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WEAKLY PERIODIC GROUND STATES AND GIBBS MEASURES FOR THE ISING MODEL WITH COMPETING INTERACTIONS ON THE CAYLEY TREE

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We introduce the notion of a weakly periodic configuration. For the Ising model with competing interactions, we describe the set of all weakly periodic ground states corresponding to normal divisors of indices 2 and 4 of the group representation of the Cayley tree. In addition, we study new Gibbs measures for the Ising model.

Keywords: Cayley tree, Gibbs measure, Ising model, weakly periodic ground state

1. Introduction

It is known that the phase diagram of Gibbs measures for a Hamiltonian is close to the phase diagram of isolated (stable) ground states of this Hamiltonian. At low temperatures, a periodic ground state corresponds to a periodic Gibbs measure [1], [2]. The notion of a weakly periodic Gibbs measure was introduced in [3], and the set of such measures for the Ising model was described there. The problem of describing weakly periodic ground states therefore arises.

Here, we introduce the notion of a weakly periodic ground state and describe the sets of such states for the Ising model with competing interactions (see Hamiltonian (2) below). This model was studied at the physical level in [4], where the phase diagram of states of the system was given for specific values of the parameters. The problem of describing translation-invariant Gibbs measures was solved in [5]. The results in [6] permit explicitly describing the region of the parameters of model (2) where periodic Gibbs measures exist. All periodic ground states and a set of nonperiodic ground states for model (2) were described in [7], and the contour method was used there to prove that at sufficiently low temperatures, there exist two Gibbs measures corresponding to translation-invariant boundary conditions. Here, we describe weakly periodic ground states and the Gibbs measures for model (2).

This paper is organized as follows. In Sec. 2, we present necessary definitions, the statement of the problem, and the required knowledge in [7]. In Sec. 3, we describe weakly periodic ground states. Each weakly periodic configuration corresponds to a quadratic matrix $\mathcal{B}$. We reduce the problem of describing weakly periodic ground states to the problem of solving a system of linear equations for the elements of $\mathcal{B}$. In Sec. 3.1, we study weakly periodic ground states corresponding to arbitrary normal divisors of index 2, and in Sec. 3.2, we consider a family of normal divisors of index 4 and describe the corresponding ground states. In Sec. 4, we present new weakly periodic Gibbs measures for the Ising model, significantly improving the result in [3]. In Sec. 5, we discuss the obtained results.

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2. Preliminaries

2.1. Cayley tree. Let $\tau^k = (V, L)$, $k \geq 1$, be the Cayley tree of order k, i.e., an infinite tree such that exactly $k{+}1$ edges issue from each of its vertices, where V is the set of vertices and L is the set of edges of τ^k. It is known that τ^k can be represented as G_k, i.e., as a free product of $k{+}1$ cyclic second-order groups with generators $a_1, a_2, \dots, a_{k+1}$. Therefore, G_k can be considered instead of V. For an arbitrary point $x^0 \in G_k$, we set

$$W_n = \{x \in G_k \colon d(x^0, x) = n\}, \qquad V_n = \bigcup_{m=0}^{n} W_m, \qquad L_n = \{\langle x, y\rangle \in L \colon x, y \in V_n\}, \tag{1}$$

where $d(x, y)$ is the distance between x and y on the Cayley tree, i.e., the number of edges in the path connecting x to y.

Let $S(x)$ be the set of direct descendants of a point x: if $x \in W_n$, then $S(x) = \{y \in W_{n+1} \colon d(x, y) = 1\}$. We note that for any $x \in G_k$, the set $\{y \in G_k \colon d(x, y) = 1\} \setminus S(x)$ has a unique element; we let $x_\downarrow$ denote it.

2.2. Configuration space. Let $\Phi = \{-1, 1\}$, and let $\sigma \in \Omega = \Phi^V$ be a configuration, i.e., $\sigma = \{\sigma(x) \in \Phi \colon x \in V\}$. Let $A \subset V$. We let Ω_A denote the space of configurations defined on the set A and taking values in Φ.

We assume that a group of spatial transitions acts on Ω. We define an F_k-*periodic configuration* as a configuration $\sigma(x)$ that is invariant under a subgroup $F_k \subset G_k$ of finite index, i.e., $\sigma(yx) = \sigma(x)$ for all $x \in G_k$ and $y \in F_k$. For a given periodic configuration, the index of the subgroup is called the *period of the configuration*, and a configuration invariant under all transitions is said to be *translation invariant*.

Let $G_k/F_k = \{H_1, \dots, H_r\}$ be the quotient group, where F_k is a normal divisor of an index $r \geq 1$. A configuration $\sigma(x)$ is said to be F_k-weakly periodic if $\sigma(x) = a_{ij}$ for all $x_\downarrow \in H_i$ and $x \in H_j$, where $a_{ij} \in \Phi$, $i, j = 1, \dots, r$.

2.3. Model with competing interactions. We present necessary definitions and results in [7]. We consider the Hamiltonian of the Ising model with competing interactions

$$H(\sigma) = J_1 \sum_{\langle x,y\rangle} \sigma(x)\sigma(y) + J_2 \sum_{x,y \in V \colon d(x,y)=2} \sigma(x)\sigma(y), \tag{2}$$

where $J_1, J_2 \in \mathbb{R}$ and $\sigma \in \Omega$. We set

$$H(\sigma, \varphi) = J_1 \sum_{\langle x,y\rangle} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y)) + J_2 \sum_{x,y \in V \colon d(x,y)=2} (\sigma(x)\sigma(y) - \varphi(x)\varphi(y)), \tag{3}$$

where $J = (J_1, J_2) \in \mathbb{R}^2$.

Let M be the set of unit balls in V and σ_b be the restriction of a configuration σ to a ball $b \in M$. The energy of the configuration σ_b is defined as

$$U(\sigma_b) \equiv U(\sigma_b, J) = \frac{1}{2} J_1 \sum_{\langle x,y\rangle,\, x,y \in b} \sigma(x)\sigma(y) + J_2 \sum_{x,y \in b \colon d(x,y)=2} \sigma(x)\sigma(y), \tag{4}$$

where $J = (J_1, J_2) \in \mathbb{R}^2$. Configurations σ_b and $\sigma'_{b'}$ belong to the same class (are equivalent, $\sigma'_{b'} \sim \sigma_b$) if $U(\sigma_b) = U(\sigma'_{b'})$.

We let $|A|$ denote the number of elements of a set A.

Lemma [7]**.** 1. *We have* $U(\sigma_b) \in \{U_0, U_1, \dots, U_{k+1}\}$ *for all* σ_b, *where*

$$U_i = \left(\frac{k+1}{2} - i\right)J_1 + \left[\frac{k(k+1)}{2} + 2i(i-k-1)\right]J_2, \quad i = 0, 1, \dots, k+1. \tag{5}$$

2. *Let* $\mathcal{C}_i = \Omega_i \cup \Omega_i^-$, $i = 0, \dots, k+1$, *where*

$$\Omega_i = \{\sigma_b \colon \sigma_b(c_b) = +1, \ |\{x \in b \setminus \{c_b\} \colon \sigma_b(x) = -1\}| = i\},$$

$$\Omega_i^- = \{-\sigma_b = \{-\sigma_b(x), x \in b\} \colon \sigma_b \in \Omega_i\},$$

and c_b *is the center of the ball* b. *Then* $U(\sigma_b) = U_i$ *for all* $\sigma_b \in \mathcal{C}_i$.

Definition. A configuration φ is called a *ground state* of the Hamiltonian H if

$$U(\varphi_b) = \min\{U_0, U_1, \dots, U_{k+1}\} \quad \text{for all } b \in M. \tag{6}$$

Let $U_i(J) = U(\sigma_b, J)$ for $\sigma_b \in \mathcal{C}_i$, $i = 0, 1, \dots, k+1$. For a fixed $m = 0, 1, \dots, k+1$, we set

$$A_m = \{J \in \mathbb{R}^2 \colon U_m(J) = \min\{U_0(J), U_1(J), \dots, U_{k+1}(J)\}\}. \tag{7}$$

It is easy to verify that

$$A_0 = \{J \in \mathbb{R}^2 \colon J_1 \le 0, \ J_1 + 2kJ_2 \le 0\},$$

$$A_m = \{J \in \mathbb{R}^2 \colon J_2 \ge 0, \ 2(2m-k-2)J_2 \le J_1 \le 2(2m-k)J_2\}, \quad m = 1, 2, \dots, k,$$

$$A_{k+1} = \{J \in \mathbb{R}^2 \colon J_1 \ge 0, \ J_1 - 2kJ_2 \ge 0\},$$

and $\mathbb{R}^2 = \bigcup_{i=0}^{k+1} A_i$. For any A_i and A_j, $i \ne j$, we have

$$A_i \cap A_j = \begin{cases} \{J \colon J_1 = 2(2i-k)J_2, \ J_2 \ge 0\}, & j = i+1, \quad i = 0, 1, \dots, k, \\ (0,0), & 1 < |i-j| < k+1, \\ \{J \colon J_1 = 0, \ J_2 \le 0\}, & |i-j| = k+1. \end{cases} \tag{8}$$

Let

$$B = A_0 \cap A_{k+1}, \qquad B_i = A_i \cap A_{i+1}, \quad i = 0, \dots, k,$$

$$\widetilde{A}_0 = A_0 \setminus (B \cup B_0), \qquad \widetilde{A}_{k+1} = A_{k+1} \setminus (B \cup B_k),$$

$$\widetilde{A}_i = A_i \setminus (B_{i-1} \cup B_i), \quad i = 1, \dots, k.$$

We let $\mathrm{GS}(H)$ denote the set of all ground states of the Hamiltonian H (see (3)). We introduce the notation $\bar{\sigma} = -\sigma = \{-\sigma(x), \ x \in V\}$ for $\sigma = \{\sigma(x), \ x \in V\} \in \Omega$.

Theorem 1 [7]**.** *The following assertions hold*:

1. *If* $J = (0,0)$, *then* $\mathrm{GS}(H) = \Omega$.

2. *If* $J \in \widetilde{A}_i$, $i = 0, \dots, k+1$, *then* $\mathrm{GS}(H) = \{\sigma^{(i)}, \bar{\sigma}^{(i)}\}$.

3. *If* $J \in B_i \setminus \{(0,0)\}$, $i = 0, \dots, k$, *then* $\mathrm{GS}(H) = \{\sigma^{(i)}, \bar{\sigma}^{(i)}, \sigma^{(i+1)}, \bar{\sigma}^{(i+1)}\} \cup S_i$, *where* S_i *contains at least countably many nonperiodic ground states.*

4. *If* $J \in B \setminus \{(0,0)\}$, *then* $\mathrm{GS}(H) = \{\sigma^{(0)}, \bar{\sigma}^{(0)}, \sigma^{(k+1)}, \bar{\sigma}^{(k+1)}\}$, *where* $\sigma^{(i)}$ *and* $\bar{\sigma}^{(i)}$, $i = 0, \dots, k+1$, *are periodic ground states such that* $\sigma_b^{(i)}, \bar{\sigma}_b^{(i)} \in \mathcal{C}_i$ *for all* $b \in M$, *i.e.,* $\sigma^{(0)}$ *and* $\bar{\sigma}^{(0)}$ *are translation invariant, and* $\sigma^{(i)}$ *and* $\bar{\sigma}^{(i)}$, $i = 1, \dots, k+1$, *are periodic states with the period* 2.

We note that the weakly periodic ground states (which do not coincide with periodic ground states) belong to the set S_i. Our main goal here is to describe the set of weakly periodic ground states explicitly.

3. Weakly periodic ground states

Let $\mathcal{H}$ be an arbitrary normal divisor of a finite index $r \geq 2$, and let $G_k/\mathcal{H} = \{H_1, \dots, H_r\}$ be the quotient group. We set

$$\mathcal{I} = \mathcal{I}(\mathcal{H}) = \{(i,j) \in \{1, \dots, r\}^2 \colon \exists x \in H_i,\ \exists y \in H_j \text{ such that } d(x,y) = 1\},$$

$$\mathcal{I}_i = \mathcal{I}_i(\mathcal{H}) = \{j \in \{1, \dots, r\} \colon (i,j) \in \mathcal{I}\}.$$

We associate each $\mathcal{H}$-weakly periodic configuration σ, i.e.,

$$\sigma(x) = a_{ij}, \quad x_{\downarrow} \in H_i, \quad x \in H_j, \quad (i,j) \in \mathcal{I}, \tag{9}$$

with a quadratic matrix $\mathcal{B} = \mathcal{B}(\sigma) = \{b_{ij}\}_{i,j=1}^{r}$ with the elements

$$b_{ij} = \begin{cases} a_{ij}, & (i,j) \in \mathcal{I}, \\ 0, & (i,j) \notin \mathcal{I}. \end{cases}$$

Let $S_1(x)$ be the set of all nearest neighbors of a point x, and let

$$q_j(x) = |S_1(x) \cap H_j|, \quad j = 1, \dots, r, \qquad Q(x) = (q_1(x), \dots, q_r(x)).$$

We note that for any $x \in G_k$, there exists a permutation π_x of the coordinates of the vector $Q(e)$ such that

$$\pi_x Q(e) = Q(x). \tag{10}$$

The following theorem imposes conditions on the elements of the matrix $\mathcal{B}(\sigma)$ under which σ is a ground state.

Theorem 2. 1. *An* $\mathcal{H}$*-weakly periodic configuration* σ *is* $\mathcal{H}$*-periodic if and only if the matrix* $\mathcal{B}(\sigma)$ *consists of equal rows.*

2. *An* $\mathcal{H}$*-weakly periodic configuration* σ *is a ground state of the Hamiltonian* H *if and only if there exists* $j \in \{0, \dots, k\}$ *such that* $J_1 = 2(2j-k)J_2$, $J_2 \geq 0$, *and*

$$a_{pm} - a_{nm} + \sum_{s=1}^{r} q_s(x) a_{ns} = (k - 2j \pm 1) a_{mn}, \tag{11}$$

where $x \in H_n$, $m \in \mathcal{I}_p$, *and* $n \in \mathcal{I}_m$.

Proof. 1. Assertion 1 follows from (9) and the definition of $\mathcal{B}$.

2. The condition $J_1 = 2(2j-k)J_2$, $J_2 \geq 0$, follows from assertion 3 in Theorem 1 and from (8). We note that for $J_1 = 2(2j-k)J_2$, the configuration σ is a ground state if there exists $j \in \{0,\dots,k\}$ such that $\sigma_b \in \mathcal{C}_j \cup \mathcal{C}_{j+1}$ for all $b \in M$. This means that for any x and for $\sigma(x) = 1$ or $\sigma(x) = -1$, the configuration $\sigma_{S_1(x)}$ contains either j or $j+1$ or either $k+1-j$ or $k-j$ elements -1. The sum of all values of $\sigma(y)$, $y \in S_1(x)$, is therefore $(k-2j\pm 1)\sigma(x)$, which implies the statement in the theorem.

Remark 1. System (11) consists of linear equations, and it is necessary to find $a_{ij} \in \{-1,1\}$. The best method for finding such solutions is to consider all possible versions $a_{ij} \in \{-1,1\}$, $j \in \mathcal{I}_i$ (the number of versions does not exceed $2^{|\mathcal{I}|}$) and to check which of them satisfies system (11) for some $j \in \{0,\dots,k\}$. Here, we solve system (11) for normal divisors of indices 2 and 4.

3.1. Case of index 2. Let $H_A = \{x \in G_k\colon \sum_{j\in A} w_j(x) \text{ is an even number}\}$, $A \subset \{1,2,\dots,k+1\}$, where $w_j(x)$ is the number of letters a_j in the word x. It is clear that H_A is a normal divisor of index 2. Let $G_k/H_A = \{H_1, H_2\}$ be the quotient group, where $H_1 = H_A$ and $H_2 = G_k \setminus H_A$. Then the H_A-weakly periodic configuration has the form

$$\varphi(x) = \pm\begin{cases} a_{11}, & x_{\downarrow} \in H_1, \quad x \in H_1, \\ a_{12}, & x_{\downarrow} \in H_1, \quad x \in H_2, \\ a_{21}, & x_{\downarrow} \in H_2, \quad x \in H_1, \\ a_{22}, & x_{\downarrow} \in H_2, \quad x \in H_2. \end{cases} \tag{12}$$

Let $|A| = i$. Then system (11) has the form

$$\begin{aligned}
&a_{11} + (k-i)a_{11} + ia_{12} = (k-2j\pm 1)a_{11},\\
&a_{21} + (k-i)a_{11} + ia_{12} = (k-2j\pm 1)a_{11},\\
&a_{12} + (k-i+1)a_{11} + (i-1)a_{12} = (k-2j\pm 1)a_{21},\\
&a_{22} + (k-i+1)a_{11} + (i-1)a_{12} = (k-2j\pm 1)a_{21},\\
&a_{11} + (k-i)a_{21} + ia_{22} = (k-2j\pm 1)a_{12},\\
&a_{21} + (k-i)a_{21} + ia_{22} = (k-2j\pm 1)a_{12}\\
&a_{12} + (k-i+1)a_{21} + (i-1)a_{22} = (k-2j\pm 1)a_{22},\\
&a_{22} + (k-i+1)a_{21} + (i-1)a_{11} = (k-2j\pm 1)a_{22},
\end{aligned} \tag{13}$$

where $i \in \{0,\dots,k+1\}$ and $j \in \{0,\dots,k\}$.

We note that the vector $(a_{11}, a_{12}, a_{21}, a_{22}) \in \{(\pm1,\pm1,\pm1,\pm1)\}$ takes 16 values. Substituting the coordinates of these vectors in system (13), we obtain the following theorem.

Theorem 3. *Let* $|A| = i$.

1. *If* $i \neq (k+1)/2$, *then each* H_A*-weakly periodic ground state is* H_A*-periodic or translation invariant, i.e., corresponds to one of the vectors* $\pm(1,1,1,1)$ *and* $\pm(-1,1,-1,1)$.

2. *For* $i = (k+1)/2$ *and* $J_1 = 2J_2$, $J_2 \geq 0$, *there exist two* H_A*-weakly periodic* (*nonperiodic*) *ground states corresponding to the vectors* $\pm(1,-1,-1,1)$.

3.2. Case of index 4. Let $H_A = \{x \in G_k\colon \sum_{j\in A} w_j(x) \text{ is an even number}\}$, $A \subset \{1, 2, \dots, k+1\}$, and $G_k^{(2)} = \{x \in G_k\colon |x| \text{ is an even number}\}$, where $|x|$ is the length of the word x. Then $G_k^{(4)} = H_A \cap G_k^{(2)}$ is a normal divisor of index 4. Let $G_k/G_k^{(4)} = \{H_1, H_2, H_3, H_4\}$ be the quotient group.

We note that $\mathcal{I}(G_k^{(4)}) = \{(1,3), (3,1), (1,4), (4,1), (2,3), (3,2), (2,4), (4,2)\}$, and any $G_k^{(4)}$-weakly periodic configuration therefore has the form

$$\varphi(x) = a_{ij} \quad \text{if } x_\downarrow \in H_i, \quad x \in H_j, \quad (i,j) \in \mathcal{I}(G_k^{(4)}).$$

Each weakly periodic configuration is thus associated with an eight-dimensional vector $(a_{ij}, (i,j) \in \mathcal{I})$. In this case, it is easy to verify the following assertion.

Theorem 4. *Let $|A| = i$.*

1. *If $i \neq (k+1)/2$, then each $G_k^{(4)}$-weakly periodic ground state is periodic and translation invariant.*

2. *If $i = (k+1)/2$ and $J_1 = 2J_2$, $J_2 \geq 0$, then there are four $G_k^{(4)}$-weakly periodic (nonperiodic) ground states $\pm\varphi'$ and $\pm\varphi''$ corresponding to $\pm(1,1,-1,-1,-1,-1,1,1)$ and $\pm(-1,1,1,-1,-1,1,-1,1)$:*

$$\pm\varphi'(x) = \pm\begin{cases} +1, & x_\downarrow \in H_2, \quad x \in H_4, \\ +1, & x_\downarrow \in H_4, \quad x \in H_2, \\ -1, & x_\downarrow \in H_1, \quad x \in H_4, \\ -1, & x_\downarrow \in H_4, \quad x \in H_1, \\ -1, & x_\downarrow \in H_3, \quad x \in H_2, \\ -1, & x_\downarrow \in H_2, \quad x \in H_3, \\ +1, & x_\downarrow \in H_1, \quad x \in H_3, \\ +1, & x_\downarrow \in H_3, \quad x \in H_1, \end{cases} \qquad \pm\,\varphi''(x) = \pm\begin{cases} -1, & x_\downarrow \in H_2, \quad x \in H_4, \\ +1, & x_\downarrow \in H_4, \quad x \in H_2, \\ +1, & x_\downarrow \in H_1, \quad x \in H_4, \\ -1, & x_\downarrow \in H_4, \quad x \in H_1, \\ -1, & x_\downarrow \in H_3, \quad x \in H_2, \\ +1, & x_\downarrow \in H_2, \quad x \in H_3, \\ -1, & x_\downarrow \in H_1, \quad x \in H_3, \\ +1, & x_\downarrow \in H_3, \quad x \in H_1. \end{cases}$$

Corollary 1. *If k is an even number, then each weakly periodic ground state is periodic.*

It follows from (5) and (8) that Theorems 3 and 4 imply the following assertion.

Corollary 2. *For the ground states constructed in Theorems 3 and 4, the energy of these configurations in any ball of radius 1 is given by $-(k+1)J_2/2$, $J_2 \geq 0$.*

Remark 2. We note that the matrices of the configurations constructed in Theorem 3 are symmetric or skew-symmetric. The matrix $\mathcal{B}(\varphi')$ is also symmetric, and the matrix $\mathcal{B}(\varphi'')$ is skew-symmetric.

The following conjecture therefore seems to hold.

Conjecture 1. *Let an $\mathcal{H}$-weakly periodic configuration φ be a ground state for Hamiltonian (2). Then the matrix $\mathcal{B}(\varphi)$ is either symmetric or skew-symmetric.*

The following example shows that it does not follow from the symmetry or skew-symmetry of the matrix $\mathcal{B}(\varphi)$ that φ is a ground state.

Example. We consider

$$\varphi_0(x)=\begin{cases}+1, & x_\downarrow\in H_2, \quad x\in H_4,\\ +1, & x_\downarrow\in H_4, \quad x\in H_2,\\ +1, & x_\downarrow\in H_1, \quad x\in H_4,\\ +1, & x_\downarrow\in H_4, \quad x\in H_1,\\ +1, & x_\downarrow\in H_3, \quad x\in H_2,\\ +1, & x_\downarrow\in H_2, \quad x\in H_3,\\ -1, & x_\downarrow\in H_1, \quad x\in H_3,\\ -1, & x_\downarrow\in H_3, \quad x\in H_1,\end{cases} \qquad \mathcal{B}(\varphi_0)=\begin{pmatrix}0&0&-1&1\\0&0&1&1\\-1&1&0&0\\1&1&0&0\end{pmatrix}.$$

The matrix $\mathcal{B}(\varphi_0)$ is symmetric, but it is easy to verify that φ_0 is not a ground state.

Theorems 3 and 4 also imply the following conjecture.

Conjecture 2. *Let $\mathcal{H}$ be an arbitrary normal divisor of finite index. An $\mathcal{H}$-weakly periodic configuration φ is a ground state of Hamiltonian* (2) *if and only if $\varphi\in\mathcal{C}_{(k+1)/2}\cup\mathcal{C}_{(k+1)/2+1}$ and $J_1=2J_2$, $J_2\geq 0$.*

4. New weakly periodic Gibbs measures

We describe new weakly periodic Gibbs measures for the Ising model (i.e., model (2) with $J_2=0$). The results in this section supplement the results in [3].

The problem of describing weakly periodic Gibbs measures in several special cases was reduced in [3] to the problem of describing solutions of the equation

$$\pm x=-(k-1)f(x,\theta)+f(kf(x,\theta),\theta), \tag{14}$$

where $f(x,\theta)=\operatorname{arctanh}(\theta\tanh x)$ and $\theta=\tanh(J\beta)$. Lemma 1 in [3] can also be used to obtain the equations

$$\pm x=(k-1)f(x,\theta)+f(kf(x,\theta),\theta). \tag{15}$$

Because $f(x,-\theta)=-f(x,\theta)$, Eqs. (15) can be reduced to Eqs. (14) by the change $\theta=-\theta$. The solutions of Eqs. (15) are therefore obtained from the solutions of (14) by replacing θ with $-\theta$.

Theorem 2 in [3] can therefore be supplemented with new solutions, and we obtain the following theorem.

Theorem 5. *Let $\alpha=(1-\theta)/(1+\theta)$. For $k=4$, there exist critical values $\alpha_{\mathrm{cr}}(\approx 0.152)$ and $\alpha_{\mathrm{c}}=3/5$ such that*

1. *there exist seven weakly periodic Gibbs measures for $\alpha\in[0,\alpha_{\mathrm{cr}})\cup(\alpha_{\mathrm{cr}}^{-1},+\infty)$,*
2. *there exist five weakly periodic Gibbs measures for $\alpha=\alpha_{\mathrm{cr}},\alpha_{\mathrm{cr}}^{-1}$,*
3. *there exist three weakly periodic Gibbs measures for $\alpha\in(\alpha_{\mathrm{cr}},\alpha_{\mathrm{c}})\cup(\alpha_{\mathrm{c}}^{-1},\alpha_{\mathrm{cr}}^{-1})$, and*
4. *there exists one weakly periodic Gibbs measure for $\alpha\in[\alpha_{\mathrm{c}},\alpha_{\mathrm{c}}^{-1}]$.*

Remark 3. 1. In all cases in Theorem 5, one of the weakly periodic measures is translation invariant, which corresponds to the solution $x = 0$ of Eqs. (14) and (15). All the other measures are $H_{\{1\}} \cap G_4^{(2)}$-weakly periodic.

2. From $\alpha_c = (1 - \theta_c)/(1 + \theta_c) = 3/5$, we obtain the critical value θ_c of the phase transition for the Ising model, i.e., $|\theta_c| = 1/k$ for $k = 4$ (see, e.g., [8]).

3. Five Gibbs measures among the seven in assertion 1 in Theorem 5 correspond to the stable solutions of Eqs. (14) and (15). The well-known methods (see, e.g., [8]–[10]) can therefore be used to prove that at least five of them are extreme (indecomposable).

The following conjecture is formulated based on computer calculations and Theorem 5.

Conjecture 3. *1. For the Ising model on the Cayley tree of order $k \geq 4$, the statements in Theorem 5 hold for the critical values $\alpha_{cr} = \alpha_{cr}(k)$ and $\alpha_c = (k - 1)/(k + 1)$.*

2. The estimate $\alpha_{cr} = (1 - \theta_{cr})/(1 + \theta_{cr}) < (\sqrt{k} - 1)/(\sqrt{k} + 1)$ holds for α_{cr}, i.e., $|\theta_{cr}| < |\theta_c^{SG}| = 1/\sqrt{k}$, where θ_c^{SG} is the critical value for the spin glass model and the second critical value for the Ising model, below which the measure corresponding to $x = 0$ is an extreme measure (*see* [11], [12]).

5. Discussion

There are two approaches used to describe the Gibbs measures on the Cayley tree. The first approach, based on the theory of Markov random fields, permits describing the set of special Gibbs measures (called Markov chains in [8] and [13], entrance laws in [14] and [15], boundary laws in [9], and splitting Gibbs measures in [16]). The second approach, based on the contour method, gives Gibbs measures corresponding to some boundary conditions (ground states) [7], [17]. The relation between the measures constructed using these two approaches is not clear in the general case. Such a relation for the Ising model was described in [10].

Our results here show that the weakly periodic ground states for model (2) exist only for $J_1 = 2J_2$. Therefore, for $J_2 = 0$, i.e., for the usual Ising model (for $J_1 \neq 0$), there do not exist weakly periodic ground states. It is therefore not clear what boundary configurations are associated with the weakly periodic measures described in Theorem 5.

The problem of describing weakly periodic Gibbs measures for model (2) (for $J_2 \neq 0$) is rather complicated. The point is that to solve this problem, it is necessary to solve the system of functional equations (4) given in [6] in the class of weakly periodic functions, which is a very complicated problem. On the other hand, the results in Sec. 3 can be useful for describing the weakly periodic Gibbs measures of model (2) ($J_2 \neq 0$) by the contour method.

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