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THE 2×2 MATRIX SCHLESINGER SYSTEM AND THE BELAVIN–POLYAKOV–ZAMOLODCHIKOV SYSTEM

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We show that the Belavin–Polyakov–Zamolodchikov equation of the minimal model of conformal field theory with the central charge $c = 1$ for the Virasoro algebra is contained in a system of linear equations that generates the Schlesinger system with 2×2 matrices. This generalizes Suleimanov's result on the Painlevé equations. We consider the properties of the solutions, which are expressible in terms of the Riemann theta function.

Keywords: Belavin–Polyakov–Zamolodchikov equation, Schlesinger system, Painlevé equation, Garnier system

1. Systems of linear equations accompanying the Painlevé and Schlesinger equations

The Schlesinger system [1] for the matrices $A_i = A_i(t_1, \dots, t_m)$

$$\begin{aligned} \frac{\partial A_j}{\partial t_i} &= \frac{[A_i, A_j]}{t_i - t_j}, \quad i \neq j, \\ \frac{\partial A_i}{\partial t_i} &= -\sum_{j\neq i} \frac{[A_i, A_j]}{t_i - t_j}, \end{aligned} \tag{1}$$

where $i, j = \overline{1, m}$ (the second group of equations can be replaced with the condition that the matrix $A_1 + \dots + A_m = A_\infty$ is constant in $t_1, \dots, t_m$), was discovered as the compatibility condition for the system

$$\begin{aligned} \Psi_x &= A\Psi, \qquad A = \sum_{i=1}^{m} \frac{A_i}{x - t_i}, \\ \Psi_{t_i} &= -\frac{A_i}{x - t_i}\Psi. \end{aligned} \tag{2}$$

All algebraic integrals of motion of system (1) are known. They are the traces $\operatorname{tr} A_i^k$, $k = 1, 2, \dots$ (equivalently, the characteristic polynomials of the matrices A_i). A closed form $\omega = \sum H_i\, dt_i$ was found in [2],

$$H_i = \frac{1}{2} \operatorname*{res}_{x=t_i} (\operatorname{Tr} A^2) \equiv \sum_{j\neq i}^{m} \frac{\operatorname{tr}(A_j A_i)}{t_i - t_j},$$

and the τ-function $(\log \tau)_{t_i} = H_i$ was also determined.

Here, we consider only one aspect of the problem of the complete integrability of system (1) in terms of special functions. Namely, in the case of 2×2 matrices A_i, we seek a second-order linear equation for

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the function $\Phi = \tau\Psi$, all of whose coefficients are rational functions of $x, t_1, \dots, t_m$. We also consider the question of using linear superpositions of its solutions to integrate Eqs. (2) and (1).

We henceforth assume that A_i are 2×2 matrices and that

$$\operatorname{tr} A_i = 0, \qquad \det A_i = -\Delta_i, \qquad \det A_\infty = -\Delta_\infty.$$

We note the relation $\Delta_i = \delta_i^2/4$ with the difference δ_i of the eigenvalues of A_i. Because

$$\det A = -\sum_i \left[\frac{H_i}{x-t_i} + \frac{\Delta_i}{(x-t_i)^2}\right],$$

it follows from system (2) that

$$\Psi_{xx} = (A^2 + A_x)\Psi = \sum_i \left[\frac{H_i}{x-t_i} + \frac{\Delta_i}{(x-t_i)^2} - \frac{A_i}{(x-t_i)^2}\right]\Psi,$$

$$\sum_i \frac{1}{x-t_i}\Psi_{t_i} = \Psi_{xx} - \sum_i \left[\frac{\Delta_i}{(x-t_i)^2} + \frac{H_i}{x-t_i}\right]\Psi.$$

For $\Phi = \tau\Psi$, this implies that

$$\sum_i \frac{1}{x-t_i}\Phi_{t_i} = \Phi_{xx} - \sum_i \frac{\Delta_i}{(x-t_i)^2}\Phi. \tag{3}$$

Moreover,

$$\sum_i (x-t_i)\Psi_{t_i} = -A_\infty\Psi, \qquad \Psi_x + \sum_i \Psi_{t_i} = 0.$$

Using the relations

$$\sum_i H_i = 0, \qquad \sum_i (t_i H_i + \Delta_i) = \Delta_\infty,$$

which follow from the expansion $\det A \sim -\Delta_\infty x^{-2}$ as $x \to \infty$, we obtain the equations

$$\sum_i (x-t_i)\Phi_{t_i} = \left(\sum_i \Delta_i - \Delta_\infty - A_\infty\right)\Phi, \qquad \sum_i \Phi_{t_i} = -\Phi_x. \tag{4}$$

The coefficients of system (3), (4) are rational functions of x and t_i and are uniquely determined by fixing the values Δ_i and A_∞ of the algebraic integrals of system (1). We use a substitution $\Phi \to g\Phi$ with a matrix g independent of x and t_i to transform the matrix A_∞ into the Jordan normal form

$$A_\infty = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad A_\infty = \frac{1}{2}\begin{pmatrix} k_\infty & 0 \\ 0 & -k_\infty \end{pmatrix}.$$

The first component of the vector $\Phi = (\varphi, \psi)^{\mathrm{T}}$ then satisfies the relations

$$\sum_i \frac{1}{x-t_i}\varphi_{t_i} = \varphi_{xx} - \sum_i \frac{\Delta_i}{(x-t_i)^2}\varphi, \qquad \sum_i (x-t_i)\varphi_{t_i} = \rho\varphi, \qquad \sum_i \varphi_{t_i} = -\varphi_x,$$

where $\rho = \sum_i \Delta_i - \Delta_\infty - k_\infty/2$.

Equations (3) have appeared in various papers. In constructing the "minimal" models of two-dimensional quantum field theory with the conformal symmetry group, Belavin, Polyakov, and Zamolodchikov in particular obtained the equation (see formula (5.17) in [3])

$$\sum_i \frac{1}{x-t_i}\Phi_{t_i} = \varkappa\Phi_{xx} - \sum_i \frac{\Delta_i}{(x-t_i)^2}\Phi,$$

where the central charge of the Virasoro algebra is $c = -6\varkappa + 13 - 6\varkappa^{-1}$. Equation (3) corresponds to the value $c = 1$. Additional conditions (4) also appear in [3], [4].

In [5], [6], the Schlesinger systems for matrices of arbitrary size were obtained from the Knizhnik–Zamolodchikov equations in the semiclassical limit. A similar turning from the equations in [3], [4] to the Painlevé VI equation as $\varkappa \to 0$ was reported in [7]. The relation between the equations in [3] and [8] by means of an integral transformation together with a coordinate change introduced in [9] was considered in [10].

We obtained the results described before becoming aware of [11]–[13], where equations of parabolic type that are part of the Lax representation of the Painlevé I–VI equations were obtained. For comparison, we give an equivalent form of the Fuchs system [14], eliminating f from which gives the Painlevé VI equation for u. Introducing the variables

$$t = \frac{t_3 - t_1}{t_2 - t_1}, \qquad z = \frac{x - t_1}{t_2 - t_1}, \qquad f = z^{-\delta_1/2}(z-1)^{-\delta_2/2}(z-t)^{-\delta_3/2}\varphi$$

reduces system (3), (4) with three variables t_i to the single equation

$$\frac{t(t-1)}{z(z-1)(z-t)} f_t = f_{zz} + \left(\frac{1+\delta_1}{z} + \frac{1+\delta_2}{z-1} + \frac{\delta_3}{z-t}\right) f_z + \frac{\rho'}{z(z-1)} f,$$

and the first equation in system (2) becomes

$$f_{zz} + \left(\frac{1+\delta_1}{z} + \frac{1+\delta_2}{z-1} + \frac{1+\delta_3}{z-t} - \frac{1}{z-u}\right) f_z + \frac{\rho' z^2 + Pz + Q}{z(z-1)(z-t)(z-u)} f = 0.$$

The absence of the variable u in the first equation allows simplifying the proof of Theorem 2 in [15]. Equation (3) was found similarly, by eliminating the variables A_i from system (2) based on the theory in [16]. We note the thematic similarity between the change of variables and integral transformations [17], [18] for Eqs. (3) and (2). For example, the parameter shift $(\delta_1, \delta_2, \delta_3, k_\infty) \to (\delta_1, \delta_2 - 1, \delta_3 - 1, k_\infty)$ can be obtained by applying the Schlesinger transformation to f or by several changes of variables and integral Euler transformations. Comparing the results $f_1 = f_2$ gives an integro-differential equation for the function f.

2. Hyperelliptic theta functions and solutions of the Belavin–Polyakov–Zamolodchikov equation

We use the theory of Abelian functions [19]–[22]. On a hyperelliptic curve Γ of genus g

$$\mu^2 = R(x) = (x - t_0)(x - t_1)\cdots(x - t_{2g})$$

(we can take $t_0 = 0$ and $t_{2g} = 1$ without restricting the generality), we choose a Weierstrass canonical basis of cycles: the cycle a_i goes around the cut t_{2i-2}, t_{2i-1} on the upper sheet, and the cycle b_i goes from this cut to the cut from t_{2g} to ∞ on the upper and lower sheets. For clarity, the points $t_0, \dots, t_{2g}$ are arranged in increasing order from 0 to 1 in the real case. The cycle basis a_j, b_j determines the integrals of the first kind v and the period matrix B:

$$v_i(x) = \int_\infty^x \frac{(c_{i1}\gamma^{g-1} + \cdots + c_{ig})d\gamma}{\sqrt{R(\gamma)}}, \qquad \oint_{a_j} dv_i = \delta_{ij}, \qquad \oint_{b_j} dv_i = B_{ij}.$$

We introduce theta functions with characteristics

$$\Theta[p,q](z;B) = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_g=-\infty}^{\infty} e^{\pi i \langle n+p,(n+p)B+2(z+q)\rangle}, \qquad \Theta = \Theta[0,0].$$

We substitute any constants p and q and the quantities $z = v(x)+m_1B+m_2$, where $[m_1, m_2]$ is a half-integral characteristic such that $\Theta(z;B) \not\equiv 0$. Following [16], [23], we construct a pair of functions multivalued in x:

$$f(x) = \frac{\Theta[p,q](z;B)}{\Theta(z;B)}, \qquad f^-(x) = \frac{\Theta[-p,-q](z;B)}{\Theta(z;B)}.$$

The monodromy transformations for the functions f and f^- are independent of t_i, which can be verified using the properties of the integrals $v(x)$ and the theta function. The functions f and f^- are interchanged as x goes around ∞. As x goes around the cycles a_j and b_j, we correspondingly have

$$z_i \to z_i + \delta_{ij}, \qquad f \to e^{2\pi i p_j} f, \qquad f^- \to e^{-2\pi i p_j} f^-,$$

$$z_i \to z_i + B_{ij}, \qquad f \to e^{-2\pi i q_j} f, \qquad f^- \to e^{2\pi i q_j} f^-.$$

Henceforth, we limit ourself to the case ($t_0 = 0$, $t_{2g} = 1$)

$$f(x) = \frac{\Theta[p,q](v(x) - v(t_0) - v(t_{2g}); B)}{\Theta(v(x) - v(t_0) - v(t_{2g}); B)}. \tag{5}$$

We set $J = \{2, \ldots, 2g-2\}$ if $g > 1$ and $J = \varnothing$ if $g = 1$. According to the Riemann vanishing theorem [19], function (5) either is identically zero or has exactly g zeroes $(x,\mu) = (U_i, \sqrt{R(U_i)}\,)$, $i = \overline{1,g}$, and exactly g poles $x = \infty$ and $x = t_i$, $i \in J$. Then $\log f(x)$ is an integral of the third kind:

$$\log \frac{f(x)}{f(1)} = \frac{1}{2}\int_1^x \left[\sum_{i=1}^{g} \left(\frac{\sqrt{R(U_i)} + \sqrt{R(\gamma)}}{(\gamma - U_i)\sqrt{R(\gamma)}} + \frac{P_i \gamma^{i-1}}{\sqrt{R(\gamma)}} \right) - \sum_J \frac{1}{\gamma - t_i} \right] d\gamma. \tag{6}$$

Using this expression for $\log f$, we can write the isomonodromy deformation equations for f, the Garnier system [24], [25], a scalar version of system (2). The equation

$$f_{xx} + \left[\sum_{i\in I} \frac{1/2}{x - t_i} + \sum_{i \in J} \frac{3/2}{x - t_i} - \sum_{i=1}^{2g-1} \frac{1}{x - u_i} \right] f_x = \sum_{i=1}^{2g-1} \left(\frac{\beta_i}{x - u_i} + \frac{\alpha_i}{x - t_i} \right) \frac{f}{x(x-1)}, \tag{7}$$

where $I = \{0, 2g\} \cup \{1, 3, \ldots, 2g-1\}$, corresponds to the deformed equation $\Psi_x = A\Psi$. In our study of the Garnier system, we limit ourself to deducing and studying the properties of the analogues of relations (3) and (4).

Theorem. *Functions* (5) ($t_0 = 0$, $t_{2g} = 1$) *satisfy the equation*

$$\sum_{i=1}^{2g-1} \frac{t_i(t_i - 1)}{x(x-1)(x-t_i)} f_{t_i} = f_{xx} + \frac{1}{2}\left[\sum_{i=0}^{g} \frac{1}{x - t_{2i}} - \sum_{i=1}^{g} \frac{1}{x - t_{2i-1}} \right] f_x. \tag{8}$$

We prove the theorem in Appendix A.

We discuss the possibility of using linear superpositions of solutions of Eq. (8) to construct solutions of Fuchs-type equations (7). Averaging family (5) over the parameters q gives

$$\int_0^1 \cdots \int_0^1 \frac{\Theta[0,q](z;B)}{\Theta(z;B)} e^{-2\pi i\langle n,q\rangle} dq = \frac{e^{\pi i\langle n,nB+2z\rangle}}{\Theta(z;B)}. \tag{9}$$

The functions of family (9) with integer n are branches of the function $\Theta^{-1}(z;B)$ obtained when x traverses the cycles $b_1,\dots,b_g$ $n_1,\dots,n_g$ times.

The Riemann function $\Theta(z;B)$, $z=v(x)-v(0)-v(1)$, can be considered the generating function of families (9) and (5) and Eqs. (7) whose monodromy group has an Abelian commutant (the transformations f and f^- with $2g$ parameters p and q are given above). All other cases of Eqs. (7), including those that are interesting because of the uniformization problem for hyperelliptic curves, are apparently not covered by linear combinations of functions (9) with complex n for reasons given in Appendix B.

Presumably, a "noncommutative" family of functions consisting of the solutions of Eq. (8) satisfying Eq. (7) with an arbitrary monodromy group (there are $4g-2$ parameters) generates all solutions of Eq. (8) by linear superposition. Possibly, it suffices to consider monodromy groups with a particular relation between the generators: $M_{2i-1}=M_{2i-2}^{-1}$, $i=\overline{1,g}$, $M_\infty=M_{2g}^{-1}$ (here there are $2g-1$ parameters in the Riemann–Hilbert problem on the union of nonintersecting intervals). This is analogous to how we used only a part of the characteristics $[0,q]$ instead of all $[p,q]$ in constructing (9) from family (5).

The problem of constructing "noncommutative" analogues for family (9) and $\Theta^{-1}(z;B)$ is presumably related to constructing "conformal blocks" in the theory in [3]. We note a similarity between the τ-function of the Schlesinger system and the "conformal blocks" F of the minimal model with the central charge $c=1$:

$$\tau=\frac{\Theta[p,q](0;B)}{r(t)\Theta(0;B)}, \qquad F=\frac{e^{\pi i\langle n,nB\rangle}}{r(t)\Theta(0;B)},$$

where $r(t)$ is an algebraic factor. The function τ was constructed in [16]. The expression for F in the elliptic case $g=1$ was calculated in [26] using an infinite-dimensional integral; the hyperelliptic case was considered in [27], [28].

Appendix A

A.1. Elliptic case (7) has been investigated many times in connection with the Painlevé VI equation in the Picard and Hitchin cases (see, e.g., [16], [23]). Here, we discuss solutions (5) and (9) of Eq. (8) for $g=1$. It turns out that they correspond to the fundamental solutions $\theta[p,q](z,\tau)$ and $e^{\pi i(n^2\tau+2nz)}$ of the heat equation. The argument τ is standard notation; it replaces B in formula (5) here and should not be confused with the τ-function. In what follows, the normalized elliptic integrals are given in the form

$$z=\frac{1}{2K}\int_\infty^x \frac{d\gamma}{2\sqrt{R(\gamma)}}, \qquad \tau=\frac{1}{K}\int_\infty^0 \frac{d\gamma}{2\sqrt{R(\gamma)}}, \qquad K=\int_\infty^1 \frac{d\gamma}{2\sqrt{R(\gamma)}},$$

$R(x)=x(x-1)(x-t)$. We use the Jacobi notation: the theta function has the indices 1, 2, 3, and 4 corresponding to the characteristics $[1/2,1/2]$, $[1/2,0]$, $[0,0]$, and $[0,1/2]$; the theta constants $\theta_i(0,\tau)$ are denoted by θ_i.

Refining the statement of the theorem, we show that the replacement $f=\theta_1^{-1}(z,\tau)\psi$ transforms the equation

$$\frac{t(t-1)}{x(x-1)(x-t)} f_t = f_{xx} + \frac{1}{2}\left(\frac{1}{x}+\frac{1}{x-1}-\frac{1}{x-t}\right) f_x \tag{10}$$

into the heat equation $4\pi i\psi_\tau = \psi_{zz}$. We use the definitions of z and τ and the equations

$$4\pi i\frac{\partial\theta_1(z,\tau)}{\partial\tau} = \frac{\partial^2\theta_1(z,\tau)}{\partial z^2}, \qquad \frac{d\tau}{dt} = \frac{i\pi}{4K^2t(t-1)}, \qquad \frac{\partial z}{\partial x} = \frac{1}{4K\sqrt{R(x)}}.$$

We verify that substituting $\theta_1^{-1}(z,\tau)$ for the function f in (10) gives the equation

$$\frac{\partial\log\theta_1(z,\tau)}{\partial z} = \int_1^x \frac{-K(\gamma-t)+2t(t-1)K_t}{\sqrt{\gamma(\gamma-1)(\gamma-t)}}d\gamma,$$

which becomes an identity after simplification. Differentiating it with respect to z gives the relation

$$\frac{\partial^2\log\theta_1(z,\tau)}{\partial z^2} = -4K^2x + 4K^2t + 8t(t-1)KK_t,$$

which is transformed into an identity of two expressions for an elliptic function if we write the right-hand side in terms of z and τ using the inversion formulas [29]–[31]

$$x = \frac{\theta_2^2}{\theta_3^2}\frac{\theta_4^2(z,\tau)}{\theta_1^2(z,\tau)}, \qquad t = \frac{\theta_2^4}{\theta_3^4}, \qquad K = \frac{\pi}{2}\theta_3^2.$$

We note that changing the variable from z to x gives an equation for the function $f = \theta_1^{-1}(z,\tau)$:

$$\frac{f_{xx}}{f} - \frac{f_x^2}{f^2} + \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-t}\right)\frac{f_x}{f} = \frac{x-t-2t(t-1)K_tK^{-1}}{4x(x-1)(x-t)},$$

whose compatibility with (10) can be verified directly. We can finish the argument with a complete change of variables, but it is easier to use the following argument. As x traverses the b cycle n times (around the pair of points 0, ∞ or 1, t), the function $\theta_1^{-1}(z,\tau)$ is multiplied by $\psi_n = e^{\pi i(n^2\tau+2nz)}$. Because the coefficients in Eq. (10) are rational with respect to x and t, the functions $\theta_1^{-1}(z,\tau)\psi_n$ are its solutions. Therefore, the replacement $f = \theta_1^{-1}(z,\tau)\psi$ transforms (10) into an equation satisfied by the family ψ_n with integer n.

A.2. We prove the theorem in Sec. 2 using an analogue of the heat equation,

$$4\pi i\frac{\partial\psi}{\partial B_{jj}} = \frac{\partial^2\psi}{\partial z_j^2}, \qquad 2\pi i\frac{\partial\psi}{\partial B_{jk}} = \frac{\partial^2\psi}{\partial z_j\partial z_k}, \quad j,k=\overline{1,g}. \tag{11}$$

The lemmas given below are based on the following propositions from the theory of hyperelliptic curves. The first proposition is equivalent to Lemma 4.1 in [16]:

$$\sum_{i=0}^{2g}\frac{1}{x-t_i}\frac{\partial B_{jk}}{\partial t_i} = 4\pi i\frac{\partial v_j}{\partial x}\frac{\partial v_k}{\partial x}, \quad j,k=\overline{1,g}. \tag{12}$$

According to the second proposition, the normalized integrals of the first kind $v_j(x)$, defined in Sec. 2, are expressed in terms of the periods of the integral of the third kind I_γ^x:

$$\int_{x,-}^{x,+} dv_j = \frac{1}{2\pi i}\left(\oint_{b_j} dI_\gamma^x - \sum_{k=1}^{g} B_{jk}\oint_{a_k} dI_\gamma^x\right), \qquad dI_\gamma^x = \frac{\sqrt{R(x)}}{\gamma-x}\frac{d\gamma}{\sqrt{R(\gamma)}}. \tag{13}$$

This identity can be verified by comparing the two sides and the integral along the boundary of the Riemann surface cut along all cycles [19], [32]

$$\frac{1}{2\pi i}\oint_{\partial\Gamma'} v_j(\gamma)dI_\gamma^x$$

or by using the bilinear period relations from Abel's work [33]. Yet another method for calculating is given in the section on the hyperelliptic Riemann constants vector (the Christoffel method) in [20].

Lemma 1. *The vector* $z = v(x) - v(t_0) - v(t_{2g})$ *satisfies the equation*

$$\sum_{i=0}^{2g} \frac{1}{x - t_i} \frac{\partial z}{\partial t_i} = \frac{\partial^2 z}{\partial x^2} - F \frac{\partial z}{\partial x}, \qquad F = \frac{R'(x)}{2R(x)} + \sum_{k=1}^{g} \frac{\partial v_k}{\partial x} \left[\oint_{a_k} dI_\gamma^x + 4\pi i \mu_k \right],$$

where the numbers μ_k *are the coefficients of the period decomposition of the half-period*

$$v_j(t_0) + v_j(t_{2g}) = \sum_k \mu_k B_{kj} + \nu_j.$$

Proof. We find the derivatives of the integrals v_j from (13) using relations that hold for any cycle:

$$\frac{\partial}{\partial t_i} \oint dI_\gamma^x = -\frac{1}{2} \frac{\sqrt{R(x)}}{x - t_i} \oint \frac{d\gamma}{(\gamma - t_i)\sqrt{R(\gamma)}},$$

$$\frac{\partial}{\partial x} \oint dI_\gamma^x = \frac{1}{2} \sum_{i=0}^{2g} \frac{\sqrt{R(x)}}{x - t_i} \oint \frac{d\gamma}{(\gamma - t_i)\sqrt{R(\gamma)}}.$$

After substituting the derivatives of v_j in the considered equation and using identities (12), we obtain the final form of the proposition.

Lemma 2. *Let* z *and* F *be the same as in Lemma 1. Then all solutions* $\psi(z, B)$ *of system* (11) *are solutions of the equation*

$$\sum_{i=0}^{2g} \frac{1}{x - t_i} \psi_{t_i} = \psi_{xx} - F\psi_x. \tag{14}$$

Proof. The required equation follows from the expressions for the derivatives ψ_{t_i}, ψ_x, and ψ_{xx} in terms of the derivatives of ψ with respect to B_{jk} and z_j with formulas (12) and (11) and Lemma 1 taken into account.

Lemma 3. *Functions* (5), (9) *satisfy the equation*

$$\sum_{i=0}^{2g} \frac{1}{x - t_i} f_{t_i} = f_{xx} + \frac{1}{2} \left[\sum_{i \in J} \frac{1}{x - t_i} - \sum_{i \in I} \frac{1}{x - t_i} \right] f_x,$$

where the sets I *and* J *are the same as in Eq.* (7).

Proof. According to Lemma 2, the functions

$$\psi = \Theta[p, q](z; B), \qquad \psi = e^{\pi i \langle n, nB + 2z \rangle}$$

satisfy Eq. (14). Substituting $\psi = f \cdot \Theta(z; B)$ in (14) and taking into account that the solution $\psi = \Theta(z; B)$ corresponds to $f = 1$, we obtain an equation for f of the form

$$\sum_{i=0}^{2g} \frac{1}{x - t_i} f_{t_i} = f_{xx} - F_1 f_x.$$

To find the function F_1, we use the expression for some nonconstant solutions f of family (5) in terms of x and t_i, which follow from solving the Jacobi inversion problem. The needed formulas are given in [22] (Theorem 5.3(3) in Chap. 3a; also see [19]–[21]):

$$\frac{\prod_{i\in J}\sqrt{t_i-t_k}}{\prod_{i\in I}^{i\neq k}\sqrt{t_i-t_k}}(x-t_k)=\frac{\Theta^2[\eta_k](z;B)}{\Theta^2(z;B)},\quad k\in I,$$

where $z=v(x)-v(t_0)-v(t_{2g})$ and the characteristic η_k corresponds to the period $v(t_k)$.

Lemma 3 directly implies the theorem formulated in Sec. 2 because functions (5) are homogeneous by definition:

$$\sum_{i=0}^{2g}(x-t_i)f_{t_i}=0,\qquad \sum_{i=0}^{2g}f_{t_i}+f_x=0.$$

The systems of equations satisfied by theta functions with an argument equal to a sum of integrals of the first kind can be related to Eq. (3). We give the corresponding propositions in Appendix C.

Appendix B

We consider Eqs. (7) and (8) in the case $g=2$. We set $t_0=0$ and $t_{2g}=1$, replace $t_1\to\epsilon t_1$, $t_2\to t_2\epsilon^{-1}$, and $t_3\to 1-\epsilon t_3$, and take the limit $\epsilon\to 0$ in (7) and (8):

$$f_{xx}+\left[\frac{1}{x}+\frac{1}{x-1}-\sum_{i=1}^{3}\frac{1}{x-u_i}\right]f_x=\left(\sum_{i=1}^{3}\frac{\beta_i}{x-u_i}+k_1+\frac{k_2x+k_3}{x(x-1)}\right)\frac{f}{x(x-1)},\tag{15}$$

$$\frac{-t_1f_{t_1}}{x^2(x-1)}-\frac{t_2f_{t_2}}{x(x-1)}+\frac{t_3f_{t_3}}{x(x-1)^2}=f_{xx}.\tag{16}$$

Equation (15) is integrable in hypergeometric functions; the corresponding isomonodromy deformation system is Liouville integrable. Any solution of (15) is a linear combination of four solutions of (16) of the form

$$f_{n,m,l}=t_1^{n^2-n}t_3^{m^2-m}t_2^{-l^2+l}P(n,m,l;x),$$

where $P(n,m,l;x)$ satisfies the relation

$$P_{xx}=\left(\frac{-n(n-1)}{x^2(x-1)}+\frac{l(l-1)}{x(x-1)}+\frac{m(m-1)}{x(x-1)^2}\right)P.$$

In the limit $\epsilon\to 0$, functions (5) give particular solutions of (16). For example, taking the limit $\epsilon\to 0$ of functions (5) (multiplied beforehand by an appropriate quantity depending on ϵ) with half-integer $[p,q]$ gives the solutions

$$0,\quad 1,\quad (t_1^{-1}t_2)^{1/4}x^{1/2},\quad (t_3^{-1}t_2)^{1/4}(x-1)^{1/2},\quad (t_1t_3)^{-1/4}[x(x-1)]^{1/2},\quad x,\quad x-1.$$

It is clear that it is difficult to find the limit as $\epsilon\to 0$ of functions (6) for arbitrary $[p,q]$ (there are many divergent quantities); we obtain an expression of the form

$$\log f=\frac{1}{2}\int\left[\frac{\pm U_1(U_1-1)+x(x-1)}{x(x-1)(x-U_1)}+\frac{\pm U_2(U_2-1)+x(x-1)}{x(x-1)(x-U_2)}+\frac{V_2x+V_1}{x(x-1)}\right]dx,$$

i.e., depending on the sign and up to multiplication by a function of t,

$$f=x^n(x-1)^m,\qquad f=x^n(x-1)^m(x-U),\qquad f=x^n(x-1)^m(x-U_1)(x-U_2),$$

where $U = U_1$ or $U = U_2$. The limits as $\epsilon \to 0$ of functions (5) are presumably linear combinations of three solutions of (16) of the form

$$f_{n,m} = t_1^{n^2-n} t_3^{m^2-m} t_2^{-(n+m)^2+n+m} x^n (x-1)^m.$$

The functions of the family $f_{n,m}$ satisfy the equation

$$\left\{ac\,\partial_x + \left(\frac{1}{x} + \frac{1}{x-1} - \partial_x\right)\frac{(a+b+c)^2}{4} - \frac{a+b+c}{2}\left(\frac{a}{x} + \frac{c}{x-1}\right)\right\}f = 0, \tag{17}$$

where $a = t_1\partial_{t_1}$, $b = t_2\partial_{t_2}$, and $c = t_3\partial_{t_3}$. Therefore, the closure of the linear envelope of the family $f_{n,m}$ does not contain all solutions of (16).

The assumption that the closure of the linear envelope of family (9) for $g = 2$ contains all solutions of Eq. (8) or solutions of Eq. (7) whose monodromy group has a non-Abelian commutant contradicts the obvious fact that the corresponding functions $f_{n,m,l}$ and solutions of (15) do not satisfy (17).

The existence of Eq. (17) for the limit form of functions (5) presumably means that functions (5) satisfy a third- or fourth-order linear equation with coefficients independent of p and q, which does not follow from (8). To derive this linear equation, we must replace the variables in system (11):

$$4\pi i\frac{\partial\psi}{\partial B_{11}} = \frac{\partial^2\psi}{\partial z_1^2}, \qquad 2\pi i\frac{\partial\psi}{\partial B_{12}} = \frac{\partial^2\psi}{\partial z_1\partial z_2}, \qquad 4\pi i\frac{\partial\psi}{\partial B_{22}} = \frac{\partial^2\psi}{\partial z_2^2}.$$

Appendix C

C.1. From system (2), we derive equations for the matrix $M = \tau\Psi^{-1}(y)\Psi(x)$:

$$\sum_i M_{t_i} + M_x + M_y = 0, \qquad \sum_i t_i M_{t_i} + xM_x + yM_y = \left(\Delta_\infty - \sum_i \Delta_i\right)M,$$

$$\sum_i\left(\frac{1}{x-t_i} + \frac{1}{y-x}\right)M_{t_i} = M_{xx} - \sum_i \frac{\Delta_i}{(x-t_i)^2}M,$$

$$\sum_i\left(\frac{1}{y-t_i} + \frac{1}{x-y}\right)M_{t_i} = M_{yy} - \sum_i \frac{\Delta_i}{(y-t_i)^2}M.$$

C.2. In the case with three variables t_i, we eliminate t_1 and t_2 from the equations for M. We set $t_1 = 0$, $t_2 = 1$, and $t_3 = t$. Then the function $Y = (x-y)^{-1}M$ is a solution of the equations

$$\frac{t(t-1)Y_t}{x(x-1)(x-t)} + \frac{y(y-1)Y_y}{x(x-1)(x-y)} = Y_{xx} + \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-y}\right)Y_x - \\ - \left(\frac{\Delta_0}{x^2} + \frac{\Delta_1}{(x-1)^2} + \frac{\Delta_3}{(x-t)^2} + \frac{\Delta_\infty - 1 - \Delta_1 - \Delta_2 - \Delta_3}{x(x-1)}\right)Y,$$

$$[x(x-1)(x-t)Y_x]_x - \left((\Delta_\infty - 1)x + \Delta_1\frac{t}{x} + \Delta_2\frac{1-t}{x-1} + \Delta_3\frac{t(t-1)}{x-t}\right)Y = \\ = [y(y-1)(y-t)Y_y]_y - \left((\Delta_\infty - 1)y + \Delta_1\frac{t}{y} + \Delta_2\frac{1-t}{y-1} + \Delta_3\frac{t(t-1)}{y-t}\right)Y.$$

The variables are separated in the last equation. The substitution

$$z = \frac{xy}{t}, \qquad v = \frac{(x-t)(y-t)}{t(x-1)(y-1)}, \qquad Y = \frac{L}{z-1},$$

is also known to separate the variables in the same equation and to eliminate t:

$$(z-1)^2\biggl[(zL_z)_z-\frac{\Delta_\infty z-\Delta_1}{z(z-1)}L\biggr]=(v-1)^2\biggl[(vL_v)_v-\frac{\Delta_2 v-\Delta_3}{v(v-1)}L\biggr].$$

The appearance of hyperelliptic functions here in connection with the Lamé and Heun equations was discussed in Sec. 15.5.3 in [30] and in [34], [35].

C.3. The Painlevé I equation $u''=6u^2+t$ follows from the system

$$\Psi_x=\begin{pmatrix} u' & 2(x-u)\\ 2(x^2+ux+u^2)+t & -u'\end{pmatrix}\Psi,\qquad \Psi_t=\begin{pmatrix}0 & 1\\ x+2u & 0\end{pmatrix}\Psi.$$

Writing this system in terms of the components $\tau\Psi=(\psi,\psi_1)^{\mathrm{T}}$, where the τ-function is defined standardly, $(\log\tau)'=u'^2/2-2u^3-tu$, we obtain the equation [11]

$$2\psi_t=\psi_{xx}-(4x^3+2tx)\psi.$$

The function $Y=(x-y)^{-1}\tau\Psi(y)^{-1}\Psi(x)$ satisfies the equation

$$4Y_t=Y_{xx}+Y_{yy}+2(x-y)^{-1}(Y_x-Y_y)-(4x^3+2tx+4y^3+2ty)Y,$$

$$Y_{xx}-Y_{yy}=[4(x^3-y^3)+2t(x-y)]Y.$$

Changing the variables as $v=x+y$ and $z=(x-y)^2+2t$ allows eliminating t from the second equation:

$$2Y_t=Y_{vv}+4(z-2t)Y_{zz}+2Y_z-(v^2+3z-4t)\frac{vY}{2},\qquad 8Y_{zv}=(3v^2+z)Y.$$

Applying the Laplace transformation

$$f=\int e^{-kz}Y\,dz,$$

we obtain the relation

$$f_k=-8kf_v+3v^2f,$$

$$2f_t=f_{vv}+(32k^3-12kv)f_v+(4v^3+2tv-8k^2t-12k^2v^2-6k)f.$$

The first equation is exactly solvable,

$$f=F(\zeta)e^{128k^5/5-16vk^3+3v^2k},\qquad \zeta=4k^2-v,$$

and the second equation implies $2F_t=F_{\zeta\zeta}-(4\zeta^3+2t\zeta)F$, encountered above.

C.4. The derivatives $f_{x_ix_j}$ of the function $f=\Theta^{-1}(v(x_1)+\cdots+v(x_g)+K;B)$, where the vector of Riemann constants with the initial point ∞ is equal to the sum of the integrals to the points t_{2i},

$$K=-v(t_0)-v(t_2)-\cdots-v(t_{2g}),$$

can be expressed linearly in terms of $f_{x_1},\dots,f_{x_g}$, and f_{t_i} with algebraic coefficients. A part of these equations forms the Belavin–Polyakov–Zamolodchikov system [10] with the central charge $c=1$:

$$\sum_{i=0}^{2g}\frac{1}{x_j-t_i}f_{t_i}=f_{x_jx_j}-\frac12\sum_{i=0}^{2g}\frac{1}{x_j-t_i}f_{x_j}+\sum_{k\ne j}\frac{f_{x_j}-f_{x_k}}{x_j-x_k},\quad j=\overline{1,g},$$

which follows from lemmas similar to the ones in Appendix A and from a solution of the Jacobi inversion problem expressed by the formulas [19]–[22]

$$\sqrt{\frac{(-1)^{i+1}}{R'(t_i)}}\prod_{j=1}^{g}(x_j-t_i)=\frac{\Theta^2[\eta_i](z;B)}{\Theta^2(z;B)},\quad i=\overline{0,2g}.$$

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