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QUANTUM FOURIER TRANSFORM AND TOMOGRAPHIC RÉNYI ENTROPIC INEQUALITIES

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We show that the Rényi entropy associated with spin tomograms of quantum states satisfies new inequalities that depend on the quantum Fourier transform. We obtain the limiting inequality for the von Neumann entropy of quantum spin states and a new kind of entropy associated with the quantum Fourier transform. We consider possible connections with subadditivity and strong subadditivity conditions for tomographic entropies and von Neumann entropies.

Keywords: uncertainty relation, entropy, quantum tomography, quantum Fourier transform

1. Introduction

We previously formulated an approach to quantum information theory based on introducing the so-called probability representation [1]. In this approach, the elements of quantum information theory (such as qubits, qudits, and the operators describing qudit states and relations between them) are given in the form of functions called *operator symbols*. For the operator symbols, the product rule called the star product is determined using an integral nonlocal kernel [2]. For both pure and mixed qudit states, the density operator symbols are standard probability distribution functions. Because the qudit states for multipartite systems are described by standard probability distributions, all the characteristics of the distributions, including Shannon entropy [3] and Rényi entropy [4], can be used to introduce entropies corresponding to the density operator symbols (such as operator-symbol Rényi entropy and operator-symbol relative entropy) into quantum information theory. Because Shannon entropy is the limiting case of Rényi entropy, the corresponding analogue of Shannon entropy determined by the density operator symbol and the properties of this entropy can be obtained in quantum information theory in the framework of the proposed probability representation.

Quantum mechanics and quantum information theory have a certain fundamental feature distinguishing them from their classical counterparts: uncertainty relations. The uncertainty relations by Heisenberg [5] and Schrödinger [6] and Robertson [7], [8] written for conjugate variables like positions and momenta were also accompanied by the so-called entropic uncertainty relations. The entropic uncertainty relations for continuous variables were written in the form of inequalities for the Shannon entropy associated with the position and momentum probability densities in [9]–[11].

In the approach based on the probability representation, the subadditivity and strong subadditivity conditions in quantum information theory were obtained for the probability distributions describing qudit quantum states (so-called spin tomograms) [1]. Some relations between these inequalities and the subadditivity and strong subadditivity conditions were also clarified. Indeed, in the approach based on the probability representation, the tomographic map of the unitary group $U(n)$ onto the simplex was con-

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structed, and the notion of Shannon entropy, Rényi entropy, and other entropies for the unitary group were introduced using this map.

The entropic uncertainty relations for finite-dimensional quantum systems were obtained in the form of inequalities for Shannon entropies associated with the probability distributions related to measuring noncommuting observables [12]–[16]. The uncertainty relations for the Rényi entropy for the position and momentum distributions and also for finite-dimensional systems with measurable observables related by the quantum Fourier transform were obtained in [17]. For bipartite and tripartite systems, there exist the known inequalities for von Neumann entropy called the subadditivity and strong subadditivity conditions [18], [19].

Our aim here is to extend the study of entropic inequalities, such as the subadditivity and strong subadditivity conditions, obtained previously in [1] in the framework of the probability representation for quantum information theory and to find new entropic inequalities for spin tomograms analogous to the entropic inequalities discussed in [12]–[17]. For continuous variables, the entropic inequalities for quantum symplectic tomograms were discussed in [20]–[25]. The essential aspect of the new inequalities is their close relation to the properties of the quantum Fourier transform, which was discussed in [26] and has been used in quantum information theory (see, e.g., [27]–[29]). The quantum Fourier transform is an important element of the theory and plays a key role in quantum computing and quantum information processing.

We obtain entropic inequalities that impose certain constraints on unitary spin tomograms relating them to the number $\log N$, where N is the Hilbert space dimension. The number $\log N$ has appeared in entropic inequalities in [12]–[14], [16], but a new element here is that $\log N$ is directly associated with the tomographic probability distribution determining the qudit state in the probability representation for quantum information theory.

2. Spin tomograms

We can interpret a given N-dimensional Hilbert space of a spin system as either the state space for one particle with the spin $j=(N-1)/2$ or the space of a multipartite spin system with $j_1=(n_1-1)/2$, $j_2=(n_2-1)/2$, $\dots$, $j_M=(n_M-1)/2$ in the case of the product representation of the number as $N=n_1n_2\cdots n_M$. The $N\times N$ density matrix ρ of the quantum state can be represented by the unitary tomogram of the spin state [30].

In the case of a spin state with $j=(N-1)/2$, the tomogram is defined by the relation

$$w(m,u)=\langle m|u^\dagger\rho u|m\rangle, \tag{1}$$

where ρ is the density matrix, u is an $N\times N$ unitary matrix, and the half-integers $m=-j,-j+1,\dots,j$ are values of the spin projection on the z axis. The tomogram $w(m,u)$ is a nonnegative probability distribution function of a random spin variable satisfying the normalization condition

$$\sum_{m=-j}^{j} w(m,u)=1 \tag{2}$$

and the equality

$$\int w(m,u)\,du=1, \tag{3}$$

where du is the Haar measure on the unitary group with the normalization $\int du=1$. The important property of tomogram (1) is that it is bijectively related to the density matrix ρ, i.e., $\rho\leftrightarrow w(m,u)$ [1]. This means that the quantum state is known if the tomogram is known [31], [32].

3. Quantum Fourier transform

The $N \times N$ symmetric unitary matrix F with the elements $F_{jk} = e^{2\pi i jk/N}/\sqrt{N}$, $j,k = 0,1,\ldots,N-1$, which are characters of an irreducible representation of the cyclic group C_N, can be used to provide an invertible map of the normalized complex vector $\vec{a}$ with the components a_k to the complex vector $\vec{a}^{\,(f)}$ with the components $a_k^{(f)}$:

$$a_k^{(f)} = \sum_{j=0}^{N-1} F_{kj} a_j, \qquad a_k = \sum_{j=0}^{N-1} (F^\dagger)_{kj} a_j^{(f)}. \tag{4}$$

The matrix F with the elements F_{kj} satisfies the equality

$$F^N = 1. \tag{5}$$

Map (4) is called the quantum Fourier transform for the cyclic group C_N.

If we use the index $m = -j, -j+1, \ldots, j$ for the spin projection, then we can define the quantum Fourier transform operator $\widehat{F}$ as

$$\widehat{F}|m\rangle = \sum_{m'=-j}^{j} F_{m'm}|m'\rangle, \tag{6}$$

where the symmetric matrix

$$F_{m'm} = \langle m'|\widehat{F}\,|m\rangle \tag{7}$$

has the form

$$F_{m'm} = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & a & a^2 & \cdots & a^{N-1} \\ 1 & a^2 & a^4 & \cdots & a^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a^{N-1} & a^{N-2} & \cdots & a \end{pmatrix}, \qquad a = e^{2\pi i/N}. \tag{8}$$

Hence, the unitary quantum Fourier transform operator is

$$\widehat{F} = \sum_{m=-j}^{j} \sum_{m'=-j}^{j} F_{m'm}|m'\rangle\langle m|. \tag{9}$$

In view of (5), we have $(\widehat{F})^N = \hat{1}$, where $\hat{1}$ is the identity operator.

4. Shannon and Rényi tomographic entropies

Using standard definitions in probability theory, we can introduce the Shannon tomographic entropy [1], [3], [33] and the Rényi tomographic entropy [1], [4]. The Shannon tomographic entropy determined by the tomographic density operator symbol is the function on the unitary group

$$H_u = -\sum_{m=-j}^{j} w(m,u)\log w(m,u). \tag{10}$$

The Rényi tomographic entropy determined by the tomographic density operator symbol is also a function on the unitary group but depends on an extra parameter,

$$R_u = \frac{1}{1-q}\log \sum_{m=-j}^{j} \big(w(m,u)\big)^q. \tag{11}$$

In the case of two spin tomograms $w_1(m,u)$ and $w_2(m,u)$, we define the relative q-entropy determined by the tomographic density operator symbol

$$H_q\big(w_1(u)|w_2(u)\big) = -\sum_{m=-j}^{j} w_1(m,u)\log_q \frac{w_2(m,u)}{w_1(m,u)}, \tag{12}$$

$$\log_q x = \frac{x^{1-q}-1}{1-q}, \quad x>0, \quad q>0, \qquad \log_{q\to 1} x = \log x. \tag{13}$$

The relative tomographic q-entropy is nonnegative.

As $q \to 1$, we have $R_u \to H_u$, and the tomographic relative q-entropy becomes the relative entropy,

$$H\big(w_1(u)|w_2(u)\big) = -\sum_{m=-j}^{j} w_1(m,u)\log \frac{w_2(m,u)}{w_1(m,u)}. \tag{14}$$

As shown in [1], the minimum of the tomographic Rényi entropy over the unitary group is equal to the quantum Rényi entropy

$$\min R_u = \frac{1}{1-q}\log \operatorname{Tr}\rho^q. \tag{15}$$

Relative entropy (12) is also a nonnegative function for any admissible deformation parameter q.

The minimum of the entropy H_u given by (10) over the unitary group is equal to the von Neumann entropy [1], [33], i.e.,

$$\min H_u = -\operatorname{Tr}\rho\log\rho. \tag{16}$$

5. Shannon entropic inequalities in measuring noncommutative observables

We consider known entropic inequalities [12]–[16] appearing in the problem of measuring two observables $\hat{A}$ and $\widehat{B}$ in a finite Hilbert space. Let the spectral decompositions of the Hermitian operators $\hat{A}$ and $\widehat{B}$ be

$$\hat{A} = \sum_k A_k|a_k\rangle\langle a_k|, \qquad \widehat{B} = \sum_k B_k|b_k\rangle\langle b_k|, \quad k=1,\dots,N, \tag{17}$$

where A_k and B_k are eigenvalues of the observables and $|a_k\rangle$ and $|b_k\rangle$ are their orthonormal systems of eigenvectors.

For a pure state $|\psi\rangle$, we have two probability distributions

$$p_k = |\langle a_k|\psi\rangle|^2, \qquad q_k = |\langle b_k|\psi\rangle|^2. \tag{18}$$

The corresponding Shannon entropies related to these two distributions are

$$H_p = -\sum_k p_k\log p_k, \qquad H_q = -\sum_k q_k\log q_k. \tag{19}$$

They satisfy the inequality obtained in [34]

$$H_p + H_q \ge -2\log\frac{1}{2}(1+c), \tag{20}$$

where the parameter c is determined by the maximum value of the scalar product modulus,

$$c = \max_{j,k}|\langle a_j|b_k\rangle|. \tag{21}$$

The stronger inequality

$$H_p + H_q \geq -2\log c \tag{22}$$

was proved in [12] (also see [35], where a proof was attempted).

For observables $\hat{A}$ and $\widehat{B}$ with eigenvectors having mutually unbiased bases $|a_k\rangle$ and $|b_k\rangle$ [14], [26],

$$\max|\langle a_i|b_j\rangle| = \frac{1}{\sqrt{N}}, \tag{23}$$

inequality (22) is $H_p + H_q \geq \log N$ [14]. Hence, the Hilbert space dimension N appears in the entropic inequality.

The problem of mutually unbiased bases is related to the geometry of finite Hilbert spaces [36], [37]. It was widely discussed in connection with constructing the Wigner function for a finite Hilbert space and in quantum cryptography [27], [38]–[41].

The entropic inequalities for the Shannon entropy can also be obtained in studying the problem of measuring several noncommutative observables with orthonormal sets of eigenvectors satisfying condition (23) (see [14], [15]). In [13], the entropic inequalities for the Tsallis entropy for continuous variables were obtained based on the Sobolev inequalities, and the analogous entropic uncertainty relations for the Rényi entropy for a finite Hilbert space and continuous variables were presented in [17].

6. Known inequalities for bipartite and tripartite systems

The entropies determinable for a density operator symbol satisfy some known inequalities obtained in [1]. For example, if the spin system consists of the spins j_1 and j_2, then the basis in the tensor product space is

$$|m_1 m_2\rangle = |m_1\rangle|m_2\rangle. \tag{24}$$

In this case, the tomogram is the joint probability distribution of two random spin projections $m_1 = -j_1, -j_1+1, \ldots, j_1$ and $m_2 = -j_2, -j_2+1, \ldots, j_2$ depending on the $(2j_1{+}1)(2j_2{+}1)\times(2j_1{+}1)(2j_2{+}1)$ unitary matrix u. The tomogram is

$$w(m_1, m_2, u) = \langle m_1 m_2|u^\dagger \rho(1,2) u|m_1 m_2\rangle, \tag{25}$$

where $\rho(1,2)$ is the density matrix of the bipartite system with the elements

$$\rho(1,2)_{m_1 m_2, m_1' m_2'} = \langle m_1 m_2|\rho(1,2)|m_1' m_2'\rangle. \tag{26}$$

For the tomogram, we can introduce the Shannon entropy

$$H_{12}(u) = -\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2} w(m_1, m_2, u)\log w(m_1, m_2, u), \tag{27}$$

which satisfies the subadditivity condition for all elements of the unitary group,

$$H_{12}(u) \leq H_1(u) + H_2(u), \tag{28}$$

where $H_1(u)$ and $H_2(u)$ are the Shannon entropies associated with the subsystem tomograms

$$w_1(m_1, u) = \sum_{m_2=-j_2}^{j_2} w(m_1, m_2, u), \qquad w_2(m_2, u) = \sum_{m_1=-j_1}^{j_1} w(m_1, m_2, u) \tag{29}$$

by the relation

$$H_k(u) = -\sum_{m_k=-j_k}^{j_k} w_k(m_k,u)\log w_k(m_k,u), \quad k=1,2. \tag{30}$$

In view of the relation between the von Neumann and tomographic entropies, inequality (28) implies the known inequality [1], which is the subadditivity condition for the corresponding von Neumann entropy for a bipartite system

$$S_{12} \le S_1 + S_2, \tag{31}$$

where

$$S_k = -\operatorname{Tr}\rho_k\log\rho_k, \qquad k=1,2, \qquad \rho_1 = -\operatorname{Tr}_2\rho(1,2), \qquad \rho_2 = -\operatorname{Tr}_1\rho(1,2).$$

For a tripartite spin system with the spins j_1, j_2, and j_3 and the density matrix $\rho(1,2,3)$, the spin tomogram can be written as

$$w(m_1,m_2,m_3,u) = \langle m_1m_2m_3|u^\dagger\rho(1,2,3)u|m_1m_2m_3\rangle. \tag{32}$$

The Shannon entropy $H_{123}(u)$ is associated with this tomogram. This entropy satisfies the strong subadditivity condition on the unitary group [1]

$$H_{123}(u) + H_2(u) \le H_{12}(u) + H_{23}(u), \tag{33}$$

where

$$H_{123}(u) = -\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\sum_{m_3=-j_3}^{j_3} w(m_1,m_2,m_3,u)\log w(m_1,m_2,m_3,u)$$

and the entropies $H_{12}(u)$, $H_{23}(u)$, and $H_2(u)$ are defined using the projected tomograms

$$\begin{aligned} w_{12}(m_1,m_2,u) &= \sum_{m_3=-j_3}^{j_3} w(m_1,m_2,m_3,u),\\ w_{23}(m_2,m_3,u) &= \sum_{m_1=-j_1}^{j_1} w(m_1,m_2,m_3,u),\\ w_2(m_2,u) &= \sum_{m_1=-j_1}^{j_1} w_{12}(m_1,m_2,u). \end{aligned} \tag{34}$$

New inequality (33) does not contradict the known strong subadditivity condition for the von Neumann entropy [18], [19]

$$S_{123} + S_2 \le S_{12} + S_{23}, \qquad S_{123} = -\operatorname{Tr}\rho_{123}\log\rho_{123}, \tag{35}$$

and S_{12} and S_{23} are the von Neumann entropies for the reduced density matrices $\rho(1,2) = \operatorname{Tr}_3\rho(1,2,3)$ and $\rho(2,3) = \operatorname{Tr}_1\rho(1,2,3)$.

Inequalities (28) and (33) are new inequalities for composite finite-dimensional quantum systems obtained in [1].

7. New inequalities for Rényi tomographic entropies

In this section, we continue the study of tomographic entropies following the analysis in [1] and derive new inequalities for spin tomographic entropies related to the quantum Fourier transform. For continuous conjugate variables (position and momentum), the inequalities for the Rényi entropy associated with the probability densities in position and momentum were derived in [17]. These inequalities were used to obtain new integral inequalities for symplectic and optical tomograms in [20], [22]–[25]. The analogue of the uncertainty relation for the Rényi entropies for an N-dimensional Hilbert space was obtained in [17] in the form

$$\frac{1}{1-\alpha}\log\sum_{k=1}^{N}\widetilde{p}_k^{\alpha}+\frac{1}{1-\beta}\log\sum_{l=1}^{N}p_l^{\beta}\geq\log N, \tag{36}$$

where

$$\widetilde{p}_k=|\widetilde{a}_k|^2,\qquad p_l=|a_l|^2,\qquad \frac{1}{\alpha}+\frac{1}{\beta}=2,$$

and the complex numbers $\widetilde{a}_k$ and a_l are related by the quantum Fourier transform

$$\widetilde{a}_k=\frac{1}{\sqrt{N}}\sum_{l=1}^{N}\exp\biggl(\frac{2\pi ikl}{N}\biggr)a_l. \tag{37}$$

Below, we use inequalities (36) to obtain new inequalities for the Shannon and Rényi entropies associated with unitary spin tomograms. The spin tomogram of a state with the density operator ρ can be represented as a probability column vector on the unitary group with the components $w_m(u)$. We introduce another N-vector with the components $p_m(u)=\sqrt{w_m(u)}$. Applying inequality (36) to these vectors and taking the relation

$$\biggl|\sum_{m'=-j'}^{j}F_{mm'}\sqrt{w(m',u)}\biggr|=\sqrt{w_F(m,u)} \tag{38}$$

into account, where $F_{mm'}$ is given by (8) and $w_F(m,u)$ is the probability distribution, we obtain the inequality

$$\frac{1}{1-\alpha}\log\sum_{m=-j}^{j}w(m,u)^{\alpha}+\frac{1}{1-\beta}\log\sum_{m=-j}^{j}w_F(m,u)^{\beta}\geq\log N. \tag{39}$$

Using the definition of the spin tomogram for a pure state $|\psi\rangle$, we also obtain another, similar inequality

$$\frac{1}{1-\alpha}\log\sum_{m=-j}^{j}w(m,u)^{\alpha}+\frac{1}{1-\beta}\log\sum_{m=-j}^{j}w(m,Fu)^{\beta}\geq\log N, \tag{40}$$

where F is the quantum Fourier transform matrix. We can conjecture that inequality (40) also holds for a mixed state.

For Rényi entropy (11), we have the inequality for each unitary matrix

$$R_\alpha(u)+R_\beta(Fu)\geq\log N. \tag{41}$$

The unitary spin tomogram of a particle with the spin j for the state with the $N\times N$ density matrix ρ, where $N=2j+1$, must satisfy inequality (40). As $\alpha\to 1$ and $\beta\to 1$, we obtain the inequalities for the Shannon entropy of the spin state

$$H(u)+H(Fu)\geq\log N. \tag{42}$$

Moreover, the inequality

$$H(u) + H_F(u) \geq \log N \tag{43}$$

holds, where $H_F(u)$ is the Shannon entropy associated with the probability distribution $w_F(m,u)$.

For the minimum value of the Shannon entropy realized for the unitary matrix u_0, we have the von Neumann entropy $H(u_0) = S_{\rm vN}$. Inequality (42) written for u_0,

$$H(u_0) + H(Fu_0) \geq \log N, \tag{44}$$

provides the inequality for the von Neumann entropy

$$S_{\rm vN} + S(Fu_0) \geq \log N, \tag{45}$$

where $S(Fu_0)$ is a new entropy. It has the following physical meaning. If the density operator of the quantum spin state is given in the form of the spectral decomposition

$$\hat{\rho} = \sum_{q=-j}^{j} \lambda_q |q\rangle\langle q|, \tag{46}$$

then we can identify the eigenstate $|q\rangle$ of the density operator $\hat{\rho}$ with the "position" state. In the approach with mutually unbiased bases and the Wigner function for a finite Hilbert space [26], [27], [38]–[42], the states $|p\rangle = \widehat{F}|q\rangle$, where $\widehat{F}$ is the Fourier transform operator, are interpreted as "momentum" eigenstates. The matrix elements $\langle p|\widehat{F}|q\rangle = F_{pq}$ provide the matrix F, which coincides with the Fourier transform matrix.

We thus have the interpretation of the new inequality the same as in the case of continuous variables. The new entropy $S(Fu_0)$ in (45) is the Shannon entropy for the "momentum" distribution if we identify the standard von Neumann entropy with the Shannon entropy for the "position" distribution.

We consider the example of a qubit state with the density matrix

$$\rho = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{47}$$

The position operator $\hat{q}$ is the matrix σ_z, and the momentum operator $\hat{p}$ is the matrix σ_x. The two position eigenvectors $|q\rangle$ and two momentum eigenvectors $|p\rangle$ are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The matrix F is

$$F = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{48}$$

and the matrix $u_0 = 1$. Inequality (45) is saturated because $S_{\rm vN} = 0$, $S(F) = \log 2$, $S_{\rm vN} + S(F) = \log 2 \geq \log 2$. The inequality for the Rényi entropy is also saturated,

$$R_\alpha(u_0) + R_\beta(Fu_0) = \log 2 \geq \log 2. \tag{49}$$

In the considered example, the vectors $|q\rangle$ and $|p\rangle$ form what is called *mutually unbiased bases* [39]–[42].

We note that there are Shannon entropic uncertainty relations for distributions associated with set of mutually unbiased bases [15] and with pairs of orthogonal bases [12]. In the case where mutually unbiased bases are related by the quantum Fourier transform, our result (44) coincides with the result in [12].

Group average Shannon and Rényi entropies were introduced in [1]. Because the Haar measure is invariant, we can conclude that the group average Shannon tomographic entropy satisfies the inequality

$$\overline{H} = \int H(u)\,du \geq \frac{1}{2}\log N. \tag{50}$$

For group average Rényi entropy (11), we also have

$$\overline{R}_{\alpha\beta} = \int R_\alpha(u)\,du + \int R_\beta(u)\,du \geq \log N, \qquad \frac{1}{\alpha} + \frac{1}{\beta} = 2. \tag{51}$$

To illustrate the obtained inequalities, we now discuss the mixed state of a qubit with the diagonal density matrix

$$\rho = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad a + b = 1. \tag{52}$$

Inequality (42) can then be interpreted as follows. The von Neumann entropy of this state is

$$S_{\mathrm{vN}} = -a\log a - b\log b. \tag{53}$$

The density matrix acted on by quantum Fourier transform (48) becomes

$$F^\dagger \rho F = \begin{pmatrix} 1/2 & (a-b)/2 \\ (a-b)/2 & 1/2 \end{pmatrix}. \tag{54}$$

Its tomographic entropy satisfies the equality $H(Fu_0) = \log 2$, $u_0 = 1$. Inequality (42) becomes

$$-a\log a - b\log b + \log 2 \geq \log 2, \tag{55}$$

and this means that the von Neumann entropy is nonnegative. But inequality (43) gives a better estimate because the number $\log 2$ is replaced with a smaller number. In fact, the tomographic probability vector of the qubit state

$$\vec{w} = \begin{pmatrix} a \\ b \end{pmatrix} \tag{56}$$

is associated with the probability amplitude vector with positive components

$$\vec{W} = \begin{pmatrix} \sqrt{a} \\ \sqrt{b} \end{pmatrix}. \tag{57}$$

Taking the quantum Fourier transform of this vector, we then obtain the column vector

$$\vec{W}_F = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{a} + \sqrt{b} \\ \sqrt{a} - \sqrt{b} \end{pmatrix}. \tag{58}$$

The probability distribution vector associated with this probability amplitude vector is

$$\vec{w}_F = \begin{pmatrix} 1/2 + \sqrt{ab} \\ 1/2 - \sqrt{ab} \end{pmatrix}. \tag{59}$$

We apply the inequality relating Shannon entropies to the two vectors (57) and (59) and thus obtain

$$-a\log a - b\log b - \left(\frac{1}{2}+\sqrt{ab}\right)\log\left(\frac{1}{2}+\sqrt{ab}\right) - \left(\frac{1}{2}-\sqrt{ab}\right)\log\left(\frac{1}{2}-\sqrt{ab}\right) \geq \log 2 \tag{60}$$

or

$$S_{\rm vN} - \left(\frac{1}{2}+\sqrt{ab}\right)\log\left(\frac{1}{2}+\sqrt{ab}\right) - \left(\frac{1}{2}-\sqrt{ab}\right)\log\left(\frac{1}{2}-\sqrt{ab}\right) \geq \log 2. \tag{61}$$

This inequality is not entirely obvious, although we know that $S_{\rm vN} \geq 0$.

Some inequalities for the unitary matrix can be obtained. We consider the $N\times N$ unitary matrix u_{jk}. We have the inequality

$$-\sum_{j=1}^{N}\left(|u_{jk}|^2\log|u_{jk}|^2 + |(Fu)_{jk}|^2\log|(Fu)_{jk}|^2\right) \geq \log N \tag{62}$$

or

$$-\sum_{j=1}^{N}\sum_{k=1}^{N}\left(|u_{jk}|^2\log|u_{jk}|^2 + |(Fu)_{jk}|^2\log|(Fu)_{jk}|^2\right) \geq N\log N, \tag{63}$$

where F_{jk} is the Fourier transform matrix. Integrating inequality (62) over the unitary group with the Haar measure normalized as in (3), we obtain the inequality

$$-\int\sum_{j=1}^{N}|u_{jk}|^2\log|u_{jk}|^2\,du \geq \frac{1}{2}\log N. \tag{64}$$

We demonstrated in the example of a qubit that for tomograms of spin states related by the quantum Fourier transform, we have constraints in the form of inequalities for Shannon tomographic entropies. Analogous constraints for Rényi tomographic entropies can also be demonstrated.

8. Conclusions

We showed that there exist several inequalities for Shannon and Rényi entropies associated with spin quantum state tomograms. These inequalities provide new relations for information theory in addition to the known subadditivity and strong subadditivity conditions. The physical and information theory meaning of the obtained inequalities needs further study.

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