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TOPOLOGICAL EXPANSION OF THE β-ENSEMBLE MODEL AND QUANTUM ALGEBRAIC GEOMETRY IN THE SECTORWISE APPROACH

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We construct the solution of the loop equations of the β-ensemble model in a form analogous to the solution in the case of the Hermitian matrices $\beta = 1$. The solution for $\beta = 1$ is expressed in terms of the algebraic spectral curve given by $y^2 = U(x)$. The spectral curve for arbitrary β converts into the Schrödinger equation $\big((\hbar\partial)^2 - U(x)\big)\psi(x) = 0$, where $\hbar \propto \big(\sqrt{\beta} - 1/\sqrt{\beta}\big)/N$. The basic ingredients of the method based on the algebraic solution retain their meaning, but we use an alternative approach to construct a solution of the loop equations in which the resolvents are given separately in each sector. Although this approach turns out to be more involved technically, it allows consistently defining the $\mathcal{B}$-cycle structure for constructing the quantum algebraic curve (a D-module of the form $y^2 - U(x)$, where $[y, x] = \hbar$) and explicitly writing the correlation functions and the corresponding symplectic invariants $\mathcal{F}_h$ or the terms of the free energy in an $1/N^2$-expansion at arbitrary $\hbar$. The set of "flat" coordinates includes the potential times t_k and the occupation numbers $\widetilde{\epsilon}_\alpha$. We define and investigate the properties of the $\mathcal{A}$- and $\mathcal{B}$-cycles, forms of the first, second, and third kinds, and the Riemann bilinear identities. These identities allow finding the singular part of $\mathcal{F}_0$, which depends only on $\widetilde{\epsilon}_\alpha$.

Keywords: Schrödinger equation, Bergman kernel, correlation function, Riemann identity, flat coordinates, Riccati equation

Contents

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1. Introduction

In contemporary mathematical physics, the notion of quantum surfaces is rather often encountered, appearing in many different guises. Having no intention to describe all problems in which quantization of the space–time coordinates themselves occurs (which pertains mainly to string or brane models), we nevertheless stress that the main feature of most, if not all, of these models is that the consideration is commonly restricted to the simple geometry of the sphere or torus. The observables in these theories are not the coordinates themselves, which cease to commute with each other and satisfy some postulated quantum algebras, but objects related to representations of these algebras, because only these objects admit a classical interpretation. We here propose a new approach for describing these so-called "quantum surfaces," namely, we begin with solutions of the standard one-dimensional Schrödinger equation with a polynomial potential and construct a higher-genus quantum surface (which is an analogue of a classical hyperelliptic Riemann surface) for which we can define analogues of all the main notions of algebraic geometry.

This paper is an "alternative version" of our paper where we introduced the notion of quantum algebraic geometry [1]. In both versions, the origin of quantum algebraic geometry is the same: the Schrödinger equation $\big((\hbar\partial)^2 - U(x)\big)\psi(x) = 0$. The principal difference is that we here define all quantities starting from the one-point resolvents differently in different sectors, i.e., these resolvents are constructed based on different solutions of the Schrödinger equation in different Stokes sectors of the complex plane. This allows rigorously defining the integrations over $\mathcal{A}$- and $\mathcal{B}$-cycles and also presenting a self-consistent procedure for constructing the correlation functions and symplectic invariants.

The correlation functions $W_n^{(h)}(x_1,\dots,x_n)$ and the symplectic invariants $\mathcal{F}_h$ for any planar algebraic curve given by a polynomial equation

$$\mathcal{E}(x,y) = \sum_{i,j} \mathcal{E}_{i,j} x^i y^j = 0$$

were defined in [2], [3]. The invariants $\mathcal{F}_h(\mathcal{E})$ are described in terms of algebraic geometry based on the Riemann surface of the equation $\mathcal{E}(x,y) = 0$. On the matrix model side, these invariants are terms of the $1/N^2$-expansion (the genus expansion) of the free energy calculated in [4] for the one-matrix model and in [5] for the two-matrix model.

We introduce the notion of a "quantum curve" for which $\mathcal{E}(x,y)$ is a noncommutative polynomial of x and y:

$$\mathcal{E}(x,y) = \sum_{i,j} \mathcal{E}_{i,j} x^i y^j, \qquad [y,x] = \hbar.$$

The notion of a quantum curve is also known as *D-modules*, i.e., a quotient of the space of functions by $\operatorname{Ker}\mathcal{E}(x,y)$, where $y = \hbar\partial/\partial x$.

Our construction is based on functions $\psi(x)$ satisfying $\mathcal{E}(x,\hbar\partial_x)\psi(x) = 0$, and we show that all the basic notions of algebraic geometry can be consistently defined within this construction. Although some objects, branch points for example, lose their explicit meaning, we can define cycles, forms, Bergman kernels, period matrices and the corresponding Abel maps and also other objects consistently. It is nevertheless quite surprising that almost all relations of classical algebraic geometry, for instance, the Riemann bilinear identity, the modified Rauch variational formula, and the topological recurrence relations defining the correlation functions and symplectic invariants, retain their significance for $\hbar \neq 0$.

The symplectic invariants $\mathcal{F}_h$ were first introduced for the solution of loop equations arising in the 1-Hermitian random matrix model [2], [4]. They were later generalized to other Hermitian multimatrix models [5], [6].

The models corresponding to the quantum surface are the *β-ensembles* classified by the exponent β. The three Wigner ensembles (see [7] with the change $\beta \to \beta/2$) correspond to $\beta = 1$ (Hermitian matrix case), $\beta = 1/2$ (real symmetric matrix case), and $\beta = 2$ (real self-dual quaternion matrix case), but we can easily define a β-ensemble eigenvalue model for any real value of β as the N-fold integral of the form

$$\int d\lambda_1 \cdots d\lambda_N |\Delta(\lambda)|^{2\beta} e^{-N\sqrt{\beta}\sum_{j=1}^N V(\lambda_j)},$$

where Δ is the Vandermonde determinant.

The solution in [2] was generalized in [8] to the β-ensemble models, but the solution was given as a double half-infinite sum for $\beta = O(1)$ at large N,

$$\mathcal{F} = \sum_{h,k=0}^{\infty} N^{2-2h-k}\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right)^{k}\mathcal{F}_{h,k}.$$

The coefficients $\mathcal{F}_{h,k}$ in this series were computed in [8].

Here as in [1], we assume that $\hbar = (\sqrt{\beta} - 1/\sqrt{\beta}\,)/N$, and we therefore perform an (infinite) resummation in the above formula; the free-energy expansion then takes the standard form

$$\mathcal{F} = \sum_{h=0}^{\infty} N^{2-2h}\mathcal{F}_h(\hbar).$$

The coefficients $\mathcal{F}_{h,k}$ in [8] can be recovered by computing the semiclassical small $\hbar$-expansion of $\mathcal{F}_h(\hbar)$. We demonstrate that $\mathcal{F}_h(\hbar)$ is the natural generalization of the symplectic invariants in [3] for a "quantum spectral curve" $\mathcal{E}(x,y)$ with $[y,x] = \hbar$.

We also define analogues of the multipoint resolvents

$$W_n(x_1,\ldots,x_n) = N^{-n}\left\langle \sum_{j=1}^{N}(x_1-\lambda_j)^{-1}\cdots\sum_{j=1}^{N}(x_n-\lambda_j)^{-1}\right\rangle_{\text{conn}},$$

where the angle brackets denote averaging with the weight $|\Delta(\lambda)|^{2\beta}e^{-N\sqrt{\beta}\sum_{j=1}^{N}V(\lambda_j)}$ and the subscript "conn" means that we select the connected part of the correlator. These resolvents themselves admit the $1/N^2$-expansion in the form

$$W_n(x_1,\ldots,x_n) = \sum_{h=0}^{\infty} N^{2-2h-n}W_n^{(h)}(x_1,\ldots,x_n).$$

Here, we calculate all the terms $W_n^{(h)}$ using the modified diagram technique.

The main tool used to study the β-eigenvalue model is the loop equation method. We obtain loop equations from the invariance of an integral under a special change of variables. Loop equations for the β-eigenvalue model were obtained in [9], [10], and we solve them here order by order of the perturbation theory in $1/N^2$ with a fixed $\hbar$.

Models of the indicated type recently received a new impulse for development from the Alday, Gaiotto, and Tachikawa (AGT) conjecture [11] relating Nekrasov's instanton function [12] to conformal blocks of the Liouville theory. These conformal blocks can in turn be described by a matrixlike model (see [13], [14]). The relation to the Nekrasov parameters $\epsilon_{1,2}$ is explicit: $\epsilon_1\epsilon_2 \sim 1/N^2$ and $\epsilon_1/\epsilon_2 \sim \beta$. Therefore, using the approach developed here, we can construct *nonperturbative* solutions of Nekrasov's formulas in ϵ_1/ϵ_2. Here, we investigate only the case of polynomial potentials; the generalization to the realistic logarithmic potentials appearing in the AGT conjecture will be the subject of a subsequent publication.

This paper has the following structure. We collect the generalities on the Stokes phenomenon pertaining to solutions of the Schrödinger equation in Sec. 2. We describe our quantum Riemann surface in Sec. 3, where we introduce $\mathcal{A}$- and $\mathcal{B}$-cycles, filling fractions $\widetilde{\epsilon}_i$, and the first-kind functions (analogues of holomorphic and Krichever–Whitham meromorphic differentials) and also the system of flat coordinates and the Riemann period matrix. In Sec. 4, we introduce the recursion kernels and the second- and third-kind (bi)differentials. In Sec. 5, we go beyond the leading approximation in $1/N^2$ and construct correlation functions of all orders using a Feynman-like diagram technique. We reveal the origin of our recursive procedure in Sec. 6, where we develop the variations with respect to the set of flat coordinates in detail. A summary of the results is contained in Sec. 7. In the next two sections (completely new compared with [1]), we investigate the link to the β-ensemble models (Sec. 8) and construct the free-energy terms based on this analysis (Sec. 9). In Appendixes A–C, we prove the three main theorems in Sec. 5 concerning properties of the correlation functions, and Appendix D contains a new formula expressing $\mathcal{F}_0$ in terms of the filling fractions $\widetilde{\epsilon}_i$. In matrix models, the singular term has the structure $(\widetilde{\epsilon}_i^2/2)\log\widetilde{\epsilon}_i$. In the quantum geometry, this term turns out to be proportional to the integral of $\log\Gamma(\widetilde{\epsilon}_i)$, which is the first actual example of calculations in the case of quantum Riemann surfaces.

2. Schrödinger equation and resolvents

2.1. Solutions of the Schrödinger equation. We begin with the Schrödinger equation

$$\hbar^2\psi''(x) = U(x)\psi(x), \tag{2.1}$$

where $U(x)$ is a polynomial of even degree $2d$ for which we define the polynomial "potential" $V(x)$ of degree $d+1$ by

$$V'(x) = 2(\sqrt{U})_+ = \sum_{k=0}^{d} t_{k+1}x^k, \tag{2.2}$$

where $(\,\cdot\,)_+$ denotes the positive part of the Laurent series. In the matrix model language (see Sec. 8), $t_1,\dots,t_{d+1}$ are called the *times* associated with the potential $V(x)$. We also define the polynomial of degree $d-1$

$$P(x) = \frac{(V')^2(x)}{4} - U(x) - \hbar\frac{V''(x)}{2}. \tag{2.3}$$

Finally, we introduce

$$t_0 = \lim_{x\to\infty}\frac{xP(x)}{V'(x)}, \tag{2.4}$$

which is the normalized total number of eigenvalues (particles) or the temperature. The remaining coefficients of P are fixed by introducing the "filling fractions" ϵ_i below.

2.1.1. Stokes sectors. A function $\psi(x)$ that is a solution of the Schrödinger equation exhibits the Stokes phenomenon: although $\psi(x)$ is an entire function, its asymptotic behavior is discontinuous near ∞, where it has an essential singularity. Let $\theta_0 = \operatorname{Arg}(t_{d+1})$ be the argument of the leading coefficient of the potential $V(x)$. We define the Stokes rays

$$L_k = \left\{x\colon\ \operatorname{Arg}(x) = -\frac{\theta_0}{d+1} + \pi\frac{k+1/2}{d+1}\right\},$$

along which $\operatorname{Re}V(x)$ vanishes asymptotically, together with the corresponding Stokes sectors

$$S_k = \left\{x\colon\ \operatorname{Arg}(x) \in \left] -\frac{\theta_0}{d+1} + \pi\frac{k-1/2}{d+1},\ -\frac{\theta_0}{d+1} + \pi\frac{k+1/2}{d+1}\right[\right\},$$

i.e., S_k is the sector between L_{k-1} and L_k.

We note that asymptotically, $\operatorname{Re}V(x) > 0$ in even sectors and $\operatorname{Re}V(x) < 0$ in odd sectors.

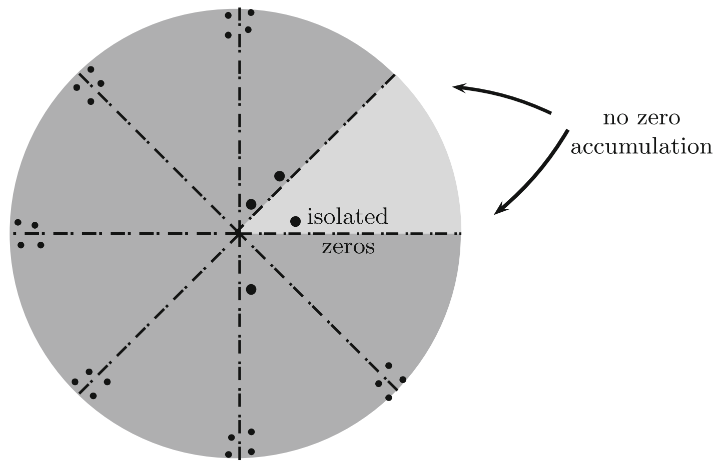


Fig. 1. Example of the Stokes sector partition and structure of zeros for the Schrödinger equation solution $\psi(x)$ that decreases in the light-colored sector and increases in all other sectors (the degree of the potential $V(x)$ is four).

2.1.2. The Stokes phenomenon: Decreasing solution. Investigations of the Schrödinger equation show that $\psi(x)$ is an entire function with the large-x expansion in each sector S_k

$$\psi(x) \underset{S_k}{\sim} e^{\pm V(x)/2\hbar} x^{C_k} \left(A_k + \frac{B_k}{x} + \dots \right), \tag{2.5}$$

where the sign $\pm$ can change in the passage from one sector to another as can the numbers $A_k, B_k, C_k, \dots$ (and in the general case, all the coefficients of the series in $1/x^j$ at infinity).[1] In each sector S_k, there exists a unique solution that decreases exponentially along each asymptotic direction inside the sector. We now separate solutions in the even and odd sectors and consider the set $\{\psi_\alpha(x)\}$ of solutions each of which decreases in the corresponding even sector. We thus introduce a *sectorwise* system of solutions of the Schrödinger equation.

The Stokes theorem is a useful result, stating that if the asymptotic value of $\psi(x)$ is exponentially small in some sector, then the same asymptotic series expansion (2.5) holds in the two adjacent sectors (and $\psi(x)$ therefore increases exponentially in those two sectors).

In the general position case, (i.e., for a potential $U(x)$ of general form), the solution $\psi_\alpha(x)$ decreases only in the sector S_α and is exponentially large in all other sectors (see Fig. 1). But if the Schrödinger potential $U(x)$ is special, then there may exist several sectors where $\psi_\alpha(x)$ is exponentially small (which means that $\psi_{\alpha_1}(x) = \psi_{\alpha_2}(x)$ for some $\alpha_1 \neq \alpha_2$).

In what follows, we mainly consider the general position case and therefore assume all the functions ψ_α are different unless otherwise stated.

The case studied in [15] is the most degenerate case, where the same solution ψ is exponentially small in $d+1$ sectors.

2.1.3. Zeros of ψ. Every $\psi_\alpha(x)$ is an entire function with an essential singularity at ∞ and with isolated zeros $s_i^{(\alpha)}$. The number of these zeros can be finite or infinite. In the latter case, zeros can accumulate only near ∞ and only along the Stokes rays L_j bordering the sectors (see Fig. 1). This zero accumulation along the ray L_j occurs if and only if $\psi_\alpha(x)$ is exponentially large on both sides of the ray. Therefore, no accumulation of zeros of the function $\psi_\alpha(x)$ occurs along the lines that border the α sector,

[1] The corresponding series is asymptotic and hence cannot be continued analytically to other sectors.

and this function can therefore have only a finite number of zeros inside the "larger" sector when we join the α sector with the adjacent parts of the two neighboring sectors.

If $U(x)$ is general, then each $\psi_\alpha(x)$ has an infinite number of zeros, the zeros accumulate at ∞ along all rays L_j with $j \neq \alpha, \alpha-1$. An important property of any solution ψ_α is that

$$\operatorname*{Res}_{s_i^{(\alpha)}} \frac{1}{\psi_\alpha^2(x)} = 0. \tag{2.6}$$

In [1], we define the genus of the Schrödinger equation to be related to the number of rays of zero accumulation of a selected function ψ_0. But this definition depends on the scheme, and we can in principle obtain different genera for the same function $U(x)$. The clear understanding of this is still lacking; a possible explanation is that we actually deal with different sections of an ambient infinite-genus quantum surface.

2.1.4. Sheets. In sector S_α, we have the asymptotic expansion

$$\psi_\alpha(x) \sim e^{-\hbar V(x)/2} x^{t_0/\hbar}\biggl(A_\alpha + \frac{B_\alpha}{x} + \dots\biggr),$$

and the function ψ_α has the same asymptotic expansion in the two adjacent sectors. We consider an α sheet of the quantum Riemann surface to be the union of these three sectors with a possible analytic continuation into a bounded domain of the complex plane. We consider only the sheets enumerated by even α and, in contrast to [1], consider them on an equal footing: they are all equivalent in the approach developed here. Sheets obviously overlap; we must introduce boundaries (cuts) between them.

2.2. Resolvent. The first ingredient of our strategy is to define a resolvent similar to the one in matrix models. We define the resolvent sectorwise: for $x \in S_\alpha$,

$$\omega(\overset{\alpha}{x}) = \hbar \frac{\psi_\alpha'(x)}{\psi_\alpha(x)} + \frac{V'(x)}{2}, \tag{2.7}$$

where a quantity defined sectorwise is indicated by setting the sector index above the variable as for the argument of the resolvent. It follows from this definition that $\omega(x)$ has simple poles at zeros of ψ_α in the corresponding sector. The boundaries between sectors overlap, but we fix them more explicitly in what follows (see the picture of the partition of the complex plane by $\mathcal{A}$-cycles). A straightforward computation then gives

$$\omega(\overset{\alpha}{x}) \sim \frac{t_0}{x} + O\biggl(\frac{1}{x^2}\biggr), \quad x \to \infty_\alpha, \infty_{\alpha\pm1},$$

i.e., the resolvent in each sheet has the asymptotic properties of a standard matrix-model resolvent.

The main property of $\omega(\overset{\alpha}{x})$ is that it satisfies the Riccati equation. We have

$$\begin{aligned} V'(x)\omega(\overset{\alpha}{x}) - \omega^2(\overset{\alpha}{x}) - \hbar\omega'(\overset{\alpha}{x}) &= \frac{(V')^2(x)}{4} - \hbar^2 \frac{\psi_\alpha''(x)}{\psi_\alpha(x)} - \hbar\frac{V''(x)}{2} = \\ &= \frac{(V')^2(x)}{4} - U(x) - \hbar\frac{V''(x)}{2} = P(x), \end{aligned} \tag{2.8}$$

where $P(x)$ is a polynomial of degree $d-1$ in x, and this polynomial is the same for all sheets of the quantum Riemann surface introduced below.

3. Quantum Riemann surface

In this section, we define the notions of $\mathcal{A}$- and $\mathcal{B}$-cycles and the first-kind differentials dual to them.

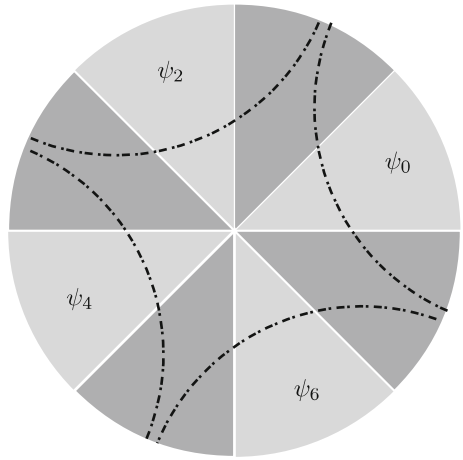


Fig. 2. The original integration contour $\mathcal{C}_{\mathrm{D}}$.

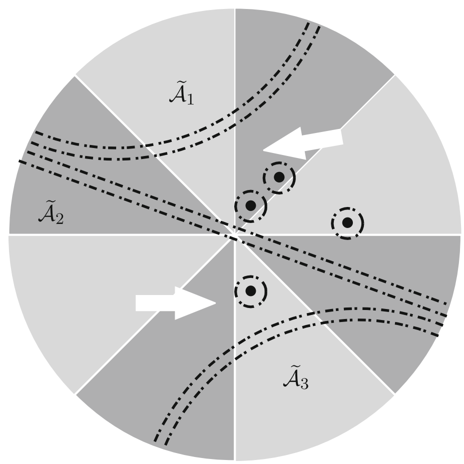


Fig. 3. Example of pushing the contour $\mathcal{C}_{\mathrm{D}}$ from infinities to the set of $\widetilde{\mathcal{A}}$-cycles.

3.1. The integration contour $\mathcal{C}_{\mathrm{D}}$ and the set of $\mathcal{A}$- and $\mathcal{B}$-cycles. In papers on matrix models (at an early stage before coming to residues at the branch points), we have the special integration contour $\mathcal{C}_{\mathrm{D}}$ that encircles all the singularities of resolvents, not considering all other possible singular points. The analogue of such a contour in our case is the union of $d+1$ contours, one per sheet, that pairwise coincide in far asymptotic domains of odd Stokes sectors and separate all the zeros of the function ψ_α from the infinity ∞_α (which is always possible because we have a finite number of zeros in each sheet). We have

$$\oint_{\mathcal{C}_{\mathrm{D}}} dx\, f(x) \equiv \sum_{\alpha} \int_{\infty_{\alpha-1}}^{\infty_{\alpha+1}} dx\, f(\overset{\alpha}{x}) \tag{3.1}$$

for *any* function $f(\overset{\alpha}{x})$ that has no asymptotic zero accumulation along the boundary rays of the sector S_α (see Fig. 2). Here and hereafter, we assume that $f(\overset{\alpha}{x})$ may depend on a finite number of derivatives of $\psi_\alpha(x)$; the symbol $f(\overset{\alpha}{x})$ then indicates that we substitute the solution $\psi_\alpha(x)$ as an argument.

We now deform the integration contour $\mathcal{C}_{\mathrm{D}}$ pushing it through the "middle" part of the complex plane and taking the residues at the zeros $s_i^{(\alpha)}$ of the corresponding functions ψ_α as shown in Fig. 3. On the way, we might break some contours, representing them as the unions of newly introduced contours all of which are stretched between different asymptotic directions. As a result, we obtain a system of exactly $2d$ contours in which (not considering the residues at zeros $s_i^{(\alpha)}$) all the contours are pairwise identified and represent edges of d "cuts." As a result, we obtain a complete system of d cuts $\widetilde{\mathcal{A}}_i$, $i = 1, \dots, d$, that separate all the odd-numbered infinities[2] and determine the corresponding sheets of the quantum Riemann surface. If the functions $\psi_\alpha(x)$ coincide for some sheets, then we can identify these sheets. We note that we definitely have an arbitrariness in constructing this system of cuts; we can also arbitrarily assign the residues inside the sheet to belong to one of several contours bounding this sheet.

We call the cut separating two sheets a *cycle* $\widetilde{\mathcal{A}}_\alpha$, and it is characterized by four indices: α_+ and α_- are indices of the sheets separated by this cut (they are even numbered in our classification); $\widetilde{\alpha}_+$ and $\widetilde{\alpha}_-$ are indices of infinities that are asymptotic for this cut (they are odd numbered).

Into correspondence with each complete set $\{\widetilde{\mathcal{A}}_\alpha\}_{\alpha=1}^d$ of $\widetilde{\mathcal{A}}$-cycles, we uniquely assign the set $\{\widetilde{\mathcal{B}}_\alpha\}_{\alpha=1}^d$ of $\widetilde{\mathcal{B}}$-cycles that go pairwise between the even-numbered infinities (α_+ and α_-) such that the intersection index $\widetilde{\mathcal{A}}_\alpha \circ \widetilde{\mathcal{B}}_\beta = \delta_{\alpha,\beta}$.

Definition 3.1. We define the integrals over the cycles $\widetilde{\mathcal{A}}_\alpha$ and the conjugate cycle $\widetilde{\mathcal{B}}_\alpha$ as (see Fig. 4)

$$\oint_{\widetilde{\mathcal{A}}_\alpha} dx\, f(x) \overset{\text{def}}{=} \int_{\infty_{\widetilde{\alpha}_-}}^{\infty_{\widetilde{\alpha}_+}} dx\, \bigl(f(\overset{\alpha_+}{x}) - f(\overset{\alpha_-}{x})\bigr) + \sum \operatorname*{Res}_{s_i^{(\alpha\pm)}(\alpha)} f(\overset{\alpha_\pm}{x}), \tag{3.2}$$

$$\oint_{\widetilde{\mathcal{B}}_\alpha} dx\, f(x) \overset{\text{def}}{=} \int_{\infty_{\alpha_-}}^{\infty_{\alpha_+}} dx\, \bigl(f(\overset{\alpha_+}{x}) - f(\overset{\alpha_-}{x})\bigr), \tag{3.3}$$

where the residues in the first expression are taken at those zeros of $\psi_{\alpha_\pm}$ that are assigned to the corresponding contour.

Because the prescription for the sheet assignment follows from definitions (3.2) and (3.3) of the cycle integrals, we omit the sheet labels in the corresponding integrands in what follows.

Remark 3.1. The assignment of residues in the α sheet to the contours bounding this sheet is arbitrary; we therefore have a (discrete) ambiguity in definition (3.2) of the $\widetilde{\mathcal{A}}$-cycle integrals. But the notion of the integral over $\mathcal{C}_{\mathrm{D}}$ is well defined and is independent of the choice of the $\widetilde{\mathcal{A}}$-cycles. Obviously,

$$\oint_{\mathcal{C}_{\mathrm{D}}} dx\, f(x) = \sum_{i=1}^{d} \oint_{\widetilde{\mathcal{A}}_i} dx\, f(x).$$

We now introduce the "genuine" $\mathcal{A}$- and $\mathcal{B}$-cycles, which are direct analogues of the set of $\mathcal{A}$- and $\mathcal{B}$-cycles on a standard Riemann surface. For this, we select one among the $\widetilde{\mathcal{A}}$-cycles, for example, the cycle $\widetilde{\mathcal{A}}_d$ and the conjugate cycle $\widetilde{\mathcal{B}}_d$. We then identify $\mathcal{A}_i = \widetilde{\mathcal{A}}_i$ and $\mathcal{B}_i = \widetilde{\mathcal{B}}_i - \widetilde{\mathcal{B}}_d$, $i = 1, \dots, d-1$, in the sense of Definition 3.1, i.e.,

$$\begin{aligned} \oint_{\mathcal{A}_i} dx\, f(x) &\overset{\text{def}}{=} \oint_{\widetilde{\mathcal{A}}_i} dx\, f(x), \\ \oint_{\mathcal{B}_i} dx\, f(x) &\overset{\text{def}}{=} \oint_{\widetilde{\mathcal{B}}_i} dx\, f(x) - \oint_{\widetilde{\mathcal{B}}_d} dx\, f(x), \quad i = 1, \dots, d-1, \end{aligned} \tag{3.4}$$

and we call the number $g = d - 1$ of independent $\mathcal{A}$- and $\mathcal{B}$-cycles the *genus* of the quantum Riemann surface.

The newly introduced $\mathcal{A}$- and $\mathcal{B}$-cycles again satisfy the standard intersection formula $\mathcal{A}_\alpha \cap \mathcal{B}_\beta = \delta_{\alpha,\beta}$, and most of our construction features depend only on the homology class of the paths $\mathcal{A}_\alpha$ and $\mathcal{B}_\alpha$ at the asymptotic infinities, but in the intermediate considerations, it is useful to choose a representative, the intersection point P_α, $\mathcal{A}_\alpha \cap \mathcal{B}_\alpha = \{P_\alpha\}$.

[2] In what follows, we identify an infinity "point" with the corresponding number with the related asymptotic direction.

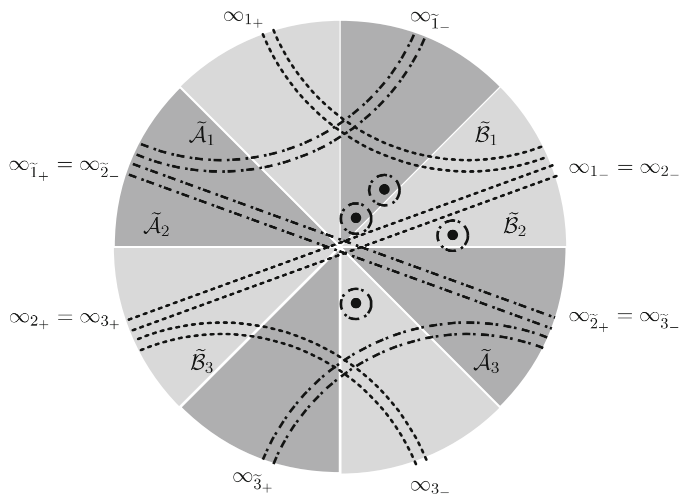


Fig. 4. The pattern of $\widetilde{\mathcal{A}}$-cycles (dashed lines) and $\widetilde{\mathcal{B}}$-cycles (dotted lines) for the example in Fig. 3.

3.2. Filling fractions. In random matrix models, the notion of filling fractions is just the $\mathcal{A}$-cycle integrals of the resolvent. If the $\mathcal{A}$-cycles are chosen to be in the physical sheet (which is possible, for example, in the hyperelliptic case), then the discontinuity of the resolvent along the corresponding cuts determines the eigenvalue density, and the $\mathcal{A}$-cycle integrals determine the portions of eigenvalues lying on the corresponding interval of the eigenvalue distribution. They are therefore called the filling fractions.

In the case of the quantum surface, we define the "filling fractions" $\widetilde{\epsilon}_\alpha$ as

$$\widetilde{\epsilon}_\alpha = \frac{1}{2i\pi} \oint_{\widetilde{\mathcal{A}}_\alpha} dx\, \omega(x) \stackrel{\text{def}}{=} \int_{\infty_{\tilde{\alpha}_-}}^{\infty_{\tilde{\alpha}_+}} dx \left(\omega(\overset{\alpha_+}{x}) - \omega(\overset{\alpha_-}{x})\right), \quad \alpha = 1, \ldots, d. \tag{3.5}$$

We note that this definition depends on where we place the contours and (in the case where a sheet is bounded by more than one $\widetilde{\mathcal{A}}$-cycle) we also have a freedom to assign residues inside the sheet to different $\widetilde{\mathcal{A}}$-cycles. Therefore, the filling fractions are defined up to integers times $\hbar$.

For the difference in the right-hand side of (3.5), we have

$$\omega(\overset{\alpha_+}{x}) - \omega(\overset{\alpha_-}{x}) = \frac{w_{\alpha_+,\alpha_+}}{\psi_{\alpha_+}(x)\psi_{\alpha_-}(x)},$$

where $w_{\alpha_+,\alpha_+} = \psi'_{\alpha_+}\psi_{\alpha_-} - \psi'_{\alpha_-}\psi_{\alpha_+}$ is the Wronskian of the two solutions. Therefore, this difference decreases exponentially in sectors where both the solutions ψ_{α_+} and ψ_{α_-} increase, and we can identify the asymptotic domains of the $\mathcal{A}$-cycle integrals with "branch points."

We have $\sum_{\alpha=1}^{d} \widetilde{\epsilon}_\alpha = t_0$, which follows from the simple fact that summing the integrals over $\widetilde{\mathcal{A}}$-cycles is equivalent to integrating over $\mathcal{C}_{\mathrm{D}}$. This also means that we should take only $d-1=g$ of the variables $\widetilde{\epsilon}_1, \ldots, \widetilde{\epsilon}_d$ as independent variables if we regard t_0 as an independent variable, and we naturally choose these g variables ϵ_α to be the filling fractions corresponding to the cycles $\mathcal{A}_\alpha$ of the quantum Riemann surface.

Remark 3.2. In the case $g = -1$ in [1], the only filling fraction is $\widetilde{\epsilon}_d = t_0$, and it is given by the (finite) sum of residues of the function ω at the zeros s_i,

$$\widetilde{\epsilon}_d = t_0 = \sum_i \operatorname*{Res}_{s_i} \omega = \hbar\, \#\{s_i\},$$

and t_0 is hence discrete in this case. For $g \geq 0$, the variables $\widetilde{\epsilon}_\alpha$, $\alpha = 1, \ldots, g$, and t_0 can take arbitrary, not necessarily integer, values.

3.3. First-kind functions. After defining the cycles, the next important step is to define the first-, second-, and third-kind differentials. We begin by defining the first-kind differentials.

Let h_k, $k = 1, \dots, d-1$, be a basis in the complex vector space of polynomials of degree $\le d-2$. We introduce the functions

$$v_k(\overset{\alpha}{x}) = \frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} dx'\, h_k(x')\psi_\alpha^2(x'). \tag{3.6}$$

We use the same polynomial $h_k(x')$ for all sheets of the Riemann surface.

We note that because every $\psi_\alpha(x)$ is a solution of the Schrödinger equation, $v_k(\overset{\alpha}{x})$ has double poles with zero residues at the $s_j^{(\alpha)}$ (at the zeros of ψ_α) and behaves as $O(1/x^2)$ in the sector S_α and inside all the sectors where ψ_α is exponentially large (if the polynomial $h_k(x')$ has a degree less than $d-2$). Therefore, the integrals

$$I_{k,\alpha} = \oint_{\mathcal{A}_\alpha} dx\, v_k(x), \quad \alpha = 1, \dots, g, \quad k = 1, \dots, d-1,$$

are well defined in the general case. If the matrix $I_{k,\alpha}$ with $k = 1, \dots, d-1$ has the full rank (which we assume in what follows), then we can choose the canonically normalized basis of h_k such that

$$I_{k,\alpha} = \delta_{k,\alpha}. \tag{3.7}$$

The functions $v_k(x)$, $k = 1, \dots, g$ are therefore natural analogues of canonically normalized holomorphic forms (first-kind differentials). We now extend this notion to the *meromorphic* (Whitham–Krichever) differentials [16]. For this, we consider the following basis h_k, $k = 1, 2, \dots$, in the space of polynomials of arbitrary order. The first $d-1$ elements of this basis are the original polynomials h_k, each of which has a degree not exceeding $d-2$. Each polynomial h_k with $k > d-1$ has exactly the degree $k-1$ and must be chosen on the following grounds. We define the functions $v_k(\overset{\alpha}{x})$ with $k > d-1$ exactly as in (3.6). Now let h_k be a polynomial of arbitrary (fixed) degree $k-1$. The coefficients of h_k, $k \ge d-1$, are uniquely fixed by the normalization conditions:

the residue condition
$$\oint_{\mathcal{C}_{\mathrm{D}}} \frac{dx}{x^l} v_k(x) = \delta_{l,k-d}, \quad l = 0, 1, \dots, \quad k \ge d-1, \tag{3.8}$$

and the normalizing condition
$$\oint_{\mathcal{A}_\alpha} dx\, v_k(x) = 0, \quad \alpha = 1, \dots, d-1, \quad k \ge d. \tag{3.9}$$

Remark 3.3. Although the functions $v_k(\overset{\alpha}{x})$ generally increase as x^{k-d} as $x \to \infty$, integral (3.8) and also normalizing condition (3.9) are well defined for any finite l and k because the difference $v_k(\overset{\alpha_+}{x}) - v_k(\overset{\alpha_-}{x})$ is exponentially small as $x \to \infty_{\tilde\alpha_\pm}$ for any k and we can integrate it along $\mathcal{C}_{\mathrm{D}}$ weighted by any polynomially increasing function. Integral (3.8) is therefore a natural analogue of the residue at infinity of order $l+1$.

3.4. Riemann matrix of periods. An interesting quantity in standard algebraic geometry is the Riemann matrix of periods provided by integrals of the holomorphic differentials over $\mathcal{B}$-cycles. An analogous "quantum" Riemann period matrix $\tau_{i,j}$, $i, j = 1, \dots, g$ is

$$\tau_{\alpha,i} \overset{\text{def}}{=} \oint_{\mathcal{B}_\alpha} dx\, v_i(x).$$

We note that this definition makes sense because $v_i(x)$, $i = 1, \dots, g$, behaves as $O(1/x^2)$ in the sectors that are asymptotic for the $\mathcal{B}$-cycles. Because the residues of $v_i(\overset{\alpha}{x})$ vanish at all zeros $s_j^{(\alpha)}$, these integrals depend only on the homology class of $\mathcal{B}$-cycles.

Like for the classical Riemann matrix of periods, we have the following property.

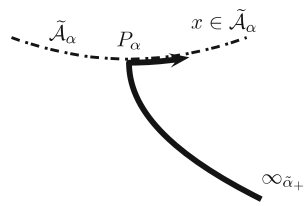


Fig. 5. The path of integration with respect to the variable x' in the expression for the recursion kernel $K(\overset{\alpha}{x}, y)$.

Theorem 3.1. *The period matrix τ is symmetric: $\tau_{i,j} = \tau_{j,i}$.*

This result follows from Theorem 4.8 below because

$$\oint_{\mathcal{B}_\beta} dx \oint_{\mathcal{B}_\alpha} dz\, B(x,z) = 2i\pi \oint_{\mathcal{B}_\beta} dx\, v_\alpha(x) = 2i\pi\tau_{\beta,\alpha},$$

after which we can apply the equality $B(\overset{\alpha}{x}, \overset{\beta}{z}) = B(\overset{\beta}{z}, \overset{\alpha}{x})$ (see Theorem 4.9 on the symmetry of the Bergman kernel).

4. Recursion kernels

One of the key geometric objects in [3] and [15] is the "recursion kernel" $K(x,z)$. It was used to construct a solution of loop equations in the context of matrix models [5]. We use its analogue $K(x,z)$ below to construct the third- and second-kind differentials.

4.1. The recursion kernel. We first define the kernel

$$\widehat{K}(\overset{\alpha}{x}, z) = \frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} \frac{dx'}{x'-z} \psi_\alpha^2(x'), \tag{4.1}$$

and for each $\alpha = 1, \ldots, g$, we define

$$\hbar C_\alpha(z) = \oint_{\mathcal{A}_\alpha} dx\, \widehat{K}(x,z) \equiv \int_{\infty_{\tilde{\alpha}_-}}^{\infty_{\tilde{\alpha}_+}} dx\, \big(K(\overset{\alpha_+}{x}, z) - K(\overset{\alpha_-}{x}, z)\big). \tag{4.2}$$

In these expressions, we must also specify the integration contours with respect to the variable x' from the infinities $\infty_{\alpha_\pm}$ to the point x on the cycle $\mathcal{A}_\alpha$. We assume that these contours go first from the corresponding infinity along the part of the adjoint cycle $\mathcal{B}_\alpha$ that lies in the sheet $\alpha_\pm$ until it reaches the intersection point P_α; after this point, we integrate along the cycle $\mathcal{A}_\alpha$ towards the final point x (see Fig. 5).

To find the domain of the function $\widehat{K}(\overset{\alpha}{x}, z)$, we slightly deform the integration contours over edges of the $\widetilde{\mathcal{A}}$-cycles as shown in Fig. 6; then for the variable z lying in the domain that is "inner" with respect to integrations from infinities for all the functions $\psi_{\gamma_\pm}$, i.e., for the domain that is separated from all the infinities $\infty_{\gamma_\pm}$ by the drawn apart edges of the $\widetilde{\mathcal{A}}$-cycles, the kernel $\widehat{K}(\overset{\alpha}{x}, z)$ is well defined (and it develops logarithmic cuts if we push the variable z through the boundary of the sheet S_α).

We now need to describe the analytic properties of the introduced functions. For a fixed x, the kernel $\widehat{K}(\overset{\alpha}{x}, z)$ is defined for z in the crosshatched domain in Fig. 6.

Taking an integration path between ∞_α and x, we find that $\widehat{K}(\overset{\alpha}{x}, z)$ is defined for z outside this path. Across the path $]\infty_\alpha, x]$, $\widehat{K}(\overset{\alpha}{x}, z)$ has a discontinuity with respect to z:

$$\delta_z \widehat{K}(\overset{\alpha}{x}, z) = \frac{2i\pi}{\hbar} \frac{\psi_\alpha^2(z)}{\psi_\alpha^2(x)}.$$

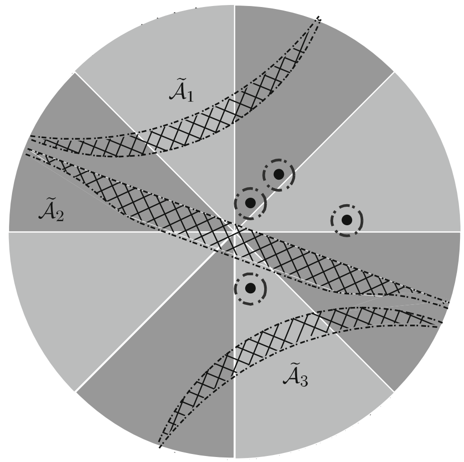


Fig. 6. The domain of variable z (crosshatched) in (4.1) (we slightly deform the $\widetilde{\mathcal{A}}$-cycle integrals).

A similar statement is true for $C_\alpha(z)$: when z crosses the line of the cycle $\mathcal{A}_\beta$, we have

$$\delta_z C_\beta(z) = \frac{2i\pi\psi^2_{\beta_\pm}(z)}{\hbar}\int_z^{\infty_{\tilde{\beta}_\pm}} \frac{dx''}{\psi^2_{\beta_\pm}(x'')}. \tag{4.3}$$

We now define the recursion kernel $K(\overset{\alpha}{x}, z)$, which is the main ingredient in our construction.

Definition 4.1. The recursion kernel $K(\overset{\alpha}{x}, z)$ is

$$K(\overset{\alpha}{x}, z) = \widehat{K}(\overset{\alpha}{x}, z) - \sum_{j=1}^{d-1} v_j(\overset{\alpha}{x})C_j(z).$$

It is defined for z in the crosshatched domain in Fig. 6.

Theorem 4.1. *The kernel K has the following properties*:

For a fixed z, $K(\overset{\alpha}{x}, z) \sim O(x^{-2})$ as $x \to \infty$ in all sectors (if the function $\psi_\alpha(x)$ increases in all the sectors except S_α).

The normalization condition is

$$\oint_{\mathcal{A}_j} dx\, K(x, z) = 0, \quad j = 1, \ldots, d-1. \tag{4.4}$$

At the zeros $s_j^{(\alpha)}$ of ψ_α, $K(\overset{\alpha}{x}, z)$ has double poles with zero residues.

Proof. The first statement follows from the asymptotic behavior of $v_k(\overset{\alpha}{x})$ given by (3.6) and the kernel $\widehat{K}(\overset{\alpha}{x}, z)$, $\widehat{K}(\overset{\alpha}{x}, z) \sim O(x^{-d})$. The second statement follows from the definition of the kernel K, and the third statement follows from (2.6).

Theorem 4.2. *We have $K(\overset{\alpha}{x}, z) \underset{z\to\infty}{\sim} O(z^{-d})$ in all sectors at infinity. More precisely, we have*

$$K(\overset{\alpha}{x}, z) \sim -\sum_{k=d-1}^{\infty} \frac{K_k(\overset{\alpha}{x})}{z^{k+1}},$$

where

$$K_k(\overset{\alpha}{x}) = \widehat{K}_k(\overset{\alpha}{x}) - \sum_{j=1}^{g} v_j(\overset{\alpha}{x}) \oint_{\mathcal{A}_j} dx'\, \widehat{K}_k(x'), \tag{4.5}$$

$$\widehat{K}_k(\overset{\alpha}{x}) = \frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} dx' (x')^k \psi_\alpha^2(x'), \tag{4.6}$$

Proof. We can expand $\widehat{K}(\overset{\alpha}{x}, z)$ in a series as

$$\widehat{K}(\overset{\alpha}{x}, z) \sim -\sum_{k=0}^{\infty} \frac{\widehat{K}_k(\overset{\alpha}{x})}{z^{k+1}},$$

where $\widehat{K}_k(\overset{\alpha}{x})$ is defined in (4.6), and (4.5) is therefore satisfied for $K_k(\overset{\alpha}{x})$. For $k \le d-2$, $(x')^k$ can be represented as a linear combination of polynomials $h_j(x')$,

$$(x')^k = \sum_{\beta=1}^{d-1} b_{k,\beta} h_\beta(x'),$$

and from the normalizing condition, we immediately obtain

$$\oint_{\mathcal{A}_\alpha} dx'\, \widehat{K}_k(x') = b_{k,\alpha},$$

and therefore $K_k(x) = 0$ for $k \le d-2$, which implies that $K(x,z) = O(z^{-d})$. The theorem is proved.

4.2. Third-kind differential: The kernel $G(\overset{\alpha}{x}, \overset{\beta}{z})$. We now define the second important kernel, which is an analogue of the third-kind differential. The kernel $G(\overset{\alpha}{x}, \overset{\beta}{z})$ is

$$G(\overset{\alpha}{x}, \overset{\beta}{z}) = -\hbar\psi_\beta^2(z)\, \partial_z \frac{K(\overset{\alpha}{x}, z)}{\psi_\beta^2(z)} = 2\hbar \frac{\psi_\beta'(z)}{\psi_\beta(z)} K(\overset{\alpha}{x}, z) - \hbar\, \partial_z K(\overset{\alpha}{x}, z). \tag{4.7}$$

Integrating by parts gives

$$\begin{aligned} G(\overset{\alpha}{x}, \overset{\beta}{z}) = & -\frac{1}{x-z} + \frac{2}{\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} \frac{dx'}{x'-z}\, \psi_\alpha^2(x') \left(\frac{\psi_\alpha'(x')}{\psi_\alpha(x')} - \frac{\psi_\beta'(z)}{\psi_\beta(z)} \right) - \\ & - \hbar \sum_{j=1}^{d-1} v_j(\overset{\alpha}{x}) \psi_\beta^2(z)\, \partial_z \frac{C_j(z)}{\psi_\beta^2(z)}. \end{aligned}$$

In what follows, we are often in a situation where we take two integration contours $\mathcal{C}_{\mathrm{D}_x}$ and $\mathcal{C}_{\mathrm{D}_z}$ and must interchange the order of integration (or the order in which these two contours intersect the $\mathcal{B}$-cycles). It is then obvious from the definition of the $\mathcal{A}$-cycles that we must interchange the variables x and z within the same sector, and we therefore need permutation relations for $G(\overset{\alpha}{x}, \overset{\beta}{z})$ with $\alpha = \beta$. As $x \to z$, we then find that $G(\overset{\alpha}{x}, \overset{\alpha}{z}) \sim 1/(z-x)$, i.e., there is a simple pole with the unit residue at $z = x$. Because the combination

$$\frac{1}{x'-z} \left(\frac{\psi_\alpha'(x')}{\psi_\alpha(x')} - \frac{\psi_\alpha'(z)}{\psi_\alpha(z)} \right)$$

is regular at $x' = z$, interchanging the order of integration over $\mathcal{C}_{\mathrm{D}_x}$ and $\mathcal{C}_{\mathrm{D}_z}$ then just gives the residue at $z = x$; no logarithmic cut occurs.

Theorem 4.3. *The function $G(\overset{\alpha}{x},\overset{\beta}{z})$ is analytic in x with a simple pole at $x=z$ with the residue -1 for $\alpha=\beta$, with double poles at the $s_j^{(\alpha)}$ (zeros of $\psi_\alpha(x)$) with zero residues, and possibly with an essential singularity at ∞.*

The function $G(\overset{\alpha}{x},\overset{\beta}{z})$ is analytic in z with a simple pole at $z=x$ with the residue $+1$ for $\alpha=\beta$, with simple poles at $z=s_j^{(\beta)}$, and with a discontinuity across $\mathcal{A}_\gamma$-cycles with $\gamma=1,\dots,g$ (this discontinuity has opposite signs depending on which line of the cycle $\mathcal{A}_\gamma$, γ_+ or γ_-, we cross; no discontinuity occurs when crossing the last cycle $\widetilde{\mathcal{A}}_d$):

$$\delta_z G(\overset{\alpha}{x},\overset{\beta_\pm}{z})=\mp 2i\pi v_\beta(\overset{\alpha}{x}).$$

We also have

$$\oint_{\mathcal{A}_\alpha} dx\, G(x,\overset{\beta}{z})=0.$$

Proof. All the discontinuities of $K(\overset{\alpha}{x},z)$ except those arising in expression (4.3) for $C_j(z)$ are proportional to $\psi_\alpha^2(z)$ and vanish in $G(x,z)$ given by (4.7). The discontinuity of $C_j(z)$ gives $\mp 2\pi i$, and the discontinuity of $G(\overset{\alpha}{x},\overset{\beta}{z})$ is therefore $\delta_z G(\overset{\alpha}{x},\overset{\beta_\pm}{z})=\mp 2i\pi v_\beta(\overset{\alpha}{x})$.

Because $K(\overset{\alpha}{x},z)$ is regular at $z=s_j^{(\beta)}$, it is clear that $G(\overset{\alpha}{x},\overset{\beta}{z})$ has simple poles at $z=s_j^{(\beta)}$ with the residue $-2\hbar K(x,s_j^{(\beta)})$.

In the variable x, $K(\overset{\alpha}{x},z)$ has double poles at $x=s_j^{(\alpha)}$ with zero residues, and this property also holds for $G(\overset{\alpha}{x},\overset{\beta}{z})$.

The property that the $\mathcal{A}$-integral vanishes follows immediately from (4.4). The theorem is proved.

Theorem 4.4. *As $x\to\infty_\alpha$ for any α, we have the asymptotic behavior $G(\overset{\alpha}{x},\overset{\beta}{z})=O(1/x^2)$. At large z in the sector S_γ, we have*

$$\lim_{z\to\infty_\gamma} G(\overset{\alpha}{x},\overset{\beta}{z})=G(\overset{\alpha}{x},\infty_\beta)=\eta_{\gamma,\beta}t_{d+1}K_{d-1}(\overset{\alpha}{x}),\tag{4.8}$$

where $\eta_{\gamma,\beta}=\pm 1$ depending on the asymptotic behavior of $\psi_\beta\sim e^{\pm V/2\hbar}$ in the sheet S_γ.

Proof. The large-x behavior $G(\overset{\alpha}{x},\overset{\beta}{z})$ follows from Theorem 4.1. The large-z behavior is given by Theorem 4.2, i.e., $G(\overset{\alpha}{x},\overset{\beta}{z})\sim\pm V'(z)K(\overset{\alpha}{x},z)\sim\pm t_{d+1}K_{d-1}(\overset{\alpha}{x})$. The sign depends on the behavior of the solution in this sector.

4.3. The Bergman kernel $B(\overset{\alpha}{x},\overset{\beta}{z})$. In classical algebraic geometry, the Bergman kernel is the fundamental second-kind bidifferential; it is the derivative of a third-kind differential. Using the same definition as in [15], we set

$$B(\overset{\alpha}{x},\overset{\beta}{z})=-\frac{1}{2}\partial_z G(\overset{\alpha}{x},\overset{\beta}{z}).$$

We call the kernel B the "quantum" Bergman kernel.

Theorem 4.5. *The quantum Bergman kernel $B(\overset{\alpha}{x},\overset{\beta}{z})$ is an analytic function of x. For $\alpha=\beta$, it has a double pole at $x=z$ in both the variables x and z with a zero residue, has double poles in x and in z at the respective zeros $s_j^{(\alpha)}$ and $s_j^{(\beta)}$ with zero residues, and possibly has an essential singularity at ∞. Differentiation eliminates the discontinuity in the kernel G across $\mathcal{A}$-cycles, and $B(\overset{\alpha}{x},\overset{\beta}{z})$ is hence defined analytically in the whole complex plane.*

Proof. These properties follow from those of $G(\overset{\alpha}{x},\overset{\beta}{z})$ in Theorem 4.3. In particular, the only discontinuity of $G(\overset{\alpha}{x},\overset{\beta}{z})$ is along the $\mathcal{A}$-cycles and is independent of z. Therefore, $B(\overset{\alpha}{x},\overset{\beta}{z})$ is continuous there.

Theorem 4.6. *We have the asymptotic behaviors* $B(\overset{\alpha}{x},\overset{\beta}{z}) = O(1/x^2)$ *as* $x \to \infty$ *in all sectors and* $B(\overset{\alpha}{x},\overset{\beta}{z}) = O(1/z^2)$ *as* $z \to \infty$ *in all sectors.*

Proof. Such behavior of the kernel follows from the large-x and large-z behaviors of $G(\overset{\alpha}{x},\overset{\beta}{z})$.

Theorem 4.7. *The kernel* B *satisfies the loop equations*

$$\left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}+\partial_x\right)\left(B(\overset{\alpha}{x},\overset{\beta}{z})-\frac{1}{2(x-z)^2}\right)+\partial_z\frac{\psi'_\alpha(x)/\psi_\alpha(x)-\psi'_\beta(z)/\psi_\beta(z)}{x-z}=P_2^{(0)}(x,\overset{\beta}{z}), \tag{4.9}$$

where $P_2^{(0)}(x,\overset{\beta}{z})$ *is a polynomial in* x *of a degree not exceeding* $d-2$*, and*

$$\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\left(B(\overset{\alpha}{x},\overset{\beta}{z})-\frac{1}{2(x-z)^2}\right)+\partial_x\frac{\psi'_\alpha(x)/\psi_\alpha(x)-\psi'_\beta(z)/\psi_\beta(z)}{x-z}=\widetilde{P}_2^{(0)}(\overset{\alpha}{x},z), \tag{4.10}$$

where $\widetilde{P}_2^{(0)}(\overset{\alpha}{x},z)$ *is a polynomial in* z *of a degree not exceeding* $d-2$*.*

Proof. We begin by proving the first loop equation for $B(\overset{\alpha}{x},\overset{\beta}{z})$. Let

$$\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z})=\frac{1}{2}\partial_z\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}-\partial_z\right)\widehat{K}(\overset{\alpha}{x},z),$$

i.e., we have

$$B(\overset{\alpha}{x},\overset{\beta}{z})=\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z})-\sum_{j=1}^{d-1}v_j(\overset{\alpha}{x})\oint_{\mathcal{A}_j}dx''\,\widehat{B}(x'',\overset{\beta}{z}).$$

Because $h_j(x)=\big(2\psi'_\alpha(x)/\psi_\alpha(x)+\partial_x\big)v_j(\overset{\alpha}{x})$ is itself a polynomial of degree not exceeding $d-2$, it suffices to prove Eq. (4.9) for $\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z})$. We have

$$\begin{aligned}\left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}+\partial_x\right)\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z}) &= \frac{1}{2}\partial_z\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}-\partial_z\right)\frac{1}{x-z}=\\ &= -\frac{1}{(x-z)^3}+\partial_z\frac{\psi'_\beta(z)}{\psi_\beta(z)(x-z)}\end{aligned}$$

and therefore

$$\left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}+\partial_x\right)\left(\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z})-\frac{1}{2(x-z)^2}\right)+\partial_z\frac{\psi'_\alpha(x)/\psi_\alpha(x)-\psi'_\beta(z)/\psi_\beta(z)}{x-z}=0.$$

This proves Eq. (4.9) with

$$P_2^{(0)}(x,\overset{\beta}{z})=-\sum_{j=1}^{g}h_j(x)\oint_{\mathcal{A}_j}dx''\,\widehat{B}(x'',\overset{\beta}{z}).$$

We now prove the second loop equation for $B(\overset{\alpha}{x},\overset{\beta}{z})$. We have

$$\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z})=\frac{1}{2}\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\partial_z\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}-\partial_z\right)\widehat{K}(\overset{\alpha}{x},z),$$

where the differential operator in the right-hand side is

$$\widehat{U}(z)=-\frac{1}{2}\partial_z^3+\frac{2}{\hbar^2}U(z)\partial_z+\frac{1}{\hbar^2}U'(z) \tag{4.11}$$

and is therefore independent of the solution $\psi_\beta(z)$ with which we started. This operator is just the Gelfand–Dikii operator [17]. We then obtain

$$\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z})=\frac{1}{\psi_\alpha^2(x)}\int_{\infty_\alpha}^{x}dx'\,\psi_\alpha^2(x')\left(-\frac{3}{(x'-z)^4}+\frac{2U(z)}{(x'-z)^2}+\frac{U'(z)}{x'-z}\right).$$

Integrating the first term by parts three times, introducing $Y_\alpha(x)=\psi'_\alpha(x)/\psi_\alpha(x)$ (and taking into account that $Y'_\alpha+Y_\alpha^2=U$), we obtain

$$\begin{aligned}\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z})&=\frac{1}{(x-z)^3}-\frac{\partial}{\partial x}\frac{Y_\alpha(x)}{x-z}+\\&\quad+\frac{1}{\psi_\alpha^2(x)}\int_{\infty_\alpha}^{x}dx\,\psi_\alpha^2(x')\left(2\frac{U(z)-U(x')}{(x'-z)^2}+\frac{U'(z)+U'(x')}{x'-z}\right).\end{aligned}$$

This implies that

$$\begin{aligned}&\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)}+\partial_z\right)\left(\widehat{B}(\overset{\alpha}{x},\overset{\beta}{z})-\frac{1}{2(x-z)^2}\right)+\frac{\partial}{\partial x}\frac{Y_\alpha(x)-Y_\beta(z)}{x-z}=\\&\qquad=\frac{1}{\psi_\alpha^2(x)}\int_{\infty_\alpha}^{x}dx'\,\psi_\alpha^2(x')\left(2\frac{U(z)-U(x')}{(x'-z)^2}+\frac{U'(z)+U'(x')}{x'-z}\right).\end{aligned} \tag{4.12}$$

The obtained expression is obviously a polynomial in z. The expression in parentheses in the right-hand side of (4.12) is a skew-symmetric polynomial in x' and z of a degree not exceeding $2d-2$. Moreover, all the terms with $(x')^k$ with $k\le d-2$ become linear combinations of $v_j(\overset{\alpha}{x})$ after integration and vanish identically when we apply the projection to the subspace of zero $\mathcal{A}$-cycle integrals. Therefore, the minimum power of x' that contributes to the answer is $(x')^{d-1}$. But there is no term $(x')^{d-1}z^{d-1}$ in the parentheses because it would contradict the skew-symmetricity. The first nonzero term that might contribute is proportional to $(x')^{d-1}z^{d-2}$, which obviously means that the obtained polynomial $\widetilde{P}_2^{(0)}(\overset{\alpha}{x},z)$ has a degree not exceeding $d-2$ in z.

Theorem 4.8. *For each $\alpha=1,\dots,g$, we have*

$$\oint_{\mathcal{A}_i}dx\,B(x,\overset{\beta}{z})=0,\qquad \oint_{\mathcal{A}_j}dz\,B(\overset{\alpha}{x},z)=0, \tag{4.13}$$

and

$$\oint_{\mathcal{B}_j}dz\,B(\overset{\alpha}{x},z)=2i\pi v_j(\overset{\alpha}{x}). \tag{4.14}$$

Proof. The vanishing of $\mathcal{A}$-cycle integrals in the x variable is by construction. For the z variable, we have

$$\oint_{\mathcal{A}_\beta} dz\, B(\overset{\alpha}{x}, z) = \int_{\infty_{\widetilde{\beta}_-}}^{\infty_{\widetilde{\beta}_+}} dz\, \big(B(\overset{\alpha}{x}, \overset{\beta_+}{z}) - B(\overset{\alpha}{x}, \overset{\beta_-}{z})\big) =$$

$$= -\frac{1}{2}\big(G(\overset{\alpha}{x}, \overset{\beta_+}{\infty}_{\widetilde{\beta}_+}) - G(\overset{\alpha}{x}, \overset{\beta_-}{\infty}_{\widetilde{\beta}_+}) - G(\overset{\alpha}{x}, \overset{\beta_+}{\infty}_{\widetilde{\beta}_-}) + G(\overset{\alpha}{x}, \overset{\beta_-}{\infty}_{\widetilde{\beta}_-})\big),$$

and the asymptotic conditions for the function G in all four cases in the right-hand side of the equality are the same. From Theorem 4.4, we therefore conclude that the result is zero.

We now turn to the integral over a cycle $\widetilde{\mathcal{B}}_\beta$:

$$\oint_{\widetilde{\mathcal{B}}_\beta} dy\, B(\overset{\alpha}{x}, y) = \text{discontinuity of } G(\overset{\alpha}{x}, y) \text{ at } \widetilde{\mathcal{A}}_\beta -$$

$$-\frac{1}{2}\big(G(\overset{\alpha}{x}, \overset{\beta_+}{\infty}_{\beta_+}) - G(\overset{\alpha}{x}, \overset{\beta_-}{\infty}_{\beta_+}) - G(\overset{\alpha}{x}, \overset{\beta_+}{\infty}_{\beta_-}) + G(\overset{\alpha}{x}, \overset{\beta_-}{\infty}_{\beta_-})\big) =$$

$$= 2\pi i(1-\delta_{\beta,d}) v_\beta(\overset{\alpha}{x}) + 2K_{d-1}(\overset{\alpha}{x}),$$

where we again use the asymptotic conditions in Theorem 4.4. This formula implies that if we integrate $\mathcal{B}_j$ for $j = 1, \ldots, d-1$, which is the difference of integrals over the cycles $\widetilde{\mathcal{B}}_j$ and $\widetilde{\mathcal{B}}_d$, then we obtain formula (4.14). The theorem is proved.

The main property of the Bergman kernel is given in the following theorem.

Theorem 4.9. *The kernel* $B(\overset{\alpha}{x}, \overset{\beta}{z})$ *is symmetric,* $B(\overset{\alpha}{x}, \overset{\beta}{z}) = B(\overset{\beta}{z}, \overset{\alpha}{x})$.

Proof. The proof uses the fact that $B(\overset{\alpha}{x}, \overset{\beta}{z})$ satisfies the loop equation in the two variables. We have

$$\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)} + \partial_z\right)\left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \partial_x\right)\left(B(\overset{\alpha}{x}, \overset{\beta}{z}) - \frac{1}{2(x-z)^2}\right) =$$

$$= \left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)} + \partial_z\right)\left(P_2^{(0)}(x, \overset{\beta}{z}) - \partial_z \frac{\psi'_\alpha(x)/\psi_\alpha(x) - \psi'_\beta(z)/\psi_\beta(z)}{x-z}\right) =$$

$$= \left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \partial_x\right)\left(\widetilde{P}_2^{(0)}(\overset{\alpha}{x}, z) - \partial_x \frac{\psi'_\alpha(x)/\psi_\alpha(x) - \psi'_\beta(z)/\psi_\beta(z)}{x-z}\right).$$

We then obtain

$$\left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)} + \partial_z\right)P_2^{(0)}(x, \overset{\beta}{z}) - \left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \partial_x\right)\widetilde{P}_2^{(0)}(\overset{\alpha}{x}, z) =$$

$$= \left(2\frac{\psi'_\beta(z)}{\psi_\beta(z)} + \partial_z\right)\partial_z \frac{\psi'_\alpha(x)/\psi_\alpha(x) - \psi'_\beta(z)/\psi_\beta(z)}{x-z} -$$

$$- \left(2\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \partial_x\right)\partial_x \frac{\psi'_\alpha(x)/\psi_\alpha(x) - \psi'_\beta(z)/\psi_\beta(z)}{x-z} =$$

$$= 2\frac{U(x) - U(z)}{(x-z)^2} - \frac{U'(x) + U'(z)}{x-z},$$

and hence

$$(x-z)^2\left(2\frac{\psi_\beta'(z)}{\psi_\beta(z)}+\partial_z\right)P_2^{(0)}(x,\overset{\beta}{z})+2U(z)+(x-z)U'(z)=$$

$$=(x-z)^2\left(2\frac{\psi_\alpha'(x)}{\psi_\alpha(x)}+\partial_x\right)\widetilde{P}_2^{(0)}(\overset{\alpha}{x},z)+2U(x)+(z-x)U'(x)\stackrel{\mathrm{def}}{=}R(x,z). \tag{4.15}$$

Here, the left-hand side is a polynomial in x, and the right-hand side is in turn a polynomial in z. Therefore, $R(x,z)$ is a polynomial in both variables of a degree not exceeding d in each variable. Moreover, we must have $R(x,x)=2U(x)$. Therefore, we must have

$$R(x,z)=\frac{1}{\hbar^2}\left(\frac{1}{2}V'(x)V'(z)-\hbar\frac{V'(x)-V'(z)}{x-z}-P(x)-P(z)\right)+(x-z)^2\widetilde{R}(x,z),$$

where $\widetilde{R}(x,z)$ is a polynomial in both variables of a degree not exceeding $d-2$ in each variable. Substituting this polynomial in (4.15) and using the symmetry under $x\leftrightarrow z$, we obtain

$$\left(2\frac{\psi_\beta'(z)}{\psi_\beta(z)}+\partial_z\right)\left(P_2^{(0)}(x,\overset{\beta}{z})-\widetilde{P}_2^{(0)}(\overset{\beta}{z},x)\right)=\widetilde{R}(x,z)-\widetilde{R}(z,x). \tag{4.16}$$

We can then decompose the right-hand side in the basis $h_i(x)h_j(z)$,

$$\widetilde{R}(x,z)-\widetilde{R}(z,x)=\sum_{i,j=1}^{d-1}(\widetilde{R}_{i,j}-\widetilde{R}_{j,i})h_i(x)h_j(z).$$

Applying the integral operator

$$f(z)\mapsto\frac{1}{\psi_\beta^2(z)}\int_{\infty_\beta}^{z}dz'\,\psi_\beta^2(z')f(z') \tag{4.17}$$

to differential equation (4.16), we obtain

$$P_2^{(0)}(x,\overset{\beta}{z})-\widetilde{P}_2^{(0)}(\overset{\alpha}{z},x)=\sum_{i,j=1}^{d-1}(\widetilde{R}_{i,j}-\widetilde{R}_{j,i})h_i(x)\,v_j(\overset{\beta}{z})+A_1(x),$$

where $A_1(x)$ is some integration constant.

Using loop equations (4.7) we then subtract and obtain

$$\left(2\frac{\psi_\alpha'(x)}{\psi(x)}+\partial_x\right)\left(B(\overset{\alpha}{x},\overset{\beta}{z})-B(\overset{\beta}{z},\overset{\alpha}{x})\right)=P_2^{(0)}(x,\overset{\beta}{z})-\widetilde{P}_2^{(0)}(\overset{\beta}{z},x),$$

and applying integral operator (4.17) with respect to the variable x in the sheet S_α, we obtain

$$B(\overset{\alpha}{x},\overset{\beta}{z})-B(\overset{\beta}{z},\overset{\alpha}{x})=\sum_{i,j=1}^{d-1}(\widetilde{R}_{i,j}-\widetilde{R}_{j,i})v_i(\overset{\alpha}{x})v_j(\overset{\beta}{z})+A(x)+\widetilde{A}(z),$$

where $A(x)$ and $\widetilde{A}(z)$ the integration constants.

Further, the large-x and large-z behavior of B implies that $A(x)=\widetilde{A}(z)=0$, and therefore

$$B(\overset{\alpha}{x},\overset{\beta}{z})-B(\overset{\beta}{z},\overset{\alpha}{x})=\sum_{i,j}(\widetilde{R}_{i,j}-\widetilde{R}_{j,i})v_i(\overset{\alpha}{x})v_j(\overset{\beta}{z}). \tag{4.18}$$

Now using Theorem 4.8 for all i and j,

$$\oint_{\mathcal{A}_i}dx\,B(x,\overset{\beta}{z})=\oint_{\mathcal{A}_j}dz\,B(\overset{\alpha}{x},z)=0,$$

we obtain $\widetilde{R}_{i,j}=\widetilde{R}_{j,i}$ for all i and j, which completes the proof of the symmetricity of the Bergman kernel.

We therefore see that our "quantum Bergman kernel" has all the features of the standard Bergman kernel associated with a Riemann surface: it is symmetric, has no discontinuities, and has the double pole with zero residue at coinciding arguments (which in our case corresponds to coinciding arguments on the *same* sheet S_k). Using all these kernels, we can then generalize the recursive procedure in [2], [5], [15] and define the correlation functions (see the next section).

4.4. Meromorphic forms and the Riemann bilinear identity. A meromorphic form $\mathcal{R}(\overset{\alpha}{x})$ is defined as

$$\mathcal{R}(\overset{\alpha}{x}) = \frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} dx'\, r(x')\psi_\alpha^2(x'), \tag{4.19}$$

where $r(x)$ is a rational function of x that behaves as at most $O(x^{d-2})$ at large x and whose poles r_i are such that

$$\operatorname*{Res}_{r_i} \psi_\alpha^2(x)\, r(x) = 0.$$

The holomorphic forms $v_j(\overset{\alpha}{x})$ and the kernels $G(\overset{\alpha}{x}, \overset{\beta}{z})$ and $B(\overset{\alpha}{x}, \overset{\beta}{z})$ are meromorphic forms of x.

A meromorphic form $\mathcal{R}(\overset{\alpha}{x})$ defined by (4.19) has poles at $x = r_i$, the poles of $r(x)$. The degree of these poles is one less than that of $r(x)$. The form $\mathcal{R}(\overset{\alpha}{x})$ has double poles with zero residues at the $s_i^{(\alpha)}$ and behaves as $O(x^{-2})$ in all sectors (also having an accumulation of poles along the rays L_i of accumulations of zeros of ψ_α). We note that the integrals $\oint_{\mathcal{A}_\alpha} dx\, \mathcal{R}(x)$ are well defined.

We prove the following theorem (**Riemann bilinear identity**).

Theorem 4.10. *For z in the sector S_α (outside the crosshatched domain in Fig. 6), we have the representation formula for the meromorphic form $\mathcal{R}(\overset{\alpha}{z})$*

$$\begin{aligned}\mathcal{R}(\overset{\alpha}{z}) = &-\sum_\beta \sum_{r_i \in S_\beta} \operatorname*{Res}_{y^\beta = r_i} G(\overset{\alpha}{z}, \overset{\beta}{y})\mathcal{R}(\overset{\beta}{y}) - \\ &-\sum_\beta \sum_{s_k^{(\beta)} \in S_\beta} \operatorname*{Res}_{y^\beta = s_k^{(\beta)}} G(\overset{\alpha}{z}, \overset{\beta}{y})\mathcal{R}(\overset{\beta}{y}) + \sum_{j=1}^{g} v_j(\overset{\alpha}{z}) \oint_{\mathcal{A}_j} dy\, \mathcal{R}(y).\end{aligned} \tag{4.20}$$

Proof. We begin with the integral of $G(\overset{\alpha}{z}, y)\mathcal{R}(y)$ over $\mathcal{C}_{\mathrm{D}}$ (Fig. 2)

$$\int_{\mathcal{C}_{\mathrm{D}}} dy\, G(\overset{\alpha}{z}, y)\mathcal{R}(y) = \sum_\beta \int_{\infty_{\beta-1}}^{\infty_{\beta+1}} dy\, G(\overset{\alpha}{z}, \overset{\beta}{y})\mathcal{R}(\overset{\beta}{y}).$$

This integral is identically zero because of the asymptotic conditions $G(\overset{\alpha}{z}, \overset{\beta}{y}) \to 1/y$ as $y \to \infty_\beta$ and $\mathcal{R}(\overset{\beta}{y}) \sim 1/y^2$ as $y \to \infty_\beta$ and because no accumulation of zeros occurs for the function $\mathcal{R}(\overset{\beta}{y})$ on the boundaries between the sectors S_β and $S_{\beta\pm 1}$. We can then push the integration contours through the complex plane towards the $\widetilde{\mathcal{A}}$-cycles as in Fig. 3. The residues at the points r_i and $s_k^{(\beta)}$ give the two double summations in (4.20). The residue at the point $x = y$ in the sector S_α gives the left-hand side because it follows from Theorem 4.3 that $G(\overset{\alpha}{z}, \overset{\alpha}{y}) = 1/(z-y) + \text{regular terms}$. It remains to consider the integrals along the $\widetilde{\mathcal{A}}$-cycles. For this, we note that our integrations over y are outer with respect to the integrations over the variable x along $\mathcal{A}$-cycles in formula (4.2) for the factors C_j, and when we push the integration over y through that over x, we have the discontinuity of $G(\overset{\alpha}{z}, \overset{\beta_\pm}{y})$, which is equal to $2\pi i v_j(\overset{\alpha}{z})$. No such discontinuity occurs for the cycle $\widetilde{\mathcal{A}}_d$. All these discontinuities are independent of y, and the contour integral of the product hence factors for each cycle $\mathcal{A}_j$ and gives the last term in (4.20).

To evaluate the remaining integrals inside the crosshatched domain in Fig. 6, we recall that $G(\overset{\alpha}{x},\overset{\beta}{y}) = \psi_\beta^2(y)\partial_y\big(K(\overset{\alpha}{x},y)/\psi_\beta^2(y)\big)$. Integrating by parts, we then obtain

$$\int_{\infty_{\tilde\beta_-}}^{\infty_{\tilde\beta_+}} dz\,\big(G(\overset{\alpha}{x},\overset{\beta_+}{z})\mathcal{R}(\overset{\beta_+}{z}) - G(\overset{\alpha}{x},\overset{\beta_-}{z})\mathcal{R}(\overset{\beta_-}{z})\big) =$$

$$= -\int_{\infty_{\tilde\beta_-}}^{\infty_{\tilde\beta_+}} dz\,\big(K(\overset{\alpha}{x},z)r(z) - K(\overset{\alpha}{x},z)r(z)\big) = 0,$$

and these contributions vanish for all the cycles $\mathcal{A}_\beta$. The theorem is proved.

5. Correlation functions: Diagram representation

In this section, we define the sectorwise versions of the quantum correlation functions considered in [1] (deformations of "classical" correlation functions introduced in [2], [3], [5]). Our definitions follow from (non-Hermitian) eigenvalue models (see Sec. 8), but they are also applicable in the general setting of an arbitrary Schrödinger equation.

5.1. The definition and the properties of correlation functions. We define the functions $W_n^{(h)}(\overset{\alpha_1}{x_1},\dots,\overset{\alpha_n}{x_n})$, called the *n-point correlation functions of "genus" h* by the recurrence relations

$$W_1^{(0)}(\overset{\alpha}{x}) = \omega(\overset{\alpha}{x}), \qquad W_2^{(0)}(\overset{\alpha_1}{x_1},\overset{\alpha_2}{x_2}) = B(\overset{\alpha_1}{x_1},\overset{\alpha_2}{x_2}), \tag{5.1}$$

$$W_{n+1}^{(h)}(\overset{\alpha_0}{x_0},J) = \oint_{\mathcal{C}_{\mathrm{D}_x}} dx\,K(\overset{\alpha_0}{x_0},x)\bigg(\overline{W}_{n+2}^{(h-1)}(x,x,J) + \\ + \sum_{r=0}^{h}\sum_{I\subset J}{}' W_{|I|+1}^{(r)}(x,I)W_{n-|I|+1}^{(h-r)}(x,J\setminus I)\bigg), \tag{5.2}$$

where $J = \{x_1,\dots,x_n\}$ and the symbol $\sum\sum'$ means that we exclude the terms $r=0$, $I=\varnothing$; $r=0$, $I=\{x_i\}$; $r=h$, $I=J\setminus\{x_i\}$; and $r=h$, $I=J$. Here, the integration over the contour $\mathcal{C}_{\mathrm{D}_x}$ is defined in (3.1), and

$$\overline{W}_n^{(h)}(\overset{\alpha_1}{x_1},\dots,\overset{\alpha_n}{x_n}) = W_n^{(h)}(\overset{\alpha_1}{x_1},\dots,\overset{\alpha_n}{x_n}) - \frac{\delta_{n,2}\delta_{h,0}\delta_{\alpha_1,\alpha_2}}{2(x_1-x_2)^2}. \tag{5.3}$$

The point x_0 in these expressions is outside the integration contour $\mathcal{C}_{\mathrm{D}_x}$ for x, and all the x_i are outside the $\mathcal{A}$-cycles of the projection integrals.

The main property of the introduced correlation functions is that these quantities solve the loop equations in the $1/N^2$-expansion. We also prove the following properties.

Theorem 5.1. *Each function $W_n^{(h)}(\overset{\alpha_1}{x_1},\dots,\overset{\alpha_n}{x_n})$ with $2-2h-n<0$ is analytic in all its arguments with poles only as $x_i \to s_j^{(\alpha_i)}$. It vanishes at least as $O(1/x_i^2)$ as $x_i\to\infty_{\alpha_i}$ and has no discontinuities across $\mathcal{A}$-cycles. Consequently, we have the equality*

$$\int_{\mathcal{C}_{\mathrm{D}_x}} dx\,W_{n+1}^{(h)}(x,J) = t_0\delta_{n,0}\delta_{h,0}. \tag{5.4}$$

Proof. We proceed by recursion on $2h+n$. The analyticity is obvious. The theorem is already proved for $W_2^{(0)}$. We suppose that it holds for $2g+n$ and prove it for $W_{n+1}^{(h)}(x_0,x_1,\dots,x_n)$. To prove the asymptotic behavior, we note that Definition 4.1 implies, first, that the term $\int_{\mathcal{C}_{\mathrm{D}_y}} dy\, \widehat{K}(\overset{\alpha}{x}, y) U_n^{(h)}(y, J)$, where we introduce the notation

$$U_n^{(h)}(\overset{\alpha}{x}, J) = \overline{W}_{n+2}^{(h-1)}(\overset{\alpha}{x}, \overset{\alpha}{x}, J) + \sum_{r=0}^{h} {\sum_{I\subset J}}' \overline{W}_{|I|+1}^{(r)}(\overset{\alpha}{x}, I)\overline{W}_{n-|I|+1}^{(h-r)}(\overset{\alpha}{x}, J\setminus I) \tag{5.5}$$

for brevity, is of the order of $\psi_\alpha^{-2}(x)\int_{\infty_\alpha}^{x} dx'\, \psi_\alpha^2(x')/(x')^2 \sim x^{-d-2}$, and the leading contribution hence comes from the terms proportional to $v_j(\overset{\alpha}{x}) \sim x^{-2}$, which completes the proof of the theorem.

We also have the following simple lemma, which follows from equality (5.4) and from the normalization conditions for the kernel $K(\overset{\alpha}{x}, y)$.

Lemma 5.1. *For all $(n, h) \neq (0, 0)$, we have*

$$\oint_{\widetilde{\mathcal{A}}_\alpha} dx\, W_{n+1}^{(h)}(x, J) = 0.$$

We now formulate the first of our main theorems.

Theorem 5.2. *For $2 - 2h - n < 0$, $W_n^{(h)}$ satisfies the loop equation. This means that the function*

$$\begin{aligned} P_{n+1}^{(h)}(x; \overset{\alpha_1}{x_1}, \ldots, \overset{\alpha_n}{x_n}) = \hbar\biggl(2\frac{\psi_\alpha'(x)}{\psi_\alpha(x)} + \partial_x\biggr)\overline{W}_{n+1}^{(h)}(\overset{\alpha}{x}, \overset{\alpha_1}{x_1}, \ldots, \overset{\alpha_n}{x_n}) + \\ + \sum_{r=0}^{h} {\sum_{I\subset J}}' \overline{W}_{|I|+1}^{(r)}(\overset{\alpha}{x}, I)\overline{W}_{n-|I|+1}^{(h-r)}(\overset{\alpha}{x}, J\setminus I) + \overline{W}_{n+2}^{(h-1)}(\overset{\alpha}{x}, \overset{\alpha}{x}, J) + \\ + \sum_j \partial_{x_j}\biggl(\frac{\overline{W}_n^{(h)}(\overset{\alpha}{x}, J\setminus\{x_j\})\delta_{\alpha,\alpha_j} - \overline{W}_n^{(h)}(\overset{\alpha_j}{x_j}, J\setminus\{x_j\})}{x - x_j}\biggr) \end{aligned} \tag{5.6}$$

is a polynomial in the variable x of a degree not exceeding $d - 2$ and is independent of the choice of the sector S_α.

The proof is in Appendix A.

Theorem 5.3. *Each $W_n^{(h)}$ is a symmetric function of all its arguments.*

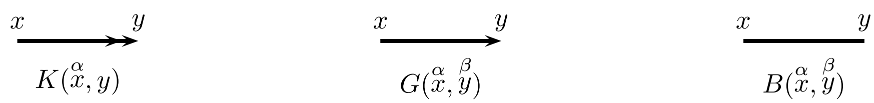
The special case $W_3^{(0)}$ is proved in Appendix B, and the theorem is proved in Appendix C.

Theorem 5.4. *For $2 - 2h - n < 0$, $W_n^{(h)}(\overset{\alpha_1}{x_1}, \ldots, \overset{\alpha_n}{x_n})$ is homogeneous of degree $2 - 2h - n$,*

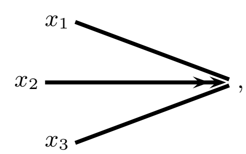
$$\begin{aligned} &\biggl(\hbar\frac{\partial}{\partial\hbar} + \sum_{j=1}^{d+1} t_j\frac{\partial}{\partial t_j} + \sum_{i=1}^{g}\epsilon_i\frac{\partial}{\partial\epsilon_i} + t_0\frac{\partial}{\partial t_0}\biggr)W_n^{(h)}(\overset{\alpha_1}{x_1}, \ldots, \overset{\alpha_n}{x_n}) = \\ &= (2 - 2h - n)W_n^{(h)}(\overset{\alpha_1}{x_1}, \ldots, \overset{\alpha_n}{x_n}). \end{aligned}$$

Proof. Under a change $t_k \to \lambda t_k$, $\hbar \to \lambda\hbar$, $\epsilon_i \to \lambda\epsilon_i$, and $t_0 \to \lambda t_0$, the Schrödinger equation remains unchanged, and ψ is therefore unchanged. The kernel K is changed to K/λ, and nothing else is changed. By the recurrence relation, $W_n^{(h)}$ is then multiplied by λ^{2-2h-n}.

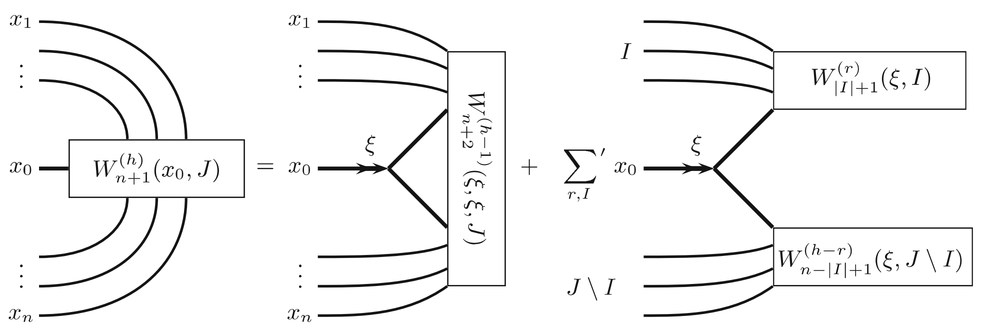
5.2. Diagram representation. The diagram representation for the correlation functions structurally coincides with the one for the correlation functions in one- and two-matrix models [2], [4], [5]. We introduce the three kinds of propagators



and assume the partial ordering from "infinity" to "$\mathcal{A}$-cycles" to be from left to right in graphical expressions. We represent the terms $W_n^{(h)}(J)$ as graphs with three-valent vertices. We assign its own variable ξ to each inner vertex and assume that the integration over this variable is along the contour $\mathcal{C}_{\mathrm{D}}$. The order of integration depends on which vertex is closer to the "$\mathcal{A}$-cycles": we begin by integrating at the innermost vertex. We also have n outer legs (one-valent vertices) corresponding to the points $\overset{\alpha_i}{x_i}$, $i = 1, \ldots, n$. They are assumed to be outside all the inner integrations. For example, the term $W_3^{(0)}(x_1, x_2, x_3)$ then has the form



and recurrence relation (5.2) becomes



We now formulate the diagram technique for constructing the functions $W_n^{(h)}(J)$ for $n > 0$ and $2g - 2 + n > 0$. It is formally the same as the one in [2], [5]. In the given order, all diagrams contribute with the corresponding automorphism multipliers constructed according to the following rules:

The diagram for $W_n^{(h)}(J)$ contains exactly n external legs and h loops.

We segregate one variable, for example, x_1, and take all the maximum connected rooted subtrees starting at the vertex x_1 and not going to any other external leg.

We associate the directed propagators $K(x, y)$ with all the edges of the rooted subtree; the direction is always from the root to branches.

All other propagators that comprise exactly h inner propagators and $n-1$ remaining external legs are $B(x, y)$ if the vertices x and y are distinct and $\overline{B}(x, x)$ for the loop composed of a single propagator.

Each rooted tree establishes a partial ordering on the set of three-valent vertices of the diagram; we allow the inner propagators $B(x, y)$ to connect only comparable vertices (a vertex is comparable to itself).

6. Deformations

In this section, we consider the variations of correlation functions $W_n^{(h)}$ under infinitesimal variations of the Schrödinger potential $U(x)$ or $\hbar$. Infinitesimal variations of the resolvent $\omega(x)$ can be decomposed in the basis of "meromorphic forms" $v_k(\overset{\alpha}{x})$, $k = 1, \ldots$. We set these forms to be dual to special cycles with the duality kernel being the Bergman kernel. It turns out that the classical $\hbar = 0$ formulas retain their form for $\hbar \neq 0$.

6.1. Variation of the resolvent. We consider an infinitesimal polynomial variation $U \to U + \delta U$, $\hbar \to \hbar + \delta\hbar$. Because $U = (V')^2/4 - \hbar V''/2 - P$, we have

$$\delta U = \frac{V'}{2}\delta V' - \frac{\hbar}{2}\delta V'' - \frac{\delta\hbar}{2}\delta V'' - \delta P.$$

We can also consider variations of $V'(z)$ with respect to the higher times t_k, $k = 1, \ldots,$

$$\delta V'(x) = \sum_{k=1} \delta t_k \, x^{k-1}.$$

Then for $k \le d+1$, the polynomial δP has a degree not exceeding $d-1$, and for $k > d+1$, δP has a degree not exceeding $k-2$.

Computing $\delta\big(\psi'_\alpha(x)/\psi_\alpha(x)\big)$, we obtain

$$\delta\left(\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}\right) = \frac{1}{\hbar^2\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} dx'\, \psi_\alpha^2(x')\left(\delta U(x') - 2\frac{\delta\hbar}{\hbar}U(x')\right), \tag{6.1}$$

and for $\omega(\overset{\alpha}{x}) = V'(x)/2 + \hbar\psi'_\alpha(x)/\psi_\alpha(x)$, we have

$$\delta\omega(\overset{\alpha}{x}) = \frac{\delta V'(x)}{2} + \delta\hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \frac{1}{\hbar\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} dx'\, \psi_\alpha^2(x')\left(\delta U(x') - 2\frac{\delta\hbar}{\hbar}U(x')\right). \tag{6.2}$$

6.2. Variations with respect to "flat" coordinates. We choose a system of "flat" coordinates $\epsilon_1, \ldots, \epsilon_{d-1}, t_0, t_1, \ldots$ on the genus-$(d{-}1)$ manifold.

6.2.1. Variations with respect to the filling fractions. For the filling fraction $\delta\epsilon_\alpha$, we have $\delta V' = 0$ and hence $\delta U(x) = -\delta P(x)$, where $\deg \delta P \le d-2$. We can therefore decompose it in the basis h_α:

$$\delta P(x) = \sum_{\alpha'} c_{\alpha'} h_{\alpha'}.$$

From (6.2), we hence have $\delta\omega(x) = -\sum_{\alpha'} c_{\alpha'} v_{\alpha'}(x)\, dx$, and because $2i\pi\epsilon_{\alpha'} = \oint_{\mathcal{A}_{\alpha'}} \omega$, we obtain

$$2i\pi\delta_{\alpha,\alpha'} = \oint_{\mathcal{A}_{\alpha'}} \delta\omega = -\sum_{\alpha''} \oint_{\mathcal{A}_{\alpha'}} dx\, c_{\alpha''} v_{\alpha''}(x) = -c_{\alpha'}.$$

Therefore, $\delta U(x)/\delta_{\epsilon_\alpha} = 2i\pi h_\alpha(x)$ and

$$\delta_{\epsilon_\alpha}\omega(\overset{\beta}{x}) = 2i\pi v_\alpha(\overset{\beta}{x})\, dx = \oint_{\mathcal{B}_\alpha} dz\, B(\overset{\beta}{x}, z).$$

The flat coordinate ϵ_α is dual to the holomorphic form v_α, which is in turn dual to the cycle $\mathcal{B}_\alpha$:

$$\epsilon_\alpha = \frac{1}{2i\pi}\oint_{\mathcal{A}_\alpha} \omega, \qquad \delta_{\epsilon_\alpha}\omega = 2i\pi v_\alpha(x)\, dx = \oint_{\mathcal{B}_\alpha} dz\, B(z).$$

6.2.2. Variation with respect to t_0. We have $\delta U(x) = -\delta P(x) = -t_{d+1}\, x^{d-1} + Q(x)$, where $\deg Q \le d-2$. Using Eq. (6.2), we obtain

$$\delta\omega(\overset{\alpha}{x}) = \frac{1}{\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} dx' \left(-t_{d+1}(x')^{d-1} + Q(x')\right)\psi_\alpha^2(x'),$$

and the polynomial Q must be chosen such that $\oint_{\mathcal{A}_i} \delta\omega = 0$. We therefore have

$$\begin{aligned}\delta\omega(\overset{\alpha}{x}) &= -t_{d+1}K_{d-1}(\overset{\alpha}{x}) = \\ &= -t_{d+1}\left(\widehat{K}_{d-1}(\overset{\alpha}{x}) - \sum_{\beta=1}^{d-1} v_\beta(\overset{\alpha}{x}) \oint_{\mathcal{A}_\beta} dx'\, \widehat{K}_{d-1}(x')\right) = v_d(\overset{\alpha}{x}),\end{aligned}$$

where

$$\widehat{K}_k(\overset{\alpha}{x}) = \frac{1}{\psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} dx'\, (x')^k \psi_\alpha^2(x')$$

and $K_k(\overset{\alpha}{x})$ is the kth term in the large-z expansion of

$$K(\overset{\alpha}{x}, z) = -\sum_{k=0}^{\infty} \frac{K_k(\overset{\alpha}{x}, z)}{z^{k+1}}$$

in Theorem 4.2. From Theorem 4.4, we have $G(x, \infty_\alpha) = \eta_\alpha t_{d+1} K_{d-1}(\overset{\alpha}{x})$. It hence follows that

$$\delta_{t_0}\omega(\overset{\alpha}{x}) = \frac{1}{2}\left(G(\overset{\alpha}{x}, \infty_{\tilde{d}_+}) - G(\overset{\alpha}{x}, \infty_{\tilde{d}_-})\right) = \int_{\infty_{\tilde{d}_-}}^{\infty_{\tilde{d}_+}} dz\, B(\overset{\alpha}{x}, z).$$

The integral in this expression is taken over the last cycle $\widetilde{B}_d$.

The flat coordinate t_0 is then dual to the third-kind meromorphic form $-2G(\overset{\alpha}{x}, \infty)$, which is in turn dual to the cycle $[\infty_{\tilde{d}_-}, \infty_{\tilde{d}_+}]$:

$$t_0 = \oint_{\mathcal{C}_{\mathrm{D}}} dz\, \omega(z), \qquad \delta_{t_0}\omega(\overset{\alpha}{x}) = \int_{\infty_{\tilde{d}_-}}^{\infty_{\tilde{d}_+}} dz\, B(\overset{\alpha}{x}, z).$$

6.2.3. Variations with respect to t_k: The two-point correlation function. Because

$$t_k = \oint_{\mathcal{C}_{\mathrm{D}}} \frac{dz}{z^k} \hbar \frac{\psi'(z)}{\psi(z)}, \quad k = 0, 1, \ldots,$$

the conditions $\partial t_k / \partial t_r = \delta_{k,r}$ and $\partial t_k / \partial \epsilon_\beta = 0$ imply

$$\oint_{\mathcal{C}_{\mathrm{D}}} \frac{dz}{z^k} \frac{\partial}{\partial t_r}\left(\hbar \frac{\psi'(z)}{\psi(z)}\right) = \delta_{k,r}, \qquad \oint_{\mathcal{A}_\beta} dz\, \frac{\partial}{\partial t_r}\left(\hbar \frac{\psi'(z)}{\psi(z)}\right) = 0,$$

and from general variational form (6.1), we conclude that (cf. (3.8) and (3.9))

$$\frac{\partial}{\partial t_r}\left(\hbar \frac{\psi_\alpha'(x)}{\psi_\alpha(x)}\right) = v_{d+r}(\overset{\alpha}{x}).$$

We formulate the following lemma.

Lemma 6.1. *We have the equalities*

$$v_{d+r}(\overset{\alpha}{x}) = \frac{1}{2i\pi}\oint_{\mathcal{C}_{\rm D}>x} dz\,\frac{z^r}{r}\,B(\overset{\alpha}{x},z), \quad r=1,2,\dots, \tag{6.3}$$

where $\mathcal{C}_{\rm D} > x$ means that the contour $\mathcal{C}_{\rm D}$ separates x from all infinities ∞_β, $\beta = 1,2,\dots$.

Proof. That the expression in (6.3) has the desired structure follows from the explicit form of the kernel B. We need only verify the normalization conditions. It is obvious that the $\mathcal{A}$-cycle integrals vanish. It remains to prove the equality

$$\delta_{d,l} = \frac{1}{2i\pi}\oint_{\mathcal{C}_{\rm D}}\frac{dx}{x^l}v_{d+r}(x) = \frac{1}{(2i\pi)^2}\oint_{\mathcal{C}_{\rm D}>x} dz\oint_{\mathcal{C}_{\rm D}}\frac{dx}{x^l}\frac{z^r}{r}\,B(z,x).$$

Interchanging the order of integration contours and taking into account that $x^{-l}B(z,x)\sim x^{-l-2}$ as $x\to\infty$, we conclude that the only nonzero contribution comes from the double pole at $x=z$, and consequently

$$\frac{1}{2i\pi}\oint_{\mathcal{C}_{\rm D}}\frac{dx}{x^l}\left(\frac{\partial}{\partial z}\frac{z^r}{r}\right)\Big|_{z=x} = \frac{1}{2i\pi}\oint_{\mathcal{C}_{\rm D}} dx\,x^{r-l-1} = \delta_{r,l}.$$

We now define the *loop insertion operator*

$$\frac{\partial}{\partial V(y)} = \sum_{r=1}^{\infty} r y^{-r-1}\frac{\partial}{\partial t_r},$$

applying which to $\hbar\psi'/\psi$, we obtain

$$\frac{\partial}{\partial V(y)}\left(\hbar\,\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}\right) = \sum_{r=1}^{\infty} y^{-r-1}\oint_{y>\mathcal{C}_{\rm D}} dz\,B(\overset{\alpha}{x},z)z^r,$$

and because $\oint_{\mathcal{C}_{\rm D}} dz\,B(\overset{\alpha}{x},z)=0$, we add the term with $r=0$ into the sum, obtaining

$$\oint_{y>\mathcal{C}_{\rm D}}\frac{dz}{y-z}B(\overset{\alpha}{x},z)$$

in the right-hand side. We note that the point y lies between some infinity, ∞_β for example, and the integration contour $\mathcal{C}_{\rm D}$. Pulling the contour of integration through the point y to infinity, we obtain zero because of the asymptotic conditions for $B(\overset{\alpha}{x},\overset{\beta}{z})$. The only nonzero contribution therefore comes from the residue at $y=z$, which finally gives

$$\frac{\partial}{\partial V(y)}\left(\hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}\right) = -\frac{1}{2}B(\overset{\alpha}{x},\overset{\beta}{y}). \tag{6.4}$$

Correspondingly, because $\partial V'(x)/\partial V(y) = 1/(y-x)^2$, we obtain

$$W_2^{(0)}(\overset{\alpha}{x},\overset{\beta}{y}) := \frac{\partial}{\partial V(y)}\omega(\overset{\alpha}{x}) = -\frac{1}{2}B(\overset{\alpha}{x},\overset{\beta}{y}) + \frac{1}{2(y-x)^2}$$

for the *two-point correlation function* $W_2^{(0)}(x,y)$.

6.3. Variation of higher correlation functions. We note that for all the above variations with respect to the flat coordinates, we have a cycle $\delta\omega^*$ and a (sector-independent) function $\Lambda^*_{\delta\omega}$ such that

$$\delta\omega(\overset{\alpha}{x}) = \int_{\delta\omega^*} dz\, B(\overset{\alpha}{x}, z)\Lambda^*_{\delta\omega}(z).$$

The following theorem allows computing infinitesimal variations of any $W_n^{(h)}$ under a variation of the Schrödinger equation.

Theorem 6.1. *Under an infinitesimal deformation $U \to U + \delta U$, we have*

$$\delta W_n^{(h)}(\overset{\alpha_1}{x}_1, \dots, \overset{\alpha_n}{x}_n) = \int_{\delta\omega^*} dx'\, W_{n+1}^{(h)}(\overset{\alpha_1}{x}_1, \dots, \overset{\alpha_n}{x}_n)\Lambda^*(x'),$$

where $(\delta\omega^*, \Lambda^*_{\delta\omega})$ is the cycle dual to the deformation of the resolvent $\omega \to \omega + \delta\omega$.

Proof. We prove this theorem by induction. We begin with the loop equation for $W_n^{(h)}(\overset{\alpha}{x}, J)$:

$$\left(2\hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \hbar\partial_x\right)W_n^{(h)}(\overset{\alpha}{x}; J) + U_n^{(h)}(\overset{\alpha}{x}, \overset{\alpha}{x}; J) = P_n^{(h)}(x, J). \tag{6.5}$$

Taking a variation δ with respect to an arbitrary flat coordinate, we obtain

$$\left(2\hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \hbar\partial_x\right)\delta W_n^{(h)}(\overset{\alpha}{x}, J) + \left(2\delta\hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)}\right)W_n^{(h)}(\overset{\alpha}{x}, J) + \delta U_n^{(h)}(\overset{\alpha}{x}, \overset{\alpha}{x}; J) = \delta P_n^{(h)}(x, J),$$

where $\delta P_n^{(h)}(x, J)$ is a polynomial in x of a degree not exceeding $d-2$. Here, both

$$\delta U_n^{(h)}(\overset{\alpha}{x}, \overset{\alpha}{x}; J), \qquad 2\delta\hbar\,\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} = \int_{\delta\omega^*} dx'\, 2B(\overset{\alpha}{x}, x')\Lambda^*(x')$$

can be expressed by the induction assumption in the dual-cycle-integration form. Moreover,

$$\delta U_n^{(h)}(\overset{\alpha}{x}, \overset{\alpha}{x}; J) = \int_{\delta\omega^*} dx'\, U_{n+1}^{(h)}(\overset{\alpha}{x}, \overset{\alpha}{x}; J, x')\Lambda^*(x') - \int_{\delta\omega^*} dx'\, 2B(\overset{\alpha}{x}, x')\Lambda^*(x')\cdot W_n^{(h)}(\overset{\alpha}{x}, J) \tag{6.6}$$

because no term containing the two-point correlation function $W_2^{(0)}(\overset{\alpha}{x}, x')$ appears in $\delta U_n^{(h)}(\overset{\alpha}{x}, \overset{\alpha}{x}; J)$.

Using the loop equation of form (6.5) relating $W_{n+1}^{(h)}(\overset{\alpha}{x}; J, x')$ and $U_{n+1}^{(h)}(\overset{\alpha}{x}, \overset{\alpha}{x}; J, x')$, we observe that the second term in the right-hand side of (6.6) cancels the contribution of $2\delta\hbar\psi'_\alpha(x)/\psi_\alpha(x)$, and we obtain

$$\left(2\hbar\frac{\psi'_\alpha(x)}{\psi_\alpha(x)} + \hbar\partial_x\right)\left(\int_{\omega^*} dx'\, W_{n+1}^{(h)}(\overset{\alpha}{x}, J, x')\Lambda^*(x') - \delta W_n^{(h)}(\overset{\alpha}{x}, J)\right) =$$

$$= \delta P_n^{(h)}(x, J) - \int_{\omega^*} dx'\, P_{n+1}^{(h)}(x, J, x')\Lambda^*(x') = \sum_{i=1}^{d-1}\alpha_i(J)h_i(x),$$

where the right-hand side is a polynomial in x of a degree not exceeding $d-2$ expressed in the basis of the polynomials $h_i(x)$. Using (6.2), we obtain

$$\int_{\omega^*} dx'\, W_{n+1}^{(h)}(\overset{\alpha}{x}, J, x')\Lambda^*(x') - \delta W_n^{(h)}(\overset{\alpha}{x}, J) = \sum_{i=1}^{d-1}\alpha_i(J)v_i(x),$$

but because both $W_n^{(h)}(x, J)$ and $W_{n+1}^{(h)}(x, J, x')$ have vanishing $\mathcal{A}$-cycle integrals, it follows that $\alpha_i = 0$, i.e.,

$$\delta W_n^{(h)}(\overset{\alpha}{x}, J) = \int_{\omega^*} dx'\, W_{n+1}^{(h)}(\overset{\alpha}{x}, J, x')\Lambda^*(x').$$

The theorem is proved.

Corollary 6.1. *For all $n \geq 0$ and $h \geq 0$,*

$$\frac{\partial W_n^{(h)}(J)}{\partial \epsilon_\alpha} = \oint_{\mathcal{B}_\alpha} dx'\, W_{n+1}^{(h)}(J, x').$$

7. Classical and quantum geometry: Summary

In the following table, we summarize the comparison between items in classical algebraic geometry and their quantum counterparts.

	classical geometry ($\hbar = 0$)	**quantum geometry**
planar curve	$E(x,y) = \sum_{i,j} E_{i,j} x^i y^j$, $E(x,y) = 0$	$E(x,y) = \sum_{i,j} E_{i,j} x^i y^j$,$[y,x] = \hbar$, $E(x, \hbar\partial_x)\psi = 0$
hyperelliptic planar curve	$y^2 = U(x)$, $\deg U = 2d$	$\hbar^2 \psi'' = U\psi$, $[y,x] = \hbar$
potential	$V'(x) = 2\big(\sqrt{U(x)}\,\big)_+$	
resolvent	$\omega(x) = \dfrac{V'(x)}{2} + y$	$\omega(\overset{\alpha}{x}) = \dfrac{V'(x)}{2} + \dfrac{\hbar\psi'_\alpha(x)}{\psi_\alpha(x)}$
physical sheet(s)	$y \underset{\infty}{\sim} -\dfrac{V'(x)}{2}, \quad \omega \sim \dfrac{t_0}{x}$	combined sectors where $\dfrac{\hbar\psi'}{\psi} \underset{\infty}{\sim} -\dfrac{V'(x)}{2}, \quad \omega \sim \dfrac{t_0}{x}$
branch points	simple zeros of $U(x)$, $U(a_i) = 0$, $U'(a_i) \neq 0$, $i = 1, \ldots, 2d+2$	rays L_i of accumulations of zeros of ψ, $i = 1, \ldots, 2d+2$
double points	double zeros of $U(x)$, $U(\widehat{a}_i) = 0$, $U'(\widehat{a}_i) = 0$	rays without accumulations of zeros of ψ
genus $g = -1$	degenerate surface	$\psi e^{V/2\hbar}$ is polynomial
$\mathcal{A}_\alpha$-cycles $\alpha = 1, \ldots, g$	surround pairs of branch points	surround pairs of rays of accumulating zeros
extra $\mathcal{A}_d$-cycle	surrounds last pair of branch points	surrounds last pair of rays of accumulating zeros
$\mathcal{B}$-cycles	$\mathcal{A}_i \cap \mathcal{B}_j = \delta_{i,j}$	
holomorphic forms, first-kind differentials	$v_i(x) = -\dfrac{h_i(x)}{2\sqrt{U(x)}}$,	$v_i(\overset{\alpha}{x}) = \dfrac{1}{\hbar\psi_\alpha^2(x)} \displaystyle\int_{\infty_\alpha}^{x} dx'\, \psi_\alpha^2(x') h_i(x')$,
	h_i are polynomials, $\deg h_i \leq d-2$, normalized: $\oint_{\mathcal{A}_\alpha} dx\, v_i(x) = \delta_{\alpha,i}$, $\alpha = 1, \ldots, d-1$	
period matrix	$\tau_{i,j} = \oint_{\mathcal{B}_j} dz\, v_i(z), \quad i,j = 1, \ldots, g, \qquad \tau_{i,j} = \tau_{j,i}$	
filling fractions	$2i\pi\epsilon_\alpha = \oint_{\mathcal{A}_\alpha} dx\, \omega(x), \quad \alpha = 1, \ldots, g, \qquad \epsilon_d = t_0 - \sum_{\alpha=1}^{g} \epsilon_\alpha$	
third-kind form	$G(x,z) \underset{x \to z}{\sim} \dfrac{1}{z-x}$, $G(\overset{\alpha}{x}, \overset{\beta}{z}) = \big(2\omega(\overset{\beta}{z}) - V'(z) - \hbar\partial_z\big) K(\overset{\alpha}{x}, z)$	

	classical geometry ($\hbar = 0$)	**quantum geometry**
recursion kernel	$K(\overset{\alpha}{x}, z) = \widehat{K}(\overset{\alpha}{x}, z) - \sum_i v_i(\overset{\alpha}{x}) C_i(z)$, $C_i(z) = \oint_{\mathcal{A}_i} dx'\, \widehat{K}(x', z)$, $\widehat{K}(x, z) = \frac{1}{z-x} \frac{1}{2\sqrt{U(x)}}$	$\widehat{K}(\overset{\alpha}{x}, z) = \frac{1}{\hbar \psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} \frac{dx'}{x'-z} \psi_\alpha^2(x')$
Bergman kernel, second-kind differential	$B(\overset{\alpha}{x}, \overset{\beta}{z}) = -\frac{\partial_z G(\overset{\alpha}{x}, \overset{\beta}{z})}{2}$, $B(\overset{\alpha}{x}, \overset{\alpha}{z}) \sim \frac{1}{2(x-z)^2}$	
symmetry	$B(\overset{\alpha}{x}, \overset{\beta}{z}) = B(\overset{\beta}{z}, \overset{\alpha}{x})$	
bilinear Riemann identity	$\oint_{\mathcal{A}_i} dx\, B(x, \overset{\beta}{z}) = 0$, $\oint_{\mathcal{B}_i} dx\, B(x, \overset{\beta}{z}) = 2i\pi v_i(\overset{\beta}{z})$	
meromorphic forms	$\mathcal{R}(x)\, dx = \frac{r(x)\, dx}{2\sqrt{U(x)}}$, $r(x)$ is rational with poles r_i, $r(x) = O(x^{d-2})$,	$\mathcal{R}(\overset{\alpha}{x}) = \frac{1}{\hbar \psi_\alpha^2(x)} \int_{\infty_\alpha}^{x} dx'\, r(x') \psi_\alpha^2(x')$, $\operatorname*{Res}_{r_i} r(x)\psi^2(x) = 0$
higher correlators	$W_{n+1}^{(h)}(\overset{\alpha}{x}, J) = \sum_i \frac{1}{2i\pi} \oint_{\mathcal{C}_i} dz\, K(\overset{\alpha}{x}, z) \big(W_{n+2}^{(h-1)}(z, z, J) + {} + \sum'_{s+s'=h,\, I \cup I' = J} W_{1+\lvert I\rvert}^{(s)}(z, I) W_{1+\lvert I'\rvert}^{(s')}(z, I') \big)$, $\mathcal{C}_i$ surrounds the branch point L_i	$\sum_i \frac{1}{2i\pi} \oint_{\mathcal{C}_i} \cdots = \oint_{\mathcal{C}_{\mathrm{D}}} \cdots$
symmetry	$W_n^{(h)}(x_1, \ldots, x_n) = W_n^{(h)}(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), \quad \sigma \in S_n$	
variations and dual cycles	$U(x) \to U(x) + \delta U(x)$, $\delta U^* \colon \delta\omega(\overset{\alpha}{x}) = \int_{\delta U^*} dx' B(\overset{\alpha}{x}, x') \Lambda_{\delta U}(x')$	
$\delta V' = \sum \delta t_k x^{k-1}$	$\delta_{t_k}\omega(\overset{\alpha}{x}) = \oint_{\mathcal{C}_{\mathrm{D}}} dx' \frac{(x')^k}{k} B(\overset{\alpha}{x}, x')$	
variation δt_0	$\delta_{t_0}\omega(\overset{\alpha}{x}) = \int_{\infty_{\tilde{d}_-}}^{\infty_{\tilde{d}_+}} dx' B(\overset{\alpha}{x}, x')$	
variation $\delta\epsilon_i$	$\delta_{\epsilon_i}\omega(\overset{\alpha}{x}) = \oint_{\mathcal{B}_i} dx'\, B(\overset{\alpha}{x}, x')$	
variations of higher correlators	$\delta W_n^{(h)}(x_1, \ldots, x_n) = \int_{\delta U^*} dx'\, W_{n+1}^{(h)}(x_1, \ldots, x_n, x') \Lambda_{\delta U}(x')$	

8. Application: Matrix models

The main reason for the interest in $W_n^{(h)}$ is that they satisfy the loop equations for the random β-eigenvalue ensembles. We can therefore identify them with the correlation functions (resolvents) of these ensembles.

We consider a (possibly formal) matrix integral

$$Z = \int_{E_{N,\beta}} dM\, e^{-(N\sqrt{\beta}/t_0)\,\mathrm{tr}\, V(M)},$$

where $V(x)$ is some polynomial, $E_{N,1} = H_N$ is the set of Hermitian matrices of size N, $E_{N,1/2}$ is the set of real symmetric matrices of size N, and $E_{N,2}$ is the set of quaternion self-dual matrices of size N (see [7]).

Alternatively, we can integrate over the angular part and obtain an integral only over eigenvalues [7]:

$$Z = \int d\lambda_1 \cdots d\lambda_N\, |\Delta(\lambda)|^{2\beta} \prod_{i=1}^{N} e^{-(N\sqrt{\beta}/t_0)V(\lambda_i)}, \tag{8.1}$$

where $\Delta(\lambda) = \prod_{i<j}(\lambda_j - \lambda_i)$ is the Vandermonde determinant.

We generalize the matrix model to arbitrary values of β taking integral (8.1) as a definition of the β-model integral. For this, we take

$$\hbar = \frac{t_0}{N}\left(\sqrt{\beta} - \frac{1}{\sqrt{\beta}}\right).$$

We note that $\hbar = 0$ corresponds to the Hermitian case $\beta = 1$, and $\hbar \to -\hbar$ corresponds to $\beta \to 1/\beta$.

8.1. Correlation functions and loop equations. We define the connected correlation functions (the resolvents)

$$W_k(x_1, \ldots, x_k) = \beta^{k/2}\left\langle \sum_{i_1,\ldots,i_k} \frac{1}{x_1 - \lambda_{i_1}} \cdots \frac{1}{x_k - \lambda_{i_k}} \right\rangle_{\mathrm{conn}}$$

and

$$W_0 = \mathcal{F} = \log \mathcal{Z}.$$

When considering variations in the potential $V(x)$, we again assume that these resolvents satisfy the asymptotic conditions sectorwise, which means that they are also defined sectorwise. And we assume (this is automatically true if we consider formal matrix integrals) that there is a large-N expansion of the type (where we assume $\hbar = O(1)$)

$$W_k(\overset{\alpha_1}{x_1}, \ldots, \overset{\alpha_k}{x_k}) = \sum_{h=0}^{\infty}\left(\frac{N}{t_0}\right)^{2-2h-k} W_k^{(h)}(\overset{\alpha_1}{x_1}, \ldots, \overset{\alpha_k}{x_k}), \tag{8.2}$$

$$W_0 = \mathcal{F} = \sum_{h=0}^{\infty}\left(\frac{N}{t_0}\right)^{2-2h} W_0^{(h)} \equiv \sum_{h=0}^{\infty}\left(\frac{N}{t_0}\right)^{2-2h} \mathcal{F}_h. \tag{8.3}$$

The loop equations are obtained by integrating by parts, for example, the identity

$$0 = \sum_i \int d\lambda_1 \cdots d\lambda_N \frac{\partial}{\partial \lambda_i}\left(\frac{1}{x - \lambda_i}|\Delta(\lambda)|^{2\beta} \prod_j e^{-(N\sqrt{\beta}/t_0)V(\lambda_j)}\right) \tag{8.4}$$

gives

$$\begin{aligned}
0 = & \sum_i \left\langle \frac{1}{(x-\lambda_i)^2} + 2\beta \sum_{j\neq i} \frac{1}{x-\lambda_i}\frac{1}{\lambda_i-\lambda_j} - \frac{N\sqrt{\beta}}{t_0}\frac{V'(\lambda_i)}{x-\lambda_i} \right\rangle = \\
= & \sum_i \left\langle \frac{1}{(x-\lambda_i)^2} + \beta \sum_{j\neq i} \frac{1}{x-\lambda_i}\frac{1}{x-\lambda_j} - \frac{N\sqrt{\beta}}{t_0}\frac{V'(\lambda_i)}{x-\lambda_i} \right\rangle = \\
= & \sum_i \left\langle \frac{1-\beta}{(x-\lambda_i)^2} + \beta \sum_{j} \frac{1}{x-\lambda_i}\frac{1}{x-\lambda_j} - \frac{N\sqrt{\beta}}{t_0}\frac{V'(\lambda_i)}{x-\lambda_i} \right\rangle = \\
= & \frac{\beta-1}{\sqrt{\beta}}\, W_1'(x) + \beta\left(\frac{1}{\beta}W_1^2(x) + \frac{1}{\beta}W_2(x,x)\right) - \\
& - \frac{N\sqrt{\beta}}{t_0}\left(\frac{1}{\sqrt{\beta}}\, V'(x)W_1(x) - \sum_i \left\langle \frac{V'(x)-V'(\lambda_i)}{x-\lambda_i} \right\rangle\right).
\end{aligned}$$

We define the polynomial

$$P_1(x) = \sqrt{\beta}\sum_i \left\langle \frac{V'(x)-V'(\lambda_i)}{x-\lambda_i} \right\rangle = (V'W_1)_+.$$

We then have the loop equation in [8]

$$W_1^2(x) + \hbar\frac{N}{t_0}W_1'(x) + W_2(x,x) = \frac{N}{t_0}\big(V'(x)W_1(x) - P_1(x)\big).$$

Using expansion (8.2), we obtain the Riccati equation

$$W_1^{(0)}(x)^2 + \hbar\partial_x W_1^{(0)}(x) = V'(x)W_1^{(0)}(x) - P_1^{(0)}(x),$$

satisfied by $\omega(x) = W_1^{(0)}(x)$. The correlation functions of β-eigenvalue models therefore satisfy the topological recursion formulated in Sec. 5.1.

8.2. Variation with respect to $\hbar$. In this subsection, we use the analogy with the β-eigenvalue ensemble to suggest the possible form of the last remaining building block of our construction, which is the variation with respect to $\hbar$, the exponent of the Vandermonde determinant in (8.1). Up to irrelevant multipliers, we can consider $\hbar\partial/\partial\hbar$ instead of $\beta\partial/\partial\beta$, for which we have

$$\beta\frac{\partial}{\partial\beta}\log\mathcal{Z} \sim \frac{2\beta}{\mathcal{Z}}\int d\lambda_1\cdots d\lambda_N\, \Delta(\lambda)^{2\beta}\log|\Delta(\lambda)|\prod_{i=1}^N e^{-(N\sqrt{\beta}/t_0)V(\lambda_i)}, \tag{8.5}$$

and the logarithm of the Vandermonde determinant thus appears.

It seems impossible to construct expression (8.5) only from $W_1(\overset{\alpha}{x})$, but we can use the *two-point* correlation function $W_2(\overset{\alpha}{x},\overset{\gamma}{y})$ instead. Adopting a β-model-inspired definition of $W_2(\overset{\alpha}{x},\overset{\beta}{y})$ as a two-resolvent correlation function (not necessarily connected),

$$W_2(x,y) = \frac{1}{\mathcal{Z}}\int d\lambda_1\cdots d\lambda_N \sum_{i=1}^N \frac{1}{x-\lambda_i}\sum_{j=1}^N \frac{1}{y-\lambda_j}|\Delta(\lambda)|^{\hbar N} e^{-(N\sqrt{\beta}/t_0)V(\lambda)},$$

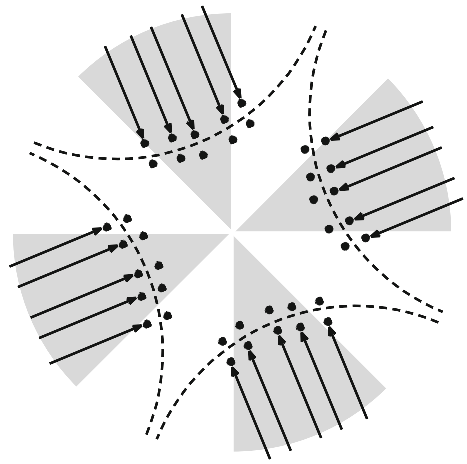


Fig. 7. The origin of the integration contour $\mathcal{C}_{\mathrm{D}}$ in the matrix-model concept. The inner dots are λ_i and the outer dots are $\lambda_i+\delta_\gamma$, $\gamma=0,2,4,6$; thin arrowed lines are the logarithmic cuts.

we then introduce the regularization (both IR and UV, if speaking in physical terms). At this point, we also split all the eigenvalues λ_i into clusters, each of which corresponds to some sector S_γ. For each term $1/(y-\lambda_i)$, we then integrate over y from Λ_γ to $x+\delta_\gamma$ along the straight lines all of which are parallel. The regularization parameters depend only on the sector number γ, and the limit of removed regularization corresponds to $\Lambda_\gamma\to\infty_\gamma$ and $\delta_\gamma\to 0$. We then obtain

$$\frac{2\beta}{\mathcal{Z}}\int_{\Lambda_\gamma}^{x+\delta_\gamma} d\xi\, W_2(\overset{\alpha}{x},\overset{\gamma}{\xi}) \sim$$

$$\sim\int d\lambda_1\cdots d\lambda_N\,\Delta(\lambda)^{2\beta}\sum_{i=1}^{N}\frac{1}{x-\lambda_i}\sum_{\gamma}\sum_{j_\gamma=1}^{\epsilon_\gamma}\int_{\Lambda_\gamma}^{x+\delta_\gamma}\frac{d\xi}{\xi-\lambda_{j_\gamma}}\prod_{i=1}^{N}e^{-(N\sqrt{\beta}/t_0)V(\lambda_i)}=$$

$$=\int d\lambda_1\cdots d\lambda_N\,\Delta(\lambda)^{2\beta}\sum_{i=1}^{N}\frac{1}{x-\lambda_i}\times$$

$$\times\sum_{\gamma}\sum_{j_\gamma=1}^{\epsilon_\gamma}\left(\log|x+\delta_\gamma-\lambda_{j_\gamma}|-\log|\Lambda_\gamma|+O\left(\frac{1}{\Lambda_\gamma}\right)\right)\prod_{i=1}^{N}e^{-(N\sqrt{\beta}/t_0)V(\lambda_i)}. \tag{8.6}$$

We now want to integrate over x to obtain expression (8.5). Obviously, we must choose the integration contour in a rather specific way: we want it to encircle all the poles in $x=\lambda_i$ in the variable x leaving all the logarithmic cuts from ∞_γ to $\lambda_i+\delta_\gamma$ in the corresponding sector outside (see Fig. 7). Given such a contour, we can then integrate over x by residues at the points λ_i (we recall that in the eigenvalue model pattern, we do not yet have boundaries between sectors inside the complex plane; they appear because of the collective effect of taking the λ poles into account by virtue of sectorwise regularization chosen). Evaluating the integral over x in (8.6) by the sum of residues at λ_i, we obtain

$$\frac{2\beta}{\mathcal{Z}}\int d\lambda_1\cdots d\lambda_N\,\Delta(\lambda)^{2\beta}\Bigg(\sum_{i=1}^{N}\sum_{\gamma}\sum_{j_\gamma=1}^{\epsilon_\gamma}\log|\lambda_i+\delta_\gamma-\lambda_{j_\gamma}|-$$

$$-N\sum_{\gamma}\epsilon_\gamma\log|\Lambda_\gamma|+O\left(\frac{1}{\Lambda_\gamma}\right)\Bigg)\prod_{i=1}^{N}e^{-(N\sqrt{\beta}/t_0)V(\lambda_i)}=$$

$$
\begin{aligned}
&= \frac{2\beta}{\mathcal{Z}} \int d\lambda_1 \cdots d\lambda_N \, \Delta(\lambda)^{2\beta} \left| \log \prod_{i \neq j} (\lambda_i - \lambda_j + \delta_\gamma) \right| - \\
&\quad - 2\beta N \sum_\gamma \epsilon_\gamma \log |\Lambda_\gamma| + 2\beta \sum_\gamma \epsilon_\gamma \log |\delta_\gamma|,
\end{aligned}
$$

where the first term in the right-hand side gives the sought integral (8.5) as $\delta_\gamma \to 0$ and the last two terms diverge in the limit of removed regularization. But these two terms depend only on the filling fractions and therefore contribute to only the potential-independent part of $\mathcal{F}_0$, and we can remove them by the proper normalization.

9. The free energy

We use the variations and Theorem 5.4 to define the $\mathcal{F}_h$.

9.1. The operator $\widehat{\boldsymbol{H}}$. Theorem 5.4 gives

$$
(2 - 2h - n - \hbar\partial_\hbar) W_n^{(h)} = \left(t_0 \partial_{t_0} + \sum_{k=1}^{d+1} t_k \partial_{t_k} + \sum_{i=1}^{g} \epsilon_i \partial_{\epsilon_i} \right) W_n^{(h)}.
$$

In Sec. 6, we expressed the derivatives of $W_n^{(h)}$ as integrals of $W_{n+1}^{(h)}$ up to the action of $\hbar\partial/\partial\hbar$,

$$
(2 - 2h - n - \hbar\partial_\hbar) W_n^{(h)} = \widehat{H}.W_{n+1}^{(h)} = \widehat{H}.\frac{\partial}{\partial V} W_n^{(h)},
$$

where $\widehat{H}$ is the linear operator acting as

$$
\widehat{H}.f(x) = t_0 \oint_{\widetilde{\mathcal{B}}_d} dx \, f(x) + \sum_{j=1}^{d+1} \int_{\mathcal{C}_{\mathrm{D}}} dx \, \frac{t_j x^j}{j} f(x) + \sum_{i=1}^{g} \epsilon_i \oint_{\mathcal{B}_i} dx \, f(x).
$$

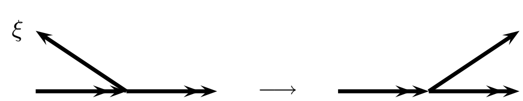
We set $W_0^{(h)} = \mathcal{F}_h$ for $n = 0$ and $h \geq 2$. The *free energy* $\mathcal{F}_h$ for $h \geq 2$ is the functions for which

$$
(2 - 2h - \hbar\partial_\hbar)\mathcal{F}_h = \widehat{H}.W_1^{(h)}.
$$

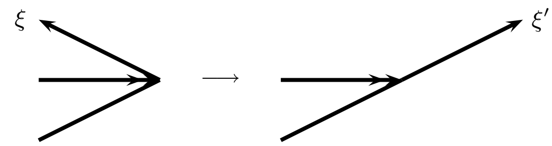
9.2. The derivative $\hbar\,\boldsymbol{\partial/\partial}\hbar$. The matrix-model considerations in the preceding section imply that constructing the derivative in $\hbar$ of the correlation function $W_n^{(h)}(J)$ would involve resolvents of the order $n+2$. In other words,

$$
\begin{aligned}
\hbar\frac{\partial}{\partial\hbar} W_n^{(h)}(J) = \int_{\mathcal{C}_{\mathrm{D}_\xi}} &\left(\int_\infty^\xi d\xi' \, W_{n+2}^{(h-1)}(\bar{\xi}', \xi, J) + \right. \\
&\left. + \sum_{r=0}^{h} \sum_{I \subseteq J} \int_\infty^\xi d\xi' \, W_{|I|+1}^{(r)}(\bar{\xi}', I) \cdot W_{n-|I|+1}^{(h-r)}(\xi, J \setminus I) \right), \qquad (9.1)
\end{aligned}
$$

where $\bar{\xi}$ must be taken to be an "innermost" variable in the sense that taking $\int^\xi d\xi' \, B(\xi', y) = G(\xi, y)$ into account, we everywhere replace



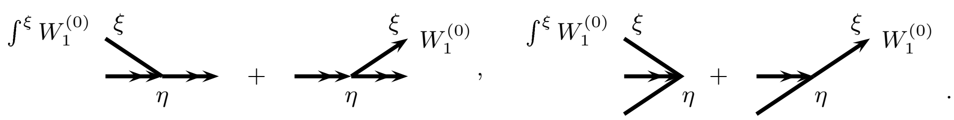
and



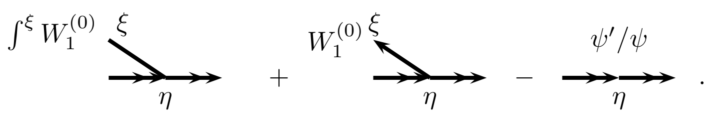
without adding *additional factors*.

We note that the sum in (9.1) ranges all cases, not necessarily stable ones, and we therefore begin by studying nonstable contributions to stable cases ($2h-2+n>0$). We note that all these contributions then come from the second term in (9.1).

9.2.1. The cases $r=0$, $I=\varnothing$ and $r=h$, $I=J$. We consider the situation where $n\geq 1$. We can then fix x_1 to be the root of all the subtrees composed from the K-propagators, and ξ can then be the variable of any of the external B-legs. The contribution to $W_n^{(h)}(J)$ then includes all the insertions



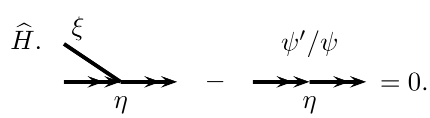
We consider the first diagram; the second is analogous to the first. Changing the integration order over ξ and η in the second diagram gives



Here, the sum of the first two terms contains the integral of the total derivative of $\int^\xi d\xi'\, W_1^{(0)}(\xi')\cdot G(\eta,\xi)$, and because

$$G(\eta,\xi)\sim O(\xi^{-1}),\qquad \int^\xi d\xi'\, W_1^{(0)}(\xi')\sim\int^\xi\frac{d\xi'}{\xi'}t_0\sim t_0\log|\xi|,$$

this contribution vanishes. Only the third contribution coming from the residue at $\xi=\eta$ survives, and this contribution is just minus the action of the $\widehat{H}$ operator on the external leg $B(\eta,\xi)$. Hence,



Therefore, the total contribution of the two cases $r=0$, $I=\varnothing$ and $r=h$, $I=J$ *exactly cancels the action of the $\widehat{H}$ operator.*

9.2.2. The case $r=0$, $I=\{x_1\}$. We begin with the identity

$$\int_{x>\mathcal{C}_{\mathrm{D}_\xi}>y} d\xi\, G(\overset{\alpha}{x},\xi)K(\xi,y)=-K(\overset{\alpha}{x},y)\tag{9.2}$$

(we recall that inequalities of the type $x > \mathcal{C}_{\mathrm{D}_\xi} > y$ indicate the mutual positions of points and integration contours). Indeed, writing $G(\overset{\alpha}{x}, \overset{\gamma}{\xi}) = \psi_\gamma^2(\xi)\partial_\xi(K(\overset{\alpha}{x}, \xi)/\psi_\gamma^2(\xi))$ and integrating by parts, we obtain

$$\sum_\gamma K(\overset{\alpha}{x}, \xi)K(\overset{\gamma}{\xi}, y)\Big|_{\infty_{\tilde{\gamma}_-}}^{\infty_{\tilde{\gamma}_+}} - \int_{x > \mathcal{C}_{\mathrm{D}_\xi} > y} d\xi\, K(\overset{\alpha}{x}, \xi)\left[\frac{1}{\xi - y} + \sum_j h_j(\xi)C_j(y)\right].$$

The substitution obviously gives zero, and only the residue at $\xi = y$ contributes in the second term, thus producing (9.2). An obvious corollary is the second convolution formula

$$\int_{x > \mathcal{C}_{\mathrm{D}_\xi} > y} d\xi\, G(\overset{\alpha}{x}, \xi)B(\xi, \overset{\beta}{y}) = -B(\overset{\alpha}{x}, \overset{\beta}{y}). \tag{9.3}$$

In the case $r = 0$, $I = \{x_1\}$, we have the diagram

$$x_1 \xrightarrow[\xi]{} \boxed{W_n^{(h)}(\xi, J \setminus \{x_1\})} = -W_n^{(h)}(J).$$

9.2.3. The case $\boldsymbol{r = h}$, $\boldsymbol{I = J \setminus \{x_n\}}$. Here, we need another identity,

$$\int_{x,y > \mathcal{C}_{\mathrm{D}_\xi}} d\xi\, G(\overset{\alpha}{x}, \xi)B(\xi, \overset{\beta}{y}) = 0. \tag{9.4}$$

To obtain it, we represent the functions G and B in terms of the kernel K, i.e., we have

$$\begin{aligned}
&\int_{x,y > \mathcal{C}_{\mathrm{D}_\xi}} d\xi\, G(\overset{\alpha}{x}, \xi)B(\xi, \overset{\beta}{y}) = \\
&\quad = \int_{x,y > \mathcal{C}_{\mathrm{D}_\xi}} d\xi \left(\partial_\xi - 2\frac{\psi_\gamma'(\xi)}{\psi_\gamma(\xi)}\right)K(\overset{\alpha}{x}, \xi)\,\partial_\xi\left(\partial_\xi - 2\frac{\psi_\gamma'(\xi)}{\psi_\gamma(\xi)}\right)K(\overset{\beta}{y}, \xi) = \\
&\quad = K(\overset{\alpha}{x}, \xi)B(\overset{\beta}{y}, \overset{\gamma}{\xi})\Big|_{\infty_-}^{\infty_+} - \int_{x,y > \mathcal{C}_{\mathrm{D}_\xi}} d\xi\, K(\overset{\alpha}{x}, \xi)\left(\partial_\xi + 2\frac{\psi_\gamma'(\xi)}{\psi_\gamma(\xi)}\right)\partial_\xi\left(\partial_\xi - 2\frac{\psi_\gamma'(\xi)}{\psi_\gamma(\xi)}\right)K(\overset{\beta}{y}, \xi).
\end{aligned}$$

Here, the substitution gives zero, and the third-order differential operator acting on the kernel $K(\overset{\beta}{y}, \xi)$ is again Gelfand–Dikii operator (4.11), which is independent of the sector γ. The integrand is also obviously regular at all zeros of the ψ functions, and the total integration over $\widetilde{\mathcal{A}}$-cycles therefore just gives zero.

Therefore, the contribution of the case $r = h$, $I = J \setminus \{x_n\}$ is zero, and the total contribution of all the unstable cases together with the action of the operator $\widehat{H}$ just gives the original contribution $W_n^{(h)}(J)$ taken with the opposite sign.

9.3. Examples of applying $\hbar\,\partial/\partial\hbar$.

9.3.1. Reconstructing $\boldsymbol{W_n^{(0)}(J)}$. We now use formula (9.1) to reconstruct the correlation function $W_n^{(0)}(J)$. In the zero-genus case, we need only take the contributions of nonconnected subdiagrams (the second term in (9.1)) into account. We choose the root of the first subdiagram, $\int_\infty^\xi d\xi'\, W_{|I|+1}^{(r)}(\bar{\xi}', I)$, at x_1; the point $\bar{\xi}$ is then the end of some other (nonrooted) leg $G(\eta, \bar{\xi})$ of the first diagram. For the second diagram, we choose the root at the end ξ of the leg with the corresponding propagator $K(\xi, \rho)$. As a result of integrating over ξ, using (9.2), we find that these two diagrams are sewed along the propagator $K(\eta, \rho)$, thus producing a *connected* diagram with the maximum subtree of propagators K rooted at the external point x_1. We can now ask *how many times* the given diagram can be obtained as a composition of two diagrams in formula (9.1). We obtain this diagram by first breaking it into two parts by cutting some of the internal arrowed lines (also including the external line $K(x_1, \kappa)$ if we take the nonstable contributions into account) and then sewing again along the same line. Obviously, we obtain this diagram as many times as the total number of arrowed lines (with the minus sign from (9.2)), i.e., $2-n$ for $W_n^{(0)}(J)$. Hence, using definition (9.1) for the action of $\hbar\,\partial/\partial\hbar$, we obtain

$$\left(\hbar\frac{\partial}{\partial\hbar} + \widehat{H}.\frac{\partial}{\partial V}\right)W_n^{(0)}(J) = (2-n)W_n^{(0)}(J),$$

which is a particular case of formula (9.1).

9.3.2. Acting on $\overline{W}_2^{(0)}(x_1, x_2)$. Here, we consider the action on a nonstable correlation function

$$\overline{W}_2^{(0)}(x_1, x_2) = B(\overset{\alpha_1}{x_1}, \overset{\alpha_2}{x_2}) - \frac{\delta_{\alpha_1,\alpha_2}}{(x_1-x_2)^2}.$$

Excluding the terms that compensate the action of $\widehat{H}$, we find that the action of $\hbar\,\partial/\partial\hbar$ gives

$$\int_{x_1>\mathcal{C}_{\mathrm{D}_\xi}>x_2} d\xi\, G(\overset{\alpha_1}{x_1}, \xi)\left(B(\xi, \overset{\alpha_2}{x_2}) - \frac{1}{(\xi-x_2)^2}\right) - \int_{x_1,x_2>\mathcal{C}_{\mathrm{D}_\xi}} \frac{d\xi}{x_1-\xi} B(\overset{\alpha_2}{x_2}, \xi) + (x_1 \leftrightarrow x_2) =$$
$$= -2B(\overset{\alpha_1}{x_1}, \overset{\alpha_2}{x_2}) + B(\overset{\alpha_1}{x_1}, \overset{\alpha_2}{x_2}) + B(\overset{\alpha_2}{x_2}, \overset{\alpha_1}{x_1}) = 0,$$

which again is in accordance with formula (9.1) (the expression $(x_1 \leftrightarrow x_2)$ in the left-hand side denotes terms obtained by interchanging x_1 and x_2 in the integrals).

9.3.3. Acting on $\mathcal{F}_1$. We expect that applying formula (9.1) in the case of $\mathcal{F}_1$ gives zero, perhaps up to some irrelevant regularizing factors. Nonstable terms do not contribute; the only contribution comes from the first term in (9.1), which gives

$$\left(\hbar\frac{\partial}{\partial\hbar} + \widehat{H}.\frac{\partial}{\partial V}\right)\mathcal{F}_1 = \int_{\mathcal{C}_{\mathrm{D}_\xi}} d\xi\left(G(\xi, \bar{\xi}) - \frac{1}{\xi-\bar{\xi}}\right) =$$
$$= \int_{\mathcal{C}_{\mathrm{D}_\xi}} d\xi\left(\int_{\infty_\alpha}^{\xi+\delta_\alpha} d\xi'\,\frac{\partial}{\partial\xi}\frac{\psi_\alpha^2(\xi')/\psi_\alpha^2(\xi)-1}{\xi'-\xi} + \right.$$
$$\left. + \sum_j \int_{\infty_\alpha}^{\xi+\delta_\alpha} d\xi'\, h_j(\xi')\psi_\alpha^2(\xi')\left(\frac{C_j(\xi)}{\psi_\alpha^2(\xi)}\right)'\right).$$

Integrating by parts in the second term, we obtain $\int_{\mathcal{C}_{\mathrm{D}_\xi}} d\xi\, h_j(\xi)C_j(\xi)$ up to terms of the order $O(\delta_\alpha)$, and the integrand turns out to be independent of the sector and nonsingular at zeros of ψ_α and therefore gives zero when integrated. In the first term, integrating the term with $1/(\xi-\xi')^2$ by parts in the variable ξ' and taking into account that

$$\lim_{\xi'\to\xi}\frac{1}{\xi'-\xi}\left(\frac{\psi_\alpha^2(\xi')}{\psi_\alpha^2(\xi)}\right) = 2\frac{\psi_\alpha'(\xi)}{\psi_\alpha(\xi)},$$

we obtain

$$
\int_{\mathcal{C}_{\mathrm{D}_\xi}} d\xi \left(-2\frac{\psi'_\alpha(\xi)}{\psi_\alpha(\xi)} + \int_{\infty_\alpha}^{x+\delta_\alpha} d\xi' \frac{2\psi'_\alpha(\xi')\psi_\alpha(\xi')/\psi^2_\alpha(\xi) - 2\psi'_\alpha(\xi)\psi^2_\alpha(\xi')/\psi^3_\alpha(\xi)}{\xi' - \xi} \right) =
$$

$$
\begin{aligned}
&= \int_{\mathcal{C}_{\mathrm{D}_\xi}} d\xi \left(-2\frac{\psi'_\alpha(\xi)}{\psi_\alpha(\xi)} + \int_{\infty_\alpha}^{x+\delta_\alpha} d\xi' \left(\frac{2}{\xi' - \xi} \frac{\psi'_\alpha(\xi')\psi_\alpha(\xi')}{\psi^2_\alpha(\xi)} + \frac{\psi^2_\alpha(\xi')}{\xi' - \xi} \left(\frac{1}{\psi^2_\alpha(\xi)} \right)' \right) \right) = \\
&= \int_{\mathcal{C}_{\mathrm{D}_\xi}} d\xi \left(-2\frac{\psi'_\alpha(\xi)}{\psi_\alpha(\xi)} + \int_{\infty_\alpha}^{x+\delta_\alpha} d\xi' \left(\frac{2}{\xi' - \xi} \frac{\psi'_\alpha(\xi')\psi_\alpha(\xi')}{\psi^2_\alpha(\xi)} - \frac{\psi^2_\alpha(\xi')}{\psi^2_\alpha(\xi)} \frac{1}{(\xi' - \xi)^2} \right) \right) = \\
&= \int_{\mathcal{C}_{\mathrm{D}_\xi}} d\xi \left(-2\frac{\psi'_\alpha(\xi)}{\psi_\alpha(\xi)} + \int_{\infty_\alpha}^{x+\delta_\alpha} d\xi' \left(\frac{2}{\xi' - \xi} \frac{\psi'_\alpha(\xi')\psi_\alpha(\xi')}{\psi^2_\alpha(\xi)} + \frac{\psi^2_\alpha(\xi')}{\psi^2_\alpha(\xi)} d\frac{1}{\xi' - \xi} \right) \right) = \\
&= \int_{\mathcal{C}_{\mathrm{D}_\xi}} d\xi \left(-2\frac{\psi'_\alpha(\xi)}{\psi_\alpha(\xi)} + \int_{\infty_\alpha}^{x+\delta_\alpha} d\xi' \left(\frac{1}{\xi' - \xi} \frac{\partial}{\partial \xi'} \frac{\psi^2_\alpha(\xi')}{\psi^2_\alpha(\xi)} + \frac{\psi^2_\alpha(\xi')}{\psi^2_\alpha(\xi)} d\frac{1}{\xi' - \xi} \right) \right) = \\
&= \int_{\mathcal{C}_{\mathrm{D}_\xi}} d\xi \left(-2\frac{\psi'_\alpha(\xi)}{\psi_\alpha(\xi)} + \frac{1}{\xi' - \xi} \frac{\psi^2_\alpha(\xi')}{\psi^2_\alpha(\xi)} \bigg|_{\infty_\alpha}^{x+\delta_\alpha} \right) = \int_{\mathcal{C}_{\mathrm{D}_\xi}} d\xi \left(\frac{1}{\delta_\alpha} + O(\delta_\alpha) \right).
\end{aligned}
$$

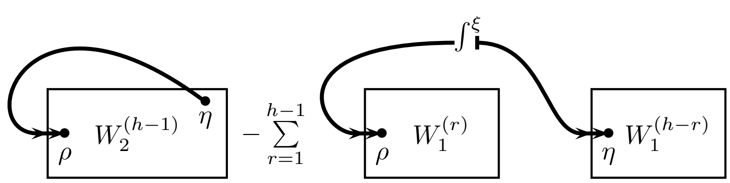
The result is hence a constant, which diverges in the limit of removed regularization but is otherwise independent of all the variables (the same phenomenon occurs when calculating the corresponding action of the $\widehat{H}$ operator on $\mathcal{F}_1$ in the standard matrix models [4], [8]).

9.3.4. Acting on $W_1^{(1)}(x)$. In the case of $W_1^{(1)}(x)$, we have two possible contributions: the one from nonstable graphs gives $W_1^{(1)}(x)$ with the (desired) factor -1, and the other would come from the first term in (9.1) originating from the $W_3^{(0)}$ term, i.e.,

$$
\int_{\mathcal{C}_{\mathrm{D}_\xi}} d\xi \int_{\mathcal{C}_{\mathrm{D}_\eta}} d\eta \, K(\overset{\alpha}{x}, \eta) G(\eta, \bar{\xi}) B(\eta, \xi),
$$

where the contour of integration over η goes between the points $\bar{\xi}$ and ξ. We set integrals of this type to be zero, which provides the last required prescription for the diagram technique describing the free energy terms $\mathcal{F}_h$.

9.4. The term $\mathcal{F}_h$. For the stable cases ($h \neq 0, 1$), we can now formulate the diagram technique for the term $\mathcal{F}_h$. We need the diagrams describing the stable terms $W_1^{(r)}(\bar{\xi})$ and $W_1^{(h-r)}(\xi)$ with $1 \le r \le h-1$ and $W_2^{(h-1)}(\xi, \bar{\xi})$:

$(2h-2)\mathcal{F}_h =$ 

.

Here, the sum in the first term ranges all the diagrams contributing to $W_2^{(h-1)}$ that have distinct vertices to which the external legs are attached; we then amputate both these legs and join the vertices η and ρ (the vertex ρ is always the first three-valent vertex in the rooted tree) by the propagator $K(\eta, \rho)$. The integration over ξ is already taken into account, and we thus obtain an extra minus sign. We cannot integrate that easily in the second term, where the integration over ξ is such that $\rho > \mathcal{C}_{\mathrm{D}_\xi} > \eta$ and the symbol $\int^\xi$ indicates that we must insert the integration

$$\int_{\rho>\mathcal{C}_{\mathrm{D}_\xi}>\eta} d\xi \int_{\infty_\alpha}^{\xi+\delta_\alpha} d\xi'\, K(\overset{\alpha}{\xi'},\rho)K(\overset{\alpha}{\xi},\eta)$$

between the integrations over the variables ρ and η.

10. Conclusion

We have defined a quantum version of algebraic geometry notions, which allows solving the loop equations in the case of an arbitrary β-ensemble.

The notion of branch points becomes "blurred." A branch point is no longer a point but an asymptotic accumulation line along which we integrate instead of taking the residue at the branch point.

Another surprising property pertains to the cohomology theory, which makes sense only if the cycle integral of any form depends only on the homology class of the cycle, i.e., we need all forms to have zero residues at the zeros s_i. This "no-monodromy" condition is automatically satisfied for our forms coming from the Schrödinger equation, and it is equivalent to the set of Bethe ansatz equations satisfied by s_i, similar to what occurs in the Gaudin model [18].

In contrast to [1], there is no explicit dependence here on the chosen sector. But even the total number of $\mathcal{A}$-cycles and the rank of the period matrix may vary depending on the choice of cuts in the complex plane. This might be because we do not have an actual finite-genus (classical) Riemann surface. Analytic continuation may never result in sewing the corresponding solutions of the Schrödinger equation, and we therefore deal with different (finite-genus) sections of an ambient infinite-genus surface. Then the genus is indeed no longer deterministic.

Using the sectorwise approach, we can define the symplectic invariants. In Appendix D, we present the first nontrivial calculation of this sort: the dependence of the leading term on the filling fractions.

Here, we restricted ourself to the case of hyperelliptic curves, i.e., second-order differential equations, which corresponds to the one-matrix model. The first straightforward generalization is to include the logarithmic potentials in the consideration, which would produce the Nekrasov functions nonperturbatively in the parameter ϵ_2/ϵ_1. A more challenging problem is to generalize this approach to linear differential equations of any order, which would correspond to a two-matrix β-ensemble model. In this case, we can also presumably define the notions of sheets, branch points, forms, and correlation functions. We also expect the preservation of the Bethe ansatz property ensuring a no-monodromy condition claiming that all cycle integrals depend only on the homology classes of cycles. The difference between the hyperelliptic case and the general case is comparable to the difference between the patterns in [2] and [5], i.e., the definition of the kernel K must be more involved and less explicit, but we postpone this discussion for further publications.

It would also be interesting to see whether the quantities $\mathcal{F}_h$ have a symplectic invariance or, more precisely, a "canonical invariance," i.e., whether they are invariant under any change $(x,y) \to (\widetilde{x},\widetilde{y})$ such that $[\widetilde{y},\widetilde{x}] = [y,x] = \hbar$.

Appendix A: Proof of Theorem 5.2

We now prove Theorem 5.2, that all $W_n^{(h)}$ satisfy the loop equation, i.e.,

$$P_{n+1}^{(h)}(x;\overset{\alpha_1}{x_1},\ldots,\overset{\alpha_n}{x_n}) = \hbar\biggl(2\frac{\psi_\alpha'(x)}{\psi_\alpha(x)} + \partial_x\biggr)\overline{W}_{n+1}^{(h)}(\overset{\alpha}{x},\overset{\alpha_1}{x_1},\ldots,\overset{\alpha_n}{x_n}) +$$

$$+\sum_{r=0}^{h}\sum_{I\subset J}{}' \overline{W}^{(r)}_{|I|+1}(\overset{\alpha}{x},I)\overline{W}^{(h-r)}_{n-|I|+1}(\overset{\alpha}{x},J\setminus I)+\overline{W}^{(h-1)}_{n+2}(\overset{\alpha}{x},\overset{\alpha}{x},J)+$$

$$+\sum_{j}\partial_{x_j}\biggl(\frac{\overline{W}^{(h)}_{n}(\overset{\alpha}{x},J\setminus\{x_j\})\delta_{\alpha,\alpha_j}-\overline{W}^{(h)}_{n}(\overset{\alpha_j}{x}_j,J\setminus\{x_j\})}{x-x_j}\biggr)$$

is a polynomial in x of a degree not exceeding $d-2$. From the definition, we have (with U from (5.5))

$$W^{(h)}_{n+1}(\overset{\alpha}{x},J)=\frac{1}{2i\pi}\oint_{\mathcal{C}}dz\,K(\overset{\alpha}{x},z)\biggl(U^{(g-1)}_{n+2}(z,z,J)+\sum_{j}B(\overset{\alpha_j}{x_j},z)W^{(h)}_{n}(z,J\setminus\{x_j\})\biggr).$$

Acting with $\hbar\bigl(2\psi'_\alpha(x)/\psi_\alpha(x)+\partial_x\bigr)$ on $K(\overset{\alpha}{x},z)$ gives

$$\frac{1}{x-z}+\sum_{j=1}^{g}h_j(x)C_j(z),$$

and the second part is obviously a polynomial satisfying the assertions in the theorem. Pulling the contour of integration over z to infinity (with x originally outside the integration contour) and taking into account that the integral at infinity vanishes because of the asymptotic conditions, we find that only the residue at $z=x$ and the residue at $z=x_j$ in the second term in the parentheses contribute. The integration result is

$$U^{(g-1)}_{n+2}(\overset{\alpha}{x},\overset{\alpha}{x},J)+\sum_{j}B(\overset{\alpha_j}{x_j},\overset{\alpha}{x})W^{(h)}_{n}(\overset{\alpha}{x},J\setminus\{x_j\})+\sum_{j}\frac{\partial}{\partial x_j}\frac{W^{(h)}_{n}(J)}{x-x_j},$$

and taking (5.3) into account, we obtain the assertions in the theorem.

Appendix B: The symmetricity of $W_3^{(0)}$

Theorem B.1. *The three-point function $W_3^{(0)}$ is symmetric.*

Proof. Introducing $Y_\alpha=-2\hbar\psi'_\alpha/\psi_\alpha$, from the definition, we obtain

$$\begin{aligned}W_3^{(0)}(\overset{\alpha_0}{x_0},\overset{\alpha_1}{x_1},\overset{\alpha_2}{x_2})&=\frac{1}{i\pi}\oint_{\mathcal{C}_{\mathrm{D}}}dx\,K(\overset{\alpha_0}{x_0},x)B(\overset{\alpha_1}{x_1},x)B(\overset{\alpha_2}{x_2},x)=\frac{1}{4i\pi}\oint_{\mathcal{C}_{\mathrm{D}}}dx\,K_0G'_1G'_2=\\&=\frac{1}{4i\pi}\oint_{\mathcal{C}_{\mathrm{D}}}dx\,K_0\bigl((\hbar K''_1+YK'_1+Y'K_1)(\hbar K''_2+YK'_2+Y'K_2)\bigr)=\\&=\frac{1}{4i\pi}\oint_{\mathcal{C}_{\mathrm{D}}}dx\,K_0\bigl(\hbar^2K''_1K''_2+\hbar Y(K'_1K''_2+K''_1K'_2)+\hbar Y'(K''_1K_2+K''_2K_1)+\\&\quad+Y^2K'_1K'_2+YY'(K_1K'_2+K'_1K_2)+(Y')^2K_1K_2\bigr),\end{aligned}$$

where we introduce the shorthand notation $K_i=K(x_i,x)$ and $G_i=G(x_i,x)$, all the derivatives are with respect to x, and we omit the sector indices.

The combinations $K_0K_1K_2f(x)$, where $f(x)$ is sector-independent ($f=1,U,U',\dots$), vanish after integration with respect to x because each K_i is also sector-independent with respect to x. We can then use the Riccati equation $Y_\alpha^2 = 2\hbar Y'_\alpha + 4U$ to replace Y_α^2 with $2\hbar Y'_\alpha$ and $Y_\alpha Y'_\alpha$ with $\hbar Y''_\alpha$, which gives

$$
\begin{aligned}
W_3^{(0)}(x_0,x_1,x_2) &= \frac{1}{4i\pi}\oint_{\mathcal{C}_{\mathrm{D}}} dx\, K_0\big(\hbar Y(K_1'K_2''+K_1''K_2') + \hbar Y'(K_1''K_2+K_2''K_1) + {}\\
&\quad + 2\hbar Y'K_1'K_2' + \hbar Y''(K_1K_2'+K_1'K_2) + (Y')^2K_1K_2\big) = \\
&= \frac{1}{4i\pi}\oint_{\mathcal{C}_{\mathrm{D}}} dx\, K_0\big(\hbar Y(K_1'K_2')' + \hbar Y'(K_1K_2)'' + \hbar Y''(K_1K_2)' + (Y')^2K_1K_2\big) = \\
&= \frac{1}{4i\pi}\oint_{\mathcal{C}_{\mathrm{D}}} dx\, (Y')^2K_0K_1K_2 + \hbar\big(Y''K_0(K_1K_2)' - (YK_0)'K_1'K_2' - (Y'K_0)'(K_1K_2)'\big) = \\
&= \frac{1}{4i\pi}\oint_{\mathcal{C}_{\mathrm{D}}} dx\, (Y')^2K_0K_1K_2 - \hbar\big((YK_0)'K_1'K_2' + Y'K_0'(K_1K_2)'\big) = \\
&= \frac{1}{4i\pi}\oint_{\mathcal{C}_{\mathrm{D}}} dx\, (Y')^2K_0K_1K_2 - \hbar YK_0'K_1'K_2' - \hbar Y'(K_0K_1'K_2' + K_0'K_1K_2' + K_0'K_1'K_2).
\end{aligned}
$$

This expression is explicitly symmetric in x_0, x_1, and x_2. The theorem is proved.

Appendix C: Proof of Theorem 5.3

We prove that each $W_n^{(h)}$ is a symmetric function of all its arguments. The special case of $W_3^{(0)}$ was proved in Appendix B. The symmetricity of the two-point correlation function $\overline{W}_2^{(0)}$ was proved in Theorem 4.9.

For technical reason, it is easier to proceed with the proof for *nonconnected* correlation functions. We introduce two types of them: the correlation function

$$\widehat{W_n^{(h)}}(I) = \sum_{\{I_1,\dots,I_k\}}\prod_{j=1}^{k} W_{n_j}^{(h_j)}(I_j), \tag{C.1}$$

where the summation ranges all partitions $\{I_1,\dots,I_k\}$ of the set I, which includes only partitions of the stable $(2h_j+n_j-2>0)$ type with $I_j\neq\varnothing$, and the correlation function

$$\widetilde{W_n^{(h)}}(I) = {\sum_{\{I_1,\dots,I_k\}}}' \prod_{j=1}^{k} W_{n_j}^{(h_j)}(I_j), \tag{C.2}$$

which moreover admits two-point correlation functions $\overline{W}_2^{(0)}$ in the sums, $2h_j+n_j-2\geq 0$, with $I_j\neq\varnothing$.

The symmetricity of all $\widehat{W_s^{(h')}}(I)$ with $s+2h'\leq n+2h$ obviously implies the symmetricity of $\widetilde{W_n^{(h)}}(I)$.

It is obvious that $\widehat{W_{n+1}^{(h)}}(\overset{\alpha_0}{x_0},\overset{\alpha_1}{x_1},\dots,\overset{\alpha_n}{x_n})$ is symmetric in $x_1,x_2,\dots,x_n$, and it therefore suffices to show that (for $n\geq 1$)

$$\widehat{W_{n+1}^{(h)}}(\overset{\alpha_0}{x_0},\overset{\alpha_1}{x_1},J) - \widehat{W_{n+1}^{(h)}}(\overset{\alpha_1}{x_1},\overset{\alpha_0}{x_0},J) = 0,$$

where $J=\{\overset{\alpha_2}{x_2},\dots,\overset{\alpha_n}{x_n}\}$.

The proof is by recursion on $-\chi = 2h-2+n$. We assume that all $\widehat{W_k^{(h')}}$ and $\widetilde{W_k^{(h')}}$ with $2h'+k-2\leq 2h+n$ are symmetric. We have

$$
\begin{aligned}
\widehat{W_{n+1}^{(h)}}(\overset{\alpha_0}{x_0},\overset{\alpha_1}{x_1},J) &= \frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathrm{D}_x}>y} dx\, K\big(\overset{\alpha_0}{x_0},x\big)\widetilde{W_{n+2}^{(h-1)}}(x,x,x_1,J) + {}\\
&\quad + \frac{1}{\pi i}\iint_{\mathcal{C}_{\mathrm{D}_x}>\mathcal{C}_{\mathrm{D}_y}} dx\,dy\, K\big(\overset{\alpha_0}{x_0},x\big)B(\overset{\alpha_1}{x_1},x)K(x,y)\widetilde{W_{n+1}^{(h-1)}}(y,J)\big).
\end{aligned}\tag{C.3}
$$

We first consider the product of functions KBK in the second term. Recalling that

$$B(\overset{\alpha_1}{x_1},\overset{\beta}{x}) = \partial_x\left(\partial_x - 2\frac{\psi'_\beta(x)}{\psi_\beta(x)}\right)K\left(\overset{\alpha_1}{x_1},x\right)$$

and integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathrm{D}_x}>y} dx\, K(\overset{\alpha_0}{x_0},x)B(\overset{\alpha_1}{x_1},x)K(x,y) = & -\frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathrm{D}_x}>y} dx\, K'_x(\overset{\alpha_0}{x_0},x)K'_x(\overset{\alpha_1}{x_1},x)K(x,y) + \\ & + \frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathrm{D}_x}>y} dx\, K'_x(\overset{\alpha_0}{x_0},x)K(\overset{\alpha_1}{x_1},x)2\,\frac{\psi'(x)}{\psi(x)}K(x,y) - \\ & - \frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathrm{D}_x}>y} dx\, K(\overset{\alpha_0}{x_0},x)K'_x(\overset{\alpha_1}{x_1},x)K'_x(x,y) + \\ & + \frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathrm{D}_x}>y} dx\, K(\overset{\alpha_0}{x_0},x)K(\overset{\alpha_1}{x_1},x)2\,\frac{\psi'(x)}{\psi(x)}K'_x(x,y).\end{aligned}$$

The first and last terms in the right-hand side are already symmetric under the replacement $x_0 \leftrightarrow x_1$, and we disregard them. Integrating by parts in the third term in the right-hand side, we obtain one more symmetric term with $K(\overset{\alpha_0}{x_0},x)K(\overset{\alpha_1}{x_1},x)K''_{xx}(x,y)$ (which we can also disregard) plus the term with $K'_x(\overset{\alpha_0}{x_0},x)K(\overset{\alpha_1}{x_1},x)K'_x(x,y)$. Combining the result with the second term, we obtain

$$\begin{aligned}&\frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathrm{D}_x}>y} dx\, K'_x(\overset{\alpha_0}{x_0},x)K(\overset{\alpha_1}{x_1},x)\left(\partial_x + 2\frac{\psi'(x)}{\psi(x)}\right)K(x,y) = \\ &\qquad = \frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathrm{D}_x}>y} dx\, K'_x(\overset{\alpha_0}{x_0},x)K(\overset{\alpha_1}{x_1},x)\left(\frac{1}{x-y} + \sum_\beta h_\beta(x)C_\beta(y)\right),\end{aligned}$$

where the integrand is the same in all sectors of x, and only the residue at $x = y$ (with the minus sign) hence contributes in the second term in (C.3), which then becomes

$$-\frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathrm{D}_y}} dy\, 2K'_y(\overset{\alpha_0}{x_0},y)K(\overset{\alpha_1}{x_1},y)\widetilde{W^{(h-1)}_{n+1}}(y,y,J). \tag{C.4}$$

For the first term in (C.3), we use the induction assumption, writing it in the form

$$\begin{aligned}&\frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathrm{D}_x}>y}\frac{dx}{2\pi i}\oint_{\mathcal{C}_{\mathrm{D}_y}} dy\, K(\overset{\alpha_0}{x_0},x)K(\overset{\alpha_1}{x_1},y)\times \\ &\qquad \times\left(2B(x,y)^2\delta_{1,2} + 4B(x,y)\widetilde{W^{(h-1)}_{n+1}}(x,y,J)' + \widetilde{W^{(h-2)}_{n+3}}(x,x,y,y,J)'\right),\end{aligned}$$

where the prime indicates that no propagators of the $B(x,y)$ type enter the expression, and no singularity occurs in the corresponding terms under interchanging the order of contour integration over x and y. The last term is again obviously symmetric under the replacement $x_0 \leftrightarrow x_1$.

The skew-symmetric part in the middle term is one-half the residue coming from the double pole $-1/(x-y)^2$ in the expression for $B(x,y)$ (it again comes with the minus sign by virtue of the choice of contour ordering), and we therefore obtain

$$\frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathrm{D}_y}} dy\, 2K'_y(\overset{\alpha_0}{x_0},y)K(\overset{\alpha_1}{x_1},y)\widetilde{W^{(h-1)}_{n+1}}(y,y,J)' + \text{symmetric term},$$

which exactly cancels the term in (C.4) in all cases except only the case $h = 1$, $n = 2$, where we use the fact that $B(x,y) = -1/(x-y)^2 + \overline{W}_2^{(0)}(x,y)$ as $x \to y$, and therefore

$$2B(x,y)^2 = 2(x-y)^{-4} - 4(x-y)^{-2}\overline{W}_2^{(0)}(x,y) + \text{regular part}.$$

The most singular first term results in the integrand $K'''K$, which is sector-independent and therefore vanishes, and the second term produces

$$\frac{1}{2\pi i}\oint_{\mathcal{C}_{\mathrm{D}_y}} dy\, 2K'_y(\overset{\alpha_0}{x_0}, y)K(\overset{\alpha_1}{x_1}, y)\overline{W}_2^{(0)}(y,y) + \text{symmetric term},$$

which cancels the last remaining term in (C.4). The theorem is proved.

Appendix D: Calculating $\partial^3\mathcal{F}_0/\partial t_0^3$ in the Gaussian case

In this appendix, we calculate the singular part of the third derivative of $\mathcal{F}_0$ and integrate the answer, which allows obtaining the singular part of the free energy $\mathcal{F}_0$. Although we calculate only the Gaussian model case explicitly, based on it, we can propose the singular part of the free energy for the model with a general potential.

In the Gaussian model case with the potential $V(x) = x^2$, we have four sectors of solutions with the asymptotic directions $\pm\infty$ and $\pm i\infty$. As the basic solutions, we take $\psi_+(x)$ and $\psi_-(x)$ that decrease at the corresponding imaginary infinities $+i\infty$ and $-i\infty$. The real axis then plays the role of the $\widetilde{\mathcal{A}}$-cycle, and the imaginary axis is the $\widetilde{\mathcal{B}}$-cycle.

We are interested in evaluating the singular part of the third-order derivative $\partial^3\mathcal{F}_0/\partial t_0^3$. We first clarify the origin of this singularity. Obviously, local singularities at finite t_0 appear when the solutions ψ_+ and ψ_- coincide, which happens when $\psi_\pm = \psi_n = H_n(ix)e^{x^2/2\hbar}$, where H_n are the Hermite polynomials and

$$\hbar^2\partial_x^2\psi_n(x) = x^2\psi_n(x) + (2n+1)\hbar\psi_n(x).$$

From Corollary 6.1, we have

$$\frac{\partial^3\mathcal{F}_0}{\partial t_0^3} = \frac{1}{(2\pi i)^3}\oint_{\mathcal{B}}\oint_{\mathcal{B}}\oint_{\mathcal{B}} dz_1\, dz_2\, dz_3\, W_3^{(0)}(z_1,z_2,z_3). \tag{D.1}$$

Because $W_3^{(0)}(\overset{\alpha_1}{z_1}, \overset{\alpha_2}{z_2}, \overset{\alpha_3}{z_3}) = \oint_{\mathcal{C}_{\mathrm{D}}} d\xi\, K(\overset{\alpha_1}{z_1}, \xi)B(\overset{\alpha_2}{z_2}, \xi)B(\overset{\alpha_3}{z_3}, \xi)$ and no singularities appear when integrating over z_2 and z_3, we find that by Theorem 4.8, each integral gives just $v_0(\overset{\alpha}{\xi})$ with $\alpha = \pm$ and

$$v_0(\overset{\pm}{\xi}) = C_0\frac{1}{\psi_\pm^2(\xi)}\int_{\pm i\infty}^{\xi} d\rho\, \psi_\pm^2(\rho) \tag{D.2}$$

with the normalization constant C_0 such that

$$\int_{-\infty}^{+\infty} d\xi[v_0(\overset{-}{\xi}) - v_0(\overset{+}{\xi})] = 1.$$

We note that even in the case where $\psi_+ = \psi_-$, the functions $v_0(\overset{+}{\xi})$ and $v_0(\overset{-}{\xi})$ differ because of different lower integration limits, their difference is just $\left(C_0\psi^{-2}(\xi)\right)\int_{-i\infty}^{+i\infty} d\rho\, \psi^2(\rho)$, and the normalization constant C_0 at $\psi_+ = \psi_- = \psi_n$ is

$$C_0 = \left(\int_{-\infty}^{+\infty}\frac{d\xi}{\psi_n^2(\xi)}\right)^{-1}\left(\int_{-i\infty}^{+i\infty} d\rho\, \psi_n^2(\rho)\right)^{-1}. \tag{D.3}$$

The remaining integral over z_1 in (D.1) develops a singularity as $\psi_\pm \to \psi_n$ because the function $K(\overset{\pm}{z}_1, \xi)$ develops a logarithmic cut on the $\widetilde{\mathcal{B}}$-cycle, and using explicit form (4.1) for the K-kernel ($K = \widehat{K}$ in this simplest case), we obtain

$$\frac{\partial^3 \mathcal{F}_0}{\partial t_0^3} = \sum_{\pm} \int_{\mathcal{C}_\xi^\pm} \frac{d\xi}{2\pi i} \int_{\pm i\infty}^{\mp i\infty} \frac{1}{\hbar} \frac{dz}{\psi_\pm^2(z)} \int_{\pm i\infty}^{z} d\rho \, \frac{\psi_\pm^2(\rho)}{\rho - \xi} v_0^2(\overset{\pm}{\xi}), \tag{D.4}$$

where the contour $\mathcal{C}_\xi^\pm$ runs between $\pm\infty$ and $\mp\infty$ encircling the point ρ. The singularity appears as ρ (and correspondingly z) tends to $-i\infty$ for ψ_+ and to $+i\infty$ for ψ_-. This singular part comes from the residue at $\xi = \rho$, and we find that the expression in (D.4) is

$$\sum_{\pm} \int_0^{\mp i\infty} \frac{1}{\hbar} \frac{dz}{\psi_\pm^2(z)} \int_0^z d\rho \, \psi_\pm^2(\rho) v_0^2(\overset{\pm}{\rho}) + \text{regular part},$$

and using explicit expressions (D.2) and (D.3) for v_0, we obtain the singular part of the third derivative $\partial^3 \mathcal{F}_0 / \partial t_0^3$

$$\text{sing.}\left(\frac{\partial^3 \mathcal{F}_0}{\partial t_0^3}\right) = \sum_{\pm} \int_0^{\mp i\infty} \frac{1}{\hbar} \frac{dz}{\psi_\pm^2(z)} \int_0^z \frac{d\rho}{\psi_\pm^2(\rho)} \left[\int_{\pm i\infty}^{\rho} ds \, \psi_\pm^2(s)\right]^2 C_0^2,$$

where the singularity occurs at the upper integration limit for z and ρ as $\psi_+, \psi_- \to \psi_n$. The term in the square brackets is nonsingular in this limit, and we can therefore replace it with its limit value, which exactly cancels the corresponding term in the normalization constant C_0 (see (D.3)). The integrals over z and ρ can be separated, and we obtain the final expression

$$\text{sing.}\left(\frac{\partial^3 \mathcal{F}_0}{\partial t_0^3}\right) = \sum_{\pm} \frac{1}{2\hbar} \left[\int_0^{\mp i\infty} \frac{dz}{\psi_\pm^2(z)}\right]^2 \left[\int_{-\infty}^{+\infty} \frac{dx}{\psi_\pm^2(x)}\right]^{-2}. \tag{D.5}$$

We now calculate t_0 as $\psi_+, \psi_- \to \psi_n$. Choosing $\psi_-(x) = \psi_+(x) \int_{-\infty}^x d\xi \, \psi_+^{-2}(\xi)$ and taking into account that the number of poles of solutions outside the $\widetilde{\mathcal{A}}$-cycle is n, we obtain

$$\begin{aligned} t_0 &= -\hbar n + \hbar \int_{-\infty}^{+\infty} dz \left(\frac{\psi_-'(z)}{\psi_-(z)} - \frac{\psi_+'(z)}{\psi_+(z)}\right) = -\hbar n + \hbar \int_{-\infty}^{+\infty} \frac{dz}{\psi_+(z)\psi_-(z)} = \\ &= -\hbar n + \hbar \int_{-\infty}^{+\infty} \left(\int_{-i\infty}^{0} \frac{d\xi}{\psi_+^2(\xi)} + \int_0^z \frac{d\xi}{\psi_+^2(\xi)}\right)^{-1} \frac{dz}{\psi_+^2(z)}. \end{aligned}$$

The first integral in the expression in parentheses diverges as $\psi_+ \to \psi_n$. Letting Λ denote this integral, we obtain

$$t_0\big|_{\psi_+ \to \psi_n} = -\hbar n + \hbar \int_{-\infty}^{+\infty} \frac{dz}{\psi_+^2(z)} \left[\int_{-i\infty}^{0} \frac{d\xi}{\psi_+^2(\xi)}\right]^{-1} + O(\Lambda^{-2}). \tag{D.6}$$

Comparing this expression with (D.5), we obtain

$$\text{sing.}\left(\frac{\partial^3 \mathcal{F}_0}{\partial t_0^3}\right) = \frac{1}{\hbar(n + t_0/\hbar)^2}, \quad n \in \mathbb{Z}_{+,0}, \tag{D.7}$$

i.e., this derivative has double poles with the coefficient $1/\hbar$ at all points $t_0 = -\hbar n$, $n = 0, 1, \ldots$. A function that exhibits such a behavior is obviously the function Γ; we hence find that up to an entire function,

$$\mathrm{sing.}\left(\frac{\partial^3 \mathcal{F}_0}{\partial t_0^3}\right) \simeq \frac{1}{\hbar}[\log\Gamma]''\left(\frac{t_0}{\hbar}\right)$$

and in turn

$$\mathrm{sing.}(\mathcal{F}_0) \simeq \hbar^2\left[\int \log\Gamma\right]\left(\frac{t_0}{\hbar}\right). \tag{D.8}$$

Turning to the asymptotic behavior of $\int dx\, \log\Gamma(x)$ at large positive x, we observe that the leading term is $(x^2/2)\log x$, which is exactly what we might expect from matrix-model-like arguments: we must be able to apply the semiclassical approximation at large positive $t_0/\hbar$, and we have the leading asymptotic behavior of the Gaussian matrix model in this regime, i.e., $\mathrm{sing.}(\mathcal{F}_0) \simeq (t_0^2/2)\log t_0$ up to polynomial terms (of a degree not exceeding two).

We can therefore propose the following conjecture.

Conjecture. *The singular part of $\mathcal{F}_0$ for any potential $V_{d+1}(x)$ has the form*

$$\hbar^2 \sum_{i=1}^{d} \frac{1}{2}\left[\int \log\Gamma\right]\left(\frac{\widetilde{\epsilon}_i}{\hbar}\right),$$

where $\widetilde{\epsilon}_i$ are the filling fractions on the cycles $\widetilde{\mathcal{A}}_i$.

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