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# RESOLVENTS AND SEIBERG–WITTEN REPRESENTATION FOR A GAUSSIAN $\beta$-ENSEMBLE

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*The exact free energy of a matrix model always satisfies the Seiberg–Witten equations on a complex curve defined by singularities of the semiclassical resolvent. The role of the Seiberg–Witten differential is played by the exact one-point resolvent in this case. We show that these properties are preserved in the generalization of matrix models to $\beta$-ensembles. But because the integrability and Harer–Zagier topological recursion are still unavailable for $\beta$-ensembles, we must rely on the ordinary Alexandrov–Mironov–Morozov/Eynard–Orantin recursion to evaluate the first terms of the genus expansion. We restrict our consideration to the Gaussian model.*

**Keywords:** matrix model, $\beta$-ensemble, integrability, Seiberg–Witten theory

## 1. Introduction

Seiberg–Witten (SW) prepotentials $\mathcal{F}(\vec{a})$ [1]–[4] are defined from the peculiar set of implicit equations

$$\vec{a} = \oint_{\vec{A}} \Omega, \qquad \frac{\partial \mathcal{F}(\vec{a})}{\partial \vec{a}} = \oint_{\vec{B}} \Omega, \tag{1}$$

where $\Omega$ is a type-$(m,0)$ analytic form (holomorphic, meromorphic, or even having essential singularities) on a family of $2m$-dimensional complex manifolds with a system of conjugate cycles $\vec{A}$ and $\vec{B}$. When system (1) is solvable (its consistency is guaranteed by the Riemann identities, for example), the $\vec{a}$ are called flat coordinates on the moduli space of the family (or, simply, the flat moduli), and $\mathcal{F}(\vec{a})$ is a "semiclassical" or Whitham $\tau$-function on this space satisfying a set of the (generalized) Witten–Dijkgraaf–Verlinde–Verlinde equations (usually as a consequence of the residue formula) [5]. This is already a classical branch of science, presented in great detail in numerous papers.

It was recently understood that although the SW equations are "semiclassical," they perfectly survive various quantization procedures. The conceptual meaning of this phenomenon is still a riddle, but the very fact is becoming established ever more reliably. The latest example is the Bohr–Sommerfeld representation [6], [7] of the Losev–Moore–Nekrasov–Shatashvili (LMNS) free energy [8] in the Nekrasov–Shatashvili limit [9] $\epsilon_2 = 0$: if we also set $\epsilon_1 = 0$, then the obtained free energy is just the ordinary SW prepotential in [1], [3], but it is remarkable that Eqs. (1) survive when at least the first "quantization parameter" $\epsilon_1$ is switched on. It was claimed in [10], [11] that the SW equations in fact survive under even further deformation, when both $\epsilon_1$ and $\epsilon_2$ are nonzero. This claim is inspired by the Alday–Gaiotto–Tachikawa (AGT)

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relations [12]–[14], which provide a matrix model representation of the LMNS partition function [15], [10]. In the course of the argument, the fundamental fact is used that the exact matrix model free energies admit the SW representation with the role of the SW differential played by the one-point resolvent

$$\Omega^{\mathrm{MM}}(z) = \rho_1(z) = \left\langle \operatorname{Tr} \frac{dz}{z-M} \right\rangle_{\mathrm{MM}}, \tag{2}$$

which is a meromorphic differential on the spectral curve $\Sigma^{\mathrm{MM}}$. Again, the SW representation is easily verified for the planar free energy (for which it was already discussed in [16], [17]), but it turns out that it also survives when all higher-genus corrections in powers of the string coupling constant $g_s$ (i.e., the t' Hooft's coupling $\Lambda = gN$) are switched on. This fact is still less known and underestimated. It was mentioned in passing in [18] and [19], but its real significance can be seen from its use in a conceptual proof of the AGT relations, in which the topological recursion procedures [18], [20] were used to construct a double deformation of the SW prepotential, where $g_s = \sqrt{-\epsilon_1\epsilon_2} \neq 0$ and $\beta = b^2 = -\epsilon_1/\epsilon_2 \neq 1$.

Our goal here to provide more illustrations of the SW representation of exact matrix model free energies and to make this crucially important technique more understandable and more convenient to use. This paper is devoted to this issue, and we intentionally avoid discussing other topics. The first illustration of this kind was already provided in the appendix to [11]; we reproduce the derivation in that work here and extend it to the case $\beta \neq 1$. We do not consider non-Gaussian models here, because that requires rather cumbersome calculations, but we will address this question in subsequent papers. Of course, an essentially non-Gaussian $\beta$-ensemble is used in the AGT relations: the open-contour Dotsenko–Fateev integral in the spirit of [10], [21], which we do not consider here. But the SW representation undoubtedly also exists there, and for arbitrary $\beta$, and in all orders of the genus expansion.

## 2. The case $\beta = 1$: A source of questions and educated guesses

The partition function is defined as

$$Z(N) = \frac{1}{N!} \int d\lambda_1 \cdots d\lambda_N \prod_{i<j} |\lambda_i - \lambda_j|^{2\beta} \exp\left[-\frac{1}{2g} \sum_i \lambda_i^2\right]. \tag{3}$$

For $\beta = 1$, it is equal to

$$Z(N) = (\sqrt{2\pi})^N \sqrt{g}^{N^2} \prod_{k=1}^{N-1} k!. \tag{4}$$

Hence, for the free energy $F = \log Z$ (up to terms quadratic and linear in $N$), we have

$$F(N) = \sum_{k=1}^{N-1} \log(k!). \tag{5}$$

It turns out [22] that $Z(N)$ is a Toda-chain $\tau$-function and $F(N)$ has SW representation (1). Namely, let $\rho_1(z)$ be the one-point resolvent of the model,

$$\rho_1(z) = \left\langle \sum_i \frac{1}{z - \lambda_i} \right\rangle. \tag{6}$$

Then the system of partial differential SW equations

$$-\frac{1}{2\pi i} \oint_{\mathrm{A}} \rho_1(z)\, dz = a, \qquad -\oint_{\mathrm{B}} \rho_1(z)\, dz = \frac{\partial \mathcal{F}_{\mathrm{SW}}}{\partial a} \tag{7}$$

determines the SW prepotential, which is equal to the free energy

$$\mathcal{F}_{\mathrm{SW}}(N) = F(N), \tag{8}$$

as can be verified using the explicit expression for the resolvent in [11], and this equality just gives the SW representation of the free energy of the matrix model.

Another fact worthy of attention is that the one-point resolvent satisfies the difference equation [23], [24]

$$\rho_1(N+1,z) + \rho_1(N-1,z) - 2\rho_1(N,z) = \frac{\partial^2}{\partial z^2}\rho_1(N,z), \tag{9}$$

which implies that its B-periods satisfy [11]

$$\Pi_{\mathrm{B}}(N+1) + \Pi_{\mathrm{B}}(N-1) - 2\Pi_{\mathrm{B}}(N) = -\frac{1}{N}. \tag{10}$$

Equation (9) is closely related to the integrability of $Z(N)$, i.e., to the Toda chain equation [22]

$$Z(N)\partial_1^2 Z(N) - \partial_1 Z(N)\partial_1 Z(N) = Z(N+1)Z(N-1), \tag{11}$$

where $\partial_1 Z(N) = \langle\sum_i \lambda_i\rangle$ and $\partial_1^2 Z(N) = \langle(\sum_i \lambda_i)^2\rangle$. Equation (10) is a weaker corollary of (9).

Knowing these facts, we naturally pose the following questions:

1. Does (8) also hold in the case $\beta \neq 1$?
2. Is there some $\beta$-deformed version of (9) and (10)?

The remainder of the paper is devoted to answering the first question affirmatively. Partial progress in answering the second question is outlined in the appendix.

## 3. Resolvents

**3.1. Ward identities: Generalities.** A powerful technique for evaluating correlators in matrix models is known by various names: the Virasoro constraints, the loop equations, and the Ward identities [25], [18]. It relies on the "general covariance" of the partition function, i.e., the invariance of the integral under an arbitrary change of the integration variables. For the eigenvalue integral, not necessarily Gaussian, the Virasoro constraints can be deduced as follows [26]. We consider the obvious identity

$$\sum_k \int d\lambda_1\cdots d\lambda_N \frac{\partial}{\partial\lambda_k}\biggl(\lambda_k^n \Delta^{2\beta}\exp\biggl[-\frac{1}{g}\sum_i V(\lambda_i)\biggr]S_{i_1}\cdots S_{i_m}\biggr) = 0, \tag{12}$$

where $S_i = \sum_a \lambda_a^i$ and $\Delta$ is the absolute value of the Vandermonde determinant. Here, $V(\lambda) = \sum_k T_k\lambda^k$; only $T_2 = 1/2$ is nonzero in the Gaussian case.

It is easily verified that

$$\sum_k \frac{\partial}{\partial\lambda_k}(\lambda_k^n\Delta^{2\beta}) = \biggl(\beta\sum_{a=0}^{n-1} S_a S_{n-1-a} + (1-\beta)nS_{n-1}\biggr)\Delta^{2\beta}, \tag{13}$$

and this is the only part of the equation that changes when $\beta$ is changed. Differentiating the potential term gives

$$\sum_k \lambda_k^n \frac{\partial}{\partial\lambda_k}\biggl[\exp\biggl(-\frac{1}{g}\sum_i V(\lambda_i)\biggr)\biggr] = \biggl(-\frac{1}{g}\sum_a V'(\lambda_a)\lambda_a^n\biggr)\exp\biggl[-\frac{1}{g}\sum_i V(\lambda_i)\biggr], \tag{14}$$

and this is the only model-dependent part of our consideration. Differentiating the remaining terms gives

$$\sum_k \lambda_k^n \frac{\partial}{\partial \lambda_k}(S_{i_1}\cdots S_{i_m}) = \sum_{j=1}^m i_j S_{i_1}\cdots S_{i_j+n-1}\cdots S_{i_m}. \tag{15}$$

Having all the ingredients of the equations, we can now write them in various forms.

*Virasoro constraints.* If we write the disconnected correlator as

$$C_{i_0,\dots,i_m} = \langle S_{i_0}\dots S_{i_m}\rangle, \tag{16}$$

then the above considerations imply that

$$\beta\sum_{a=0}^{n-1} C_{a,n-1-a,i_1,\dots,i_m} + (1-\beta)nC_{n-1,i_1,\dots,i_m} - \\ -\frac{1}{g}\sum_k kT_k C_{n-1+k,i_1,\dots,i_m} + \sum_{j=1}^m i_j C_{i_1,\dots,i_j+n-1,\dots,i_m} = 0. \tag{17}$$

*Differential* ($\widetilde{W}$) *operators.* If we work with the general partition function (with infinitely many nonfixed times), then we can write these equations as a differential equation for the (full) partition function. Namely, let the potential have the form

$$V(\lambda) = (T_0+t_0)N + \sum_{k=1}^{\infty}(T_k+t_k)\lambda^n, \tag{18}$$

where $T_k$ are background values of source fields (usually, only finitely many of them are nonzero) and $t_k$ are perturbations of these background values. The partition function can then be regarded as a formal series in $t_k$. We note that for the nonnormalized average, we can obtain

$$\langle S_a\rangle = -g\frac{\partial}{\partial t_a}\langle 1\rangle, \tag{19}$$

and Virasoro constraints (17) can hence be written as

$$\frac{\partial}{\partial t_{i_1}}\cdots\frac{\partial}{\partial t_{i_m}}\left(\sum_{k=0}^{\infty}k(T_k+t_k)\frac{\partial}{\partial t_{k-1+n}} + g(1-\beta)n\frac{\partial}{\partial t_{n-1}} + g^2\beta\sum_{a=1}^{n-1}\frac{\partial^2}{\partial t_a\partial t_{n-1-a}}\right)Z = 0. \tag{20}$$

*Loop equations.* The loop equations arise when we sum all the Virasoro constraints with the coefficients $1/z^{n+1}$ and write the resulting equation in terms of the resolvents. For this, it is convenient to rewrite the Vandermonde part of the identity as

$$\sum_k \frac{\partial}{\partial\lambda_k}(\lambda_k^n\Delta^{2\beta}) = \left(2\beta\sum_{i<j}\frac{\lambda_i^n-\lambda_j^n}{\lambda_i-\lambda_j} + \sum_a n\lambda_a^{n-1}\right)\Delta^{2\beta}. \tag{21}$$

Summing all the contributions, we now obtain

$$\beta r(z_0,z_0,z_1,\dots,z_m) + (\beta-1)\frac{\partial}{\partial z_0}r(z_0,z_1,\dots,z_m) + \\ +\sum_{j=1}^m\frac{\partial}{\partial z_j}\frac{r(z_1,\dots,z_m)-r(z_1,\dots,z_0,\dots,z_m)}{z_m-z_0} - \\ -\frac{1}{g}\sum_{k=0}^{\infty}kT_k z_0^{k-1}r(z_0,z_1,\dots,z_m) + \\ +\frac{1}{g}\sum_{k=0}^{\infty}kT_k\sum_{j=0}^{k-2}z_0^j\frac{1}{2\pi i}\oint_\infty dz\, z^{k-2-j}r(z,z_1,\dots,z_m) = 0, \tag{22}$$

where $r(z_0, \ldots, z_m)$ is the disconnected resolvent

$$r(z_0, \ldots, z_m) = \left\langle \sum_{i_0} \frac{1}{z_0 - \lambda_{i_0}} \cdots \sum_{i_m} \frac{1}{z_m - \lambda_{i_m}} \right\rangle. \tag{23}$$

To solve these equations perturbatively in $g$, we must rewrite the disconnected resolvents in terms of the connected ones. The iteration procedure then becomes well defined: at each step of the procedure, we have a system of linear equations for $\rho_{i,j}$ with a fixed value of $i+j$. The expansion in powers $k$ of $g$, as usual, counts contributions of genus $k/2$ Riemann surfaces in the string (or topological) expansion. Here, $\rho_{i,j}$ denotes the genus $j/2$ contribution to the $i$-point connected resolvent.

**3.2. Prerequisite: Particular correlators.** The Ward identities in the form of the Virasoro constraints are very useful for evaluating concrete correlators $C_{i_1,\ldots,i_m}$. The advantage of this method is that the answers are exact in $g$ and the disconnected correlators need not be rewritten in terms of the connected ones for the iteration procedure to work (this drastically simplifies the work if symbolic computer computations are used).

To give a picture of what individual correlators look like, we provide the first few one- and two-point correlators (we note that $K$ denotes the *connected* correlators, and $\Lambda \equiv Ng$):

$$\begin{aligned}
&K_k = C_k = \left\langle \sum_i \lambda_i^k \right\rangle = \left\langle\!\!\left\langle \sum_i \lambda_i^k \right\rangle\!\!\right\rangle,\\
&K_0 = \Lambda, \qquad K_2 = \Lambda(\beta\Lambda - \beta + 1),\\
&K_4 = \Lambda(2\beta^2\Lambda^2 - 5\beta^2\Lambda + 3\beta^2 + 5\beta\Lambda - 5\beta + 3),\\
&K_6 = 5\beta^3\Lambda^4 + (22\beta^2 - 22\beta^3)\Lambda^3 + (32\beta^3 - 54\beta^2 + 32\beta)\Lambda^2 +\\
&\qquad + (-15\beta^3 + 32\beta^2 - 32\beta + 15)\Lambda,\\
&K_8 = 14\beta^4\Lambda^5 + (93\beta^3 - 93\beta^4)\Lambda^4 + (234\beta^4 - 398\beta^3 + 234\beta^2)\Lambda^3 +\\
&\qquad + (-260\beta^4 + 565\beta^3 - 565\beta^2 + 260\beta)\Lambda^2 +\\
&\qquad + (105\beta^4 - 260\beta^3 + 331\beta^2 - 260\beta + 105)\Lambda, \qquad \ldots
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
&K_{k,j} = C_{k,j} - C_k C_j = \left\langle\!\!\left\langle \sum_i \sum_l \lambda_i^k \lambda_l^j \right\rangle\!\!\right\rangle,\\
&K_{1,1} = \Lambda,\\
&K_{1,3} = 3\Lambda(\beta(\Lambda - 1) + 1), \qquad K_{2,2} = 2\Lambda(\beta(\Lambda - 1) + 1),\\
&K_{1,5} = 10\beta^2\Lambda^3 + 5(5\beta - 5\beta^2)\Lambda^2 + 5(3\beta^2 - 5\beta + 3)\Lambda,\\
&K_{2,4} = 4\Lambda(\beta(\Lambda - 1)(\beta(2\Lambda - 3) + 5) + 3),\\
&K_{3,3} = 3\Lambda(\beta(\Lambda - 1)(\beta(4\Lambda - 5) + 9) + 5), \qquad \ldots.
\end{aligned} \tag{25}$$

In terms of the CFT-inspired variables $M = b\Lambda$ and $Q = b - 1/b$, $b = \sqrt{\beta}$, they have the forms

$$\begin{aligned}
&K_0 = \frac{M}{b}, \qquad K_2 = M(M-Q), \qquad K_4 = Mb(1+2M^2-5MQ+3Q^2),\\
&K_6 = Mb^2(5M(2+M^2)-(13+22M^2)Q+32MQ^2-15Q^3),\\
&K_8 = Mb^3(21+14M^4-93M^3Q+160Q^2+105Q^4-\\
&\qquad -5MQ(43+52Q^2)+M^2(70+234Q^2)), \qquad \dots
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
&K_{1,1} = \frac{M}{b}, \qquad K_{1,3} = 3M(M-Q), \qquad K_{2,2} = 2M(M-Q),\\
&K_{1,5} = 5Mb(1+2M^2-5MQ+3Q^2),\\
&K_{2,4} = 4Mb(1+2M^2-5MQ+3Q^2),\\
&K_{3,3} = 3Mb(1+4M^2-9MQ+5Q^2), \qquad \dots .
\end{aligned} \tag{27}$$

We note the remarkable simplification in comparison with (24) and (25).

**3.3. The answer for the resolvent at $\boldsymbol{\beta = 1}$.** For completeness of the picture (and in part to emphasize the relative complexity of the case $\beta \neq 1$), we begin from the well-known one-point resolvent at $\beta = 1$ [18]:

$$\rho_1 = \left\langle \sum_i \frac{1}{z-\lambda_i} \right\rangle = \sum_{k=0}^{\infty} \rho_{1,k} g^k. \tag{28}$$

The particular genus contributions are

$$\rho_{1,0}(z) = \frac{1}{2}(z-y(z)), \qquad \rho_{1,2}(z) = \frac{\Lambda}{y^5(z)}, \tag{29}$$

$$\rho_{1,4}(z) = \frac{21\Lambda(\Lambda+z^2)}{y^{11}(z)}, \qquad \rho_{1,6}(z) = \frac{11\Lambda(158\Lambda^2+558\Lambda z^2+135z^4)}{y^{17}(z)}, \qquad \dots, \tag{30}$$

where $y^2(z) = z^2 - 4\Lambda$ and all $\rho_{1,2k+1}$ vanish. General formulas for $\rho_{1,2n}$ can be obtained from the exact Harer–Zagier functions by an integral transformation (see [23], [18], [24]).

**3.4. The answer for $\boldsymbol{\rho_1}$ at arbitrary $\boldsymbol{\beta \neq 1}$.** Loop equations (22) in the case of the Gaussian model acquire a very simple form

$$\begin{aligned}
&\beta r(z_0,z_0,z_1,\dots,z_m) + (\beta-1)\frac{\partial}{\partial z_0} r(z_0,z_1,\dots,z_m) +\\
&+\sum_j \frac{\partial}{\partial z_j} \frac{r(z_1,\dots,z_j,\dots,z_m) - r(z_1,\dots,z_0,\dots,z_m)}{z_j - z_0} -\\
&-\frac{1}{g} z_0 r(z_0,z_1,\dots,z_m) + \frac{\Lambda}{g^2} r(z_1,\dots,z_m) = 0,
\end{aligned} \tag{31}$$

where $r$ denotes the disconnected resolvent

$$r(z_1,\dots,z_m)=\left\langle\sum_{i_1}\frac{1}{z_1-\lambda_{i_1}}\cdots\sum_{i_m}\frac{1}{z_m-\lambda_{i_m}}\right\rangle. \tag{32}$$

To solve this system of equations, we should rewrite the disconnected correlators in terms of the connected ones and replace the connected correlators with their Laurent expansion [27]. Thus, assuming that $\rho_1(z)=(1/g)\sum_{i=0}^{\infty}\rho_{1,i}(z)\cdot g^i$ (and the even parts of $\rho$ are hence associated with oriented surfaces, and the odd parts are associated with the nonoriented surfaces, with half-integer genera), we obtain the first few terms

$$\begin{aligned}
\rho_{1,0}(z)&=\frac{z}{2\beta}-\frac{y(z)}{2\beta}=\frac{1}{2\beta}(z-y(z)),\\
\rho_{1,1}(z)&=\frac{1/2-1/(2\beta)}{y(z)}+\frac{z/(2\beta)-z/2}{y^2(z)}=\frac{\beta-1}{2\beta y(z)}\left(1-\frac{z}{y(z)}\right),\\
\rho_{1,2}(z)&=\frac{5\beta^2\Lambda-9\beta\Lambda+5\Lambda}{y^5(z)}+\frac{\beta+1/\beta-2}{y^3(z)}+\frac{-\beta z-z/\beta+2z}{y^4(z)},\\
\rho_{1,3}(z)&=(\beta-1)\left(\frac{10-19\beta+10\beta^2}{2\beta y^5(z)}\left(1-\frac{z}{y(z)}\right)+\right.\\
&\quad\left.+\frac{5\Lambda(5-9\beta+5\beta^2)}{y^7(z)}-\frac{\Lambda z(30-43\beta+30\beta^2)}{y^8(z)}\right),\\
\rho_{1,4}(z)&=\frac{1}{y^7(z)}\left[37\beta^3-\frac{273\beta^2}{2}+199\beta+\frac{37}{\beta}-\frac{273}{2}\right]+\\
&\quad+\frac{1}{y^8(z)}\left[-37\beta^3z+\frac{273\beta^2z}{2}-199\beta z-\frac{37z}{\beta}+\frac{273z}{2}\right]+\\
&\quad+\frac{419\beta^4\Lambda-1357\beta^3\Lambda+1897\beta^2\Lambda-1357\beta\Lambda+419\Lambda}{y^9(z)}+\\
&\quad+\frac{-240\beta^4\Lambda z+824\beta^3\Lambda z-1168\beta^2\Lambda z+824\beta\Lambda z-240\Lambda z}{y^{10}(z)}+\\
&\quad+\frac{1105\beta^5\Lambda^2-3240\beta^4\Lambda^2+4375\beta^3\Lambda^2-3240\beta^2\Lambda^2+1105\beta\Lambda^2}{y^{11}(z)},\\
\rho_{1,5}(z)&=(\beta-1)\left[\frac{706-2379\beta+3367\beta^2-2379\beta^3+706\beta^4}{2\beta y^9(z)(1-z/y(z))}+\right.\\
&\quad+\frac{4351-13458\beta+18508\beta^2-13458\beta^3+4351\beta^4}{y^{11}(z)}-\\
&\quad-\frac{3\Lambda z(1530-4241\beta+5764\beta^2-4241\beta^3+1530\beta^4)}{y^{12}(z)}+\\
&\quad+\frac{55\beta\Lambda^2(221-648\beta+875\beta^2-648\beta^3+221\beta^4)}{y^{13}(z)}-\\
&\quad\left.-\frac{4\beta\Lambda^2z(3390-7883\beta+10420\beta^2-7883\beta^3+3390\beta^4)}{y^{14}(z)}\right],\\
\rho_{1,6}(z)&=\frac{1}{y^{11}(z)}\left[4081\beta^5-\frac{40405\beta^4}{2}+44699\beta^3-57155\beta^2+44699\beta+\frac{4081}{\beta}-\frac{40405}{2}\right]+
\end{aligned} \tag{33}$$

$$+\frac{1}{y^{12}(z)}\Big[-4081\beta^5 z+\frac{40405\beta^4 z}{2}-44699\beta^3 z+57155\beta^2 z-44699\beta z-$$
$$-\frac{4081z}{\beta}+\frac{40405z}{2}\Big]+\frac{1}{y^{13}(z)}\big[77597\beta^6\Lambda-340402\beta^5\Lambda+702694\beta^4\Lambda-$$
$$-878293\beta^3\Lambda+702694\beta^2\Lambda-340402\beta\Lambda+77597\Lambda\big]+$$
$$+\frac{1}{y^{14}(z)}\big[-59040\beta^6\Lambda z+269328\beta^5\Lambda z-564000\beta^4\Lambda z+707424\beta^3\Lambda z-$$
$$-564000\beta^2\Lambda z+269328\beta\Lambda z-59040\Lambda z\big]+\frac{1}{y^{15}(z)}\big[451720\beta^7\Lambda^2-$$
$$-1792898\beta^6\Lambda^2+3483419\beta^5\Lambda^2-4266464\beta^4\Lambda^2+3483419\beta^3\Lambda^2-$$
$$-1792898\beta^2\Lambda^2+451720\beta\Lambda^2\big]+\frac{1}{y^{16}(z)}\big[-189840\beta^7\Lambda^2 z+821128\beta^6\Lambda^2 z-$$
$$-1656256\beta^5\Lambda^2 z+2049936\beta^4\Lambda^2 z-1656256\beta^3\Lambda^2 z+821128\beta^2\Lambda^2 z-$$
$$-189840\beta\Lambda^2 z\big]+\frac{1}{y^{17}(z)}\big[828250\beta^8\Lambda^3-3012930\beta^7\Lambda^3+5531740\beta^6\Lambda^3-$$
$$-6644070\beta^5\Lambda^3+5531740\beta^4\Lambda^3-3012930\beta^3\Lambda^3+828250\beta^2\Lambda^3\big],$$

where $y^2(z)=z^2-4\Lambda\beta$ defines the spectral curve, which in this case is the torus with a degenerated handle (located at the infinity of the complex plane).

*The $\beta\to 1/\beta$ symmetry.* The AGT relation implies that the $\beta$-deformed matrix model should be related to some CFT with the central charge of the corresponding CFT given by

$$c=1-6\left(\sqrt{\beta}-\frac{1}{\sqrt{\beta}}\right)^2. \tag{34}$$

This suggests that there should be the symmetry $\beta\to 1/\beta$ in the matrix model although this is far from obvious in the original expression (3). Indeed, it is easily seen that if the quantities are rescaled as

$$z'=\sqrt{\beta}z,\qquad \rho'_{1,g}=(\sqrt{\beta})^{g+1}\rho_{1,g}, \tag{35}$$

then the resulting expressions $\rho'_{1,g}$ are symmetric under $\beta\to 1/\beta$.

## 4. Seiberg–Witten construction

**4.1. Ideology.** The SW construction, originally proposed to obtain the low-energy effective action in the $\mathcal{N}$=2 SUSY gauge theory, is in fact a manifestation of a more general principle.

The starting objects in the SW representation are the algebraic curve and the meromorphic differential $\lambda_{\mathrm{SW}}$ on it. Given such data, we can write the system of equations

$$\oint_{A_i}\lambda_{\mathrm{SW}}\sim a_i,\qquad \oint_{B_i}\lambda_{\mathrm{SW}}\sim\frac{\partial\mathcal{F}_{\mathrm{SW}}}{\partial a_i}, \tag{36}$$

where $A_i$ and $B_i$ form a symplectic basis of cycles on the algebraic curve and the proportionality coefficients in the equations depend slightly on the adopted setup.

It turns out that an important source of the initial SW data is the so-called eigenvalue models (EVMs). Namely, the role of the algebraic curve is played by the spectral curve of the given EVM, and the role of the SW differential is played by $\rho_1(z)\,dz$, where $\rho_1$ is the one-point resolvent. We note that the original SW construction corresponds to the zeroth order of the genus expansion of the resolvent in $g$, and taking further terms of the expansion into account corresponds to deforming (quantizing) the original SW differential and prepotential. Remarkably, not only the zeroth approximation but also all the free energy continues to satisfy the SW equations.

We fix the proportionality coefficients in the SW equations by

$$-\frac{1}{2\pi i}\oint_{A_i}\rho_1(z)\,dz = a_i, \qquad -\beta\oint_{B_i}\rho_1(z)\,dz = \frac{\partial\mathcal{F}_{\mathrm{SW}}}{\partial a_i}, \tag{37}$$

because the relation between the free energy and the SW prepotential is most transparent in such a normalization. We note that $\beta$ appears as a coefficient in the second equation [7].

**4.2. Calculation of A- and B-periods.** We now apply the SW construction to $\rho_1$ found in Sec. 3.4. The spectral curve is given by the equation

$$y^2 = z^2 - 4\Lambda\beta. \tag{38}$$

The A-cycle encircles the ramification points $-\sqrt{4\Lambda\beta}$ and $\sqrt{4\Lambda\beta}$, and the B-cycle encircles $\sqrt{4\Lambda\beta}$ and $\infty$.

Because the value of the A-period in this case is equal to the residue at infinity, the A-period receives contributions only from $\rho_{1,0}$ and $\rho_{1,1}$:

$$a = -\frac{1}{2\pi i}\oint_{-\sqrt{4\Lambda\beta}}^{\sqrt{4\Lambda\beta}}\rho(z)\,dz = N + \frac{1-\beta}{2\beta}. \tag{39}$$

At this point, we note that the dependence $a(N)$ is linear and $\partial/\partial a$ can be safely replaced with $\partial/\partial N$ in the SW equation for the B-period. We can therefore write everything in terms of $N$ to simplify the calculations in what follows.

Evaluating the B-periods is trickier. We must use the formula

$$\oint_{\sqrt{4\Lambda\beta}}^{+\infty}\frac{dz}{y^p(z)} = \frac{1}{2^{2p-3}(\Lambda\beta)^{(p-1)/2}}\frac{\Gamma(p-1)\Gamma(1-p/2)}{\Gamma(p/2)}. \tag{40}$$

To derive this formula, we must make the change of variables $z = ((2-\zeta)/\zeta)\sqrt{4\Lambda\beta}$ and note that the resulting integral is proportional to the integral representation for the Euler $B$-function:

$$\begin{aligned}\oint_{\sqrt{4\beta\Lambda}}^{+\infty}(z^2-4\beta\Lambda)^{-p/2}\,dz &= (4\beta\Lambda)^{-p/2+1/2}\oint_1^{+\infty}(w^2-1)^{-p/2}\,dw = \\ &= (4\beta\Lambda)^{-p/2+1/2}4^{-p/2}2\oint_0^1(1-\zeta)^{-p/2}\zeta^{p-2}\,d\zeta = \\ &= 2^{-2p+3}(\beta\Lambda)^{-p/2+1/2}\frac{\Gamma(1-p/2)\Gamma(p-1)}{\Gamma(p/2)}.\end{aligned} \tag{41}$$

The terms in (33) with odd powers of $z$ do not contribute to the periods, because they are total derivatives. For instance,

$$\oint_{\sqrt{4\Lambda\beta}}^{+\infty}\frac{z\,dz}{y^p(z)} = -\frac{1}{p-2}\oint_{\sqrt{4\Lambda\beta}}^{+\infty}d\biggl(\frac{1}{y^{p-2}(z)}\biggr) = 0. \tag{42}$$

We note that this is a contour integral and the contour does not pass through the singularities of the integrand.

Alternatively, we can use the fact that $y(z)$ and also its B-periods satisfy the differential equation

$$\frac{\partial}{\partial\Lambda} y^p = -2\beta p y^{p-2}. \tag{43}$$

Together with the initial conditions

$$\oint_{\mathrm{B}} \frac{dz}{y(z)} = -\log\beta\Lambda \tag{44}$$

and

$$\oint_{\mathrm{B}} y^p\,dz = 0|_{\Lambda=0}, \quad p \neq -1, \tag{45}$$

this gives the expressions for the integrals over the B-periods of odd powers of $y(z)$, which are presented in Table 1 (integrals of even powers vanish).

**Table 1**

| $n$ | $\oint_{\mathrm{B}} y^n\,dz$ | $\oint_{\mathrm{B}} y^{-n}\,dz$ |
|---|---|---|
| 1 | $-2\beta(\Lambda - \Lambda\log(\beta\Lambda))$ | $-\log(\beta\Lambda)$ |
| 3 | $-6\beta\left(\beta\Lambda^2\log(\beta\Lambda) - \frac{3\beta\Lambda^2}{2}\right)$ | $-\frac{1}{2\beta\Lambda}$ |
| 5 | $-10\beta\left(\frac{11\beta^2\Lambda^3}{3} - 2\beta^2\Lambda^3\log(\beta\Lambda)\right)$ | $\frac{1}{12\beta^2\Lambda^2}$ |
| 7 | $-14\beta\left(5\beta^3\Lambda^4\log(\beta\Lambda) - \frac{125\beta^3\Lambda^4}{12}\right)$ | $-\frac{1}{60\beta^3\Lambda^3}$ |
| 9 | $-18\beta\left(\frac{959\beta^4\Lambda^5}{30} - 14\beta^4\Lambda^5\log(\beta\Lambda)\right)$ | $\frac{1}{280\beta^4\Lambda^4}$ |
| 11 | $-22\beta\left(42\beta^5\Lambda^6\log(\beta\Lambda) - \frac{1029\beta^5\Lambda^6}{10}\right)$ | $-\frac{1}{1260\beta^5\Lambda^5}$ |
| 13 | $-26\beta\left(\frac{11979\beta^6\Lambda^7}{35} - 132\beta^6\Lambda^7\log(\beta\Lambda)\right)$ | $\frac{1}{5544\beta^6\Lambda^6}$ |

Hence, for the B-periods of $\rho_{1,i}$, we obtain the following (in the case of $\rho_{1,0}$ and $\rho_{1,1}$, we must evaluate the respective integrals for $p = -1+\epsilon$ and $1+\epsilon$ and then neglect the terms that diverge as $\epsilon \to 0$; this is safe because these terms are constant and linear in $\Lambda$):

$$\begin{aligned}
&\oint_{\mathrm{B}} \rho_{1,0}(z)\,dz = -\Lambda\log\Lambda, && \oint_{\mathrm{B}} \rho_{1,1}(z)\,dz = \frac{1-\beta}{2\beta}\log\Lambda,\\
&\oint_{\mathrm{B}} \rho_{1,2}(z)\,dz = \frac{-1+3\beta-\beta^2}{12\beta^2\Lambda}, && \oint_{\mathrm{B}} \rho_{1,3}(z)\,dz = \frac{1-\beta}{24\beta^2\Lambda^2},\\
&\oint_{\mathrm{B}} \rho_{1,4}(z)\,dz = \frac{1-5\beta^2+\beta^4}{360\beta^4\Lambda^3}, && \oint_{\mathrm{B}} \rho_{1,5}(z)\,dz = \frac{-1+\beta^3}{240\beta^4\Lambda^4},\\
&\oint_{\mathrm{B}} \rho_{1,6}(z)\,dz = \frac{-2+7\beta^2+7\beta^4-2\beta^6}{2520\beta^6\Lambda^5}.
\end{aligned} \tag{46}$$

Remarkably, although the complexity of $\rho_{1,i}$ increases very rapidly (exponentially) as $i$ increases, the complexity of their B-periods increases more slowly (linearly).

Already at this stage, it can be seen that these formulas agree with the formulas for general terms in [27],

$$\oint_{\mathrm{B}} \rho_{1,2m+2}\, dz = \sum_{s=0}^{m+1} B_{2m-2s} B_{2s} \frac{\Gamma(2m+1)}{\Gamma(2s+1)\Gamma(2m-2s+3)} \beta^{-2s} \frac{1}{N^{2m+1}}, \quad m \ge 0,$$
$$\oint_{\mathrm{B}} \rho_{1,2m+1}\, dz = \left(\frac{1}{2\beta} - \frac{1}{2\beta^{2m}}\right) \frac{B_{2m+2}(2m-1)}{(2m+1)(2m+2)} \frac{1}{N^{2m}}, \quad m \ge 1. \tag{47}$$

In [27], they were deduced from Eq. (51), which we now verify.

**4.3. Relation to free energy.** The partition function for the $\beta$-deformed Gaussian integral over eigenvalues is defined as

$$Z(N) = \frac{1}{N!} \int d\lambda_1 \, \dots \, d\lambda_N \prod_{i<j} |\lambda_i - \lambda_j|^{2\beta} \exp\left(-\frac{1}{2g} \sum_i \lambda_i^2\right) \tag{48}$$

and can be calculated explicitly. The corresponding generalization of expression (4) in the case $\beta \ne 1$ is (see [27])

$$Z(N) = (\sqrt{2\pi})^N (\sqrt{g})^{\beta N^2 + (1-\beta)N} \prod_{k=1}^{N} \frac{\Gamma(1+\beta k)}{\Gamma(1+\beta)} \cdot \frac{1}{\Gamma(N+1)}. \tag{49}$$

We are now ready to verify that the free energy

$$F(N) = \log Z \sim \sum_{k=1}^{N} \log \Gamma(1+\beta k) - \log N! \tag{50}$$

is exactly equal to the SW prepotential.

Indeed, we can calculate the derivative of $F(N)$ with respect to $N$ and apply the Euler–Maclaurin formula (see Eq. (72) in the appendix). As a result, we obtain

$$\frac{\partial}{\partial N} F\left(\frac{\Lambda}{g}\right) = \frac{1}{g} \beta \Lambda \log \Lambda + \frac{\beta - 1}{2} \log \Lambda + g \frac{1 - 3\beta + \beta^2}{12\beta\Lambda} + g^2 \frac{\beta - 1}{24\beta\Lambda^2} + g^3 \frac{-1 + 5\beta^2 - \beta^4}{360\beta^3\Lambda^3} +$$
$$+ g^4 \frac{1 - \beta^3}{240\beta^3\Lambda^4} + g^5 \frac{2 - 7\beta^2 - 7\beta^4 + 2\beta^6}{2520\beta^5\Lambda^5} + o\left(\frac{1}{\Lambda^5}\right).$$

This expression can now be compared with (45), taking the factor $-\beta$ in (37) into account. Finally, we obtain

$$F = \mathcal{F}_{\mathrm{SW}}. \tag{51}$$

The main statement in this paper is:

*The exact free energy of the Gaussian $\beta$-ensemble satisfies SW equations* (37) *with the exact resolvents playing the role of the SW differential.*

## Appendix: Toward an understanding of $\beta \ne 1$

In this appendix, we outline a few topics that are poorly understood but are crucially important for the future theory of $\beta$-ensembles.

**A.1. Integrability.** We saw that the free energy and resolvent at $\beta = 1$ satisfy integrable differential-difference equations (9) and (10). These equations are intimately related to the Toda integrable structure of the Gaussian matrix model (the Kadomtsev–Petviashvili hierarchy plays the role of the Toda hierarchy in the non-Gaussian case). In particular, the Toda equation can be written as

$$\frac{\partial^2}{\partial t_1^2} \log Z(N) = \frac{Z(N+1)Z(N-1)}{Z^2(N)}, \tag{52}$$

and, in terms of the free energy,

$$F(N+1) - 2F(N) + F(N-1) = \log\left(\frac{\partial^2}{\partial t_1^2} F(N)\right). \tag{53}$$

Differentiating with respect to $t_i$ and using the Virasoro constraints, we obtain

$$K_i(N+1) - 2K_i(N) + K_i(N-1) = \frac{i(i-1)}{N} K_{i-2}(N). \tag{54}$$

Summing these equations with the weights $1/z^{i+1}$, we obtain Eq. (9). Equation (10) can be obtained from (9) by integrating it along the B-period on the spectral curve.

A very important and intriguing question is what is the $\beta$-deformation determination of the integrability of the Toda or Kadomtsev–Petviashvili hierarchy, but it is difficult to answer directly. As we see, Eq. (10) can be $\beta$-deformed rather easily, but integrability requires more: we need a $\beta$-deformation of (9), which is still unknown.

**A.1.1. Difference equation for periods.** For $\beta = 1$, Eq. (10) is

$$\Pi_{\mathrm{B}}(\Lambda+1) - 2\Pi_{\mathrm{B}}(\Lambda) + \Pi_{\mathrm{B}}(\Lambda-1) = -\frac{1}{\Lambda}, \tag{55}$$

where $\Pi_{\mathrm{B}}(\Lambda)$ denotes the B-period of $\rho$. We can show (e.g., by expanding the left-hand side of the equation in the series in $1/\Lambda$) that for $\beta \neq 1$, this equation becomes

$$\Pi_{\mathrm{B}}\left(\Lambda + \frac{1}{\beta}\right) - \Pi_{\mathrm{B}}(\Lambda) - \Pi_{\mathrm{B}}\left(\Lambda + \frac{1-\beta}{\beta}\right) + \Pi_{\mathrm{B}}(\Lambda - 1) = -\frac{1}{\beta\Lambda}. \tag{56}$$

**A.1.2. Difference equation for resolvents.** Nevertheless, we have been unable to find a $\beta$-deformed analogue of (9), and even the corrections of the first order in $\beta - 1$ are lacking. We can say that in the required generalization, both sides of (9) are most probably deformed more strongly than in (56).

**A.2. Harer–Zagier topological recursion.** A detailed description of the Harer–Zagier functions for $\beta = 1$ can be found, for example, in [24]. Describing the matrix model correlators in terms of the resolvents has two advantages: it provides Ward identities (17) in the simple form of loop equations (22), and it reveals the important hidden structure, the spectral curve. The drawbacks are the divergence of genus expansion series and the lack of explicit formulas for the exact correlators (in terms of the coupling constant).

The last two problems can be solved, for example, by passing from the exact resolvents to the Harer–Zagier functions, where the correlators are summed with additional factorial factors, i.e., the Padé method.

Much less is known for $\beta \neq 1$. So far, we have been able to obtain the Harer–Zagier functions only for specific values of $\beta \neq 1$. Attempts to find at least the first correction in $\beta - 1$ led to some generalizations of the hypergeometric equations, which suggests that something conceptually new should be done for the results to become simple for arbitrary $\beta$. Our preliminary results are presented below.

The one-point Harer–Zagier generating function is defined as

$$\phi(z) = \frac{4\beta}{\tau^2-1}\sum_{k=0}^{\infty}\sum_{N=0}^{\infty} C_k\left(\frac{N}{\beta},\beta\right)\frac{z^k}{(2k-1)!!}\left(\frac{\tau-1}{\tau+1}\right)^N, \tag{57}$$

where $C_k(N/\beta,\beta)$ is the one-point correlator in the $\beta$-deformed matrix model with matrix size equal to $N/\beta$.

*The case* $\beta = 1$. The Harer–Zagler function has the form

$$\phi(\beta=1,z,\tau) = \frac{1}{1-\tau z^2}. \tag{58}$$

This is the classical result of Harer and Zagier. It satisfies the differential equation derived from the integrability conditions

$$\lambda\frac{\partial}{\partial\lambda}\left(\frac{(1-\lambda)^2}{\lambda}\varphi(\lambda,x)\right) = x\frac{\partial}{\partial x}(x^2\varphi(\lambda,x)), \tag{59}$$

where $\varphi = ((\tau^2-1)/4)\phi$ and $\lambda = (\tau-1)/(\tau+1)$.

It turns out that the two- and three-point Harer–Zagier functions can also be found, and they are expressed in terms of the arctan function [24], i.e., they remain elementary functions.

*The case* $\beta = 2$. The Harer–Zagier function for the $SO(N)$ matrix model has the form

$$\begin{aligned}\phi(\beta=2,z,\tau) = {}&\frac{\tau}{\tau-z-z^2\tau} + \\ &+\frac{\sqrt{z}(\tau+1)}{2(\tau-z-z^2\tau)^{3/2}}\arctan\left(\frac{2\sqrt{z}(z\tau-z-1)\sqrt{\tau-z-z^2\tau}}{1+(2-3\tau)z+(2-2\tau+2\tau^2)z^2}\right),\end{aligned} \tag{60}$$

and it satisfies

$$\left[\frac{2z+2\tau^2z+\tau}{2z} + (z-\tau+\tau^2z)\frac{\partial}{\partial z}\right]\phi(\beta=2,z,\tau) = \frac{\tau}{2z} + \frac{2+2\tau+2\tau^2z+\tau^2}{2(2z+1)(1-z\tau)} \tag{61}$$

with the initial conditions $\phi(\beta=2,z,\tau) = 1+(\tau-1)z+\dots$.

*The case* $\beta = 1/2$. The Harer–Zagier function for the $Sp(N)$ matrix model has the form

$$\phi(\beta=1/2,z,\tau) = \frac{1}{1-\tau z} + \sqrt{\frac{z}{1+\tau}}\frac{1}{(1-\tau z)^{3/2}}\arctan\left(\frac{2\sqrt{z+\tau z}\sqrt{1-z\tau}}{2-z-2\tau z}\right), \tag{62}$$

and it satisfies

$$\left[\left(\frac{1}{z}-3\tau-\frac{5}{2}\right) - z(1+\tau)\frac{\partial}{\partial z} + \frac{(1+\tau)(2-\tau z)}{z}\frac{\partial}{\partial\tau}\right]\phi\left(\beta=\frac{1}{2},z,\tau\right) = \frac{1}{z} \tag{63}$$

with the initial conditions $\phi(\beta=1/2,z,\tau) = 1+(\tau+1/2)z+\dots$.

It can be seen that in the two cases $\beta = 2, 1/2$, which correspond to classical groups, the Harer–Zagier functions are expressed in terms of arctangents. But this is not the case in the general situation.

*The case* $\beta = 3$. The Harer–Zagier function for $\beta = 3$ satisfies the differential equation

$$\begin{aligned}&(1+8z^2\tau+24z^2\tau^2+9z^3\tau-z\tau-6z-33z^2)\phi + (11z^2\tau-18z^3-2z+9z^4\tau)\frac{\partial\phi}{\partial z} + \\ &\qquad + (9z-12z^2\tau+12z^2\tau^3-9z\tau^2)\frac{\partial\phi}{\partial\tau} = 1-4z-4z\tau.\end{aligned} \tag{64}$$

At particular values of $z$, it becomes the hypergeometric equation and hence has no solutions expressed in terms of elementary functions. Consequently, what we presumably seek is some clever deformation of the arctan function from the previously described cases.

We observe that the complexity of results increases as we move further and further away from $\beta = 1$. Further work is needed to clarify the situation.

**A.3. Identities for free energy.** It turns out that for $\beta \neq 1$, the Gaussian free energy has more structure than might be expected.

**A.3.1. Definitions.** We define the partition function without the factor $1/N!$. To avoid ambiguities, we mark all the quantities in this normalization with tildes.

The partition function for the Gaussian model that we consider is

$$\widetilde{Z}(N,\beta) = \int d\lambda_1 \cdots d\lambda_N \prod_{i<j}(\lambda_i - \lambda_j)^{2\beta} \exp\biggl(-\frac{1}{2g}\sum_i \lambda_i^2\biggr) = N!Z(N,\beta). \tag{65}$$

Instead of (49), we now have

$$\widetilde{Z}(N,\beta) = (\sqrt{2\pi})^N(\sqrt{g})^{\beta N^2+(1-\beta)N} \prod_{k=1}^{N} \frac{\Gamma(1+\beta k)}{\Gamma(1+\beta)}. \tag{66}$$

The free energy is now defined as

$$\widetilde{F}(N,\beta) = \log \widetilde{Z} \sim \sum_{k=1}^{N} \log\Gamma(1+\beta k), \tag{67}$$

where the equivalence means equality up to terms quadratic and linear in the matrix size $N$ (they can be absorbed into a redefinition of $\beta$ and $g$).

**A.3.2. Difference equation.** The free energy defined above satisfies a certain difference equation. We consider

$$\widetilde{G}(N,\beta) = \widetilde{F}(N,\beta) - \widetilde{F}(N-1,\beta) = \log\Gamma(1+\beta N). \tag{68}$$

It is then obvious that

$$\widetilde{G}(N,\beta) - \widetilde{G}\biggl(N - \frac{1}{\beta}, \beta\biggr) = \log(\beta N), \tag{69}$$

which implies that

$$\frac{\partial}{\partial N}\widetilde{G}(N,\beta) - \frac{\partial}{\partial N}\widetilde{G}\biggl(N - \frac{1}{\beta}, \beta\biggr) = \frac{1}{N}. \tag{70}$$

**A.3.3. Exact relation between $\mathcal{F}_{\mathrm{SW}}$ and $\widetilde{F}$.** Comparing (56) and (70) yields

$$\mathcal{F}_{\mathrm{SW}}(N,\beta) = \widetilde{F}\biggl(N - \frac{1}{\beta}, \beta\biggr) = \sum_{k}^{N-1/\beta} \log\Gamma(1+\beta k), \tag{71}$$

and the only peculiarity is therefore the change in the upper summation limit. In the case $\beta = 1$, it becomes $N-1$ and acquires a clear physical meaning: dividing the partition function by $N!$ implies that the eigenvalues are pairwise indistinguishable bosons.

**A.3.4. Direct comparison of series.** Instead of the existence of difference equation (70), we can use simpler means. For example, we can consider expansions of the prepotential and free energy in the perturbation theory at large $N$. Looking at these expansions, we can advance several hypotheses on the relation between these quantities. One way to obtain these expansions is to use the Euler–Maclaurin formula.

*Euler–Maclaurin formula.* We need this formula in the form

$$\frac{\partial}{\partial N}\sum_{k}^{N-1} f(k) = f(N) - \frac{1}{2}f'(N) + \frac{1}{12}f''(N) - \frac{1}{720}f''''(N) - \cdots =$$

$$= \sum_{m=0}^{\infty} \frac{B_m}{m!}\,\partial^m f(N), \tag{72}$$

where $B_m$ are the Bernoulli numbers, $\sum(B_m/m!)t^m = t/(e^t-1)$. The lower summation limit is inessential because it is independent of $N$. In the following examples, we chose $k=0$:

$$f(k)=1, \qquad \frac{\partial}{\partial N}N = 1,$$

$$f(k)=k, \qquad \frac{\partial}{\partial N}\frac{N(N-1)}{2} = N - \frac{1}{2},$$

$$f(k)=k^2, \qquad \frac{\partial}{\partial N}\frac{N(N-1)(2N-1)}{6} = N^2 - N + \frac{1}{6} = N^2 - \frac{2N}{2} + \frac{2}{12},$$

$$f(k)=k^3, \qquad \frac{\partial}{\partial N}\frac{N^2(N-1)^2}{4} = N^3 - \frac{3}{2}N^2 + \frac{1}{2}N = N^3 - \frac{3N^2}{2} + \frac{6N}{12} + 0, \qquad \ldots.$$

*Various series, as is.* Here, all equalities are understood up to terms linear and constant in $N$ or $\Lambda$. Summing contributions from different genera, we find

$$\frac{\partial}{\partial N}\mathcal{F}_{\mathrm{SW}}\left(\frac{\Lambda}{g}\right) = \frac{1}{g}\beta\Lambda\log\Lambda + \frac{\beta-1}{2}\log\Lambda + g\frac{1-3\beta+\beta^2}{12\beta\Lambda} + g^2\frac{\beta-1}{24\beta\Lambda^2} +$$
$$+ g^3\frac{-1+5\beta^2-\beta^4}{360\beta^3\Lambda^3} + g^4\frac{1-\beta^3}{240\beta^3\Lambda^4} + g^5\frac{2-7\beta^2-7\beta^4+2\beta^6}{2520\beta^5\Lambda^5} + o\left(\frac{1}{\Lambda^5}\right). \tag{73}$$

Expanding $F$ at various points, we obtain

$$\frac{\partial}{\partial N}\widetilde{F}\left(\frac{\Lambda}{g}-\frac{1}{\beta}\right) = \frac{1}{g}\beta\Lambda\log\Lambda + \frac{\beta-1}{2}\log\Lambda + g\frac{1-3\beta+\beta^2}{12\beta\Lambda} + g^2\frac{\beta-1}{24\beta\Lambda^2} +$$
$$+ g^3\frac{-1+5\beta^2-\beta^4}{360\beta^3\Lambda^3} + g^4\frac{1-\beta^3}{240\beta^3\Lambda^4} + g^5\frac{2-7\beta^2-7\beta^4+2\beta^6}{2520\beta^5\Lambda^5} + o\left(\frac{1}{\Lambda^5}\right),$$

$$\frac{\partial}{\partial N}\widetilde{F}\left(\frac{\Lambda}{g}\right) = \frac{1}{g}\beta\Lambda\log\Lambda + \frac{1+\beta}{2}\log\Lambda + g\frac{1+3\beta+\beta^2}{12\beta\Lambda} - g^2\frac{1+\beta}{24\beta\Lambda^2} -$$
$$- g^3\frac{1-5\beta^2+\beta^4}{360\beta^3\Lambda^3} + g^4\frac{1+\beta^3}{240\beta^3\Lambda^4} + g^5\frac{2-7\beta^2-7\beta^4+2\beta^6}{2520\beta^5\Lambda^5} + o\left(\frac{1}{\Lambda^5}\right), \tag{74}$$

$$\frac{\partial}{\partial N}\widetilde{F}\left(\frac{\Lambda}{g}-1\right) = \frac{1}{g}\beta\Lambda\log\Lambda + \frac{1-\beta}{2}\log\Lambda + g\frac{1-3\beta+\beta^2}{12\beta\Lambda} + g^2\frac{1-\beta}{24\beta\Lambda^2} -$$
$$- g^3\frac{1-5\beta^2+\beta^4}{360\beta^3\Lambda^3} + g^4\frac{\beta^3-1}{240\beta^3\Lambda^4} + g^5\frac{2-7\beta^2-7\beta^4+2\beta^6}{2520\beta^5\Lambda^5} + o\left(\frac{1}{\Lambda^5}\right).$$

*Interpretation.* Looking at the series written above, we can easily propose the relations

$$\mathcal{F}_{\mathrm{SW}}\left(\frac{\Lambda}{g},\beta\right)=\widetilde{F}\left(\frac{\Lambda}{g}-\frac{1}{\beta},\beta\right)=-\widetilde{F}\left(-\frac{\Lambda}{g}-1,\beta\right)=-\widetilde{F}\left(\frac{\Lambda}{g},-\beta\right). \tag{75}$$

The first equality was expected from our previous analysis of the difference equation for the free energy.

It turns out that $\widetilde{F}$ with shifted arguments also satisfies the same difference equation. Indeed,

$$\begin{aligned}&-\widetilde{F}(N,-\beta)+\widetilde{F}(N-1,-\beta)=-\log\Gamma(1-\beta N),\\&-\log\Gamma(-\beta N)+\log\Gamma(1-\beta N)=\log(-\beta N).\end{aligned} \tag{76}$$

For $\widetilde{F}(-N-1,\beta)$, we must assume that the lower summation limit is less than $-N-1$ (which seems rather weird from the standpoint of common sense):

$$-\widetilde{F}(-N-1,\beta)+\widetilde{F}(-N,\beta)=-\log\Gamma(1-\beta N). \tag{77}$$

Normally, shifting the expansion point for some function does not lead to series similar to the initial series but produces something that looks completely different. The fact that this is not the case here may indicate that some mathematical structure not yet discovered is involved in this scenario. Perhaps, it is only a peculiar property of the Gaussian potential, but it might be that such equalities hold in more complicated cases. It would be interesting to see which of these unexpected identities survive generalization to non-Gaussian models.

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