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THE NUMBER OF BOUND STATES OF A ONE-PARTICLE HAMILTONIAN ON A THREE-DIMENSIONAL LATTICE

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We consider the Hamiltonian $\hat{h}_{\mu\lambda}$, $\mu, \lambda \geq 0$, describing the motion of one quantum particle on a three-dimensional lattice in an external field. We investigate the number of eigenvalues and their arrangement depending on the value of the interaction energy for $\mu \geq 0$ and $\lambda \geq 0$.

Keywords: one-particle Hamiltonian, continuous spectrum, virtual level, eigenvalue, Birman—Schwinger operator, Fredholm determinant

1. Introduction

The importance of threshold virtual levels for Hamiltonians of two-particle systems and their role in determining the number of bound states for Hamiltonians of systems of three quantum particles moving in the Euclidean space was first indicated by Efimov in [1], [2]. In [3], [4], the notion of virtual levels was introduced mathematically rigorously and was used to prove that the number of three-particle bound states is infinite (the Efimov effect).

In [5], the role of virtual levels was studied for three-particle Hamiltonians restricted to certain subspaces satisfying symmetry conditions. In particular, it was proved that the number of bound states for Hamiltonians of systems of three identical particles (fermions) is finite. In [6], [7], virtual levels of discrete two-particle Schrödinger operators $h(k)$, $k \in \mathbb{T}^3$, on the lattice $\mathbb{Z}^3$ in the case $k = 0 \in \mathbb{T}^3$ and their role in determining the number of three-particle bound states for discrete three-particle Schrödinger operators $H(K)$, $K \in \mathbb{T}^3$, associated with Hamiltonians of systems of three arbitrary identical particles on the lattice $\mathbb{Z}^3$ were investigated.

In [8], [9], the spectral properties were studied for one-particle Hamiltonians h describing the motion of one quantum particle in a potential field $\hat{v}$ and of two-particle discrete Schrödinger operators $h(k)$, $k \in \mathbb{T}^\nu$, associated with Hamiltonians of systems of two arbitrary identical particles on a lattice that interact via short-range pair potentials. In [10], the existence of bound states for some values of the quasimomentum of two particles was investigated for a certain class of Gibbs-field transfer matrices. In [8], for a wide class of discrete two-particle Schrödinger operators $h(k)$,$k \in \mathbb{T}^d$ $(d \geq 3)$, it was found that if the two-particle operator $h(0)$ has a virtual level or an eigenvalue on the threshold $z = \varepsilon_{\min}(0)$ of the essential spectrum, then the operator $h(k)$ has an eigenvalue below the threshold of the essential spectrum for all $k \neq 0$.

Here, we consider the Hamiltonian $\hat{h}_{\mu\lambda}$, $\mu, \lambda \geq 0$, describing the motion of one quantum particle on a three-dimensional lattice in an external field. We completely investigate the dependence of the number of eigenvalues of this operator on the interaction energy for $\mu \geq 0$ and $\lambda \geq 0$. We show that all eigenvalues arise either from a threshold virtual level (resonance) or from threshold eigenvalues under a variation of

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the interaction energy. We also describe the sets of parameter values $\mu, \lambda \geq 0$ for which the Hamiltonian $h_{\mu\lambda}$ has a virtual level, an eigenvalue, or both a virtual level and an eigenvalue. The set of values of the interaction energy is described (curves and a point) for which the threshold $z = 0$ is (a) a regular point of the continuous spectrum, (b) a virtual level or a two-fold eigenvalue, or (c) both a virtual level and an eigenvalue of $h_{\mu\lambda}$.

We exactly describe the interaction-energy range where the Hamiltonian $h_{\mu\lambda}$ either has no eigenvalues or has one, two, three, or four eigenvalues lying below the lower threshold. Moreover, we prove that the first negative eigenvalue of the Hamiltonian $h_{\mu\lambda}$ arises only from a threshold virtual level (resonance) under a variation of the interaction energy. This result for the continuous two-particle Schrödinger operator was revealed by Newton (see p. 1353 in [11]) and proved by Tamura (see Lemma 1.1 in [4]) using a result by Simon [12].

We note that the discrete two-particle Schrödinger operator $h(0)$ is unitarily equivalent to the one-particle Hamiltonian h and studying the spectral properties of one-particle operators therefore plays a special role in the spectral theory of many-particle operators on the lattice $\mathbb{Z}^d$, $d = 1, 2, \ldots$.

2. Coordinate representation for the one-particle Hamiltonian

Let $\ell^2(\mathbb{Z}^3)$ be the Hilbert space of square summable functions on the three-dimensional integer lattice $\mathbb{Z}^3$. In the coordinate representation, the free Hamiltonian of one quantum particle moving on the lattice $\mathbb{Z}^3$ is associated with a bounded self-adjoint operator acting in the space $\ell^2(\mathbb{Z}^3)$ according to the formula

$$(\hat{h}_0\widehat{\varphi})(x) = \sum_{s\in\mathbb{Z}^3} \hat{\varepsilon}(x-s)\widehat{\varphi}(s), \quad \widehat{\varphi} \in \ell^2(\mathbb{Z}^3),$$

where the function $\hat{\varepsilon}$ is defined on the lattice $\mathbb{Z}^3$ by the relations

$$\hat{\varepsilon}(s) = \begin{cases} 3 & \text{for } |s| = 0, \\ -1/2 & \text{for } |s| = 1, \\ 0 & \text{for } |s| > 1, \end{cases}$$

$s = (s^{(1)}, s^{(2)}, s^{(3)}) \in \mathbb{Z}^3$, $|s| = |s^{(1)}| + |s^{(2)}| + |s^{(3)}|$.

In the coordinate representation, the (total) one-particle Hamiltonian in a potential field $\hat{v}_{\mu\lambda}$ is defined as a bounded perturbation of the free Hamiltonian $\hat{h}_0$,

$$\hat{h}_{\mu\lambda} = \hat{h}_0 - \hat{v}_{\mu\lambda},$$

where $\hat{v}_{\mu\lambda}$ is the operator of multiplication by a function $\hat{v}_{\mu\lambda}$ in $\ell^2(\mathbb{Z}^3)$, i.e.,

$$(\hat{v}_{\mu\lambda}\widehat{\varphi})(x) = \hat{v}_{\mu\lambda}(x)\widehat{\varphi}(x), \quad \widehat{\varphi} \in \ell^2(\mathbb{Z}^3).$$

The function $\hat{v}_{\mu\lambda}$ is defined on $\mathbb{Z}^3$ as

$$\hat{v}_{\mu\lambda}(s) = \begin{cases} \mu & \text{for } |s| = 0, \\ \lambda/2 & \text{for } |s| = 1, \\ 0 & \text{for } |s| > 1, \end{cases}$$

where the numbers $\mu \geq 0$ and $\lambda \geq 0$ are not simultaneously zero. The operator $\hat{h}_{\mu\lambda}$ is a bounded self-adjoint operator in the Hilbert space $\ell^2(\mathbb{Z}^3)$.

Let $\ell^2_{\mathrm{e}}(\mathbb{Z}^3) \subset \ell^2(\mathbb{Z}^3)$ be the subspace of even functions on $\mathbb{Z}^3$. We note that the Hilbert space $\ell^2_{\mathrm{e}}(\mathbb{Z}^3)$ is invariant under the action of the operator $\hat{h}_{\mu\lambda}$. In what follows, the restriction $\hat{h}_{\mu\lambda}|_{\ell^2_{\mathrm{e}}(\mathbb{Z}^3)}$ of $\hat{h}_{\mu\lambda}$ to $\ell^2_{\mathrm{e}}(\mathbb{Z}^3)$ is also denoted by $\hat{h}_{\mu\lambda}$.

3. Momentum representation for the one-particle Hamiltonian

Let $\mathbb{T}^3 \equiv (-\pi;\pi]^3$ be a three-dimensional torus, i.e., a three-dimensional cube whose opposite faces are identified. We note that the operations of addition and multiplication by a real number for the elements of the set $\mathbb{T}^3 \subset \mathbb{R}^3$ are understood as operations modulo $(2\pi\mathbb{Z})^3$ in $\mathbb{R}^3$.

Let $L^2(\mathbb{T}^3)$ be the Hilbert space of square integrable functions on $\mathbb{T}^3$, let $L^2_{\mathrm{e}}(\mathbb{T}^3) \subset L^2(\mathbb{T}^3)$ be the subspace of even functions, and let

$$\mathcal{F}\colon \ell^2(\mathbb{Z}^3) \to L^2(\mathbb{T}^3), \qquad (\mathcal{F}\hat{f})(p) = (2\pi)^{-3/2} \sum_{s\in\mathbb{Z}^3} \hat{f}(s) e^{i(p,s)},$$

be the standard Fourier transformation. We note that $\mathcal{F}\bigl(\ell^2_{\mathrm{e}}(\mathbb{Z}^3)\bigr) \subset L^2_{\mathrm{e}}(\mathbb{T}^3)$. We let $\mathcal{F}_{\mathrm{e}}$ denote the restriction of $\mathcal{F}$ to $\ell^2_{\mathrm{e}}(\mathbb{Z}^3)$. It can be easily verified that $\mathcal{F}_{\mathrm{e}}\bigl(\ell^2_{\mathrm{e}}(\mathbb{Z}^3)\bigr) = L^2_{\mathrm{e}}(\mathbb{T}^3)$.

The Hamiltonian $h_{\mu\lambda} = \mathcal{F}_{\mathrm{e}} \hat{h}_{\mu\lambda} \mathcal{F}_{\mathrm{e}}^{-1}$ (see [8]) in the momentum representation is a bounded self-adjoint operator in the Hilbert space $L^2_{\mathrm{e}}(\mathbb{T}^3)$. It acts according to the formula

$$h_{\mu\lambda} = h_0 - v_{\mu\lambda},$$

where h_0 is the operator of multiplication by a function ε,

$$(h_0 f)(p) = \varepsilon(p) f(p), \qquad \varepsilon(p) = \sum_{i=1}^{3} (1 - \cos p^{(i)}), \quad f \in L^2_{\mathrm{e}}(\mathbb{T}^3), \quad p = (p^{(1)}, p^{(2)}, p^{(3)}) \in \mathbb{T}^3,$$

and $v_{\mu\lambda}$ is a nonnegative integral operator of rank $r \le 4$,

$$(v_{\mu\lambda} f)(p) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} \biggl(\mu + \lambda \sum_{i=1}^{3} \cos p^{(i)} \cos t^{(i)}\biggr) f(t)\, dt, \quad f \in L^2_{\mathrm{e}}(\mathbb{T}^3).$$

4. Continuous spectrum and a virtual level

The perturbed operator $v_{\mu\lambda}$ is an integral operator of rank $r \le 4$, and by the Weyl theorem (see [13]), the continuous spectrum $\sigma_{\mathrm{cont}}(h_{\mu\lambda})$ of the operator $h_{\mu\lambda}$ is therefore independent of the parameters $\mu, \lambda \ge 0$ and coincides with the spectrum $\sigma(h_0)$ of h_0. Hence, the equality

$$\sigma_{\mathrm{cont}}(h_{\mu\lambda}) = \sigma(h_0) = [0, 6]$$

holds.

We consider the orthonormalized system in the space $L^2_{\mathrm{e}}(\mathbb{T}^3)$

$$\alpha_0 = \frac{1}{(2\pi)^{3/2}}, \qquad \alpha_i(p) = \frac{\sqrt{2}}{(2\pi)^{3/2}} \cos p^{(i)}, \quad i = 1, 2, 3.$$

The operator $v_{\mu\lambda}$ is representable in the form

$$v_{\mu\lambda} f = \mu \alpha_0 (f, \alpha_0) + \frac{\lambda}{2} \sum_{i=1}^{3} (f, \alpha_i) \alpha_i,$$

where $(\,\cdot\,,\,\cdot\,)$ is the inner product in $L^2_{\mathrm{e}}(\mathbb{T}^3)$.

It follows from the nonnegativity of the operator $v_{\mu\lambda} \geq 0$ that the square root $v_{\mu\lambda}^{1/2} \geq 0$ exists. The operator $v_{\mu\lambda}^{1/2}$ acts in $L^2_{\mathrm{e}}(\mathbb{T}^3)$ according to the formula

$$(v_{\mu\lambda}^{1/2} f)(p) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{T}^3} v_{\mu\lambda}^{1/2}(p-q) f(q)\, dq,$$

where

$$v_{\mu\lambda}^{1/2}(p) = \frac{1}{(2\pi)^{3/2}} \sum_{s\in\mathbb{Z}^3} \hat{v}_{\mu\lambda}^{1/2}(s) e^{i(p,s)}.$$

Here, $\hat{v}_{\mu\lambda}^{1/2}$ is the square root of the positive function $\hat{v}_{\mu\lambda}$.

Let $\mathbb{C}$ be the complex plane, and let $r_0(z)$, $z \in \mathbb{C}\setminus[0,6]$, be the resolvent of h_0.

Because the function $\varepsilon(q) = \varepsilon(q^{(1)}, q^{(2)}, q^{(3)}) = \sum_{i=1}^{3}(1-\cos q^{(i)})$ is symmetric under permutations of $q^{(i)}$ and $q^{(j)}$, $i,j = 1,2,3$, the integrals

$$\int_{\mathbb{T}^3} \frac{\cos q^{(i)}\, dq}{\varepsilon(q)-z}, \qquad \int_{\mathbb{T}^3} \frac{\cos^2 q^{(i)}\, dq}{\varepsilon(q)-z}, \qquad \int_{\mathbb{T}^3} \frac{\cos q^{(i)} \cos q^{(j)}\, dq}{\varepsilon(q)-z}$$

are independent of $i,j = 1,2,3$, $i \neq j$. We set

$$\begin{aligned}
a(z) &= (\alpha_0, r_0(z)\alpha_0) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} \frac{dq}{\varepsilon(q)-z},\\
b(z) &= (\alpha_0, r_0(z)\alpha_i) = \frac{\sqrt{2}}{(2\pi)^3} \int_{\mathbb{T}^3} \frac{\cos q^{(i)}\, dq}{\varepsilon(q)-z},\\
c(z) &= (\alpha_i, r_0(z)\alpha_i) = \frac{2}{(2\pi)^3} \int_{\mathbb{T}^3} \frac{\cos^2 q^{(i)}\, dq}{\varepsilon(q)-z},\\
d(z) &= (\alpha_i, r_0(z)\alpha_j) = \frac{2}{(2\pi)^3} \int_{\mathbb{T}^3} \frac{\cos q^{(i)} \cos q^{(j)}\, dq}{\varepsilon(q)-z},
\end{aligned} \tag{4.1}$$

where $i,j = 1,2,3$, $i \neq j$, and $z < 0$.

For any fixed $\mu, \lambda \geq 0$ and $z \in \mathbb{C}\setminus[0,6]$, we define a finite-dimensional (rank $r \leq 4$) Birman—Schwinger integral operator $G_{\mu\lambda}(z)$ acting in the space $L^2_{\mathrm{e}}(\mathbb{T}^3)$ according to the formula $G_{\mu\lambda}(z) = v_{\mu\lambda}^{1/2} r_0(z) v_{\mu\lambda}^{1/2}$. We represent $G_{\mu\lambda}(z)$ in the form

$$\begin{aligned}
G_{\mu\lambda}(z) f = \biggl(\mu a(z)(f,\alpha_0) + \sqrt{\frac{\mu\lambda}{2}}\, b(z) \sum_{i=1}^{3} (f,\alpha_i)\biggr)\alpha_0 +{}\\
+ \sum_{i=1}^{3} \biggl[\sqrt{\frac{\mu\lambda}{2}}\, b(z)(f,\alpha_0) + \frac{\lambda}{2} c(z)(f,\alpha_i) + \frac{\lambda}{2} d(z) \sum_{i\neq j=1}^{3} (f,\alpha_j)\biggr]\alpha_i.
\end{aligned} \tag{4.2}$$

Because the function ε has a nondegenerate minimum at $q = 0$ and $\varepsilon(0) = \min_{q\in\mathbb{T}^3} \varepsilon(q)$, we have the finite limits

$$\lim_{z\to 0-} a(z) = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} \frac{dq}{\varepsilon(q)}, \qquad \lim_{z\to 0-} b(z) = \frac{\sqrt{2}}{(2\pi)^3} \int_{\mathbb{T}^3} \frac{\cos q^{(i)}\, dq}{\varepsilon(q)},$$

$$\lim_{z\to 0-} c(z) = \frac{2}{(2\pi)^3} \int_{\mathbb{T}^3} \frac{\cos^2 q^{(i)}\, dq}{\varepsilon(q)}, \qquad \lim_{z\to 0-} d(z) = \frac{2}{(2\pi)^3} \int_{\mathbb{T}^3} \frac{\cos q^{(i)} \cos q^{(j)}\, dq}{\varepsilon(q)}.$$

Therefore, although the resolvent $r_0(z)$ does not exist at $z = 0$, the bounded self-adjoint operator $G_{\mu\lambda}(0)$ is defined as

$$G_{\mu\lambda}(0)f = \left(\mu a(0)(f, \alpha_0) + \sqrt{\frac{\mu\lambda}{2}}\, b(0) \sum_{i=1}^{3} (f, \alpha_i)\right)\alpha_0 +$$
$$+ \sum_{i=1}^{3} \left[\sqrt{\frac{\mu\lambda}{2}}\, b(0)(f, \alpha_0) + \frac{\lambda}{2} c(0)(f, \alpha_i) + \frac{\lambda}{2} d(0) \sum_{i\neq j=1}^{3} (f, \alpha_j)\right]\alpha_i.$$

Remark 1. Clearly, the operator $h_{\mu\lambda}$ has an eigenvalue $z \leq 0$, i.e., $\operatorname{Ker}(h_{\mu\lambda} - zI) \neq 0$ if and only if the compact operator $G_{\mu\lambda}(z)$ in $L^2_{\mathrm{e}}(\mathbb{T}^3)$ has an eigenvalue equal to 1 and there is a function $\psi \in \operatorname{Ker}(G_{\mu\lambda}(z) - I)$ such that

$$f(\,\cdot\,) = \frac{(v^{1/2}_{\mu\lambda}\psi)(\,\cdot\,)}{\varepsilon(\,\cdot\,) - z} \in L^2_{\mathrm{e}}(\mathbb{T}^3).$$

In this case, $f \in \operatorname{Ker}(h_{\mu\lambda} - zI)$. Moreover, if $z < 0$, then

$$\dim \operatorname{Ker}(h_{\mu\lambda} - zI) = \dim \operatorname{Ker}(G_{\mu\lambda}(z) - I), \tag{4.3}$$
$$\operatorname{Ker}(h_{\mu\lambda} - zI) = \left\{ f\colon f(\,\cdot\,) = \frac{(v^{1/2}_{\mu\lambda}\psi)(\,\cdot\,)}{\varepsilon(\,\cdot\,) - z},\ \psi \in \operatorname{Ker}(G_{\mu\lambda}(z) - I)\right\}.$$

In the case of the threshold eigenvalue $z = 0$, relation (4.3) must be replaced with the inequality

$$\dim \operatorname{Ker}(h_{\mu\lambda}) \leq \dim \operatorname{Ker}(G_{\mu\lambda}(0) - I).$$

Definition 1. The threshold $z = 0$ of the continuous spectrum $\sigma_{\mathrm{cont}}(h_{\mu\lambda})$ is called a *singular point of the continuous spectrum* (SPCS) of the operator $h_{\mu\lambda}$ if the number 1 is an eigenvalue of the operator $G_{\mu\lambda}(0)$. And if 1 is not an eigenvalue of $G_{\mu\lambda}(0)$, then $z = 0$ is called a *regular point of the continuous spectrum* (RPCS) of $h_{\mu\lambda}$.

Definition 2. The operator $h_{\mu\lambda}$ is said to have a *virtual level* (on the left threshold of the continuous spectrum) if the number 1 is a (simple or multiple) eigenvalue of the operator $G_{\mu\lambda}(0)$ and if at least one (up to a constant) of the corresponding eigenfunctions ψ satisfies the condition

$$\frac{v^{1/2}_{\mu\lambda}\psi}{\varepsilon} \in L^1_{\mathrm{e}}(\mathbb{T}^3) \setminus L^2_{\mathrm{e}}(\mathbb{T}^3).$$

This means that

$$1 \leq \dim \operatorname{Ker}(G_{\mu\lambda}(0) - I) \geq \dim \operatorname{Ker}(h_{\mu\lambda}) + 1.$$

Remark 2. Our definition of a virtual level is equivalent to the definition of a virtual level for the two-particle Schrödinger operator in the Euclidean space and on a lattice.

Remark 3. By definition, the virtual level $z = 0$ is an SPCS of the operator $h_{\mu\lambda}$. An SPCS can be an eigenvalue of $h_{\mu\lambda}$ (see assertions B1 and B3 in the theorem below). Moreover, it can simultaneously be a virtual level and an eigenvalue of $h_{\mu\lambda}$ (see assertion B2 in the theorem).

5. Statement of the main results

For any fixed values of the parameters $\mu, \lambda \geq 0$, the determinant of the operator $h_{\mu\lambda} - zI$ is understood as the Fredholm determinant of the operator $I - G_{\mu\lambda}(z)$,

$$\Delta(\mu, \lambda; z) := \det(h_{\mu\lambda} - zI) := \det\bigl(I - G_{\mu\lambda}(z)\bigr). \tag{5.1}$$

Clearly, $\Delta(\mu, \lambda; \cdot)$ is an analytic function in the domain $\mathbb{C} \setminus [0, 6]$ for all $\mu, \lambda \geq 0$.

Lemma 1. *For all $\mu, \lambda \geq 0$ and $z \in \mathbb{C} \setminus [0, 6]$, the representations*

$$\Delta(\mu, \lambda; z) = \Delta^{(1)}(\mu, \lambda; z)\bigl(\Delta^{(22)}(\lambda; z)\bigr)^2, \tag{5.2}$$

$$\Delta(\mu, 0; z) = 1 - \mu a(z), \qquad \Delta(0, \lambda; z) = \Delta^{(21)}(\lambda; z)\bigl(\Delta^{(22)}(\lambda; z)\bigr)^2 \tag{5.3}$$

hold, where

$$\Delta^{(1)}(\mu, \lambda; z) = \Delta(\mu, 0; z)\Delta^{(21)}(\lambda; z) - \frac{3\mu\lambda}{2} b^2(z), \tag{5.4}$$

$$\Delta^{(21)}(\lambda; z) = 1 - \frac{\lambda}{2}\bigl(c(z) + 2d(z)\bigr), \qquad \Delta^{(22)}(\lambda; z) = 1 - \frac{\lambda}{2}\bigl(c(z) - d(z)\bigr). \tag{5.5}$$

Let $\mu^0 = \bigl(a(0)\bigr)^{-1}$, $\lambda_1^0 = \bigl(3\sqrt{2}\, b(0)/2\bigr)^{-1}$, and $\lambda_2^0 = 2\bigl(c(0) - d(0)\bigr)^{-1}$. We note that $\lambda_2^0 > \lambda_1^0$ (see Corollary 1 below).

For every $\mu > \mu^0$ and $\lambda > \lambda_1^0$, the operators $h_{\mu 0}$ and $h_{0\lambda}$ have the respective negative eigenvalues $\zeta_1(\mu)$ and $\zeta_2(\lambda)$ (see Proposition 3 below). We set $\zeta_{\min}(\mu, \lambda) = \min\{\zeta_1(\mu), \zeta_2(\lambda)\}$ and $\zeta_{\max}(\mu, \lambda) = \max\{\zeta_1(\mu), \zeta_2(\lambda)\}$. Let (see Fig. 1)

$$G_0 = \left\{(\mu, \lambda) \in \mathbb{R}_+^2 \colon \Delta^{(1)}(\mu, \lambda) = 1 - \mu a(0) - \frac{\sqrt{2}}{2}\lambda(3 - \mu) b(0) > 0,\ 0 < \mu < \mu^0,\ 0 < \lambda < \lambda_1^0\right\},$$

$$G_{11}^{(0)} = \{(\mu, \lambda) \in \mathbb{R}_+^2 \colon \Delta^{(1)}(\mu, \lambda) = 0,\ 0 < \mu < \mu^0,\ 0 < \lambda < \lambda_1^0\},$$

$$G_1 = \{(\mu, \lambda) \in \mathbb{R}_+^2 \colon \Delta^{(1)}(\mu, \lambda) < 0\},$$

$$G_{12}^{(0)} = \left\{(\mu, \lambda) \in \mathbb{R}_+^2 \colon \Delta^{(1)}(\mu, \lambda) = 0,\ \mu > 3,\ \lambda > \frac{\sqrt{2} a(0)}{b(0)}\right\},$$

$$G_2 = \{(\mu, \lambda) \in \mathbb{R}_+^2 \colon \Delta^{(1)}(\mu, \lambda) > 0,\ \mu > \mu^0,\ \lambda > \lambda_1^0\}.$$

The following theorem provides a complete presentation of the existence and the number of virtual levels or negative eigenvalues for all $\mu, \lambda \geq 0$.

Theorem. *The following assertions hold*:

A1. *Let $0 < \lambda \leq \lambda_1^0$ and $(\mu, \lambda) \in G_0$. Then the operator $h_{\mu\lambda}$ has no eigenvalues in the interval $(-\infty, 0)$, and the point $z = 0$ is an RPCS of the operator $h_{\mu\lambda}$.*

A2. *Let $0 < \lambda \leq \lambda_1^0$ and $(\mu, \lambda) \in G_{11}^{(0)}$. Then $z = 0$ is a virtual level of $h_{\mu\lambda}$, and this operator has no eigenvalues in the interval $(-\infty, 0)$.*

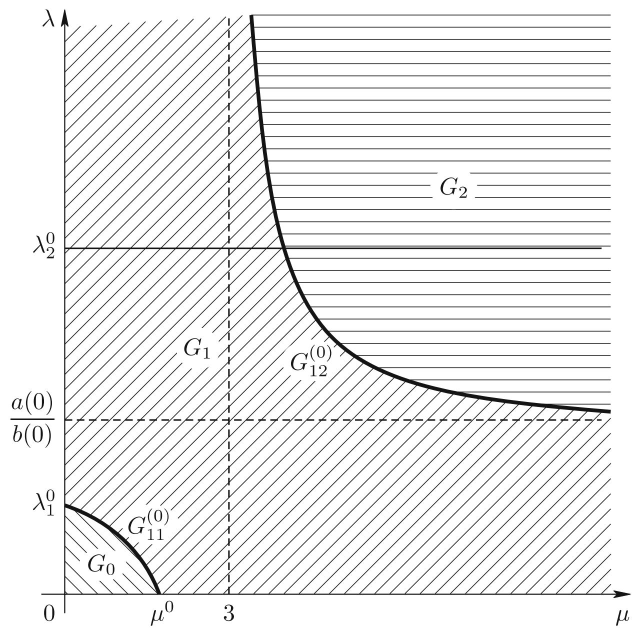

Fig. 1

A3. *Let $0 < \lambda < \lambda_2^0$ and $(\mu, \lambda) \in G_1$. Then the operator $h_{\mu\lambda}$ has a single eigenvalue $z^{(1)}(\mu, \lambda) < 0$. In this case, the inequality $z^{(1)}(\mu, \lambda) < \zeta_{\min}(\mu, \lambda)$ holds, and $z = 0$ is an RPCS of $h_{\mu\lambda}$.*

B1. *Let $\lambda_1^0 < \lambda < \lambda_2^0$ and $(\mu, \lambda) \in G_{12}^{(0)}$. Then $z = 0$ is a virtual level of $h_{\mu\lambda}$. Moreover, this operator has a single eigenvalue $z^{(1)}(\mu, \lambda) < 0$.*

B2. *Let $\lambda_1^0 < \lambda < \lambda_2^0$ and $(\mu, \lambda) \in G_2$. Then the operator $h_{\mu\lambda}$ has two eigenvalues $z^{(1)}(\mu, \lambda) < 0$ and $z^{(4)}(\mu, \lambda) < 0$. In this case, the inequalities*

$$z^{(1)}(\mu, \lambda) < \zeta_{\min}(\mu, \lambda) \leq \zeta_{\max}(\mu, \lambda) < z^{(4)}(\mu, \lambda)$$

hold, and $z = 0$ is an RPCS of the operator $h_{\mu\lambda}$.

C1. *Let $\lambda = \lambda_2^0$ and $(\mu, \lambda) \in G_1$. Then the operator $h_{\mu\lambda}$ has a single eigenvalue $z^{(1)}(\mu, \lambda) < 0$, and zero is a two-fold eigenvalue of this operator.*

C2. *Let $\lambda = \lambda_2^0$ and $(\mu, \lambda) \in G_{12}^{(0)}$. Then $z = 0$ is a virtual level and a two-fold eigenvalue of the operator $h_{\mu\lambda}$. Moreover, this operator has a single eigenvalue $z^{(1)}(\mu, \lambda) < 0$.*

C3. *Let $\lambda = \lambda_2^0$ and $(\mu, \lambda) \in G_2$. Then $z = 0$ is a two-fold eigenvalue of $h_{\mu\lambda}$. Moreover, the operator $h_{\mu\lambda}$ has two eigenvalues $z^{(1)}(\mu, \lambda) < 0$ and $z^{(4)}(\mu, \lambda) < 0$. In this case, the inequalities*

$$z^{(1)}(\mu, \lambda) < \zeta_{\min}(\mu, \lambda) \leq \zeta_{\max}(\mu, \lambda) < z^{(4)}(\mu, \lambda)$$

hold.

D1. *Let $\lambda > \lambda_2^0$ and $(\mu, \lambda) \in G_1$. Then the operator $h_{\mu\lambda}$ has three eigenvalues (counting multiplicities) $z^{(1)}(\mu, \lambda) < 0$ and $z^{(2)}(\lambda) = z^{(3)}(\lambda) < 0$. In this case, $z^{(1)}(\mu, \lambda) < z^{(2)}(\lambda) = z^{(3)}(\lambda)$, and $z = 0$ is an RPCS of $h_{\mu\lambda}$.*

D2. *Let* $\lambda > \lambda_2^0$ *and* $(\mu, \lambda) \in G_{12}^{(0)}$. *Then* $z = 0$ *is a virtual level of the operator* $h_{\mu\lambda}$. *Moreover, the operator* $h_{\mu\lambda}$ *has three eigenvalues (counting multiplicities)* $z^{(1)}(\mu, \lambda) < 0$ *and* $z^{(2)}(\lambda) = z^{(3)}(\lambda) < 0$. *In this case,* $z^{(1)}(\mu, \lambda) < z^{(2)}(\lambda) = z^{(3)}(\lambda)$.

D3. *Let* $\lambda > \lambda_2^0$ *and* $(\mu, \lambda) \in G_2$. *Then the operator* $h_{\mu\lambda}$ *has four eigenvalues (counting multiplicities)* $z^{(1)}(\mu, \lambda) < 0$, $z^{(2)}(\lambda) = z^{(3)}(\lambda) < 0$, *and* $z^{(4)}(\mu, \lambda) < 0$. *In this case,* $z^{(1)}(\mu, \lambda) < z^{(2)}(\lambda) = z^{(3)}(\lambda)$, $z^{(1)}(\mu, \lambda) < z^{(4)}(\mu, \lambda)$, *and* $z = 0$ *is an RPCS of* $h_{\mu\lambda}$.

Remark 4. It is said in assertion A2 in the theorem that the threshold $z = 0$ is a virtual level of $h_{\mu\lambda}$ for some $\mu, \lambda \geq 0$. It is said in assertion C1 that the threshold $z = 0$ is a two-fold eigenvalue. It is said in assertion C2 that $z = 0$ is simultaneously a virtual level and an eigenvalue.

Remark 5. Assertions A2 and A3 in the theorem mean that the first negative eigenvalue of $h_{\mu\lambda}$ is generated only by a threshold virtual level under a variation of $\mu, \lambda \geq 0$. It is said in assertions B1 and B2 that the second negative eigenvalue of $h_{\mu\lambda}$ is also generated by a threshold virtual level. Assertions C3 and D1 say that the negative eigenvalues of $h_{\mu\lambda}$ are generated by the multiple threshold eigenvalue $z = 0$ under a variation of $\mu, \lambda \geq 0$. Assertions C2, D2, and D3 say that the negative eigenvalues of $h_{\mu\lambda}$ are generated by both a threshold virtual level and the threshold values and eigenvalues $z = 0$.

Remark 6. Moreover, it is said in the theorem that the range of the parameters $\mu, \lambda \geq 0$ where the threshold $z = 0$ is either a virtual level or an eigenvalue is some curve. The range of the parameters $\mu, \lambda \geq 0$ where the threshold $z = 0$ is a virtual level and an eigenvalue of the operator $h_{\mu\lambda}$ is a single point.

Remark 7. A similar theorem describes the dependence of the number of eigenvalues and their arrangement on the parameters μ and λ for all $\mu, \lambda \in \mathbb{R}$. In this case, the eigenvalues of $h_{\mu\lambda}$ are located both to the left and to the right of the continuous spectrum. In the case $\mu, \lambda \leq 0$, the eigenvalues of $h_{\mu\lambda}$ are only to the right of the continuous spectrum.

6. The proofs of the main results

The remainder of the paper is devoted to the proof of the theorem.

We let $L_4 \subset L_{\mathrm{e}}^2(\mathbb{T}^3)$ denote the four-dimensional subspace spanned by the vectors 1 and $\cos p^{(i)}$, $i = 1, 2, 3$. We note that the operator $v_{\mu\lambda}$ maps the Hilbert space $L_{\mathrm{e}}^2(\mathbb{T}^3)$ to the subspace L_4.

Proof of Lemma 1. As follows from (4.2), the operator $G_{\mu\lambda}(z)$ maps the whole space $L_{\mathrm{e}}^2(\mathbb{T}^3)$ to the subspace L_4 invariant under the action of $G_{\mu\lambda}(z)$. Therefore, the restriction $G_{\mu\lambda}(z)|_{L_4}$ of $G_{\mu\lambda}(z)$ to L_4 is represented in the matrix form

$$G_{\mu\lambda}(z)|_{L_4} = \begin{pmatrix} \mu a(z) & \frac{\lambda}{\sqrt{2}} b(z) & \frac{\lambda}{\sqrt{2}} b(z) & \frac{\lambda}{\sqrt{2}} b(z) \\ \frac{\mu}{\sqrt{2}} b(z) & \frac{\lambda}{2} c(z) & \frac{\lambda}{2} d(z) & \frac{\lambda}{2} d(z) \\ \frac{\mu}{\sqrt{2}} b(z) & \frac{\lambda}{2} d(z) & \frac{\lambda}{2} c(z) & \frac{\lambda}{2} d(z) \\ \frac{\mu}{\sqrt{2}} b(z) & \frac{\lambda}{2} d(z) & \frac{\lambda}{2} d(z) & \frac{\lambda}{2} c(z) \end{pmatrix}.$$

It hence follows that the Fredholm determinant of the operator $\big(I-G_{\mu\lambda}(z)\big)|_{L_4}$ has the form

$$\det\big(\big(I-G_{\mu\lambda}(z)\big)\big|_{L_4}\big)=\Delta(\mu,\lambda;z)=\begin{vmatrix}1-\mu a(z) & -\frac{\lambda}{\sqrt{2}}b(z) & -\frac{\lambda}{\sqrt{2}}b(z) & -\frac{\lambda}{\sqrt{2}}b(z)\\ -\frac{\mu}{\sqrt{2}}b(z) & 1-\frac{\lambda}{2}c(z) & -\frac{\lambda}{2}d(z) & -\frac{\lambda}{2}d(z)\\ -\frac{\mu}{\sqrt{2}}b(z) & -\frac{\lambda}{2}d(z) & 1-\frac{\lambda}{2}c(z) & -\frac{\lambda}{2}d(z)\\ -\frac{\mu}{\sqrt{2}}b(z) & -\frac{\lambda}{2}d(z) & -\frac{\lambda}{2}d(z) & 1-\frac{\lambda}{2}c(z)\end{vmatrix},$$

where I is the identity operator in L_4. Calculating this determinant, we obtain representation (5.2).

The relation between the eigenvalues of the self-adjoint operator $h_{\mu\lambda}$ and the zeros of the Fredholm determinant $\Delta(\mu,\lambda;z)$ is established by the following lemma (see [14]).

Lemma 2. *For any $\mu,\lambda\geq 0$, a number $z\in\mathbb{C}\setminus[0,6]$ is an m-fold eigenvalue of the operator $h_{\mu\lambda}$ if and only if it is an m-fold zero of the function $\Delta(\mu,\lambda;z)=0$.*

Proposition 1. 1. *The functions a, b, c, and d are analytic in $\mathbb{C}\setminus[0,6]$, positive, and monotonically increasing in the interval $(-\infty,0)$, and the inequality $c(z)>d(z)$ holds for $z\in(-\infty,0)$.*

2. *The asymptotic expansions*

$$\begin{aligned} a(z)&=a(0)-\frac{\sqrt{2}}{4\pi}\sqrt{-z}+O(-z),\quad z\to 0-,\\ b(z)&=b(0)-\frac{1}{2\pi}\sqrt{-z}+O(-z),\quad z\to 0-,\\ c(z)&=c(0)-\frac{\sqrt{2}}{2\pi}\sqrt{-z}+O(-z),\quad z\to 0-,\\ d(z)&=d(0)-\frac{\sqrt{2}}{2\pi}\sqrt{-z}+O(-z),\quad z\to 0-,\end{aligned}\tag{6.1}$$

hold for a, b, c, and d.

Proof. 1. The positivity of the functions a and c defined in (4.1) follows straightforwardly from the nonnegativity of the integrands and from the monotonicity of the Lebesgue integral.

We represent b and d in the forms

$$\begin{aligned} b(z)&=\frac{\sqrt{2}}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi}\frac{\cos q^{(1)}}{A-\cos q^{(1)}}dq^{(1)}\right]dq^{(2)}\,dq^{(3)},\\ d(z)&=\frac{2}{(2\pi)^3}\int_{-\pi}^{\pi}\left[\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{\cos q^{(1)}\cos q^{(2)}\,dq^{(1)}dq^{(2)}}{B-(\cos q^{(1)}+\cos q^{(2)})}\right]dq^{(3)},\end{aligned}$$

where $A=3-\cos q^{(2)}-\cos q^{(3)}-z>0$ and $B=3-\cos q^{(3)}-z>0$. Representing the integrals in the square brackets as sums of integrals over the closed intervals $[-\pi,0]$ and $[0,\pi]$, changing the variables

$q^{(1)} := q^{(1)} - \pi$ and $q^{(2)} := q^{(2)} - \pi$, and using the identity $\cos(x-\pi) = -\cos x$, we obtain

$$\int_{-\pi}^{\pi} \frac{\cos q^{(1)}\, dq^{(1)}}{A - \cos q^{(1)}} = \int_0^{\pi} \frac{2\cos^2 q^{(1)}\, dq^{(1)}}{A^2 - \cos^2 q^{(1)}} > 0,$$

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\cos q^{(1)} \cos q^{(2)}\, dq^{(1)}\, dq^{(2)}}{B - (\cos q^{(1)} + \cos q^{(2)})} =$$

$$= \int_0^{\pi}\int_0^{\pi} \frac{8B\cos^2 q^{(1)} \cos^2 q^{(2)}\, dq^{(1)}\, dq^{(2)}}{[B^2 - (\cos q^{(1)} + \cos q^{(2)})^2][B^2 - (\cos q^{(1)} - \cos q^{(2)})^2]} > 0.$$

The positivity of b and d follows from the nonnegativity of the integrand and the monotonicity of the Lebesgue integral.

The relation $c(z) > d(z)$ is implied by the Cauchy–Bounjakowsky inequality,

$$d(z) = \frac{2}{(2\pi)^3}\int_{\mathbb{T}^3} \frac{\cos q^{(1)} \cos q^{(2)}\, dq}{\varepsilon(q) - z} = \frac{2}{(2\pi)^3}\int_{\mathbb{T}^3} \frac{\cos q^{(1)}}{\sqrt{\varepsilon(q)-z}}\frac{\cos q^{(2)}}{\sqrt{\varepsilon(q)-z}} dq <$$

$$< \frac{2}{(2\pi)^3}\left(\int_{\mathbb{T}^3} \frac{\cos^2 q^{(1)}\, dq}{\varepsilon(q)-z}\right)^{1/2}\left(\int_{\mathbb{T}^3} \frac{\cos^2 q^{(2)}\, dq}{\varepsilon(q)-z}\right)^{1/2} = c(z).$$

The derivatives of the functions a, b, c, and d are positive, and these functions are therefore monotonically increasing in the interval $(-\infty, 0)$.

The proof of assertion 2 is similar to the proof of Lemma 3.5 in [15].

The assertions in Proposition 1 imply the following corollary.

Corollary 1. *The functions $c-d$ and $c+2d$ are positive and monotonically increasing in the interval $(-\infty, 0)$, and the relations (asymptotic expansions)*

$$\Delta^{(22)}(\lambda; z) = \Delta^{(22)}(\lambda; 0) + O(-z), \quad z \to 0-, \tag{6.2}$$

$$\Delta^{(1)}(\mu,\lambda; z) = 1 - \mu a(0) - \frac{\sqrt{2}}{2}\lambda(3-\mu)b(0) + + \frac{\sqrt{2}}{4\pi}[\mu + 3\lambda - \mu\lambda]\sqrt{-z} + O(-z), \quad z \to 0-, \tag{6.3}$$

hold for all $\mu, \lambda \geq 0$.

Proof. The positivity of c and d and the inequality $c(z) > d(z)$ imply the positivity of $c-d$ and $c+2d$ in the interval $(-\infty, 0)$. Because the derivatives of $c-d$ and $c+2d$ are positive, these functions are monotonically increasing in $(-\infty, 0)$.

Formulas (5.5) and (6.1) give (6.2), and (5.4) and (6.1) imply

$$\Delta^{(1)}(\mu,\lambda; z) = \big(1 - \mu a(0)\big)\left(1 - \frac{\lambda}{2}\big(c(0) + 2d(0)\big)\right) - \frac{3\mu\lambda}{2} b^2(0) + + \frac{\sqrt{2}}{4\pi}\left[(\mu + 3\lambda) - \left(3a(0) - 3\sqrt{2}\, b(0) + \frac{1}{2}\big(c(0) + 2d(0)\big)\right)\mu\lambda\right]\sqrt{-z} + + O(-z), \quad z \to 0-. \tag{6.4}$$

Expressions (4.1) and (6.1) imply the relations

$$a(0) - \frac{\sqrt{2}}{2}b(0) = \frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} \frac{1-\cos q^{(i)}\,dq}{\varepsilon(q)} =$$

$$= \frac{1}{3}\frac{1}{(2\pi)^3}\int_{\mathbb{T}^3} \frac{\sum_{i=1}^3 (1-\cos q^{(i)})\,dq}{\varepsilon(q)} = \frac{1}{3}, \tag{6.5}$$

$$c(0) + 2d(0) = \frac{2}{(2\pi)^3}\int_{\mathbb{T}^3} \frac{\cos^2 q^{(1)} + 2\cos q^{(1)}\cos q^{(2)}\,dq}{\varepsilon(q)} =$$

$$= \frac{2}{(2\pi)^3}\int_{\mathbb{T}^3} \frac{\cos^2 q^{(1)} + \cos q^{(1)}\cos q^{(2)} + \cos q^{(1)}\cos q^{(3)}\,dq}{\varepsilon(q)} =$$

$$= -\frac{2}{(2\pi)^3}\int_{\mathbb{T}^3} \frac{\cos q^{(1)}(3 - \cos q^{(1)} - \cos q^{(2)} - \cos q^{(3)} - 3)\,dq}{\varepsilon(q)} =$$

$$= -\frac{2}{(2\pi)^3}\int_{\mathbb{T}^3} \cos q^{(1)}\,dq + 3\frac{2}{(2\pi)^3}\int_{\mathbb{T}^3} \frac{\cos q^{(1)}\,dq}{\varepsilon(q)} = 3\sqrt{2}\,b(0). \tag{6.6}$$

Using (6.4)–(6.6), we derive (6.3).

We note that by Lemma 2 and representation (5.2), investigating the zeros of $\Delta(\mu,\lambda;\,\cdot\,)$ reduces to investigating the zeros of $\Delta^{(1)}(\mu,\lambda;\,\cdot\,)$ and $\Delta^{(22)}(\lambda;\,\cdot\,)$.

Proposition 2. 1. *Let $0 < \lambda < \lambda_2^0$. Then $\Delta^{(22)}(\lambda;\,\cdot\,)$ has no zeros in $(-\infty, 0]$.*
2. *Let $\lambda = \lambda_2^0$. Then $z = 0$ is the only zero of $\Delta^{(22)}(\lambda;\,\cdot\,)$ in $(-\infty, 0]$.*
3. *Let $\lambda > \lambda_2^0$. Then $\Delta^{(22)}(\lambda;\,\cdot\,)$ has a single zero $z^{(2)}(\lambda) < 0$.*

Proof. 1. By Proposition 1, the function $\Delta^{(22)}(\lambda;\,\cdot\,)$ with a fixed $\lambda > 0$ is continuous and decreases monotonically on the interval $(-\infty, 0]$. Moreover, the relations

$$\lim_{z\to-\infty} \Delta^{(22)}(\lambda; z) = 1, \qquad \lim_{z\to 0-} \Delta^{(22)}(\lambda; z) = \Delta^{(22)}(\lambda; 0) = 1 - \frac{\lambda}{\lambda_2^0}$$

hold. Consequently, $\Delta^{(22)}(\lambda; 0) > 0$ for $0 < \lambda < \lambda_2^0$. It follows that

$$\Delta^{(22)}(\lambda; z) \geq \Delta^{(22)}(\lambda; 0) > 0$$

for all $z \leq 0$.

2. Let $\lambda = \lambda_2^0$. Then the relations

$$\lim_{z\to 0-} \Delta^{(22)}(\lambda; z) = \Delta^{(22)}(\lambda; 0) = 1 - \frac{\lambda}{\lambda_2^0} = 0,$$

$$\Delta^{(22)}(\lambda; z) > \Delta^{(22)}(\lambda; 0) = 0$$

hold for all $z < 0$.

3. Let $\lambda > \lambda_2^0$. The function $\Delta^{(22)}(\lambda;\,\cdot\,)$ is continuous and monotonically decreasing on the interval $(-\infty, 0]$, and the relations

$$\lim_{z\to 0-} \Delta^{(22)}(\lambda; z) = \Delta^{(22)}(\lambda; 0) = 1 - \frac{\lambda}{\lambda_2^0} < 0$$

hold. Therefore, there is a single number $z^{(2)}(\lambda) < 0$ such that

$$\Delta^{(22)}\big(\lambda; z^{(2)}(\lambda)\big) = 0.$$

Proposition 3. 1. *If* $0 < \mu < \mu^0$ *or* $0 < \lambda < \lambda_1^0$, *then* $\Delta(\mu, 0; \cdot)$ *or* $\Delta^{(21)}(\lambda; \cdot)$ *respectively has no zeros on* $(-\infty, 0]$.

2. *If* $\mu = \mu^0$ *or* $\lambda = \lambda_1^0$, *then* $z = 0$ *is the only zero respectively of* $\Delta(\mu, 0; \cdot)$ *or* $\Delta^{(21)}(\lambda; \cdot)$ *on* $(-\infty, 0]$.

3. *If* $\mu > \mu^0$ *or* $\lambda > \lambda_1^0$, *then* $\Delta(\mu, 0; \cdot)$ *or* $\Delta^{(21)}(\lambda; \cdot)$ *respectively has the only zero* $\zeta_1(\mu) < 0$ *or* $\zeta_2(\lambda) < 0$, *i.e.*,

$$\Delta\big(\mu, 0; \zeta_1(\mu)\big) = 0 \quad or \quad \Delta^{(21)}\big(\lambda; \zeta_2(\lambda)\big) = 0. \tag{6.7}$$

The proof of Proposition 3 is similar to that of Proposition 2.

Lemma 3. *The operator* $h_{\mu\lambda}$ *has at most four eigenvalues* (*counting multiplicities*) *to the left of the point* $z = 0$.

Proof. Let $z_0 \in (-\infty, 0)$. The range of the operator $G_{\mu\lambda}(z_0)$ is the subspace L_4, and its dimension does not exceed four. Therefore, $G_{\mu\lambda}(z_0)$ has at most four eigenvalues (counting multiplicities) in the interval $(1, +\infty)$. By the Birman—Schwinger principle, the number of eigenvalues of $h_{\mu\lambda}$ to the left of the point $z < 0$ coincides with the number of eigenvalues of $G_{\mu\lambda}(z)$ to the right of the point 1 (see [16], [17]).

Corollary 2. *For all* $\mu, \lambda \geq 0$, *the function* $\Delta^{(1)}(\mu, \lambda; \cdot)$ *has at most two zeros in the interval* $(-\infty, 0]$.

Proof. It follows from the assertions in Lemmas 2 and 3 that for any fixed $\mu, \lambda \geq 0$, the number of zeros of the function $\Delta(\mu, \lambda; \cdot)$ on the interval $(-\infty, 0]$ does not exceed four, and relation (5.5) and Proposition 2 imply that $\Delta^{(22)}(\lambda; \cdot)$ can have a single zero on $(-\infty, 0]$. It follows from (5.2), (5.4), and (5.5) that $\Delta^{(1)}(\mu, \lambda; \cdot)$ can have two zeros on the interval $(-\infty, 0]$.

Proposition 4. 1. *Let* $(\mu, \lambda) \in G_0$. *Then the function* $\Delta^{(1)}(\mu, \lambda; \cdot)$ *has no zeros on the interval* $(-\infty, 0]$.

2. *Let* $(\mu, \lambda) \in G_{11}^{(0)}$. *Then* $z = 0$ *is the only zero of* $\Delta^{(1)}(\mu, \lambda; \cdot)$ *on* $(-\infty, 0]$.

3. *Let* $(\mu, \lambda) \in G_1$. *Then the function* $\Delta^{(1)}(\mu, \lambda; \cdot)$ *has a single zero* $z^{(1)}(\mu, \lambda) < 0$. *In this case, we have* $z^{(1)}(\mu, \lambda) < \zeta_{\min}(\mu, \lambda)$.

4. *Let* $(\mu, \lambda) \in G_{12}^{(0)}$. *Then* $\Delta^{(1)}(\mu, \lambda; \cdot)$ *has only two zeros* $z = 0$ *and* $z = z^{(1)}(\mu, \lambda) < 0$.

5. *Let* $(\mu, \lambda) \in G_2$. *Then* $\Delta^{(1)}(\mu, \lambda; \cdot)$ *has only two zeros* $z^{(1)}(\mu, \lambda) < 0$ *and* $z^{(4)}(\mu, \lambda) < 0$. *In this case, the inequalities*

$$z^{(1)}(\mu, \lambda) < \zeta_{min}(\mu, \lambda) \leq \zeta_{\max}(\mu, \lambda) < z^{(4)}(\mu, \lambda)$$

hold.

Proof. 1. Let $z \leq 0$. The functions $\Delta(\mu, 0; \cdot)$ and $\Delta^{(21)}(\lambda; \cdot)$ decrease monotonically on $(-\infty, 0]$, and the inequalities

$$\Delta(\mu, 0; z) \geq \Delta(\mu, 0; 0) > 0, \qquad \Delta^{(21)}(\lambda; z) \geq \Delta^{(21)}(\lambda; 0) > 0, \qquad \text{and} \qquad b(z) \leq b(0)$$

hold by Proposition 3. Hence, in view of (5.4) and also by condition 1 in the proposition, we obtain

$$\Delta^{(1)}(\mu, \lambda; z) \geq \Delta(\mu, 0; 0)\Delta^{(21)}(\lambda; 0) - \frac{3\mu\lambda}{2} b^2(0) = \Delta^{(1)}(\mu, \lambda; 0) > 0.$$

Consequently, $\Delta^{(1)}(\mu, \lambda; \cdot)$ has no zeros on the interval $(-\infty, 0]$.

2. It follows from (6.3) and by condition 2 that the equality

$$\lim_{z \to 0-} \Delta^{(1)}(\mu, \lambda; z) = \Delta^{(1)}(\mu, \lambda; 0) = 0$$

holds. By the proof of assertion 1, we have

$$\Delta^{(1)}(\mu,\lambda;z) > \Delta^{(1)}(\mu,\lambda;0) = 0 \quad \text{for } z \in (-\infty,0).$$

Therefore, $\Delta^{(1)}(\mu,\lambda;\,\cdot\,)$ has no zeros on $(-\infty,0)$.

3. Let $z \le 0$. The functions $\Delta(\mu,0;\,\cdot\,)$ and $\Delta^{(21)}(\lambda;\,\cdot\,)$ decrease monotonically on $(-\infty,0]$, and therefore

$$\Delta(\mu,0;z) > \Delta\big(\mu,0;\zeta_1(\mu)\big) = 0, \qquad \Delta^{(21)}(\lambda;z) > \Delta^{(21)}\big(\lambda;\zeta_2(\lambda)\big) = 0$$

for all $z < \zeta_{\min}(\mu,\lambda)$. Hence, by Proposition 1, it follows that the inequality

$$\frac{\partial \Delta^{(1)}(\mu,\lambda;z)}{\partial z} = -\mu a'(z)\Delta^{(21)}(\lambda;z) - \frac{\lambda}{2}\big(c'(z)+2d'(z)\big)\Delta(\mu,0;z) - 3\mu\lambda b(z)b'(z) < 0$$

holds for all $z < \zeta_{\min}(\mu,\lambda)$. This means that $\Delta^{(1)}(\mu,\lambda;\,\cdot\,)$ decreases monotonically on $\big(-\infty,\zeta_{\min}(\mu,\lambda)\big)$. Formulas (5.3), (5.5), and (6.7) imply the relations

$$\begin{aligned}
\Delta^{(1)}\big(\mu,\lambda;\zeta_{\min}(\mu,\lambda)\big) &= -\frac{3\mu\lambda}{2}b^2\big(\zeta_{\min}(\mu,\lambda)\big) < 0,\\
\Delta^{(1)}\big(\mu,\lambda;\zeta_{\max}(\mu,\lambda)\big) &= -\frac{3\mu\lambda}{2}b^2\big(\zeta_{\max}(\mu,\lambda)\big) < 0.
\end{aligned} \tag{6.8}$$

Relations (5.3)–(5.5) and (4.1) imply the equality

$$\lim_{z\to-\infty} \Delta^{(1)}(\mu,\lambda;z) = 1.$$

Therefore, there is a unique number $z^{(1)}(\mu,\lambda) < \zeta_{\min}(\mu,\lambda) < 0$ such that

$$\Delta^{(1)}\big(\mu,\lambda;z^{(1)}(\mu,\lambda)\big) = 0.$$

To prove assertion 3, we show that the function $\Delta^{(1)}(\mu,\lambda;\,\cdot\,)$ has no zeros on the closed interval $[\zeta_{\min}(\mu,\lambda),0]$, namely,

$$\Delta^{(1)}(\mu,\lambda;z) < 0 \quad \text{for } z \in [\zeta_{\min}(\mu,\lambda),0]. \tag{6.9}$$

We assume the contrary, i.e., let the inequality $\Delta^{(1)}(\mu,\lambda;\zeta) \ge 0$ hold for some $\zeta \in [\zeta_{\min}(\mu,\lambda),0]$. Then, by the analyticity of the function $\Delta^{(1)}(\mu,\lambda;\,\cdot\,)$ and in view of inequality (6.8), the function $\Delta^{(1)}(\mu,\lambda;\,\cdot\,)$ must have at least two zeros (counting multiplicities) on the closed interval $[\zeta_{min}(\mu,\lambda),0]$, and the function $\Delta^{(1)}(\mu,\lambda;\,\cdot\,)$ must have at least three zeros on the interval $(-\infty,0]$ by (5.4), which contradicts the assertion in Corollary 2.

By Corollary 2, it follows from inequality (6.9) that $\Delta^{(1)}(\mu,\lambda;\,\cdot\,)$ has no zeros on $[\zeta_{\min}(\mu,\lambda),0]$, which proves assertion 3 in the proposition.

4. By analogy with the proof of assertion 2 and by condition 4 in the proposition, we have

$$\lim_{z\to0-} \Delta^{(1)}(\mu,\lambda;z) = \Delta^{(1)}(\mu,\lambda;0) = 0.$$

The function $\Delta^{(1)}(\mu,\lambda;\,\cdot\,)$ has the only zero $z^{(1)}(\mu,\lambda) < \zeta_{\min}(\mu,\lambda)$ on $(-\infty,0)$.

5. As already proved above, the function $\Delta^{(1)}(\mu,\lambda;\,\cdot\,)$ has a single zero on the interval $\big(-\infty,\zeta_{\min}(\mu,\lambda)\big)$ (see the proof of assertion 3). By the condition in assertion 5, we have

$$\lim_{z\to0-} \Delta^{(1)}(\mu,\lambda;z) > 0.$$

Hence, in view of (6.7), arguing as above, we conclude that the function $\Delta^{(1)}(\mu,\lambda;\,\cdot\,)$ has the only zero $z^{(4)}(\mu,\lambda)$ on the interval $(\zeta_{\max}(\mu,\lambda),0]$. (It would contradict Corollary 2 if it had more than one zero.) Therefore, $z^{(1)}(\mu,\lambda)$ and $z^{(4)}(\mu,\lambda)$ are zeros of $\Delta^{(1)}(\mu,\lambda;\,\cdot\,)$ on $(-\infty,0]$, which proves assertion 5.

The next lemma is important for proving the existence of virtual levels for the operator $h_{\mu\lambda}$.

Lemma 4. *The eigenfunctions f_1, f_2, f_3, and f_4 of the operator $h_{\mu\lambda}$ that correspond to the eigenvalues $z^{(1)}$, $z^{(2)}$, $z^{(3)}$, and $z^{(4)}$ ($z^{(2)} = z^{(3)}$) have the forms*

$$f_1(p) = \frac{\bigl(\sqrt{\mu}\,\alpha_0 + \sqrt{\lambda/2}\,k(z^{(1)})\sum_{i=1}^3 \alpha_i(p)\bigr)C}{\varepsilon(p) - z^{(1)}}, \tag{6.10}$$

$$f_2(p) = \frac{\bigl(\alpha_1(p) - \alpha_2(p)\bigr)C}{\varepsilon(p) - z^{(2)}}, \qquad f_3(p) = \frac{\bigl(\alpha_2(p) - \alpha_3(p)\bigr)C}{\varepsilon(p) - z^{(2)}}, \tag{6.11}$$

$$f_4(p) = \frac{\bigl(\sqrt{\mu}\,\alpha_0 + \sqrt{\lambda/2}\,k(z^{(4)})\sum_{i=1}^3 \alpha_i(p)\bigr)C}{\varepsilon(p) - z^{(4)}}, \tag{6.12}$$

where $C = \text{const} \neq 0$ and $k(z) = \Delta(\mu, 0; z)/\bigl(3\sqrt{\mu\lambda/2}\,b(z)\bigr)$.

Proof. By the Birman—Schwinger principle, the operator $h_{\mu\lambda}$, $\mu, \lambda \geq 0$, has an eigenvalue if and only if the homogeneous equation

$$G_{\mu\lambda}(z)\psi = \psi, \qquad \text{i.e.,} \qquad \bigl(I - G_{\mu\lambda}(z)\bigr)\big|_{L_4}\psi = 0, \qquad \psi = (\psi_0, \psi_1, \psi_2, \psi_3) \in \mathbb{C}^4, \tag{6.13}$$

has a nonzero solution, where $\psi_i = (\psi, \alpha_i)$, $i = 0, 1, 2, 3$. This equation is equivalent to the system of homogeneous linear equations

$$\begin{aligned}
&\Delta(\mu, 0; z)\psi_0 - \sqrt{\frac{\mu\lambda}{2}}\,b(z)\psi_1 - \sqrt{\frac{\mu\lambda}{2}}\,b(z)\psi_2 - \sqrt{\frac{\mu\lambda}{2}}\,b(z)\psi_3 = 0,\\
&-\sqrt{\frac{\mu\lambda}{2}}\,b(z)\psi_0 + \left(1 - \frac{\lambda}{2}c(z)\right)\psi_1 - \frac{\lambda}{2}d(z)\psi_2 - \frac{\lambda}{2}d(z)\psi_3 = 0,\\
&-\sqrt{\frac{\mu\lambda}{2}}\,b(z)\psi_0 - \frac{\lambda}{2}d(z)\psi_1 + \left(1 - \frac{\lambda}{2}c(z)\right)\psi_2 - \frac{\lambda}{2}d(z)\psi_3 = 0,\\
&-\sqrt{\frac{\mu\lambda}{2}}\,b(z)\psi_0 - \frac{\lambda}{2}d(z)\psi_1 - \frac{\lambda}{2}d(z)\psi_2 + \left(1 - \frac{\lambda}{2}c(z)\right)\psi_3 = 0.
\end{aligned} \tag{6.14}$$

Consequently, system (6.14) has a nonzero solution for $\mu, \lambda \geq 0$ if and only if the system of equations

$$\begin{aligned}
&\Delta(\mu, 0; z)\psi_0 - \sqrt{\frac{\mu\lambda}{2}}\,b(z)\psi_1 - \sqrt{\frac{\mu\lambda}{2}}\,b(z)\psi_2 - \sqrt{\frac{\mu\lambda}{2}}\,b(z)\psi_3 = 0,\\
&-\sqrt{\frac{\mu\lambda}{2}}\,b(z)\psi_0 + \left(1 - \frac{\lambda}{2}c(z)\right)\psi_1 - \frac{\lambda}{2}d(z)\psi_2 - \frac{\lambda}{2}d(z)\psi_3 = 0,\\
&\Delta^{(22)}(\lambda; z)\psi_1 - \Delta^{(22)}(\lambda; z)\psi_2 = 0, \qquad \Delta^{(22)}(\lambda; z)\psi_2 - \Delta^{(22)}(\lambda; z)\psi_3 = 0
\end{aligned} \tag{6.15}$$

has a nonzero solution.

Let $\lambda \geq \lambda_2^0$. Then (see assertions 2 and 3 in Proposition 2) the function $\Delta^{(22)}(\lambda; \cdot)$ has the only zero $z = z^{(2)}(\lambda)$ on the interval $(-\infty, 0]$, i.e., $\Delta^{(22)}\bigl(\lambda; z^{(2)}(\lambda)\bigr) = 0$, and we have $\Delta^{(1)}\bigl(\mu, \lambda; z^{(2)}(\lambda)\bigr) \neq 0$ by Proposition 4. Therefore, (6.15) is equivalent to the system of equations

$$\begin{aligned}
&\Delta\bigl(\mu, 0; z^{(2)}(\lambda)\bigr)\psi_0 - \sqrt{\frac{\mu\lambda}{2}}\,b\bigl(z^{(2)}(\lambda)\bigr)\psi_1 - \sqrt{\frac{\mu\lambda}{2}}\,b\bigl(z^{(2)}(\lambda)\bigr)\psi_2 - \sqrt{\frac{\mu\lambda}{2}}\,b\bigl(z^{(2)}(\lambda)\bigr)\psi_3 = 0,\\
&-\sqrt{\frac{\mu\lambda}{2}}\,b\bigl(z^{(2)}(\lambda)\bigr)\psi_0 - \frac{\lambda}{2}d\bigl(z^{(2)}(\lambda)\bigr)\psi_1 - \frac{\lambda}{2}d\bigl(z^{(2)}(\lambda)\bigr)\psi_2 - \frac{\lambda}{2}d\bigl(z^{(2)}(\lambda)\bigr)\psi_3 = 0.
\end{aligned} \tag{6.16}$$

Because $d\big(z^{(2)}(\lambda)\big) > 0$, system (6.16) is equivalent to the equation

$$\left(\Delta\big(\mu, 0; z^{(2)}(\lambda)\big) + \mu \frac{b^2\big(z^{(2)}(\lambda)\big)}{d\big(z^{(2)}(\lambda)\big)}\right)\psi_0 = 0$$

for all $\mu, \lambda \geq 0$.

Taking

$$\Delta\big(\mu, 0; z^{(2)}(\lambda)\big) + \mu \frac{b^2\big(z^{(2)}(\lambda)\big)}{d\big(z^{(2)}(\lambda)\big)} \neq 0$$

into account (see assertion 1 in Proposition 1 and Proposition 4), we conclude that $\psi_0 = 0$ for

$$\mu \neq \frac{d\big(z^{(2)}(\lambda)\big)}{a\big(z^{(2)}(\lambda)\big)d\big(z^{(2)}(\lambda)\big) - b^2\big(z^{(2)}(\lambda)\big)}.$$

Therefore, the first equation in (6.14) gives

$$-\sqrt{\frac{\mu\lambda}{2}}\, b\big(z^{(2)}(\lambda)\big)(\psi_1 + \psi_2 + \psi_3) = 0.$$

Assertion 1 in Proposition 1 implies that $b\big(z^{(2)}(\lambda)\big) > 0$. Therefore,

$$\psi_1 + \psi_2 + \psi_3 = 0, \quad \text{i.e.,} \quad \psi_1 = -(\psi_2 + \psi_3) \quad \text{for all } \psi_2, \psi_3 \in \mathbb{C}. \tag{6.17}$$

By (4.2) and (6.17) and with the notation $\psi_i = (\psi, \alpha_i)$, $i = 0, 1, 2, 3$, the general solution of Eq. (6.13) becomes

$$\psi(p) = \sum_{i=1}^{3} \psi_i \alpha_i(p) = \big(\alpha_2(p) - \alpha_1(p)\big)\psi_2 + \big(\alpha_3(p) - \alpha_1(p)\big)\psi_3. \tag{6.18}$$

Let $\psi_3 = 0$. Then (6.17) gives $\psi_1 = -\psi_2$, and (6.18) implies that

$$\psi_2(p) = \big(\alpha_1(p) - \alpha_2(p)\big)C, \qquad C = \text{const} \neq 0. \tag{6.19}$$

Let $\psi_1 = 0$. Arguing similarly, we obtain

$$\psi_3(p) = \big(\alpha_2(p) - \alpha_3(p)\big)C, \qquad C = \text{const} \neq 0. \tag{6.20}$$

If $0 < \lambda < \lambda_2^0$ and $(\mu, \lambda) \in G_{11}^{(0)} \cup G_1$ or $(\mu, \lambda) \in G_{12}^{(0)} \cup G_2$, then (see assertions 2–5 in Proposition 4) the function $\Delta^{(1)}(\mu, \lambda; \cdot)$ has the respective zeros $z = z^{(1)}(\mu, \lambda) \leq 0$ or $z = z^{(1)}(\mu, \lambda) < 0$ and $z = z^{(4)}(\mu, \lambda) \leq 0$ on the interval $(-\infty, 0]$, i.e, $\Delta^{(1)}\big(\mu, \lambda; z^{(1)}(\mu, \lambda)\big) = \Delta^{(1)}\big(\mu, \lambda; z^{(4)}(\mu, \lambda)\big) = 0$. By assertion 1 in Proposition 2, we have $\Delta^{(22)}\big(\lambda; z^{(1)}(\mu, \lambda)\big) > 0$ or $\Delta^{(22)}\big(\lambda; z^{(4)}(\mu, \lambda)\big) > 0$. It follows from (6.15) that

$$\psi_1 = \psi_2 = \psi_3 = \frac{\Delta(\mu, 0; z)}{3\sqrt{\mu\lambda/2}\, b(z)}\psi_0.$$

Arguing as in the case $\lambda \geq \lambda_2^0$, we verify that the eigenfunctions of $G_{\mu\lambda}(z)$ corresponding to the eigenvalue 1 have the forms

$$\psi_1(p) = \left(\alpha_0 + k(z^{(1)}) \sum_{i=1}^{3} \alpha_i(p)\right)C, \qquad C = \text{const} \neq 0, \tag{6.21}$$

for $z = z^{(1)}(\mu, \lambda)$ and

$$\psi_4(p) = \left(\alpha_0 + k(z^{(4)}) \sum_{i=1}^{3} \alpha_i(p)\right) C, \qquad C = \text{const} \neq 0, \qquad k(z) = \frac{\Delta(\mu, 0; z)}{3\sqrt{\mu\lambda/2}\, b(z)}, \tag{6.22}$$

for $z = z^{(4)}(\mu, \lambda)$.

According to Remark 1, the eigenfunction of $h_{\mu\lambda}$ corresponding to an eigenvalue $z \in \mathbb{C} \setminus [0, 6]$ has the form

$$f(p) = \frac{(v_{\mu\lambda}^{1/2} \psi)(p)}{\varepsilon(p) - z}, \quad p \in \mathbb{T}^3.$$

Hence, by (6.21) and the relations

$$(\alpha_0, 1) = \frac{1}{\alpha_0}, \qquad (\alpha_i, 1) = 0, \qquad (\alpha_i, \alpha_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases}$$

we obtain (6.10).

Relations (6.11) and (6.12) are verified similarly.

Remark 8. It follows from (6.5), (6.6), and (6.10)–(6.12) for $z = 0$ that

$$\begin{aligned} &f_1(0) = f_4(0) = \frac{3 - \mu}{3\sqrt{\mu}\, b(0)\varepsilon(0)} C \neq 0, \quad C \neq 0, \\ &f_2(0) = f_3(0) = 0. \end{aligned} \tag{6.23}$$

Proof of the theorem. A1. Let $z \leq 0$. Then by condition A1 in the theorem and according to assertions 1 in Propositions 2 and 4, the inequalities $\Delta^{(1)}(\mu, \lambda; z) > 0$ and $\Delta^{(22)}(\lambda; z) > 0$ hold. Hence, in view of (5.2), we obtain $\Delta(\mu, \lambda; z) > 0$. By Lemma 2, the operator $h_{\mu\lambda}$ has no eigenvalues in the interval $(-\infty, 0)$, and the point $z = 0$ is an RPCS of the operator $h_{\mu\lambda}$.

A2. Let $z = 0$. It follows from assertion 2 in Proposition 4 that $\Delta^{(1)}(\mu, \lambda; 0) = 0$, and by (6.23), solution (6.10) of the equation $h_{\mu\lambda} f = 0$ satisfies the condition

$$f_1(0) = \frac{3 - \mu}{3\sqrt{\mu}\, b(0)\varepsilon(0)} C \neq 0, \quad C \neq 0.$$

Therefore, $f_1 \in L_{\mathrm{e}}^1(\mathbb{T}^3) \setminus L_{\mathrm{e}}^2(\mathbb{T}^3)$. From assertion 1 in Proposition 2, we conclude that $\Delta^{(22)}(\lambda; z) > 0$, and by Lemma 2, it follows from (5.2) that the operator $h_{\mu\lambda}$ has no eigenvalues in the interval $(-\infty, 0)$.

A3. Let $z \leq 0$. According to assertion 3 in Proposition 4, there is a unique number $z^{(1)}(\mu, \lambda) < \zeta_{\min}(\mu, \lambda) < 0$ such that $\Delta^{(1)}\big(\mu, \lambda; z^{(1)}(\mu, \lambda)\big) = 0$, and assertion 1 in Proposition 2 implies that $\Delta^{(22)}(\lambda; z) > 0$. By Lemma 2, it follows from representation (5.2) that the operator $h_{\mu\lambda}$ has the only eigenvalue $z^{(1)}(\mu, \lambda) < \zeta_{\min}(\mu, \lambda) < 0$.

The other assertions in the theorem are proved similarly to the proof of A1–A3.

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