

## FRACTIONAL GENERALIZATION OF THE QUANTUM MARKOVIAN MASTER EQUATION

V. E. Tarasov\*

We propose a generalization of the quantum Markovian equation for observables. In this generalized equation, we use superoperators that are fractional powers of completely dissipative superoperators. We prove that the suggested superoperators are infinitesimal generators of completely positive semigroups and describe the properties of this semigroup. We solve the proposed fractional quantum Markovian equation for the harmonic oscillator with linear friction. A fractional power of the Markovian superoperator can be considered a parameter describing a measure of “screening” of the environment of the quantum system: the environmental influence on the system is absent for  $\alpha = 0$ , the environment completely influences the system for  $\alpha = 1$ , and we have a powerlike environmental influence for  $0 < \alpha < 1$ .

**Keywords:** fractional power of an operator, non-Hamiltonian quantum system, quantum Markovian equation, completely positive semigroup

### 1. Introduction

Fractional calculus appeared in 1695, when Leibniz described the derivative of order  $\alpha = 1/2$  [1]–[3]. Derivatives and integrals of noninteger order were studied by Leibniz, Liouville, Grunwald, Letnikov, and Riemann. Many books have now been written about fractional calculus and fractional differential equations [1], [2], [4]–[8]. Derivatives and integrals of noninteger order and fractional integro-differential equations have found many applications in recent studies in physics (see, e.g., [9]–[12] and [13]–[16]).

In quantum mechanics, observables are given by self-adjoint operators. The dynamical description of a quantum system is given by superoperators. A *superoperator* is a map that assigns one operator some other operator.

The motion of a system is naturally described in terms of the infinitesimal change of the system. The equation for a quantum observable is called the Heisenberg equation. For Hamiltonian quantum systems, the infinitesimal superoperator is defined by some form of derivation. A derivation is a linear map  $\mathcal{L}$  that satisfies the Leibnitz rule  $\mathcal{L}(AB) = (\mathcal{L}A)B + A(\mathcal{L}B)$  for any operators  $A$  and  $B$ . A fractional derivative can be defined as the fractional power of the derivative (see, e.g., [17]). It is known that the infinitesimal generator  $\mathcal{L} = 1/(i\hbar)[H, \cdot]$ , which is used for Hamiltonian systems, is a derivative of quantum observables. In [18], we regarded a fractional power  $\mathcal{L}^\alpha$  of the derivation operator  $\mathcal{L} = 1/(i\hbar)[H, \cdot]$  as a fractional derivative on a set of observables. As a result, we obtained a fractional generalization of the Heisenberg equation, which allows generalizing the notion of Hamiltonian quantum systems. We note that a fractional generalization of classical Hamiltonian systems was suggested in [19] (also see [20]). In the general case, quantum systems are non-Hamiltonian, and  $\mathcal{L}$  is not a derivation operator. For a wide class of quantum

---

\*Skobel'syn Institute of Nuclear Physics, Lomonosov Moscow State University, Moscow, Russia, e-mail: tarasov@theory.sinp.msu.ru.

---

Prepared from an English manuscript submitted by the author; for the Russian version, see *Teoreticheskaya i Matematicheskaya Fizika*, Vol. 158, No. 2, pp. 214–233, February, 2009. Original article submitted March 27, 2008; revised June 23, 2008.

systems, the infinitesimal generator  $\mathcal{L}$  is completely dissipative [21]–[24]. Therefore, it is interesting to consider a fractional generalization of the equation of motion for non-Hamiltonian quantum systems using a fractional power of a completely dissipative superoperator.

The most general change of state of a non-Hamiltonian quantum system is a quantum operation [25]–[31]. A quantum operation for a quantum system can be described starting from a unitary evolution of some closed Hamiltonian system if the quantum system is a part of the closed system [32], [33]. But situations can arise where it is difficult or impossible to find a Hamiltonian system that includes the given quantum system. As a result, the theory of non-Hamiltonian quantum systems can be considered a fundamental generalization of the quantum mechanics of Hamiltonian systems [21]–[24]. The quantum operations that describe the dynamics of non-Hamiltonian systems can be regarded as real completely positive trace-preserving superoperators on some operator space. These superoperators form a completely positive semigroup. The infinitesimal generator of this semigroup is completely dissipative. The problem of non-Hamiltonian dynamics is to obtain an explicit form for the infinitesimal generator, which is in turn connected with the problem of determining the most general explicit form of this superoperator. This problem was investigated in [34]–[36]. Here, we consider superoperators that are fractional powers of completely dissipative superoperators. We prove that the suggested superoperators are infinitesimal generators of completely positive semigroups. The quantum Markovian equations with a completely dissipative superoperator are the most general form of the Markovian equation describing the nonunitary evolution of a density operator that is trace preserving and completely positive. We consider a fractional generalization of the quantum Markovian equation, which is solved for the harmonic oscillator with friction. We can assume that other solutions and properties described in [37]–[45] can also be considered for fractional generalizations of the quantum Markovian equation and the Gorini–Kossakowski–Sudarshan equation [34], [35].

A fractional power of infinitesimal generator can be considered a parameter describing a measure of “screening” of the environment. Using the interaction representation of the quantum Markovian equation, we consider a fractional power  $\alpha$  of the non-Hamiltonian part of the infinitesimal generator. We obtain the Heisenberg equation for Hamiltonian systems in the limit as  $\alpha \rightarrow 0$ . In the case  $\alpha = 1$ , we have the usual quantum Markovian equation. As a result, we can distinguish the following cases: (1) absence of the environmental influence ( $\alpha = 0$ ), (2) complete environmental influence ( $\alpha = 1$ ), and (3) powerlike screening of the environmental influence ( $0 < \alpha < 1$ ). The physical interpretation of the fractional quantum Markovian equation can be connected with an existence of a powerlike “screening” of the environmental influence. Quantum computations by quantum operations with mixed states (see, e.g., [30]) can be controlled by this parameter. We assume that there exist stationary states of open quantum systems [37], [40], [45]–[49] that depend on the fractional parameter. We note that it is possible to consider quantum dynamics with a low fractal dimension by a generalization of the method proposed in [50] (also see [51], [52]).

In Sec. 2, we briefly review superoperators on an operator Hilbert space and quantum operations and introduce the notation. In Sec. 3, we consider the fractional power of a superoperator. In Sec. 4, we suggest a fractional generalization of the quantum Markovian equation. In Sec. 5, we describe the properties of the fractional semigroup. In Secs. 6 and 7, we solve the fractional equations for the quantum harmonic oscillator with and without friction.

## 2. Superoperator and quantum operations

Quantum theories essentially consist of two structures: a kinematic structure describing the initial states and observables of the system and a dynamical structure describing the change of these states and observables with time. In quantum mechanics, the states and observables can be given by operators. The dynamical description of the quantum system is given by a superoperator, which is a map from a set of operators into itself.

Let  $\mathcal{M}$  be an operator space. We let  $\mathcal{M}^*$  denote the space dual to  $\mathcal{M}$ . Hence,  $\mathcal{M}^*$  is the set of all linear functionals on  $\mathcal{M}$ . The classic denotations for an element of  $\mathcal{M}$  are  $|B\rangle$  and  $B$ . The symbols  $\langle A|$  and  $\omega$  denote the elements of  $\mathcal{M}^*$ . By the Riesz–Fréchet theorem, any linear continuous functional  $\omega$  on an operator Hilbert space  $\mathcal{M}$  has the form  $\omega(B) = \langle A|B\rangle$  for all  $B \in \mathcal{M}$ , where  $|A\rangle$  is an element in  $\mathcal{M}$ . Therefore, the element  $A$  can be considered not only an element  $|A\rangle$  of  $\mathcal{M}$  but also an element  $\langle A|$  of the dual space  $\mathcal{M}^*$ . The symbol  $\langle A|B\rangle$  for a value of the functional  $\langle A|$  on the operator  $|B\rangle$  is the graphic combination of the symbols  $\langle A|$  and  $|B\rangle$ .

**Definition 1.** A *linear superoperator* is a map  $\mathcal{L}$  from an operator space  $\mathcal{M}$  into itself such that the relation

$$\mathcal{L}(aA + bB) = a\mathcal{L}(A) + b\mathcal{L}(B)$$

is satisfied for all  $A, B \in D(\mathcal{L}) \subset \mathcal{M}$ , where  $D(\mathcal{L})$  is the domain of  $\mathcal{L}$  and  $a, b \in \mathbb{C}$ .

A superoperator  $\mathcal{L}$  assigns each operator  $A \in D(\mathcal{L})$  the operator  $\mathcal{L}(A)$ .

**Definition 2.** Let  $\mathcal{L}$  be a superoperator on  $\mathcal{M}$ . An *adjoint superoperator* of  $\mathcal{L}$  is a superoperator  $\Lambda = \bar{\mathcal{L}}$  on  $\mathcal{M}^*$  such that

$$\langle \Lambda(A)|B\rangle = \langle A|\mathcal{L}(B)\rangle \quad (1)$$

for all  $B \in D(\mathcal{L}) \subset \mathcal{M}$  and  $A \in D(\Lambda) \subset \mathcal{M}^*$ .

Let  $\mathcal{M}$  be an operator Hilbert space and  $\mathcal{L}$  be a superoperator on  $\mathcal{M}$ . Then  $\langle A|B\rangle = \text{Tr}[A^\dagger B]$ , and Eq. (1) becomes

$$\text{Tr}[(\Lambda(A))^\dagger B] = \text{Tr}[A^\dagger \mathcal{L}(B)].$$

If  $\mathcal{M}$  is an operator Hilbert space, then by the Riesz–Fréchet theorem,  $\mathcal{M}$  and  $\mathcal{M}^*$  are isomorphic, and we can define the self-adjoint superoperators.

**Definition 3.** A *self-adjoint superoperator* is a superoperator  $\mathcal{L}$  on a Hilbert operator space  $\mathcal{M}$  such that  $\langle \mathcal{L}(A)|B\rangle = \langle A|\mathcal{L}(B)\rangle$  for all  $A, B \in D(\mathcal{L}) \subset \mathcal{M}$  and  $D(\mathcal{L}) = D(\bar{\mathcal{L}})$ .

Let  $\mathcal{M}$  be a normed operator space. The superoperator  $\mathcal{L}$  is said to be *bounded* if  $\|\mathcal{L}(A)\|_{\mathcal{M}} \leq c\|A\|_{\mathcal{M}}$  for some constant  $c$  and all  $A \in \mathcal{M}$ . The value

$$\|\mathcal{L}\| = \sup_{A \neq 0} \frac{\|\mathcal{L}(A)\|_{\mathcal{M}}}{\|A\|_{\mathcal{M}}}$$

is called the *norm* of the superoperator  $\mathcal{L}$ . If  $\mathcal{M}$  is a normed space and  $\mathcal{L}$  is a bounded superoperator, then  $\|\bar{\mathcal{L}}\| = \|\mathcal{L}\|$ .

In quantum theory, the class of real superoperators is the most important.

**Definition 4.** Let  $\mathcal{M}$  be an operator space and  $A^\dagger$  be an adjoint operator of  $A \in \mathcal{M}$ . A *real superoperator* is a superoperator  $\mathcal{L}$  on  $\mathcal{M}$  such that

$$[\mathcal{L}(A)]^\dagger = \mathcal{L}(A^\dagger)$$

for all  $A \in D(\mathcal{L}) \subset \mathcal{M}$  and  $A^\dagger \in D(\mathcal{L})$ .

If  $\mathcal{L}$  is a real superoperator, then  $\Lambda = \bar{\mathcal{L}}$  is real. If  $\mathcal{L}$  is a real superoperator and  $A$  is a self-adjoint operator  $A^\dagger = A \in D(\mathcal{L})$ , then the operator  $B = \mathcal{L}(A)$  is self-adjoint. Then superoperators from a set of quantum observables  $\mathcal{M}$  into itself should be real. All possible dynamics of quantum systems must be described by a set of real superoperators.

**Definition 5.** A *nonnegative* superoperator is a map  $\mathcal{L}$  from  $\mathcal{M}$  into  $\mathcal{M}$  such that  $\mathcal{L}(A^2) \geq 0$  for all  $A^2 = A^\dagger A \in D(\mathcal{L}) \subset \mathcal{M}$ . A *positive* superoperator is a map  $\mathcal{L}$  from  $\mathcal{M}$  into itself such that  $\mathcal{L}$  is nonnegative and  $\mathcal{L}(A) = 0$  if and only if  $A = 0$ .

Let  $\mathcal{M}$  denote an operator algebra. A left superoperator corresponding to  $A \in \mathcal{M}$  is a superoperator  $L_A$  on  $\mathcal{M}$  such that  $L_A(C) = AC$  for all  $C \in \mathcal{M}$ . We can think of  $L_A$  as meaning left multiplication by  $A$ . A right superoperator corresponding to  $A \in \mathcal{M}$  is a superoperator  $R_A$  on  $\mathcal{M}$  such that  $R_A(C) = CA$  for all  $C \in \mathcal{M}$ .

The most general state change of a quantum system is called a *quantum operation* [25]–[30]. A quantum operation is described by a superoperator  $\widehat{\mathcal{E}}$  that is a map on a set of density operators. If  $\rho$  is a density operator, then  $\widehat{\mathcal{E}}(\rho)$  should also be a density operator. Any density operator  $\rho_t = \rho(t)$  is a self-adjoint ( $\rho_t^\dagger = \rho_t$ ), positive ( $\rho_t > 0$ ) operator with unit trace ( $\text{Tr } \rho_t = 1$ ). Therefore, for a superoperator  $\widehat{\mathcal{E}}$  to be a quantum operation, the following conditions must be satisfied:

1. The superoperator  $\widehat{\mathcal{E}}$  is a real superoperator, i.e.,  $(\widehat{\mathcal{E}}(A))^\dagger = \widehat{\mathcal{E}}(A^\dagger)$  for all  $A$ . The real superoperator  $\widehat{\mathcal{E}}$  maps the self-adjoint operator  $\rho$  to the self-adjoint operator  $\widehat{\mathcal{E}}(\rho)$ :  $(\widehat{\mathcal{E}}(\rho))^\dagger = \widehat{\mathcal{E}}(\rho)$ .
2. The superoperator  $\widehat{\mathcal{E}}$  is a positive superoperator, i.e.,  $\widehat{\mathcal{E}}$  maps positive operators to positive operators:  $\widehat{\mathcal{E}}(A^2) > 0$  for all  $A \neq 0$  or  $\widehat{\mathcal{E}}(\rho) \geq 0$ .
3. The superoperator  $\widehat{\mathcal{E}}$  is a trace-preserving map, i.e.,  $(I|\widehat{\mathcal{E}}|\rho) = (\widehat{\mathcal{E}}^\dagger(I)|\rho) = 1$  or  $\widehat{\mathcal{E}}^\dagger(I) = I$ .

Moreover, we assume that the superoperator  $\widehat{\mathcal{E}}$  is not only positive but also completely positive [53]. The superoperator  $\widehat{\mathcal{E}}$  is a *completely positive* map from an operator space  $\mathcal{M}$  into itself if

$$\sum_{k=1}^n \sum_{l=1}^n B_k^\dagger \widehat{\mathcal{E}}(A_k^\dagger A_l) B_l \geq 0$$

for all operators  $A_k, B_k \in \mathcal{M}$  and any integer  $n$ .

Let the superoperator  $\widehat{\mathcal{E}}$  be a convex linear map on the set of density operators, i.e.,

$$\widehat{\mathcal{E}}\left(\sum_s \lambda_s \rho_s\right) = \sum_s \lambda_s \widehat{\mathcal{E}}(\rho_s),$$

where  $0 < \lambda_s < 1$  for all  $s$  and  $\sum_s \lambda_s = 1$ . Any convex linear map of density operators can be uniquely extended to a *linear* map on self-adjoint operators. We note that any linear completely positive superoperator can be represented by

$$\widehat{\mathcal{E}} = \sum_{k=1}^m \widehat{L}_{A_k} \widehat{R}_{A_k^\dagger}, \quad \widehat{\mathcal{E}}(\rho) = \sum_{k=1}^m A_k \rho A_k^\dagger.$$

If this superoperator is trace-preserving, then

$$\sum_{k=1}^m A_k^\dagger A_k = I.$$

Because all processes occur in time, it is natural to consider quantum operations  $\widehat{\mathcal{E}}(t, t_0)$  that depend on time. Let the linear superoperators  $\widehat{\mathcal{E}}(t, t_0)$  form a completely positive quantum semigroup [54] such that

$$\frac{d}{dt} \widehat{\mathcal{E}}(t, t_0) = \widehat{\Lambda}_t \widehat{\mathcal{E}}(t, t_0), \quad (2)$$

where  $\hat{\Lambda}_t$  is an infinitesimal generator of the semigroup [24], [36], [54]. The evolution of a density operator  $\rho$  is described by

$$\widehat{\mathcal{E}}(t, t_0)\rho(t_0) = \rho(t).$$

We consider quantum operations  $\widehat{\mathcal{E}}(t, t_0)$  with an infinitesimal generator  $\hat{\Lambda}$  such that the adjoint superoperator  $\mathcal{L}$  is completely dissipative, i.e.,

$$\mathcal{L}(A_k A_l) - \mathcal{L}(A_k)A_l - A_k\mathcal{L}(A_l) \geq 0$$

for all  $A_1, \dots, A_n \in D(\mathcal{L})$  such that  $A_k A_l \in D(\mathcal{L})$ . The superoperator  $\mathcal{L}$  describes the dynamics of observables of a non-Hamiltonian quantum system. The completely dissipative superoperators are infinitesimal generators of completely positive semigroups  $\{\Phi_t \mid t > 0\}$  that are adjoint to  $\{\widehat{\mathcal{E}}_t \mid t > 0\}$ , where  $\widehat{\mathcal{E}}_t = \widehat{\mathcal{E}}(t, 0)$ .

### 3. Fractional power of a superoperator

Let  $\mathcal{L}$  be a closed linear superoperator with an everywhere dense domain  $D(\mathcal{L})$  and a resolvent  $R(z, \mathcal{L})$  on the negative semiaxis and satisfy the condition

$$\|R(-z, \mathcal{L})\| \leq \frac{M}{z}, \quad z > 0, \quad M > 0. \quad (3)$$

We note that

$$R(-z, \mathcal{L}) = (zL_I + \mathcal{L})^{-1}.$$

The superoperator

$$\mathcal{L}^\alpha = \frac{\sin \pi \alpha}{\pi} \int_0^\infty dz z^{\alpha-1} R(-z, \mathcal{L}) \mathcal{L} \quad (4)$$

is defined on  $D(\mathcal{L})$  for  $0 < \alpha < 1$  and is called a *fractional power of the superoperator  $\mathcal{L}$*  [55], [56]. We note that the superoperator  $\mathcal{L}^\alpha$  allows a closure. If a closed superoperator  $\mathcal{L}$  satisfies condition (3), then  $\mathcal{L}^\alpha \mathcal{L}^\beta = \mathcal{L}^{\alpha+\beta}$  for  $\alpha > 0$ ,  $\beta > 0$ , and  $\alpha + \beta < 1$ .

Let  $\mathcal{L}$  be a closed generating superoperator of the semigroup  $\{\Phi_t \mid t \geq 0\}$ . Then the fractional power  $\mathcal{L}^\alpha$  of  $\mathcal{L}$  is given by

$$\mathcal{L}^\alpha = \frac{1}{\Gamma(-\alpha)} \int_0^\infty dz z^{-\alpha-1} (\Phi_z - L_I),$$

which is called the Balakrishnan formula.

The resolvent for the superoperator  $\mathcal{L}^\alpha$  can be found by the equation

$$\begin{aligned} R(-z, \mathcal{L}^\alpha) &= (zL_I + \mathcal{L}^\alpha)^{-1} = \\ &= \frac{\sin \pi \alpha}{\pi} \int_0^\infty dx \frac{x^\alpha}{z^2 + 2zx^\alpha \cos \pi \alpha + x^{2\alpha}} R(-x, \mathcal{L}), \end{aligned}$$

called Kato's formula. It follows from this formula that the inequality

$$\|R(-z, \mathcal{L}^\alpha)\| \leq \frac{M}{z}, \quad z > 0,$$

is satisfied with the constant  $M$  in inequality (3) for the superoperator  $\mathcal{L}$ . It follows from the inequality

$$\|zR(-z, \mathcal{L})\| = \|z(zL_I + \mathcal{L})^{-1}\| \leq M$$

for all  $z > 0$  that the superoperator  $z(zL_I + \mathcal{L})^{-1}$  is uniformly bounded in every sector of the complex plane given by the relation  $|\arg z| \leq \phi$  for  $\phi$  not greater than some number  $\pi - \psi$ ,  $0 < \psi < \pi$ . Then the superoperator  $zR(-z, \mathcal{L}^\alpha)$  is uniformly bounded in every sector of the complex plane such that  $|\arg z| \leq \phi$  for  $\phi < \pi - \alpha\psi$ .

Let  $\mathcal{L}$  be a closed generating superoperator of the semigroup  $\{\Phi_t \mid t \geq 0\}$ . Then the superoperators

$$\Phi_t^{(\alpha)} = \int_0^\infty ds f_\alpha(t, s) \Phi_s, \quad t > 0, \quad (5)$$

form a semigroup such that  $\mathcal{L}^\alpha$  is an infinitesimal generator of  $\Phi_t^{(\alpha)}$ . Equation (5) is called the Bochner–Phillips formula.

In (5), we use the function

$$f_\alpha(t, s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} dz e^{sz-tz^\alpha}, \quad (6)$$

where  $a, t > 0$ ,  $s \geq 0$ , and  $0 < \alpha < 1$ . The branch of  $z^\alpha$  is chosen such that  $\operatorname{Re} z^\alpha > 0$  for  $\operatorname{Re} z > 0$ . This branch is a one-valued function in the  $z$  plane cut along the negative real axis. This integral obviously converges by virtue of the factor  $e^{-tz^\alpha}$ . The function  $f_\alpha(t, s)$  has the following properties:

1. For all  $s > 0$ , the function  $f_\alpha(t, s)$  is nonnegative:  $f_\alpha(t, s) \geq 0$ .

2. We have the identity

$$\int_0^\infty ds f_\alpha(t, s) = 1.$$

3. For  $t > 0$  and  $x > 0$ ,

$$\int_0^\infty ds e^{-sx} f_\alpha(t, s) = e^{-tx^\alpha}.$$

4. Passing from the integration contour in (6) to the contour consisting of the two rays  $re^{-i\theta}$  and  $re^{+i\theta}$ , where  $r \in (0, \infty)$  and  $\pi/2 \leq \theta \leq \pi$ , we obtain

$$f_\alpha(t, s) = \frac{1}{\pi} \int_0^\infty dr \exp[sr \cos \theta - tr^\alpha \cos \alpha\theta] \sin(sr \sin \theta - tr^\alpha \sin \alpha\theta + \theta). \quad (7)$$

5. If  $\alpha = 1/2$ , then  $\theta = \pi$  and

$$f_{1/2}(t, s) = \frac{1}{\pi} \int_0^\infty dr e^{-sr} \sin t\sqrt{r} = \frac{t}{2\sqrt{\pi}s^{3/2}} e^{-t^2/(4s)},$$

which is a corollary of (7).

#### 4. Fractional quantum Markovian equation

The motion of a system is naturally described in terms of the infinitesimal change. This change can be described by an infinitesimal generator. One problem of the non-Hamiltonian dynamics is to obtain an explicit form of the infinitesimal generator. For this, it is necessary to find the most general explicit form of this superoperator. The problem was investigated in [34]–[36] for completely dissipative superoperators. Lindblad showed that there exists a one-to-one correspondence between the completely positive norm-continuous semigroups and completely dissipative generating superoperators [36]. Lindblad’s structural theorem gives the most general form of a completely dissipative superoperator.

**Theorem 1.** A generating superoperator  $\mathcal{L}_V$  of a completely positive unity-preserving semigroup  $\{\Phi_t = e^{-t\mathcal{L}_V} \mid t \geq 0\}$  on an operator space  $\mathcal{M}$  can be represented in the form

$$-\mathcal{L}_V(A) = -\frac{1}{i\hbar}[H, A] + \frac{1}{2\hbar} \sum_{k=1}^{\infty} (V_k^\dagger[A, V_k] + [V_k^\dagger, A]V_k), \quad (8)$$

where  $H, V_k, \sum_k V_k^\dagger, V_k^\dagger V_k \in \mathcal{M}$ .

We note that the form of  $\mathcal{L}_V$  is not uniquely fixed by (8). Indeed, formula (8) preserves its form under the changes

$$V_k \rightarrow V_k + a_k I, \quad H \rightarrow H + \frac{1}{2i\hbar} \sum_{k=1}^{\infty} (a_k^* V_k - a_k V_k^\dagger),$$

where  $a_k$  are arbitrary complex numbers.

Using  $A_t = \Phi_t(A)$ , where  $\Phi_t = e^{-t\mathcal{L}_V}$ , we obtain the equation

$$\frac{d}{dt}A_t = -\frac{1}{i\hbar}[H, A_t] + \frac{1}{2\hbar} \sum_{k=1}^{\infty} (V_k^\dagger[A_t, V_k] + [V_k^\dagger, A_t]V_k), \quad (9)$$

where  $\mathcal{L}_V$  is defined by (8). This is called the *quantum Markovian equation* for the observable  $A$ .

The Lindblad theorem gives an explicit form of the equations of motion if the following restrictions are satisfied (here  $\Lambda_V$  is adjoint to  $\mathcal{L}_V$ ):

1.  $\mathcal{L}_V$  and  $\Lambda_V$  are bounded superoperators and
2.  $\mathcal{L}_V$  and  $\Lambda_V$  are completely dissipative superoperators.

Davies extended the Lindblad result to a class of quantum dynamical semigroups with unbounded generating superoperators [57].

We consider quantum Markovian equation (9) for an observable  $A_t$ . We rewrite this equation in the form

$$\frac{d}{dt}A_t = -\mathcal{L}_V(A_t), \quad (10)$$

where  $\mathcal{L}_V$  denotes the Markovian superoperator

$$\mathcal{L}_V = L_H^- + \frac{i}{2} \sum_{k=1}^{\infty} (L_{V_k^\dagger} L_{V_k}^- - L_{V_k}^- R_{V_k}). \quad (11)$$

Here, we use the superoperators of left multiplication  $L_V$  and right multiplication  $R_V$  determined by the relations  $L_V(A) = VA$  and  $R_V(A) = AV$ . The superoperator  $L_H^-$  is a left Lie multiplication by  $A$  such that

$$L_H^-(A) = \frac{1}{i\hbar}[H, A]. \quad (12)$$

If all operators  $V_k$  are equal to zero, then  $\mathcal{L}_V = L_H^-$ , and Eqs. (10) and (11) give the Heisenberg equations for a Hamiltonian system. In the general case, the quantum system is non-Hamiltonian [24].

We obtain a fractional generalization of the quantum Markovian equation. For this, we define a fractional power for the Markovian superoperator  $\mathcal{L}_V$  in the form

$$-(\mathcal{L}_V)^\alpha = \frac{\sin \pi\alpha}{\pi} \int_0^\infty dz z^{\alpha-1} R(-z, \mathcal{L}_V) \mathcal{L}_V, \quad 0 < \alpha < 1. \quad (13)$$

The superoperator  $(\mathcal{L}_V)^\alpha$  is called a *fractional power of the Markovian superoperator*. We note that  $(\mathcal{L}_V)^\alpha(\mathcal{L}_V)^\beta = (\mathcal{L}_V)^{\alpha+\beta}$  for  $\alpha, \beta > 0$  and  $\alpha + \beta < 1$ . As a result, we obtain the equation

$$\frac{d}{dt}A_t = -(\mathcal{L}_V)^\alpha(A_t), \quad (14)$$

where  $t$ ,  $H/\hbar$ , and  $V_k/\sqrt{\hbar}$  are dimensionless variables. We call this is the *fractional quantum Markovian equation*.

If  $V_k = 0$ , then (14) gives the fractional Heisenberg equation [18] of the form

$$\frac{d}{dt}A_t = -(L_H^-)^\alpha(A_t). \quad (15)$$

The superoperator  $(L_H^-)^\alpha$  is a *fractional power of left Lie superoperator* (12). We note that this equation cannot be represented in the form

$$\frac{d}{dt}A_t = -L_{H_{\text{new}}}^-(A_t) = \frac{i}{\hbar}[H_{\text{new}}, A_t]$$

with some operator  $H_{\text{new}}$ . Therefore, quantum systems described by (15) are not Hamiltonian systems. These systems are called the *fractional Hamiltonian quantum systems* (FHQS). Usual Hamiltonian quantum systems can be considered a special case of FHQS. We note that a fractional generalization of classical Hamiltonian systems was suggested in [19], [20].

Using the operators

$$A_U(t) = U(t)A_tU^\dagger(t), \quad W_k(t) = U(t)V_kU^\dagger(t),$$

where  $U(t) = e^{1/(i\hbar)H}$ , we can write the quantum Markovian equation in the form

$$\frac{d}{dt}A_U(t) = -\tilde{\mathcal{L}}_W(A_U(t)). \quad (16)$$

The superoperator

$$\tilde{\mathcal{L}}_W = \frac{i}{2} \sum_{k=1}^{\infty} (L_{W_k^\dagger}^- L_{W_k}^- - L_{W_k}^- R_{W_k^\dagger}) \quad (17)$$

describes the non-Hamiltonian part of the evolution. Equation (16) is the quantum Markovian equation in the interaction representation. The fractional generalization of this equation is

$$\frac{d}{dt}A_U(t) = -(\tilde{\mathcal{L}}_W)^\alpha(A_U(t)). \quad (18)$$

Equation (18) is the fractional quantum Markovian equation in the interaction representation. The parameter  $\alpha$  can be considered a measure of the influence of the environment. For  $\alpha = 1$ , we have quantum Markovian equation (16). In the limit as  $\alpha \rightarrow 0$ , we obtain the Heisenberg equation for the quantum observable  $A_t$  of a Hamiltonian system. As a result, we can consider the physical interpretation of equations with a fractional power of the Markovian superoperator an influence of the environment. The following cases can be considered in quantum theory: (1) absence of the environmental influence ( $\alpha = 0$ ), (2) complete environmental influence ( $\alpha = 1$ ), and (3) powerlike screening of the environmental influence ( $0 < \alpha < 1$ ). The physical interpretation of fractional equation (18) can be connected with an existence of a powerlike screening of the environmental influence on the system.

## 5. Fractional semigroup

If we consider the Cauchy problem for Eq. (10) with the initial condition given at the time  $t = 0$  by  $A_0$ , then its solution can be written in the form  $A_t = \Phi_t A_0$ . The one-parameter superoperators  $\Phi_t$ ,  $t \geq 0$ , have the properties

$$\Phi_t \Phi_s = \Phi_{t+s}, \quad t, s > 0, \quad \Phi_0 = L_I.$$

As a result, the superoperators  $\Phi_t$  form a semigroup, and the superoperator  $\mathcal{L}_V$  is a generating superoperator of the semigroup  $\{\Phi_t \mid t \geq 0\}$ .

We consider the Cauchy problem for fractional quantum Markovian equation (14) with the initial condition given by  $A_0$ . Then its solution can be represented in the form

$$A_t(\alpha) = \Phi_t^{(\alpha)} A_0,$$

where the superoperators  $\Phi_t^{(\alpha)}$ ,  $t > 0$ , form a semigroup, which we call the *fractional semigroup*. The superoperator  $-(\mathcal{L}_V)^\alpha$  is a generating superoperator of the semigroup  $\{\Phi_t^{(\alpha)} \mid t \geq 0\}$ . We consider some properties of the fractional semigroup  $\{\Phi_t^{(\alpha)} \mid t > 0\}$ .

The superoperators  $\Phi_t^{(\alpha)}$  can be constructed in terms of  $\Phi_t$  by Bochner–Phillips formula (5), where  $f_\alpha(t, s)$  is defined in (6). If  $A_t$  is a solution of quantum Markovian equation (10), then formula (5) gives the solution

$$A_t(\alpha) = \int_0^\infty ds f_\alpha(t, s) A_s, \quad t > 0,$$

of fractional quantum Markovian equation (14).

A linear superoperator  $\Phi_t^{(\alpha)}$  is completely positive if

$$\sum_{i,j} B_i \Phi_t^{(\alpha)}(A_i^\dagger A_j) B_j \geq 0$$

for any  $A_i, B_i \in \mathcal{M}$ . The following theorem states that the fractional semigroup is completely positive.

**Theorem 2.** *If  $\{\Phi_t \mid t > 0\}$  is a completely positive semigroup of superoperator  $\Phi_t$  on  $\mathcal{M}$ , then the fractional superoperators  $\Phi_t^{(\alpha)}$  form a completely positive semigroup  $\{\Phi_t^{(\alpha)} \mid t > 0\}$ .*

**Proof.** Bochner–Phillips formula (5) gives

$$\sum_{i,j} B_i \Phi_t^{(\alpha)}(A_i^\dagger A_j) B_j = \int_0^\infty ds f_\alpha(t, s) \sum_{i,j} B_i \Phi_s(A_i^\dagger A_j) B_j$$

for  $t > 0$ . Using

$$\sum_{i,j} B_i \Phi_s(A_i^\dagger A_j) B_j \geq 0, \quad f_\alpha(t, s) \geq 0, \quad s > 0,$$

we obtain

$$\sum_{i,j} B_i \Phi_t^{(\alpha)}(A_i^\dagger A_j) B_j \geq 0.$$

**Corollary.** *If  $\Phi_t$ ,  $t > 0$ , is a nonnegative one-parameter superoperator, i.e.,  $\Phi_t(A) \geq 0$  for  $A \geq 0$ , then the superoperator  $\Phi_t^{(\alpha)}$  is nonnegative, i.e.,  $\Phi_t^{(\alpha)}(A) \geq 0$  for  $A \geq 0$ .*

Using the Bochner–Phillips formula and the property  $f_\alpha(t, s) \geq 0$ ,  $s > 0$ , we can easily prove that the superoperator  $\Phi_t^{(\alpha)}$  is nonnegative if  $\Phi_t$ ,  $t > 0$ , is a nonnegative one-parameter superoperator. This corollary can also be proved using  $B_1 = I$ ,  $A_1 = A$ , and  $A_i = B_i = 0$ ,  $i = 2, 3, \dots$ , in the proof of the theorem.

In quantum theory, the class of real superoperators is the most important. Let  $A^\dagger \in \mathcal{M}^*$  be adjoint to  $A \in \mathcal{M}$ . A real superoperator is a superoperator  $\Phi_t$  on  $\mathcal{M}$  such that  $(\Phi_t A)^\dagger = \Phi_t(A^\dagger)$  for all  $A \in D(\Phi_t) \subset \mathcal{M}$ . A quantum observable is a self-adjoint operator. If  $\Phi_t$  is a real superoperator and  $A$  is a self-adjoint operator,  $A^\dagger = A$ , then the operator  $A_t = \Phi_t A$  is self-adjoint, i.e.,  $(\Phi_t A)^\dagger = \Phi_t A$ . Let  $\mathcal{M}$  be a set of quantum observables. Then superoperators on  $\mathcal{M}$  into  $\mathcal{M}$  must be real because quantum dynamics, i.e., temporal evolutions of quantum observables, must be described by real superoperators.

**Theorem 3.** *If  $\Phi_t$  is a real superoperator, then the superoperator  $\Phi_t^{(\alpha)}$  is also real.*

**Proof.** The Bochner–Phillips formula gives

$$(\Phi_t^{(\alpha)} A)^\dagger = \int_0^\infty ds f_\alpha^*(t, s) (\Phi_s A)^\dagger, \quad t > 0.$$

Using (7), we can easily see that  $f_\alpha^*(t, s) = f_\alpha(t, s)$  is a real-valued function. Then  $(\Phi_t A)^\dagger = \Phi_t A^\dagger$  leads to  $(\Phi_t^{(\alpha)} A)^\dagger = \Phi_t^{(\alpha)}(A^\dagger)$  for all  $A \in D(\Phi_t^{(\alpha)}) \subset \mathcal{M}$ .

If  $\Phi_t$  is a superoperator on a Hilbert operator space  $\mathcal{M}$ , then an *adjoint superoperator* of  $\Phi_t$  is a superoperator  $\widehat{\mathcal{E}}_t$  on  $\mathcal{M}^*$  such that

$$(\widehat{\mathcal{E}}_t(A)|B) = (A|\Phi_t(B)) \quad (19)$$

for all  $B \in D(\Phi_t) \subset \mathcal{M}$  and  $A \in \mathcal{M}^*$ . Using the Bochner–Phillips formula, we obtain the following theorem.

**Theorem 4.** *If  $\widehat{\mathcal{E}}_t$  is an adjoint superoperator of  $\Phi_t$ , then the superoperator*

$$\widehat{\mathcal{E}}_t^{(\alpha)} = \int_0^\infty ds f_\alpha(t, s) \widehat{\mathcal{E}}_s, \quad t > 0,$$

*is an adjoint superoperator of  $\Phi_t^{(\alpha)}$ .*

**Proof.** Let  $\widehat{\mathcal{E}}_t$  be adjoint to  $\Phi_t$ , i.e., Eq. (19) is satisfied. Then

$$(\widehat{\mathcal{E}}_t^{(\alpha)} A|B) = \int_0^\infty ds f_\alpha(t, s) (\widehat{\mathcal{E}}_s A|B) = \int_0^\infty ds f_\alpha(t, s) (A|\Phi_s B) = (A|\Phi_t^{(\alpha)} B).$$

It is known that  $\widehat{\mathcal{E}}_t$  is a real superoperator if  $\Phi_t$  is real. Analogously, if  $\Phi_t^{(\alpha)}$  is a real superoperator, then  $\widehat{\mathcal{E}}_t^{(\alpha)}$  is real.

Let  $\{\widehat{\mathcal{E}}_t \mid t > 0\}$  be a completely positive semigroup such that the density operator  $\rho_t = \widehat{\mathcal{E}}_t \rho_0$  is described by

$$\frac{d}{dt} \rho_t = -\widehat{\Lambda}_V \rho_t, \quad (20)$$

where  $\widehat{\Lambda}_V$  is adjoint to the Markovian superoperator  $\mathcal{L}_V$ . The superoperator  $\widehat{\Lambda}_V$  can be represented in the form

$$\widehat{\Lambda}_V \rho_t = -\frac{1}{i\hbar} [H, \rho_t] + \frac{1}{\hbar} \sum_{k=1}^{\infty} (V_k \rho_t V_k^\dagger - (\rho_t V_k^\dagger V_k + V_k^\dagger V_k \rho_t)).$$

We note that Eq. (20) with  $V_k = 0$  gives the von Neumann equation

$$\frac{d}{dt} \rho_t = \frac{1}{i\hbar} [H, \rho_t].$$

The semigroup  $\{\widehat{\mathcal{E}}_t^{(\alpha)} \mid t > 0\}$  describes the evolution of the density operator  $\rho_t(\alpha) = \widehat{\mathcal{E}}_t^{(\alpha)} \rho_0$  by the fractional equation

$$\frac{d}{dt} \rho_t(\alpha) = -(\widehat{\Lambda}_V)^\alpha \rho_t(\alpha).$$

This is the *fractional quantum Markovian equation for the density operator*. For  $V_k = 0$ , this equation gives

$$\frac{d}{dt} \rho_t = -(-L_H^-)^\alpha \rho_t,$$

which can be called the *fractional von Neumann equation*.

## 6. Fractional equation for the harmonic oscillator

We consider a quantum harmonic oscillator such that

$$H = \frac{1}{2m} P^2 + \frac{m\omega^2}{2} Q^2, \quad V_k = 0, \quad (21)$$

where  $t$  and  $P$  are dimensionless variables. Then Eq. (14) (also see (15)) describes a harmonic oscillator. For  $A = Q$  and  $A = P$ , Eq. (14) for  $\alpha = 1$  gives

$$\frac{d}{dt} Q_t = \frac{1}{m} P_t, \quad \frac{d}{dt} P_t = -m\omega^2 Q_t.$$

The well-known solutions of these equations are

$$Q_t = Q_0 \cos \omega t + \frac{1}{m\omega} P_0 \sin \omega t, \quad P_t = P_0 \cos \omega t - m\omega Q_0 \sin \omega t. \quad (22)$$

Using these solutions and the Bochner–Phillips formula, we can obtain solutions of the fractional equations

$$\frac{d}{dt} Q_t = -(L_H^-)^\alpha Q_t, \quad \frac{d}{dt} P_t = -(L_H^-)^\alpha P_t, \quad (23)$$

where  $H$  is given by (21). The solutions of fractional equations (23) have the forms

$$Q_t(\alpha) = \Phi_t^{(\alpha)} Q_0 = \int_0^\infty ds f_\alpha(t, s) Q_s, \quad P_t(\alpha) = \Phi_t^{(\alpha)} P_0 = \int_0^\infty ds f_\alpha(t, s) P_s. \quad (24)$$

Substituting (22) in (24) gives the equations [18]

$$Q_t = Q_0 C_\alpha(t) + \frac{1}{m\omega} P_0 S_\alpha(t), \quad P_t = P_0 C_\alpha(t) - m\omega Q_0 S_\alpha(t), \quad (25)$$

where

$$C_\alpha(t) = \int_0^\infty ds f_\alpha(t, s) \cos \omega s, \quad S_\alpha(t) = \int_0^\infty ds f_\alpha(t, s) \sin \omega s.$$

Equations (25) describe solutions of fractional equations (23) for the quantum harmonic oscillator. For  $\alpha = 1/2$ , we have

$$C_{1/2}(t) = \frac{t}{2\sqrt{\pi}} \int_0^\infty ds \frac{\cos \omega s}{s^{3/2}} e^{-t^2/(4s)},$$

$$S_{1/2}(t) = \frac{t}{2\sqrt{\pi}} \int_0^\infty ds \frac{\sin \omega s}{s^{3/2}} e^{-t^2/(4s)}.$$

These functions can be represented in terms of the Macdonald function (see Sec. 2.5.37.1 in [58]), which is also called the modified Bessel function of the third kind.

It is easy to obtain the expectations

$$\langle Q_t \rangle = x_0 C_\alpha(t) + \frac{1}{m\omega} p_0 S_\alpha(t), \quad \langle P_t \rangle = p_0 C_\alpha(t) - m\omega x_0 S_\alpha(t)$$

and the dispersions

$$D_t(Q) = \frac{a^2}{2} C_\alpha^2(t) + \frac{\hbar^2}{2a^2 m^2 \omega^2} S_\alpha^2(t), \quad D_t(P) = \frac{\hbar^2}{2a^2} C_\alpha^2(t) + \frac{a^2 m^2 \omega^2}{2} S_\alpha^2(t).$$

Here, we use the coordinate representation and the pure state

$$\Psi(x) = \langle x | \Psi \rangle = \frac{1}{\sqrt{a\sqrt{\pi}}} \exp \left[ -\frac{(x-x_0)^2}{2a} + \frac{i}{\hbar} p_0 x \right]. \quad (26)$$

The expectation and dispersion are defined as usual.

## 7. Fractional quantum Markovian equation for the oscillator with friction

We consider the fractional quantum Markovian equation with  $V_k \neq 0$ . The basic assumption is that the general form of a bounded completely dissipative superoperator given by the quantum Markovian equation also holds for an unbounded completely dissipative superoperator  $\mathcal{L}_V$ . Another condition imposed on the operators  $H$  and  $V_k$  is that they are functions of the operators  $Q$  and  $P$  such that the obtained model is exactly solvable [37], [38] (also see [39], [40]). We assume that  $V_k = V_k(Q, P)$  are first-degree polynomials in  $Q$  and  $P$  and that  $H = H(Q, P)$  is a second-degree polynomial in  $Q$  and  $P$ . These assumptions are analogous to those used in classical dynamics when friction forces proportional to the velocity are considered. Then  $H$  and  $V_k$  are given in the forms

$$H = \frac{1}{2m} P^2 + \frac{m\omega^2}{2} Q^2 + \frac{\mu}{2} (PQ + QP), \quad V_k = a_k P + b_k Q, \quad (27)$$

where  $a_k$  and  $b_k$ ,  $k = 1, 2$ , are complex numbers. It is easy to obtain

$$\mathcal{L}_V Q = \frac{1}{m} P + \mu Q - \lambda Q, \quad \mathcal{L}_V P = -m\omega^2 Q - \mu P - \lambda P,$$

where

$$\lambda = \text{Im} \left( \sum_{k=1}^2 a_k b_k^* \right) = -\text{Im} \left( \sum_{k=1}^2 a_k^* b_k \right).$$

Using the matrices

$$A = \begin{pmatrix} Q \\ P \end{pmatrix}, \quad M = \begin{pmatrix} \mu - \lambda & \frac{1}{m} \\ -m\omega^2 & -\mu - \lambda \end{pmatrix},$$

we write the quantum Markovian equation for  $A_t$  as

$$\frac{d}{dt} A_t = M A_t, \quad (28)$$

where  $\mathcal{L}_V A_t = M A_t$ . The solution of (28) is

$$A_t = \Phi_t A_0 = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{L}_V^n A_0 = \sum_{n=0}^{\infty} \frac{t^n}{n!} M^n A_0.$$

The matrix  $M$  can be represented in the form  $M = N^{-1} F N$ , where  $F$  is a diagonal matrix. Let  $\nu$  be a complex parameter such that  $\nu^2 = \mu^2 - \omega^2$ . Then we have

$$N = \begin{pmatrix} m\omega^2 & \mu + \nu \\ m\omega^2 & \mu - \nu \end{pmatrix}, \quad N^{-1} = \frac{1}{2m\omega^2\nu} \begin{pmatrix} -(\mu - \nu) & \mu + \nu \\ m\omega^2 & -m\omega^2 \end{pmatrix},$$

$$F = \begin{pmatrix} -(\lambda + \nu) & 0 \\ 0 & -(\lambda - \nu) \end{pmatrix}.$$

Taking

$$\Phi_t = \sum_{n=0}^{\infty} \frac{t^n}{n!} M^n = N^{-1} \left( \sum_{n=0}^{\infty} \frac{t^n}{n!} F^n \right) N$$

into account, we obtain the superoperator  $\Phi_t$  in the form

$$\Phi_t = e^{tM} = N^{-1} e^{tF} N = e^{-\lambda t} \begin{pmatrix} \cosh \nu t + \frac{\mu}{\nu} \sinh \nu t & \frac{1}{\mu\nu} \sinh \nu t \\ -\frac{m\omega^2}{\nu} \sinh \nu t & \cosh \nu t - \frac{\mu}{\nu} \sinh \nu t \end{pmatrix}.$$

As a result, we obtain

$$Q_t = e^{-\lambda t} \left[ \cosh \nu t + \frac{\mu}{\nu} \sinh \nu t \right] Q_0 + \frac{1}{m\nu} e^{-\lambda t} \sinh(\nu t) P_0, \quad (29)$$

$$P_t = -\frac{m\omega^2}{\nu} e^{-\lambda t} \sinh(\nu t) Q_0 + e^{-\lambda t} \left[ \cosh \nu t - \frac{\mu}{\nu} \sinh \nu t \right] P_0.$$

The fractional quantum Markovian equations for  $Q_t$  and  $P_t$  are

$$\frac{d}{dt} Q_t = -(\mathcal{L}_V)^\alpha Q_t, \quad \frac{d}{dt} P_t = -(\mathcal{L}_V)^\alpha P_t, \quad (30)$$

where  $t$  and  $V_k/\sqrt{\hbar}$  are dimensionless variables. The solutions of these fractional equations are given by the Bochner–Phillips formula,

$$Q_t(\alpha) = \Phi_t^{(\alpha)} Q_0 = \int_0^\infty ds f_\alpha(t, s) Q_s, \quad t > 0, \quad (31)$$

$$P_t(\alpha) = \Phi_t^{(\alpha)} P_0 = \int_0^\infty ds f_\alpha(t, s) P_s, \quad t > 0,$$

where  $Q_s$  and  $P_s$  are given by (29) and the function  $f_\alpha(t, s)$  is defined in (6). Substituting (29) in (31) gives

$$\begin{aligned} Q_t(\alpha) &= \left[ \text{Ch}_\alpha(t) + \frac{\mu}{\nu} \text{Sh}_\alpha(t) \right] Q_0 + \frac{1}{m\nu} \text{Ch}_\alpha(t) P_0, \\ P_t(\alpha) &= -\frac{m\omega^2}{\nu} \text{Sh}_\alpha(t) Q_0 + \left[ \text{Ch}_\alpha(t) - \frac{\mu}{\nu} \text{Sh}_\alpha(t) \right] P_0, \end{aligned} \tag{32}$$

where

$$\begin{aligned} \text{Ch}_\alpha(t) &= \int_0^\infty ds f_\alpha(t, s) e^{-\lambda s} \cosh \nu s, \\ \text{Sh}_\alpha(t) &= \int_0^\infty ds f_\alpha(t, s) e^{-\lambda s} \sinh \nu s. \end{aligned}$$

For  $\alpha = 1/2$ , we have

$$\begin{aligned} \text{Ch}_{1/2}(t) &= \frac{t}{2\sqrt{\pi}} \int_0^\infty ds \frac{\cosh \nu s}{s^{3/2}} e^{-t^2/(4s) - \lambda s}, \\ \text{Sh}_{1/2}(t) &= \frac{t}{2\sqrt{\pi}} \int_0^\infty ds \frac{\sinh \nu s}{s^{3/2}} e^{-t^2/(4s) - \lambda s}. \end{aligned}$$

These functions can be represented in terms of the Macdonald function (see Sec. 2.4.17.2 in [58]) such that

$$\begin{aligned} \text{Ch}_{1/2}(t) &= \frac{t}{2\sqrt{\pi}} [V(t, \lambda, -\nu) + V(t, \lambda, \nu)], \\ \text{Sh}_{1/2}(t) &= \frac{t}{2\sqrt{\pi}} [V(t, \lambda, -\nu) - V(t, \lambda, \nu)], \end{aligned}$$

where we use the notation

$$V(t, \lambda, \nu) = \left( \frac{t^2 + 4\nu}{4\lambda} \right)^{1/4} K_{-1/2} \left( 2\sqrt{\frac{\lambda(t^2 + 4\nu)}{4}} \right),$$

where  $\text{Re } t^2 > \text{Re } \nu$ ,  $\text{Re } \lambda > 0$ , and  $K_\alpha(z)$  is the Macdonald function [1], [2].

As a result, Eqs. (32) define a solution of the fractional quantum Markovian equation for the harmonic oscillator with friction.

## 8. Conclusion

Quantum dynamics can be described by superoperators. A map assigning each operator exactly one operator is called a superoperator. It is natural to describe motion in terms of the infinitesimal change of a system. The equation of motion for a quantum observable is called the Heisenberg equation. For Hamiltonian quantum systems, the infinitesimal superoperator is some form of derivation. A linear map  $\mathcal{L}$  satisfying the Leibnitz rule  $\mathcal{L}(AB) = (\mathcal{L}A)B + A(\mathcal{L}B)$  for all operators  $A$  and  $B$  is called a derivation. It is known that the infinitesimal generator  $\mathcal{L} = 1/(i\hbar)[H, \cdot]$ , which is used for Hamiltonian systems, is a derivative of quantum observables. We can regard a fractional power  $\mathcal{L}^\alpha$  of the derivative  $\mathcal{L} = 1/(i\hbar)[H, \cdot]$  as a fractional derivative on a set of quantum observables [18]. As a result, we obtain a fractional generalization of the Heisenberg equation [18], which allows generalizing the notion of Hamiltonian quantum systems. In the general case, quantum systems are non-Hamiltonian, and  $\mathcal{L}$  is not a derivation. For a wide class of quantum systems, the infinitesimal generator  $\mathcal{L}$  is completely dissipative [21]–[24].

Here, we considered a fractional generalization of the equation of motion for non-Hamiltonian quantum systems using a fractional power of a completely dissipative superoperator. We suggested a generalization of the quantum Markovian equation for quantum observables. In this equation, we used a superoperator that is a fractional power of a completely dissipative superoperator. We proved that the suggested superoperator is an infinitesimal generator of a completely positive semigroup and described properties of this semigroup. We solved the proposed fractional quantum Markovian equation exactly for the harmonic oscillator with linear friction. A fractional power  $\alpha$  of the quantum Markovian superoperator can be considered a parameter describing a measure of “screening” of the environment. We can separate the cases where  $\alpha = 0$ , absence of the environmental influence; where  $\alpha = 1$ , complete environmental influence; and where  $0 < \alpha < 1$ , a powerlike environmental influence. A one-parameter description of a screening of the coupling between the quantum system and the environment is thus a physical interpretation of a fractional power of the quantum Markovian superoperator.

We note that the quantum Markovian equation describes a coupling between a quantum system and an environment (see [32]). Another physical interpretation of a fractional power of the infinitesimal generator is connected with Bochner–Phillips formula (5) as follows. Using the properties

$$\int_0^\infty f_\alpha(t, s) = 1, \quad f_\alpha(t, s) \geq 0, \quad s > 0,$$

we can assume that  $f_\alpha(t, s)$  is the density of a probability distribution. Then Bochner–Phillips formula (5) can be considered a smoothing of the evolution  $\Phi_t$  with respect to the time  $s > 0$ . This smoothing can be considered a screening of the environment of the quantum system.

The function  $f_\alpha(t, s)$  can be represented as the Levy distribution using a reparametrization. We note that Levy distributions are solutions of fractional equations (see, e.g., [13], [59]–[61]) that describe anomalous diffusion. It is known that quantum Markovian equations are used to describe the Brownian motion of quantum systems [37]. Perhaps, the fractional generalization of quantum Markovian equations can be used to describe anomalous processes and random walks [13]–[16] in quantum systems.

## REFERENCES

1. K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order* (Math. Sci. Eng., Vol. 111), Acad. Press, New York (1974).
2. S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, New York (1993).
3. B. Ross, “A brief history and exposition of the fundamental theory of fractional calculus,” in: *Fractional Calculus and Its Applications* (Lect. Notes Math., Vol. 457, B. Ross, ed.), Springer, Berlin (1975), pp. 1–36.
4. K. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York (1993).
5. I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution, and Some of Their Applications* (Math. Sci. Eng., Vol. 198), Acad. Press, San Diego (1999).
6. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Application of Fractional Differential Equations* (North-Holland Math. Stud., Vol. 204), Elsevier, Amsterdam (2006).
7. V. Kiryakova, *Generalized Fractional Calculus and Applications* (Pitman Research Notes Math., Vol. 301), Longman, Harlow (1993); B. Rubin, *Fractional Integrals and Potentials* (Pitman Monogr. Surv. Pure Appl. Math., Vol. 82), Longman, Harlow (1996); A. C. McBride, *Fractional Calculus and Integral Transforms of Generalized Functions* (Res. Notes Math., Vol. 31), Pitman, San Francisco (1979).
8. K. Nishimoto, *Fractional Calculus: Integrations and Differentiations of Arbitrary Order*, Descartes, Koriyama (1989).

9. G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics*, Oxford Univ. Press, Oxford (2005).
10. R. Gorenflo and F. Mainardi, “Fractional calculus: Integral and differential equations of fractional order,” in: *Fractals and Fractional Calculus in Continuum Mechanics* (CISM Courses and Lect., Vol. 378, A. Carpinteri and F. Mainardi, eds.), Springer, Wien (1997), p. 223–276.
11. B. West, M. Bologna, and P. Grigolini, *Physics of Fractal Operators*, Springer, New York (2003).
12. R. Hilfer, ed., *Applications of Fractional Calculus in Physics*, World Scientific, Singapore (2000).
13. G. M. Zaslavsky, *Phys. Rep.*, **371**, 461–580 (2002).
14. E. W. Montroll and M. F. Shlesinger, “On the wonderful world of random walks,” in: *Nonequilibrium Phenomena II: From Stochastics to Hydrodynamics* (Stud. Statist. Mech., Vol. 11, J. Lebowitz and E. Montroll, eds.), North-Holland, Amsterdam (1984), pp. 1–121.
15. R. Metzler and J. Klafter, *Phys. Rep.*, **339**, 1–77 (2000).
16. R. Metzler and J. Klafter, *J. Phys. A*, **37**, R161–R208 (2004).
17. V. E. Tarasov, *Internat. J. Math.*, **18**, 281–299 (2007).
18. V. E. Tarasov, *Phys. Lett. A*, **17**, 2984–2988 (2008).
19. V. E. Tarasov, *J. Phys. A*, **38**, 5929–5943 (2005).
20. V. E. Tarasov, *J. Phys. A*, **39**, 8409–8425 (2006).
21. A. Kossakowski, *Rep. Math. Phys.*, **3**, 247–274 (1972).
22. E. B. Davies, *Quantum Theory of Open Systems*, Acad. Press, London (1976).
23. R. S. Ingarden and A. Kossakowski, *Ann. Phys.*, **89**, 451–485 (1975).
24. V. E. Tarasov, *Quantum Mechanics of Non-Hamiltonian and Dissipative Systems*, Elsevier, Amsterdam (2008).
25. K.-E. Hellwing and K. Kraus, *Comm. Math. Phys.*, **11**, 214–220 (1969).
26. K.-E. Hellwing and K. Kraus, *Comm. Math. Phys.*, **16**, 142–147 (1970).
27. K. Kraus, *Ann. Phys.*, **64**, 311–335 (1971).
28. K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory* (Lect. Notes Phys., Vol. 190), Springer, Berlin (1983).
29. B. Schumacher, *Phys. Rev. A*, **54**, 2614–2628 (1996).
30. V. E. Tarasov, *J. Phys. A*, **35**, 5207–5235 (2002).
31. V. E. Tarasov, *J. Phys. A*, **37**, 3241–3257 (2004).
32. L. Accardi, Y. G. Lu, and I. V. Volovich, *Quantum Theory and Its Stochastic Limit*, Springer, New York (2002).
33. U. Weiss, *Quantum Dissipative Systems* (Ser. Modern Condensed Matter Phys., Vol. 2), World Scientific, Singapore (1993).
34. V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, *J. Math. Phys.*, **17**, 821–825 (1976).
35. V. Gorini, A. Frigerio, M. Verri, A. Kossakowski, and E. C. G. Sudarshan, *Rep. Math. Phys.*, **13**, 149–173 (1978).
36. G. Lindblad, *Comm. Math. Phys.*, **48**, 119–130 (1976).
37. G. Lindblad, *Rep. Math. Phys.*, **10**, 393–406 (1976).
38. A. Sandulescu and H. Scutaru, *Ann. Phys.*, **173**, 277–317 (1987).
39. A. Isar, A. Sandulescu, H. Scutaru, E. Stefanescu, and W. Scheid, *Internat. J. Mod. Phys. E*, **3**, 635–714 (1994).
40. A. Isar, A. Sandulescu, and W. Scheid, *Internat. J. Mod. Phys. E*, **10**, 2767–2779 (1996).
41. E. B. Davies, *Ann. Inst. H. Poincaré Sec. A*, **35**, 149–171 (1981).
42. D. A. Lidar, Z. Bihary, and K. B. Whaley, *Chem. Phys.*, **268**, 35–53 (2001).
43. H. Nakazato, Y. Hida, K. Yuasa, B. Militello, A. Napoli, and A. Messina, *Phys. Rev. A*, **74**, 062113 (2006).
44. K. Dietz, *J. Phys. A*, **35**, 10573–10590 (2002).
45. V. E. Tarasov, *Phys. Rev. E*, **66**, 056116 (2002).
46. E. B. Davies, *Comm. Math. Phys.*, **19**, 83–105 (1970).
47. H. Spohn, *Rep. Math. Phys.*, **10**, 189–194 (1976).
48. H. Spohn, *Lett. Math. Phys.*, **2**, 33–38 (1977).
49. C. Anastopoulos and J. J. Halliwell, *Phys. Rev. D*, **51**, 6870–6885 (1995).
50. V. E. Tarasov and G. M. Zaslavsky, *Phys. A*, **368**, 399–415 (2006).
51. A. Tofighi and H. N. Pour, *Phys. A*, **374**, 41–45 (2007).

52. A. Tofghi and A. Golestani, *Phys. A*, **387**, 1807–1817 (2008).
53. W. Arveson, *Pacific J. Math.*, **203**, 67–77 (2002).
54. R. Alicki and K. Lendi, *Quantum Dynamical Semigroups and Applications* (Lect. Notes Phys., Vol. 286), Springer, Berlin (1987).
55. E. Hille and R. S. Phillips, *Functional Analysis and Semigroups* (Amer. Math. Soc. Colloq. Publ., Vol. 31), Amer. Math. Soc., Providence, R. I. (1957).
56. K. Yosida, *Functional Analysis* (Grundlehren Math. Wiss., Vol. 123), Springer, Berlin (1965).
57. E. B. Davies, *Rep. Math. Phys.*, **11**, 169–188 (1977).
58. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series* [in Russian] (2nd ed.), Vol. 2, *Elementary Functions*, Nauka, Moscow (2002); English transl. prev. ed., Gordon and Breach, New York (1986).
59. A. I. Saichev and G. M. Zaslavsky, *Chaos*, **7**, 753–764 (1997).
60. V. E. Tarasov and G. M. Zaslavsky, *Commun. Nonlinear Sci. Numer. Simul.*, **13**, 248–258 (2008).
61. V. V. Yanovsky, A. V. Chechkin, D. Schertzer, and A. V. Tur, *Phys. A*, **282**, 13–34 (2000).