible values of σ, we find

$$\partial_u\phi(u,z)|_{u=\tilde{u}-\sigma\tau} = \begin{cases} -\pi^2\sin^{-2}\pi\tilde{u}+o(1), & \sigma=0,\\ 4\pi^2q\,\mathrm{e}(-\tilde{u})+o(q), & 0<\sigma<\dfrac{3}{4},\\ 4\pi^2q^{3/4}[\mathrm{e}(-\tilde{u})-\mathrm{e}(\tilde{u}+\tilde{z})]+o(q^{3/4}), & \sigma=\dfrac{3}{4},\\ -4\pi^2q^{3/2-\sigma}\,\mathrm{e}(\tilde{u}+\tilde{z})+o(q^{3/2-\sigma}), & \dfrac{3}{4}<\sigma<1.\end{cases}$$

Finally, using (A.7), we obtain

$$f\begin{bmatrix} m\\ n\end{bmatrix}\left(\tilde{z}+\frac{\tau}{2}\right) = \\ = \begin{cases} -\pi^2\sin^{-2}\left(\pi\dfrac{m}{N}\right)+o(1), & n=0,\\ 4\pi^2\,\mathrm{e}\left(\dfrac{m}{N}\right)\mathrm{e}\left(-\dfrac{n\tilde{z}}{N}\right)q^{n/(2N)}+o(q^{n/(2N)}), & 0<n<\dfrac{3N}{4},\\ 4\pi^2\left[\mathrm{e}\left(\dfrac{m}{N}\right)-\mathrm{e}\left(-\dfrac{n}{N}+\tilde{z}\right)\right]\mathrm{e}\left(-\dfrac{3}{4}\tilde{z}\right)q^{3/8}+o(q^{3/8}), & n=\dfrac{3N}{4},\\ -4\pi^2\,\mathrm{e}\left(-\dfrac{m}{N}+\tilde{z}\right)\mathrm{e}\left(-\dfrac{n\tilde{z}}{N}\right)q^{3(1-n/N)/2}+o(q^{3(1-n/N)/2}), & \dfrac{3N}{4}<n<N.\end{cases} \tag{A.11}$$

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