Theoretical and Mathematical Physics, **160**(1): 1031–1041 (2009)

SPECTRAL ANALYSIS OF THE ELLIPTIC SINE-GORDON EQUATION IN THE QUARTER PLANE

© **B. Pelloni***

We study the elliptic sine-Gordon equation in the quarter plane using a spectral transform approach. We determine the Riemann–Hilbert problem associated with well-posed boundary value problems in this domain and use it to derive a formal representation of the solution. Our analysis is based on a generalization of the usual inverse scattering transform recently introduced by Fokas for studying linear elliptic problems.

Keywords: elliptic sine-Gordon equation, nonlinear boundary value problem, Riemann–Hilbert problem

1. Introduction

The elliptic sine-Gordon equation is the nonlinear partial differential equation (PDE)

$$q_{xx} + q_{yy} = \sin q, \qquad q = q(x, y). \tag{1.1}$$

This equation appears, for example, in the theory of Josephson effects, superconductors, and spin waves in ferromagnets [1]. From a strictly mathematical standpoint, the elliptic sine-Gordon equation is an example of an *integrable* PDE, at least in the sense of admitting a Lax pair formulation.This means that the tools developed in connection with the so-called inverse scattering problem can in principle be used to analyze it. Indeed, the inverse scattering analysis for this integrable equation was considered in [2] for a problem posed on $\mathbb{R}^2$ with prescribed periodic behavior at infinity and in [1] for the problem on the half-plane $y \geq 0$ with all boundary values prescribed but constrained by a certain nonlinear relation. Because of the limitations of the inverse scattering approach when dealing with boundary value problems, the solution was not constructed effectively in these works. Special exact solutions for the problem posed in the whole $\mathbb{R}^2$ were found in the 1980s by various authors (see the references in [1]), but the more realistic case of boundary value problems for this equation is still essentially open.

Here, we use the Fokas generalization of the inverse scattering method to boundary value problems [3], [4]. This generalization is based on simultaneously solving both ODEs in the Lax pair and on analyzing a relation between all boundary values, called the *global relation* by Fokas. The global relation gives an algorithmic way to derive a representation of the solution under the assumption that the given boundary conditions are *admissible* in the sense that they satisfy the global relation identically. We note that the result in [1] is a particular case where admissible boundary conditions are assumed to hold and is hence contained in our general formulation.

We present the general approach and characterize the solution under the assumption that the boundary conditions are admissible. To illustrate the methodology, we consider the particular case where the problem is posed in the positive quarter plane $x \geq 0$, $y \geq 0$. This method also allows explicitly analyzing the concrete boundary value problems obtained when the boundary conditions are *linearizable* (see [5], [6]). The analysis of such cases and of the more difficult case of general boundary conditions is in process.

*Department of Mathematics, University of Reading, Reading, UK, e-mail: b.pelloni@reading.ac.uk.

Prepared from an English manuscript submitted by the author; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 160, No. 1, pp. 189–201, July, 2009.

0040-5779/09/1601-1031

2. Notation and main result

We use the following notation. For any 2×2 matrix M, we set $\widehat{\sigma_3}M = [\sigma_3, M]$ and $e^{\widehat{\sigma_3}}M = e^{\sigma_3}Me^{-\sigma_3}$, where σ_i are the usual Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The parameter $\lambda \in \mathbb{C}$ denotes the *spectral parameter*, and we introduce the functions

$$w_1(\lambda) = \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right), \qquad w_2(\lambda) = \frac{1}{2i}\left(\lambda - \frac{1}{\lambda}\right).$$

We write

$$\lambda \in (i) \qquad \text{or} \qquad \lambda \in (ij), \quad i, j = 1, 2, 3, 4,$$

to mean that λ is in the ith quadrant of the complex λ plane or the union of the ith and jth quadrants. For a 2×2 matrix $M(\lambda)$, $\lambda \in \big((i), (j)\big)$ means that the elements of M are bounded functions of λ for $\lambda \in (i)$ for the first column vector and $\lambda \in (j)$ for the second column vector.

We now define admissible boundary conditions. Let $d_1(y)$, $u_1(y)$, $d_2(x)$, and $u_2(x)$ be smooth functions on $[0, \infty)$ decaying at infinity and compatible at the corner $(x, y) = (0, 0)$, i.e., $d_1(0) = d_2(0)$, $d_1'(0) = u_2(0)$, and $d_2'(0) = u_1(0)$. We define the column vectors $\big(a_0(\lambda), b_0(\lambda)\big)^{\mathrm{T}}$ and $\big(a_1(\lambda), b_1(\lambda)\big)^{\mathrm{T}}$ by

$$\begin{pmatrix} a_0(\lambda) \\ b_0(\lambda) \end{pmatrix} = \psi_3(0, \lambda), \quad \lambda \in (12), \qquad \begin{pmatrix} a_1(\lambda) \\ b_1(\lambda) \end{pmatrix} = \psi_1(0, \lambda), \quad \lambda \in (14), \tag{2.1}$$

where ψ_1 and ψ_3 are the unique solutions of

$$\psi_1(y, \lambda)_y + w_1(\lambda) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \psi_1(y, \lambda) = iQ_0(0, y, -\lambda)\psi_1(y, \lambda),$$

$$\psi_3(x, \lambda)_x + w_2(\lambda) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \psi_3(x, \lambda) = Q_0(x, 0, \lambda)\psi_3(x, \lambda)$$

satisfying the conditions

$$\lim_{y\to\infty} \psi_1(y, \lambda) \to (1, 0)^{\mathrm{T}}, \qquad \lim_{x\to\infty} \psi_3(x, \lambda) \to (1, 0)^{\mathrm{T}},$$

where

$$Q_0(x, 0, \lambda) = \frac{id_2'(x) + u_2(x)}{2}\sigma_1 - \frac{i}{4\lambda}\sin d_2(x) \cdot \sigma_2 + \frac{i}{4\lambda}\big(1 - \cos d_2(x)\big)\sigma_3, \tag{2.2}$$

$$Q_0(0, y, \lambda) = \frac{iu_1(y) + d_1'(y)}{2}\sigma_1 - \frac{i}{4\lambda}\sin d_1(y) \cdot \sigma_2 + \frac{i}{4\lambda}\big(1 - \cos d_1(y)\big)\sigma_3. \tag{2.3}$$

Definition 1. The set of functions $\{d_1(y), u_1(y), d_2(x), u_2(x)\}$ is said to be *admissible* if

$$a_0(\lambda) \equiv a_1(\lambda), \qquad b_0(\lambda) \equiv b_1(\lambda), \quad \lambda \in (1).$$

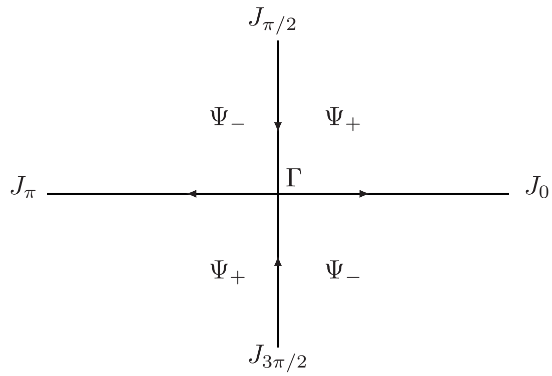

Fig. 1. The Riemann–Hilbert problem

We now formulate our main theorem.

Proposition 1. *We consider the Dirichlet boundary value problem for the elliptic sine-Gordon equation in the quarter plane*

$$q_{xx} + q_{yy} = \sin q, \qquad q = q(x,y), \quad x \geq 0, \quad y \geq 0, \tag{2.4}$$

$$q(0,y) = d_1(y), \qquad q(x,0) = d_2(x), \qquad d_1(0) = d_2(0), \tag{2.5}$$

where the functions d_1 *and* d_2 *are in the Schwarz class. We assume that there exist two functions* $u_1(y)$ *and* $u_2(x)$ *such that the set* $\{d_1(y), u_1(y), d_2(x), u_2(x)\}$ *is admissible in the sense of Definition* 1*. We also assume that* $a(\lambda) \neq 0$ *for* $\lambda \in$ (1)*.*

We consider the complex variable $z = x + iy$, $x \geq 0$, $y \geq 0$, *and define* $\Psi(z, \bar{z}, \lambda)$ *as the solution of the Riemann–Hilbert problem (see Fig.* 1*):*

$$\Psi_-(z,\bar{z},\lambda) = \Psi_+(z,\bar{z},\lambda) J(z,\bar{z},\lambda), \quad \lambda \in \mathbb{R} \cup i\mathbb{R}, \tag{2.6}$$

where

$$J(z,\bar{z},\lambda) = J^\alpha(z,\bar{z},\lambda), \qquad \text{if } \arg\lambda = \alpha, \quad \alpha = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \tag{2.7}$$

$$\begin{aligned} J^0 &= \begin{pmatrix} 1 & -\dfrac{\overline{b}_0}{a_0} e^{i\theta(z)} \\ 0 & 1 \end{pmatrix}, \qquad J^{\pi/2} = \begin{pmatrix} 1 & -\dfrac{\overline{b}_1}{a_0} e^{i\theta(z)} \\ 0 & 1 \end{pmatrix}, \\ J^\pi &= \begin{pmatrix} 1 & 0 \\ \dfrac{b_0}{\overline{a}_1} e^{-i\theta(z)} & 1 \end{pmatrix}, \qquad J^{3\pi/2} = J_\pi^{-1} J_0 J_{\pi/2}, \end{aligned} \tag{2.8}$$

the functions $a_0(\lambda)$, $a_1(\lambda)$, $b_0(\lambda)$, *and* $b_1(\lambda)$ *are given by* (2.1), $\overline{a}_0 = \overline{a_0(\lambda)}$, $\overline{b}_{0,1} = \overline{b_{0,1}(\bar{\lambda})}$, *and*

$$\theta(z) = \frac{1}{2}\left[\lambda z - \frac{\bar{z}}{\lambda}\right]. \tag{2.9}$$

Then the function $q(x,y)$ *defined by*

$$\cos q(z,\bar{z}) = 1 + \frac{2}{\pi}\frac{\partial}{\partial \bar{z}} \int_{\Gamma_-} (\Psi_+)_{12} \frac{b(\lambda)}{a(-\lambda)}\, e^{-i(\lambda z - \bar{z}/\lambda)/2}\, d\lambda, \tag{2.10}$$

where $\Gamma_- = (\mathbb{R}\cup i\mathbb{R})\cap\mathbb{C}^-$, $b(\lambda) = b_0(\lambda)$ *for* $\lambda \in \mathbb{R}^-$, $b(\lambda) = b_1(\lambda)$ *for* $\lambda \in i\mathbb{R}^-$, *and* $a(-\lambda) = a_0(-\lambda) = a_1(-\lambda)$ *for* $\lambda \in (3)$, *satisfies* $q_{xx} + q_{yy} = \sin q$ *and also the boundary conditions*

$$q(0, y) = d_1(y), \qquad q_x(0, y) = u_1(y), \qquad q(x, 0) = d_2(x), \qquad q_y(x, 0) = u_2(x).$$

The proof of this result follows the lines of the proof of Theorem 3.1 in [5] after the appropriate Riemann–Hilbert problem is determined. We present the construction of the Riemann–Hilbert problem, and in terms of its solution, we characterize the function $q(x, y)$ that satisfies the Dirichlet problem for the elliptic sine-Gordon equation.

3. The linearized problem: Modified Helmholtz equation

In the small-q limit, we find that Eq. (1.1) linearizes to the modified Helmholtz equation

$$q_{xx} + q_{yy} = q, \qquad q = q(x, y). \tag{3.1}$$

This equation was extensively analyzed using the spectral transform method proposed by Fokas [3], [7], [8]. In terms of the variables $z = x + iy$ and $\bar{z} = x - iy$, Eq. (3.1) can be written as $4q_{z\bar{z}} = q$ or, equivalently, in the form $\mu_{z\bar{z}} = \mu_{\bar{z}z}$, where μ satisfies the Lax pair

$$\mu_z - \frac{i\lambda}{2}\mu = 2q_z, \qquad \mu_{\bar{z}} + \frac{i}{2\lambda}\mu = \frac{i}{\lambda}q. \tag{3.2}$$

We note that such a μ satisfies

$$\mu \sim O\left(\frac{1}{\lambda}\right), \quad |\lambda| \to \infty.$$

The Dirichlet problem in the quarter plane. We consider modified Helmholtz equation (3.1) in the quarter plane $x \geq 0$, $y \geq 0$ with the prescribed Dirichlet boundary conditions

$$q(0, y) = d_1(y), \qquad q(x, 0) = d_2(x) \tag{3.3}$$

assuming that the solution is appropriately smooth and decays at infinity and that the boundary conditions are compatible at $x = y = 0$. Performing the spectral analysis of Lax pair (3.2), we can prove the following result [3].

Proposition 2. *The solution of Eq.* (3.1) *in the quarter plane under Dirichlet conditions* (3.3) *admits the integral representation*

$$q(z, \bar{z}) = \frac{1}{4\pi i}\left[\int_{i\mathbb{R}^+} e^{i(\lambda z - \bar{z}/\lambda)/2}\hat{q}_1(\lambda)\frac{d\lambda}{\lambda} + \int_{\mathbb{R}^+} e^{i(\lambda z - \bar{z}/\lambda)/2}\hat{q}_2(\lambda)\frac{d\lambda}{\lambda}\right], \tag{3.4}$$

where the spectral functions $\hat{q}_j(\lambda)$ *are defined by*

$$\hat{q}_1(\lambda) = \left(\lambda - \frac{1}{\lambda}\right)D_1(\lambda) - \frac{i}{2}\left(\lambda + \frac{1}{\lambda}\right)D_2(i\lambda), \quad \lambda \in i\mathbb{R}^+,$$

$$\hat{q}_2(\lambda) = i\left(\lambda + \frac{1}{\lambda}\right)D_2(-i\lambda) - \frac{i}{2}\left(\lambda + \frac{1}{\lambda}\right)D_2(i\lambda) - \left(\lambda - \frac{1}{\lambda}\right)D_1(-\lambda), \quad \lambda \in \mathbb{R}^+,$$

with

$$D_j(\lambda) = \int_0^\infty e^{(\lambda + 1/\lambda)s/2} d_j(s)\, ds, \quad j = 1, 2. \tag{3.5}$$

The characterization of the functions $\hat{q}_1$ and $\hat{q}_2$ in terms of $d_1(y)$ and $d_2(x)$ is the the Dirichlet-to-Neumann map for this problem. This characterization is based on analyzing the global relation obtained as a result of the spectral analysis of the Lax pair and on its invariance under the transformations $\lambda \to \pm 1/\lambda$. For this problem, the global relation is the explicit algebraic constraint for $\hat{q}_1$ and $\hat{q}_2$

$$\hat{q}_1(\lambda) + \hat{q}_2(\lambda) = 0, \quad \pi \leq \arg\lambda \leq \frac{3\pi}{2}. \tag{3.6}$$

4. The nonlinear problem

A Lax pair for the nonlinear problem. We write elliptic sine-Gordon equation (1.1) in terms of the variables $z = x + iy$ and $\bar{z} = x - iy$ as $q_{z\bar{z}} = (\sin q)/4$. This equation is the compatibility condition for the pair of ODEs

$$\Psi_z - \frac{i\lambda}{4}[\sigma_3, \Psi] = Q\Psi, \tag{4.1}$$

$$\Psi_{\bar{z}} + \frac{i}{4\lambda}[\sigma_3, \Psi] = \frac{1}{\lambda}\widetilde{Q}\Psi, \tag{4.2}$$

where $\lambda \in \mathbb{C}$, $\Psi = \Psi(z, \bar{z}, \lambda)$ is a nonsingular 2×2 matrix, and the matrices Q and $\widetilde{Q}$ are given by

$$\begin{aligned} Q(z, \bar{z}) &= \frac{iq_z(z, \bar{z})}{2}\sigma_1, \\ \widetilde{Q}(z, \bar{z}) &= \frac{i}{4}\big(1 - \cos q(z, \bar{z})\big)\sigma_3 - \frac{i}{4}\sin q(z, \bar{z}) \cdot \sigma_2. \end{aligned} \tag{4.3}$$

This Lax pair, which reduces to (3.2) in the linear limit, is similar but not identical to the pair used in [1], [2].

The solutions of system (4.1), (4.2) satisfy

$$\Psi(z, \bar{z}, \lambda) = I + \frac{\psi_1(z, \bar{z})}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \quad |\lambda| \to \infty. \tag{4.4}$$

In terms of the variables x and y ($\Psi(x, y)$ is denoted by the same symbol as the function in terms of the variables z and $\bar{z}$), we have

$$\begin{aligned} \Psi_x + \frac{w_2(\lambda)}{2}[\sigma_3, \Psi] &= Q_0(x, y, \lambda)\Psi, \\ \Psi_y + \frac{w_1(\lambda)}{2}[\sigma_3, \Psi] &= iQ_0(x, y, -\lambda)\Psi, \end{aligned} \tag{4.5}$$

where

$$Q_0(x, y, \lambda) = Q(x, y) + \frac{1}{\lambda}\widetilde{Q}(x, y) = \frac{i(q_x - iq_y)}{4}\sigma_1 + \frac{i}{4\lambda}(1 - \cos q)\sigma_3 - \frac{i}{4\lambda}\sin q\sigma_2 \tag{4.6}$$

and q, q_x, and q_y are considered functions of x and y. We note that this is a Lax pair with the same λ dependence in the right-hand side as the Lax pair for the hyperbolic sine-Gordon equation. In what follows, we sometimes write $Q_0(\lambda)$ for $Q_0(x, y, \lambda)$.

Matrices (4.3) and (4.6) have the following two properties:

- $\operatorname{Tr} Q = \operatorname{Tr} \widetilde{Q} = 0$;
- $\det\big(w_1(\lambda)\sigma_3/2 - iQ_0(-\lambda)\big)$ is a function of λ only through $w_1(\lambda)$ and $\det\big(w_2(\lambda)\sigma_3/2 - Q_0(\lambda)\big)$ is a function of λ only through $w_2(\lambda)$.

The latter property is crucial for characterizing linearizable boundary conditions.

Boundary value problems posed in the quarter plane. We consider the elliptic sine-Gordon equation for $(z, \bar{z})$ in a given convex polygon $\mathcal{P}$.

Lax pair (4.1), (4.2) is equivalent to the single equation

$$d\big(e^{(-i\lambda z/4 + i\bar{z}/4\lambda)\widehat{\sigma_3}}\Psi(z, \bar{z}, \lambda)\big) = W(z, \bar{z}, \lambda), \tag{4.7}$$

where W is the exact 1-form defined by

$$W(z,\bar{z},\lambda) = e^{(-i\lambda z/4 + i\bar{z}/4\lambda)\widehat{\sigma_3}}\left(Q\Psi\, dz + \frac{1}{\lambda}\widetilde{Q}\Psi\, d\bar{z}\right). \tag{4.8}$$

From this formulation, we immediately find that the solutions of system (4.1), (4.2) have the form

$$\Psi(z,\bar{z},\lambda) = I + \int_{z^*}^{z} e^{(4iz/\lambda - i\bar{z}/4\lambda)\widehat{\sigma_3}} W(\zeta,\bar{\zeta},\lambda)\, dz,$$

where z^* is an arbitrary point inside the polygon. In addition, using the convexity of the polygon, we find that the form W is closed and hence

$$\int_{\partial\mathcal{P}} W(z,\bar{z},\lambda)\, dz = 0. \tag{4.9}$$

This integral identity is the *global relation*.

We now specialize to the case where the boundary value problem is posed in the quarter plane $x > 0$, $y > 0$ and $\mathcal{P} = \mathcal{I}$ is hence the first quadrant of the complex z plane. In this case, the corners of the polygon are $z_1 = 0 + i\infty$, $z_2 = 0$, and $z_3 = \infty + i0$. Motivated by the analysis of the linear problem, we consider z_j to be the vertices of the polygon and define the particular solutions obtained when $z^* = z_j$, $j = 1,2,3$. Namely, we define

$$\Psi_j = I + \int_{z_j}^{z} e^{(i[\lambda(z-\zeta) - (\bar{z}-\bar{\zeta})/\lambda]/4)\widehat{\sigma_3}}\left(Q\Psi\, d\zeta + \frac{1}{\lambda}\widetilde{Q}\Psi\, d\bar{\zeta}\right). \tag{4.10}$$

The function Ψ_j is the unique solution of (4.8) such that $\Psi_j(z_j) = I$.

We now analyze where each solution is a bounded analytic function. The boundedness of the exponential terms involved in the definition of Ψ_j clearly depends on the location of the parameter λ in the complex plane. We note that

$$R_j = \mathrm{Re}\left(\frac{i}{4}\left[\lambda(z-\zeta) - \frac{1}{\lambda}(\bar{z}-\bar{\zeta})\right]\right) = -\frac{1}{4}\left(1 + \frac{1}{|\lambda|^2}\right)\mathrm{Im}\big(\lambda(z-\zeta)\big),$$

and we can hence explicitly compute the sector Σ_j where the exponentials e^{R_j} are bounded (as $\lambda \to \infty$ and $\lambda \to 0$). We also define the "conjugate" sectors where $-R_j$ is bounded as $\overline{\Sigma_j}$. We find that the sectors Σ_j, $j = 1,2,3$, are given explicitly by

$$\Sigma_1 = \left\{\lambda \in \mathbb{CP}^1 :\ \arg\lambda \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}, \qquad \overline{\Sigma_1} = \left\{\lambda \in \mathbb{CP}^1 :\ \arg\lambda \in \left[\frac{3\pi}{2}, \frac{\pi}{2}\right]\right\},$$

$$\Sigma_2 = \left\{\lambda \in \mathbb{CP}^1 :\ \arg\lambda \in \left[0, \frac{\pi}{2}\right]\right\}, \qquad \overline{\Sigma_2} = \left\{\lambda \in \mathbb{CP}^1 :\ \arg\lambda \in \left[\pi, \frac{3\pi}{2}\right]\right\},$$

$$\Sigma_3 = \{\lambda \in \mathbb{CP}^1 :\ \arg\lambda \in [\pi, 2\pi]\}, \qquad \overline{\Sigma_3} = \{\lambda \in \mathbb{CP}^1 :\ \arg\lambda \in [0, \pi]\}.$$

These sectors cover the complex plane. In addition, $\Sigma_1 \cap \Sigma_3 = (3)$ and $\overline{\Sigma_1} \cap \overline{\Sigma_3} = (1)$.

Taking the definition of $\widehat{\sigma_3}$ and $e^{A\widehat{\sigma_3}}$ into account, we find that the dependence of each solution Ψ_j on z and ζ has the form

$$\Psi_j = \begin{pmatrix} * & * e^{i\theta(z-\zeta)} \\ * e^{-i\theta(z-\zeta)} & * \end{pmatrix},$$

where the symbol $*$ denotes a function of λ only and $\theta(z)$ is defined by (2.9). Hence, the elements in the first column of Ψ_j are bounded for $\lambda \in \overline{\Sigma_j}$, and the elements in the second column of Ψ_j are bounded for $\lambda \in \Sigma_j$. Using the notation $\Psi = (\Psi^{(\cdot)}, \Psi^{(\cdot)})$ for a 2×2 matrix and its component column vectors, we obtain

$$\Psi_1 = (\Psi_1^{(14)}, \Psi_1^{(23)}), \qquad \Psi_2 = (\Psi_2^{(3)}, \Psi_2^{(1)}), \qquad \Psi_3 = (\Psi_3^{(12)}, \Psi_3^{(34)}),$$

where the superscript indicates the region of boundedness of each column vector (as a function of λ). Because of the trace property $\operatorname{tr} Q = \operatorname{tr} \widetilde{Q} = 0$, any solution of this Lax pair has a determinant equal to 1 and is hence invertible.

Therefore, any two solutions of system (4.1), (4.2) are related by

$$\Psi^{-1}\widetilde{\Psi} = e^{(i\lambda z/4 - i\bar{z}/4\lambda)\widehat{\sigma_3}}\rho(\lambda).$$

Because $\Psi_2(0,0,\lambda) = I$ by definition, we can write

$$\Psi_3(z,\bar{z},\lambda) = \Psi_2(z,\bar{z},\lambda)e^{(i\lambda z/4 - i\bar{z}/4\lambda)\widehat{\sigma_3}}\Psi_3(0,0,\lambda), \quad \lambda \in (\mathbb{R}^-, \mathbb{R}^+), \tag{4.11}$$

$$\Psi_1(z,\bar{z},\lambda) = \Psi_2(z,\bar{z},\lambda)e^{(i\lambda z/4 - i\bar{z}/4\lambda)\widehat{\sigma_3}}\Psi_1(0,0,\lambda), \quad \lambda \in (i\mathbb{R}^-, i\mathbb{R}^+), \tag{4.12}$$

and also

$$\Psi_3(z,\bar{z},\lambda) = \Psi_1(z,\bar{z},\lambda)e^{(i\lambda z/4 - i\bar{z}/4\lambda)\widehat{\sigma_3}}\Psi_1^{-1}(0,0,\lambda)\Psi_3(0,0,\lambda), \quad \lambda \in \big((1),(3)\big). \tag{4.13}$$

In particular, we note that this condition holds for each column vector not only along half-lines but also in a whole quadrant of the λ plane.

The global relation and the spectral functions. As in general case (4.9), the integral of the exact differential form $W(z,\bar{z},\lambda)$ given by (4.8) along the boundary of $\mathcal{I}$ vanishes:

$$\int_{\partial\mathcal{I}} e^{(-i\lambda z/4 + i\bar{z}/4\lambda)\widehat{\sigma_3}}\left(Q\Psi\, dz + \frac{1}{\lambda}\widetilde{Q}\Psi\, d\bar{z}\right) = 0.$$

Computing this integral explicitly (under the assumption that q, q_z, and $q_{\bar{z}}$ vanish as $|z| \to \infty$), we obtain

$$\int_0^\infty e^{-(i(\lambda-1/\lambda)x/4)\widehat{\sigma_3}}\left(Q + \frac{1}{\lambda}\widetilde{Q}\right)\Psi(x,x,\lambda)\, dx - i\int_0^\infty e^{((\lambda+1/\lambda)y/4)\widehat{\sigma_3}}\left(Q - \frac{1}{\lambda}\widetilde{Q}\right)\Psi(iy,-iy,\lambda)\, dy = 0. \tag{4.14}$$

This global relation holds for $\Psi = \Psi_3$ in particular. In this case, the first integral in the left-hand side is by definition

$$\int_0^\infty e^{-(i(\lambda-1/\lambda)x/4)\widehat{\sigma_3}}\left(Q + \frac{1}{\lambda}\widetilde{Q}\right)\Psi_3(x,x,\lambda)\, dx = I - \Psi_3(0,0,\lambda).$$

To compute the second integral, we note that using (4.13), we have

$$\begin{aligned} i\int_0^\infty & e^{((\lambda+1/\lambda)y/4)\widehat{\sigma_3}}\left(Q - \frac{1}{\lambda}\widetilde{Q}\right)\Psi_3(iy,-iy,\lambda)\, dy = \\ &= i\int_0^\infty e^{((\lambda+1/\lambda)y/4)\widehat{\sigma_3}}\left(Q - \frac{1}{\lambda}\widetilde{Q}\right)\Psi_1(iy,-iy,\lambda)\, dy \cdot \Psi_1^{-1}(0,0,\lambda)\Psi_3(0,0,\lambda) = \\ &= \big(\Psi_1(0,0,\lambda) - I\big)\Psi_1^{-1}(0,0,\lambda)\Psi_3(0,0,\lambda) = \Psi_3(0,0,\lambda) - \Psi_1^{-1}(0,0,\lambda)\Psi_3(0,0,\lambda). \end{aligned}$$

Hence, condition (4.14) in this case yields

$$\Psi_1^{-1}(0,0,\lambda)\Psi_3(0,0,\lambda) = I, \quad \lambda \in \big((1),(3)\big),$$

and (4.13) then implies that the two eigenfunctions coincide in their common domain:

$$\Psi_1(z,\bar{z},\lambda) = \Psi_3(z,\bar{z},\lambda), \quad \lambda \in \big((1),(3)\big). \tag{4.15}$$

We now give the following definition.

Definition 2. The *spectral functions* $S_0(\lambda)$ and $S_1(\lambda)$ are the functions defined in terms of the particular solutions Ψ_3 and Ψ_1 by

$$S_0(\lambda) = \Psi_3(0,0,\lambda), \qquad S_1 = \Psi_1(0,0,\lambda). \tag{4.16}$$

More specifically, the spectral functions S_i are defined in terms of the solutions of the integral equations

$$\Psi_3(x,0,\lambda) = I - \int_0^\infty e^{-(i(\lambda-1/\lambda)\xi/4)\widehat{\sigma_3}} Q_0(\xi,0,\lambda)\Psi_3(\xi,0,\lambda)\, d\xi, \tag{4.17}$$

$$\Psi_1(0,y,\lambda) = I - \int_0^\infty e^{((\lambda+1/\lambda)\eta/4)\widehat{\sigma_3}} iQ_0(0,\eta,-\lambda)\Psi_1(0,\eta,\lambda)\, d\eta, \tag{4.18}$$

where Q_0 is defined by (4.6). Setting

$$\psi_3(x,\lambda) = \Psi_3(x,0,\lambda), \qquad \psi_1(y,\lambda) = \Psi_1(0,y,\lambda),$$

we now have $S_0(\lambda) = \psi_3(0,\lambda)$ and $S_1(\lambda) = \psi_1(0,\lambda)$. These functions are bounded:

$$S_0(\lambda) = \big(S_0^{(12)}, S_0^{(34)}\big), \qquad S_1(\lambda) = \big(S_1^{(14)}, S_1^{(23)}\big).$$

We can write the global relation in terms of the spectral functions as

$$S_0(\lambda) = S_1(\lambda), \quad \lambda \in \big((1),(3)\big). \tag{4.19}$$

This follows from identity (4.15) for Ψ_3 and Ψ_1 in these sectors.

Properties of the spectral functions. The trace property $\operatorname{tr} Q = \operatorname{tr} \widetilde{Q} = 0$ implies that

$$\det S_0 = \det S_1 = 1. \tag{4.20}$$

We set

$$Q_0(z,\bar z,\lambda) = Q(z,\bar z) + \frac{1}{\lambda}\widetilde{Q}(z,\bar z) = \frac{iq_z}{2}\sigma_1 - \frac{i}{4\lambda}\sin q\sigma_2 + \frac{i}{4\lambda}(1-\cos q)\sigma_3.$$

For real-valued q, Q_0 has the symmetry properties

$$Q_0(\lambda)_{22} = Q_0(-\lambda)_{11} = \overline{Q_0(\bar\lambda)_{11}}, \qquad Q_0(\lambda)_{12} = Q_0(-\lambda)_{21} = -\overline{Q_0(\bar\lambda)_{21}}.$$

Hence, any solution of system (4.1), (4.2) has the same symmetry properties, and we can assume that S_0 and S_1 have the forms

$$S_0(\lambda) = \begin{pmatrix} a_0(\lambda) & -\overline{b_0(\bar\lambda)} \\ b_0(\lambda) & \overline{a_0(\bar\lambda)} \end{pmatrix} = \begin{pmatrix} a_0(\lambda) & -b_0(-\lambda) \\ b_0(\lambda) & a_0(-\lambda) \end{pmatrix},$$

$$S_1(\lambda) = \begin{pmatrix} a_1(\lambda) & -\overline{b_1(\bar\lambda)} \\ b_1(\lambda) & \overline{a_1(\bar\lambda)} \end{pmatrix} = \begin{pmatrix} a_1(\lambda) & -b_1(-\lambda) \\ b_1(\lambda) & a_1(-\lambda) \end{pmatrix}.$$

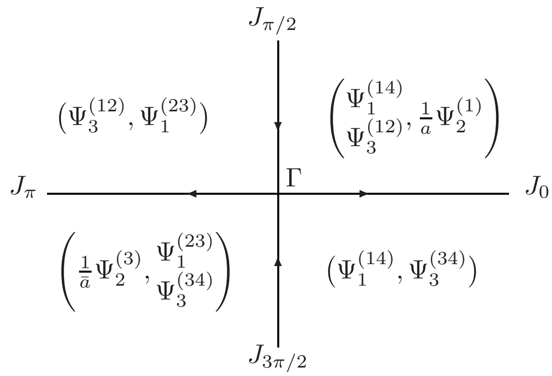

Fig. 2. Bounded eigenfunctions and the Riemann–Hilbert problem.

Global relation (4.14) becomes

$$a_0(\lambda) = a_1(\lambda), \qquad b_0(\lambda) = b_1(\lambda), \quad \lambda \in (1). \tag{4.21}$$

Hence, for $\lambda \in (1)$, we simply write $a(\lambda)$ and $b(\lambda)$ without any subscript. Similarly, for $\lambda \in (3)$, we write $a(-\lambda)$ and $b(-\lambda)$.

The Riemann–Hilbert problem. The analysis of the boundedness of each of the eigenfunctions Ψ_i shows that in each quadrant of the λ plane, there exist at least two of the column vectors defining the collection of matrix functions Ψ_j that are bounded and analytic in that quadrant. In the first and third quadrants, there are two eigenfunctions that can be selected for the respective first and second columns. Namely, we can form the matrices (see Fig. 2)

$$\left(\begin{Bmatrix} \Psi_3^{(12)} \\ \Psi_1^{(14)} \end{Bmatrix}, \Psi_2^{(1)} \right), \qquad \left(\Psi_3^{(12)}, \Psi_1^{(23)}\right),$$

$$\left(\Psi_2^{(3)}, \begin{Bmatrix} \Psi_3^{(34)} \\ \Psi_1^{(23)} \end{Bmatrix} \right), \qquad \left(\Psi_1^{(14)}, \Psi_3^{(34)}\right).$$

These matrices provide the solution of the direct problem. We note that they in fact coincide in the quadrants where two eigenfunctions are bounded. Explicitly, we have the identities for column vectors

$$\begin{aligned} \Psi_3^{(12)}(z,\bar z,\lambda) &= \Psi_1^{(14)}(z,\bar z,\lambda), \quad \operatorname{Re}\lambda \ge 0, \quad \operatorname{Im}\lambda \ge 0, \\ \Psi_3^{(34)}(z,\bar z,\lambda) &= \Psi_1^{(23)}(z,\bar z,\lambda), \quad \operatorname{Re}\lambda \le 0, \quad \operatorname{Im}\lambda \le 0. \end{aligned} \tag{4.22}$$

Equations (4.11)–(4.13) that relate the various analytic eigenfunctions can be rewritten in a form that uniquely determines a Riemann–Hilbert problem with jumps on the real and imaginary axes. Because of identity (4.22), the jumps on each of the four half-lines only involve either the first or the second column vector.

Indeed, we find

$$\Psi_-(z,\bar z,\lambda) = \Psi_+(z,\bar z,\lambda) J(z,\bar z,\lambda), \tag{4.23}$$

where using the notation $\bar{a}(\lambda)=\overline{a(\bar{\lambda})}$, we define the matrices $\Psi_\pm$ and J as

$$\Psi_+ = \begin{cases} \left(\Psi_3^{(12)}, \Psi_2^{(1)}/a(\lambda)\right), & \arg\lambda \in [0,\pi/2], \\ \left(\Psi_2^{(3)}/\bar{a}(\lambda), \Psi_3^{(34)}\right), & \arg\lambda \in [\pi, 3\pi/2], \end{cases}$$

$$\Psi_- = \begin{cases} \left(\Psi_3^{(12)}, \Psi_1^{(2)}\right), & \arg\lambda \in [\pi/2,\pi], \\ \left(\Psi_1^{(4)}, \Psi_3^{(34)}\right), & \arg\lambda \in [3\pi/2, 2\pi], \end{cases} \tag{4.24}$$

$$J(z,\bar{z},\lambda) = J^\alpha(z,\bar{z},\lambda), \quad \text{if } \arg\lambda = \alpha, \quad \alpha = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2},$$

and J^α is given by (2.8). The normalization is such that all the matrices J^α have a unit determinant.

For simplicity, we assume that $a(\lambda)\neq 0$ and the solution hence has no poles. Poles can be included by a standard procedure (see [3]). Rewriting the jump condition, we obtain

$$\Psi_+ - \Psi_- = \Psi_+ - \Psi_+ J = \Psi_+(I-J) \quad \Rightarrow \quad \Psi_+ - \Psi_- = \Psi_+\tilde{J}, \tag{4.25}$$

where $\tilde{J} = I - J$. This relation and asymptotic condition (4.4) uniquely determine a Riemann–Hilbert problem. The solution of this Riemann–Hilbert problem is given by

$$\Psi(z,\bar{z},\lambda) = I + \frac{1}{2\pi i}\int_\Gamma \frac{\Psi_+(z,\bar{z},\lambda')\tilde{J}(z,\bar{z},\lambda')}{\lambda'-\lambda}\,d\lambda', \quad \lambda \in \mathbb{R}\cup i\mathbb{R}. \tag{4.26}$$

This immediately implies

$$\psi_1 = -\frac{1}{2\pi i}\int_\Gamma \Psi_+(z,\bar{z},\lambda)\tilde{J}(z,\bar{z},\lambda)\,d\lambda, \qquad (\psi_1)_{12} = \lim_{\lambda\to\infty}(\lambda\Psi_{12}). \tag{4.27}$$

The characterization of $q(z,\bar{z})$. The last step in the construction is to derive an expression for q in terms of the solution of the Riemann–Hilbert problem. Using (4.4) in (4.1) (the first ODE in the Lax pair), we find that the λ^0 coefficient gives

$$-\frac{i}{4}[\sigma_3,\psi_1] = i\frac{q_z}{2}\sigma_1 \quad \Rightarrow \quad q_z = -(\psi_1)_{12} = -\lim_{\lambda\to\infty}(\lambda\Psi_{12}),$$

while for (4.2), the $(1,1)$ element of the λ^{-1} coefficients yields

$$\cos q(z,\bar{z}) = 1 + 4i\frac{\partial(\psi_1)_{11}}{\partial\bar{z}}.$$

In view of the explicit expression for $\tilde{J}$, this gives expression (2.10) for $\cos q(x,y)$. This concludes the proof of Proposition 1.

5. Conclusions

The formal procedure outlined in this paper represents the first step toward effectively solving boundary value problems for elliptic sine-Gordon equation (1.1) posed in a convex polygon. The main features of this procedure are solving the two ODEs in the Lax pair simultaneously and deriving and analyzing the global relation. In contrast to the case of evolution PDEs, such as the sine-Gordon equation $q_{tt} - q_{xx} = \sin q$, the global relation holds in two-dimensional region of the complex spectral plane, and this implies that the eigenfunctions of the spectral problem coincide in this region. The important consequence of this property of the spectral problem is that there are only two unknown spectral functions to be characterized, $a(\lambda)$ and $b(\lambda)$. But again in contrast to the evolution case, where the initial conditions fully characterize one of the spectral functions, no spectral function is explicitly known in the present case. It seems that to characterize the spectral data, we must determine an auxiliary Riemann–Hilbert problem for the functions $a(\lambda)$ and $b(\lambda)$ (also see [9]). When the resulting Riemann–Hilbert problem can be solved explicitly, we say that the associated boundary conditions are *linearizable*. The solution in this case will be presented in a subsequent paper. For general boundary conditions, determining this auxiliary Riemann–Hilbert problem is more complicated, and the analysis is in process.

Acknowledgments. This research was supported by the EPSRC (Grant No. EP/E022960/1).

REFERENCES

1. E. S. Gutshabash and V. D. Lipovskii, *J. Math. Sci.*, **68**, 197–201 (1994).
2. A. B. Borisov and V. V. Kiseliev, *Inverse Problems*, **5**, 959–982 (1989).
3. A. S. Fokas, *A Unified Approach to Boundary Value Problems* (CBMS-NSF Reg. Conf. Ser. in Appl. Math., Vol. 78), SIAM, Philadelphia (2008).
4. A. S. Fokas, *Proc. Roy. Soc. London Ser. A*, **453**, 1411–1443 (1997).
5. A. S. Fokas, *Comm. Math. Phys.*, **230**, 1–39 (2002).
6. A. S. Fokas, *Nonlinearity*, **17**, 1521–1534 (2004).
7. A. S. Fokas, *Proc. Roy. Soc. London Ser. A*, **457**, 371–393 (2001).
8. A. S. Fokas and E. Spence, "Novel analytical and numerical methods for elliptic boundary value problems," Preprint (2008).
9. Y. A. Antipov and A. S. Fokas, *Math. Proc. Cambridge Philos. Soc.*, **138**, 339–365 (2005).