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DESCRIPTION OF WEAKLY PERIODIC GIBBS MEASURES FOR THE ISING MODEL ON A CAYLEY TREE

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We introduce the concept of a weakly periodic Gibbs measure. For the Ising model, we describe a set of such measures corresponding to normal subgroups of indices two and four in the group representation of a Cayley tree. In particular, we prove that for a Cayley tree of order four, there exist critical values $T_c < T_{cr}$ of the temperature $T > 0$ such that there exist five weakly periodic Gibbs measures for $0 < T < T_c$ or $T > T_{cr}$, three weakly periodic Gibbs measures for $T = T_c$, and one weakly periodic Gibbs measure for $T_c < T \leq T_{cr}$.

Keywords: Cayley tree, Gibbs measure, Ising model, weakly periodic measure

1. Introduction

One of the main problems of the Ising model Hamiltonian is to describe all limiting Gibbs measures corresponding to this Hamiltonian. It is well known that for the Ising model, such measures form a nonempty convex compact subset in the set of all probability measures. The problem of completely describing the elements of this set is far from being completely solved. Some translation-invariant (see, e.g., [1]–[3]), periodic [4], [5], and continuum sets of nonperiodic [1], [6] Gibbs measures for the Ising model on a Cayley tree have already been described.

Periodic Gibbs measures for certain models with a finite interaction radius were described in [4], [5], [7]–[11], where translation-invariant and periodic measures (of period two) were mainly considered. In this paper, we introduce a more general concept of a periodic Gibbs measure and verify that such measures exist for the Ising model.

This paper is organized as follows. We give the necessary definitions and the problem statement in Sec. 2. We describe weakly periodic measures in Sec. 3. In Sec. 3.1, we study weakly periodic measures corresponding to normal subgroups of index two, and in Sec. 3.2, we consider one normal subgroup of index four whose choice ensures the minimum number of unknowns in the corresponding system of equations. In Sec. 4, we discuss the obtained results and formulate several problems that remain open.

2. Definitions and the problem statement

Let $\tau^k = (V, L)$, $k \geq 1$, be a Cayley tree of order k, i.e., an infinite tree with exactly $k+1$ edges issuing from each vertex. Here, V is the set of its vertices, and L is the set of edges τ^k. It is well known that τ^k can be represented as a free product G_k of $k+1$ cyclic groups of the second order with the generators $a_1, a_2, \dots, a_{k+1}$.

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For an arbitrary point $x^0 \in V$, we set $W_n = \{x \in V \mid d(x^0, x) = n\}$, $V_n = \bigcup_{m=0}^{n} W_m$, and $L_n = \{\langle x, y\rangle \in L \mid x, y \in V_n\}$, where $d(x, y)$ is the distance between x and y on the Cayley tree, i.e., the number of edges in the path connecting x to y.

Let $\Phi = \{-1, 1\}$, and let σ be a configuration on V, i.e., $\sigma = \{\sigma(x) \in \Phi \colon x \in V\}$, $\Omega = \Phi^V$. Let $A \subset V$. We let Ω_A denote the space of configurations defined on the set A and taking values in Φ.

We consider the Hamiltonian for the Ising model

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \sigma(x)\sigma(y), \tag{1}$$

where $J \in \mathbb{R}$ and $\langle x, y\rangle$ are nearest neighbors. Let $h_x \in \mathbb{R}$ and $x \in V$. For each n, we define the measure μ_n on Ω_{V_n}, assuming

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\biggl\{-\beta H(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x)\biggr\}, \tag{2}$$

where $\beta = T^{-1}$ is the inverse temperature, $T > 0$, $\sigma_n = \{\sigma(x), x \in V_n\} \in \Omega_{V_n}$, Z_n^{-1} is the normalization factor, and

$$H(\sigma_n) = -J \sum_{\langle x,y\rangle \in L_n} \sigma(x)\sigma(y).$$

The compatibility conditions for a sequence of measures $\mu_n(\sigma_n)$, $n \geq 1$, are given by the equality

$$\sum_{\sigma^{(n)}} \mu_n(\sigma_{n-1}, \sigma^{(n)}) = \mu_{n-1}(\sigma_{n-1}), \qquad \sigma^{(n)} = \{\sigma(x), x \in W_n\}. \tag{3}$$

Let μ_n, $n \geq 1$, be a sequence of measures on Ω_{V_n} having compatibility property (3). According to the Kolmogorov theorem, there then exists a unique limiting measure μ on $\Omega_V = \Omega$ (called the limiting Gibbs measure) such that $\mu(\sigma_n) = \mu_n(\sigma_n)$ for each $n = 1, 2, \dots$. It is well known that measures (2) satisfy (3) if and only if the set of quantities $h = \{h_x, x \in G_k\}$ is such that

$$h_x = \sum_{y \in S(x)} f(h_y, \theta), \tag{4}$$

where $S(x)$ is the set of "direct descendants" of a point $x \in V$ and $f(x, \theta) = \operatorname{arctanh}(\theta \tanh x)$, $\theta = \tanh(J\beta)$ (see [1]–[3]).

Definition 1. A set of quantities $h = \{h_x, x \in G_k\}$ is said to be $\widehat{G}_k$ *periodic* if $h_{xy} = h_x$ for any $x \in G_k$ and $y \in \widehat{G}_k$ (here, $\widehat{G}_k$ is a normal subgroup of index $r \geq 1$).

In this paper, we give a more general definition of the periodic Gibbs measure. For $x \in G_k$, we set $x_\downarrow = \{y \in G_k \colon \langle x, y\rangle\} \setminus S(x)$. Let $G_k/\widehat{G}_k = \{H_1, \dots, H_r\}$ be a quotient group.

Definition 2. A set of quantities $h = \{h_x, x \in G_k\}$ is said to be $\widehat{G}_k$ *weakly periodic* if $h_x = h_{ij}$ for $x \in H_i$ and $x_\downarrow \in H_j$ for any $x \in G_k$.

We note that the weakly periodic set of h coincides with an ordinary periodic one (see Definition 1) if the quantity h_x is independent of $x_\downarrow$.

Definition 3. A measure μ is said to be $\widehat{G}_k$ (*weakly*) *periodic* if it corresponds to the $\widehat{G}_k$-(weakly) periodic set of h. The G_k-periodic measure is said to be *translation invariant*.

The aim of our paper is to describe the set of weakly periodic Gibbs measures for the Ising model.

3. Weakly periodic measures

The level of difficulty in describing weakly periodic Gibbs measures is related to the structure and index of the normal subgroup relative to which the periodicity condition is imposed. The authors of [12] proved that in the group G_k, there is no normal subgroup of odd index different from one. Therefore, we consider normal subgroups of even indices. In this paper, we restrict ourself to the cases of indices two and four.

3.1. The case of index two. We describe $\overline{G}_k$-weakly periodic Gibbs measures for any normal subgroup $\overline{G}_k$ of index two. We note that any normal subgroup of index two of the group G_k has the form $H_A = \{x \in G_k \colon \sum_{i\in A} w_x(a_i) \text{ is an even number}\}$, where $\varnothing \neq A \subseteq N_k = \{1, 2, \ldots, k+1\}$ and $w_x(a_i)$ is the number of letters a_i in a word $x \in G_k$ [4].

Let $A \subset N_k$ and H_A be the corresponding normal subgroup of index two. We note that in the case $|A| = k+1$ ($|A|$ is the cardinality of the set A), i.e., in the case $A = N_k$, weak periodicity coincides with ordinary periodicity. Therefore, we consider $A \subset N_k$ such that $A \neq N_k$. Then, in view of (4), the H_A-weakly periodic set of h has the form

$$h_x = \begin{cases} h_1, & x \in H_A, \quad x_\downarrow \in H_A \\ h_2, & x \in H_A, \quad x_\downarrow \in G_k \setminus H_A \\ h_3, & x \in G_k\setminus H_A, \quad x_\downarrow \in H_A \\ h_4, & x \in G_k\setminus H_A, \quad x_\downarrow \in G_k\setminus H_A, \end{cases} \tag{5}$$

where the $h_i, i = \overline{1,4}$, satisfy the system of equations

$$\begin{aligned} h_1 &= |A| f(h_3,\theta) + (k - |A|) f(h_1,\theta), \\ h_2 &= (|A|-1) f(h_3,\theta) + (k+1-|A|) f(h_1,\theta), \\ h_3 &= (|A|-1) f(h_2,\theta) + (k+1-|A|) f(h_4,\theta), \\ h_4 &= |A| f(h_2,\theta) + (k - |A|) f(h_4,\theta). \end{aligned} \tag{6}$$

We consider the map $W\colon \mathbb{R}^4 \to \mathbb{R}^4$, defined as $W(h) = h'$ if

$$\begin{aligned} h_1' &= |A| f(h_3,\theta) + (k - |A|) f(h_1,\theta), \\ h_2' &= (|A|-1) f(h_3,\theta) + (k+1-|A|) f(h_1,\theta), \\ h_3' &= (|A|-1) f(h_2,\theta) + (k+1-|A|) f(h_4,\theta), \\ h_4' &= |A| f(h_2,\theta) + (k - |A|) f(h_4,\theta). \end{aligned} \tag{7}$$

We note that system (6) is equivalent to the equation $h = W(h)$. The map W has the invariant sets

$$I_1 = \{h \in \mathbb{R}^4 \colon h_1 = h_2 = h_3 = h_4\}, \qquad I_2 = \{h \in \mathbb{R}^4 \colon h_1 = h_4, h_2 = h_3\},$$
$$I_3 = \{h \in \mathbb{R}^4 \colon h_1 = -h_4, h_2 = -h_3\}.$$

Let $\alpha = (1-\theta)/(1+\theta)$.

Theorem 1. *The following assertions hold*:

1. *For the Ising model, all H_A-weakly periodic Gibbs measures on I_1 and I_2 are translation invariant.*
2. *For $|A| = k$ and $\theta > 0$, all H_A-weakly periodic Gibbs measures are translation invariant.*
3. *For $|A| = 1$ and $k = 4$, there exists a critical value $\alpha_{\rm cr}$ (≈ 0.152) such that there exist five H_A-weakly periodic Gibbs measures μ_0, $\mu_1^{\pm}$, and $\mu_2^{\pm}$ for $0 < \alpha < \alpha_{\rm cr}$, three H_A-weakly periodic Gibbs measures μ_0 and $\mu_1^{\pm}$ for $\alpha = \alpha_{\rm cr}$, and only one H_A-weakly periodic Gibbs measure μ_0 for $\alpha > \alpha_{\rm cr}$,.*
4. *For $|A| = 1$, $k > 5$, and $\theta_- < \theta < \theta_+$, $\theta_\pm = \big(k - 1 \pm \sqrt{k^2 - 6k + 1}\big)/2k$, there exist three H_A-weakly periodic Gibbs measures μ^0 and $\mu^{\pm}$ on I_3.*

Proof. 1. It suffices to show that system of equations (6) has only one root of the form $h_1 = h_2 = h_3 = h_4$. The proof is obvious for the invariant set I_1. We prove this assertion for the invariant set I_2.

Using the fact that

$$f(h,\theta) = \operatorname{arctanh}(\theta \tanh h) = \frac{1}{2}\log\frac{(1+\theta)e^{2h} + (1-\theta)}{(1-\theta)e^{2h} + (1+\theta)}$$

and introducing the notation $z_i = e^{2h_i}$, $i = \overline{1,4}$, we obtain the following system of equations instead of (6):

$$\begin{aligned}
&z_1 - z_2 = A_1(z_3 - z_1),\\
&z_1 - z_3 = A_2(z_1 - z_4) + B_2(z_3 - z_4) + C_2(z_3 - z_2),\\
&z_1 - z_4 = A_3(z_1 - z_4) + B_3(z_3 - z_2),\\
&z_2 - z_3 = A_4(z_3 - z_2) + B_4(z_1 - z_4),\\
&z_2 - z_4 = A_5(z_3 - z_2) + B_5(z_1 - z_2) + C_5(z_1 - z_4),\\
&z_3 - z_4 = A_6(z_4 - z_2),
\end{aligned} \tag{8}$$

where $A_i = (1-\alpha^2)\tilde{A}_i(z_1, z_2, z_3, z_4)$, $B_i = (1-\alpha^2)\widetilde{B}_i(z_1, z_2, z_3, z_4)$, $C_i = (1-\alpha^2)\widetilde{C}_i(z_1, z_2, z_3, z_4)$, and $\tilde{A}_i$, $\widetilde{B}_i$, and $\widetilde{C}_i$ are positive for all $i = \overline{1,6}$.

On the invariant set I_2, we have $h_2 = h_3$. As a result, for $\alpha < 1$, the equality $z_1 - z_2 = A_1(z_3 - z_1)$ implies $z_1 = z_2$.

In the antiferromagnetic case, i.e., for $\alpha > 1$, we obtain $A_i, B_i, C_i < 0$ for all $i = \overline{1,6}$. For the invariant set I_2, the equality $h_2 = h_3$ holds. From (8), we have $z_2 - z_1 = -A_1(z_3 - z_1)$, whence $z_1 = z_2$. Consequently, for any $\alpha > 0$, we have $z_1 = z_2$, whence $z_1 = z_2 = z_3 = z_4$ on I_2.

2. In the case $|A| = k$, we obtain

$$\begin{aligned}
&h_2 = (k-1)f(h_3,\theta) + f\big(kf(h_3,\theta),\theta\big),\\
&h_3 = (k-1)f(h_2,\theta) + f\big(kf(h_2,\theta),\theta\big)
\end{aligned} \tag{9}$$

from (6). We now prove that system (9) has only solutions of the form $h_2 = h_3$. We consider the case $h_2 > h_3$. From (9), we then obtain

$$h_2 - h_3 = (k-1)\big(f(h_3,\theta) - f(h_2,\theta)\big) + f\big(kf(h_3,\theta),\theta\big) - f\big(kf(h_2,\theta),\theta\big). \tag{10}$$

It is easy to verify that the function f is strictly increasing. Consequently, equality (10) cannot hold, because its left-hand side contains a positive quantity and its right-hand side contains a negative one. Equation (10) also does not hold in the case $h_2 < h_3$. Therefore, $h_2 = h_3$, which gives translation-invariant solutions of system (6).

3. The proof of the third assertion in the theorem follows from Lemma 3 in Sec. 3.2. In this case, it is necessary to analyze the solutions of the equation

$$h = 3f(h, \theta) - f\big(4f(h, \theta), \theta\big),$$

which is obtained by restricting the operator W to I_3.

4. For $|A| = 1$ and $k > 5$, we obtain

$$h_1 = g(h_1, \theta, k) \tag{11}$$

from (6), where $g(x) = g(x, \theta, k) = -f\big(kf(x, \theta), \theta\big) + (k-1)f(x, \theta)$, $x \in \mathbb{R}$. We note that $g(0) = 0$ and g is an odd bounded function. It follows from these properties that if $g'(0) > 1$, then Eq. (11) has at least three solutions. It is easy to verify that the inequality $g'(0) > 1$ is equivalent to $\theta_- < \theta < \theta_+$. In this case, Eq. (6) has three solutions

$$\big(\pm h_1^*, \pm kf(h_1^*, \theta), \mp kf(h_1^*, \theta), \mp h_1^*\big), \qquad (0, 0, 0, 0).$$

Remark 1. The measures $\mu^{\pm}$ and $\mu_i^{\pm}$, $i = 1, 2$, are H_A weakly periodic, and this provides new Gibbs measures for the Ising model. All the other measures constructed in the theorem are translation invariant.

Remark 2. If $A \subset N_k$ is such that $|A| \neq 1$ or $|A| \neq k$, then it is difficult to obtain a solution of system of equations (6) outside the invariant sets I_1 and I_2. Even for the invariant set I_3, system (6) is a system with two unknowns, which is difficult to solve.

3.2. The case of index four. Let $H_{\{a_1\}} = \{x \in G_k\colon w_x(a_1) \text{ is an even number}\}$, $G_k^{(2)} = \{x \in G_k\colon |x| \text{ is an even number}\}$, and $G_k^{(4)} = H_{\{a_1\}} \cap G_k^{(2)}$ be the corresponding normal subgroup of index four.

Remark 3. Among all normal subgroups of index four, our chosen normal subgroup $G_k^{(4)}$ is convenient because we obtain a system of equations with eight unknowns from system (4) in this case, while the number of unknowns can reach 16 for an arbitrary normal subgroup of index four.

We consider a quotient group $G_k/G_k^{(4)} = \{H_0, H_1, H_2, H_3\}$, where

$$H_0 = \{x \in G_k\colon w_x(a_1) \text{ is an even number, } |x| \text{ is an even number}\},$$

$$H_1 = \{x \in G_k\colon w_x(a_1) \text{ is an odd number, } |x| \text{ is an even number}\},$$

$$H_2 = \{x \in G_k\colon w_x(a_1) \text{ is an even number, } |x| \text{ is an odd number}\},$$

$$H_3 = \{x \in G_k\colon w_x(a_1) \text{ is an odd number, } |x| \text{ is an odd number}\}.$$

In view of (4), the $G_k^{(4)}$-weakly periodic set of h then has the form

$$h_x = \begin{cases} h_1, & x \in H_3, \quad x_\downarrow \in H_1 \\ h_2, & x \in H_1, \quad x_\downarrow \in H_3 \\ h_3, & x \in H_3, \quad x_\downarrow \in H_0 \\ h_4, & x \in H_0, \quad x_\downarrow \in H_3 \\ h_5, & x \in H_1, \quad x_\downarrow \in H_2 \\ h_6, & x \in H_2, \quad x_\downarrow \in H_1 \\ h_7, & x \in H_2, \quad x_\downarrow \in H_0 \\ h_8, & x \in H_0, \quad x_\downarrow \in H_2, \end{cases} \tag{12}$$

where the h_i, $i = \overline{1,8}$, satisfy the system of equations

$$\begin{aligned} h_1 &= (k-1)f(h_2,\theta) + f(h_4,\theta), \\ h_2 &= (k-1)f(h_1,\theta) + f(h_6,\theta), \\ h_3 &= kf(h_2,\theta), \\ h_4 &= kf(h_7,\theta), \\ h_5 &= kf(h_1,\theta), \\ h_6 &= kf(h_8,\theta), \\ h_7 &= (k-1)f(h_8,\theta) + f(h_5,\theta), \\ h_8 &= (k-1)f(h_3,\theta) + f(h_7,\theta). \end{aligned} \tag{13}$$

This system can be rewritten as $h = W(h)$, where the map $W\colon \mathbb{R}^4 \to \mathbb{R}^4$ is defined as $W(h) = h'$ if

$$\begin{aligned} h_1' &= (k-1)f(h_2,\theta) + f\big(kf(h_7,\theta),\theta\big), \\ h_2' &= (k-1)f(h_1,\theta) + f\big(kf(h_8,\theta),\theta\big), \\ h_7' &= (k-1)f(h_8,\theta) + f\big(kf(h_1,\theta),\theta\big), \\ h_8' &= (k-1)f(h_7,\theta) + f\big(kf(h_2,\theta),\theta\big). \end{aligned} \tag{14}$$

It is easy to prove the following lemma.

Lemma 1. *The map W has the invariant sets*

$$\begin{aligned} &I_1 = \{h \in \mathbb{R}^4\colon h_1 = h_2 = h_7 = h_8\}, && I_2 = \{h \in \mathbb{R}^4\colon h_1 = h_2, h_7 = h_8\}, \\ &I_3 = \{h \in \mathbb{R}^4\colon h_1 = -h_2, h_7 = -h_8\}, && I_4 = \{h \in \mathbb{R}^4\colon h_1 = h_2 = -h_7 = -h_8\}, \\ &I_5 = \{h \in \mathbb{R}^4\colon h_1 = h_7, h_2 = h_8\}, && I_6 = \{h \in \mathbb{R}^4\colon h_1 = -h_7, h_2 = -h_8\}, \\ &I_7 = \{h \in \mathbb{R}^4\colon h_1 = h_7 = -h_2 = -h_8\}, && I_8 = \{h \in \mathbb{R}^4\colon h_1 = h_8, h_2 = h_7\}, \\ &I_9 = \{h \in \mathbb{R}^4\colon h_1 = -h_8, h_2 = -h_7\}, && I_{10} = \{h \in \mathbb{R}^4\colon h_1 = h_8 = -h_2 = -h_7\}. \end{aligned}$$

We note that restricting the operator W to I_1 yields translation-invariant measures studied previously.

It is easy to verify that under certain additional conditions imposed on the variables, by restricting the operator W to the other sets I_i, $i = 2, \dots, 10$, we can reduce the system of equations $W(h) = h$ to equations for a single unknown having one of the forms

$$x = -(k-1)f(x,\theta) + f\bigl(kf(x,\theta),\theta\bigr), \tag{15}$$

$$x = (k-1)f(x,\theta) - f\bigl(kf(x,\theta),\theta\bigr). \tag{16}$$

Equation (15) reduces to the equation

$$(u^2-1)P_{2k-2}(u) = 0, \tag{17}$$

where $u = (z+\alpha)/(\alpha z+1)$ and $P_{2k-2}(u)$ is a symmetric polynomial of degree $2k-2$. It is well known that setting $u + 1/u = \xi$, we can decrease the degree of the equation $P_{2k-2}(u) = 0$ twofold, i.e., reduce this equation to the equation $P_{k-1}(\xi) = 0$, where $P_{k-1}(\xi)$ is a nonsymmetric polynomial of degree $k-1$ in the general case. But for $k \geq 6$, the equation $P_{k-1}(\xi) = 0$ cannot be solved in radicals.

We consider the case $k = 4$, where Eq. (17) has the form

$$(u^2-1)\bigl(u^6 - \alpha u^5 + u^4 + (1-\alpha)u^3 + u^2 - \alpha u + 1\bigr) = 0. \tag{18}$$

Lemma 2. *Equation* (18) *has three solutions* $u_0 = 1$, $u_1 = u_*$, *and* $u_2 = 1/u_*$ *for* $\alpha > \alpha'_{\mathrm{cr}} = 5/3$ *and the unique solution* $u_0 = 1$ *for* $0 < \alpha \leq 5/3$.

Proof. For Eq. (18), $u = 1$ is a solution. We assume that $u \neq 1$. Setting $\xi = u + 1/u > 2$, we obtain the equation

$$\xi^3 - \alpha\xi^2 - 2\xi + \alpha + 1 = 0.$$

A detailed analysis of this equation shows that this lemma holds.

Similarly, for $k = 4$, we have

$$\alpha^2(u^8-1) - \alpha u(u^6-1) + u^3(u^2-1) = 0 \tag{19}$$

from Eq. (16).

Lemma 3. *There exists a critical value* α_{cr} (≈ 0.152) *such that Eq.* (19) *has*

1. *five solutions* $u_0 = 1$, $u_1 = u_*^{(1)}$, $u_2 = 1/u_*^{(1)}$, $u_3 = u_*^{(2)}$, *and* $u_4 = 1/u_*^{(2)}$ *for* $0 < \alpha < \alpha_{\mathrm{cr}}$,
2. *three solutions* $u_0 = 1$, $u_1 = u_*^{(1)}$, *and* $u_2 = 1/u_*^{(1)}$ *for* $\alpha = \alpha_{\mathrm{cr}}$, *and*
3. *the unique solution* $u_0 = 1$ *for* $\alpha > \alpha_{\mathrm{cr}}$.

Proof. For any $\alpha > 0$, $u = 1$ is a solution of Eq. (19). Dividing Eq. (19) by $u^2 - 1$ and introducing the notation $\xi = u + 1/u$, we reduce Eq. (19) to the equation

$$\varphi(\xi) = \alpha^2\xi^3 - \alpha\xi^2 - 2\alpha^2\xi + \alpha + 1 = 0.$$

The assertions in the lemma follow from the easily verified properties of the function $\varphi(\xi)$.

Solving the equation $\varphi'(\xi)=0$, we obtain $\xi=\xi_*=\bigl(1+\sqrt{1+6\alpha^2}\,\bigr)/3\alpha$. The values α_{cr} can be found from the equation $\varphi(\xi_*)=0$, i.e., from the equation

$$f(\alpha)=9\alpha^2+27\alpha-2-2\bigl(\sqrt{6\alpha^2+1}\,\bigr)^3=0,\quad 0<\alpha<\frac{2}{5}.$$

We note that f increases in the interval $(0,2/5)$. Because $f(0)=-4$ and $f(2/5)>0$, the existence and uniqueness of a value α_{cr} such that $f(\alpha_{\mathrm{cr}})=0$ follow from the monotonicity of f. A computer calculation shows that $\alpha_{\mathrm{cr}}\approx 0.152$.

Combining Lemmas 2 and 3, we obtain the following theorem.

Theorem 2. *For $k=4$, there exist critical values $\alpha_{\mathrm{cr}}\approx 0.152$ and $\alpha'_{\mathrm{cr}}=5/3$ such that*

1. *for $0<\alpha<\alpha_{\mathrm{cr}}$, there exist five weakly periodic Gibbs measures μ_0, $\tilde\mu_1^{\pm}$, and $\tilde\mu_2^{\pm}$, which correspond to the solutions of system of equations* (13),

$$h_x=0,\qquad \pm h_x^{(i)}=\begin{cases}\pm h_*^{(i)}, & x\in H_3,\quad x_\downarrow\in H_1\\ \pm h_*^{(i)}, & x\in H_1,\quad x_\downarrow\in H_3\\ \pm 4f(h_*^{(i)},\theta), & x\in H_3,\quad x_\downarrow\in H_0\\ \mp 4f(h_*^{(i)},\theta), & x\in H_0,\quad x_\downarrow\in H_3\\ \pm 4f(h_*^{(i)},\theta), & x\in H_1,\quad x_\downarrow\in H_2\\ \mp 4f(h_*^{(i)},\theta), & x\in H_2,\quad x_\downarrow\in H_1\\ \mp h_*^{(i)}, & x\in H_2,\quad x_\downarrow\in H_0\\ \mp h_*^{(i)}, & x\in H_0,\quad x_\downarrow\in H_2,\end{cases}$$

$$h_*^{(i)}=\frac{1}{2}\log\frac{\alpha-u_*^{(i)}}{1-\alpha u_*^{(i)}},\quad i=1,2;$$

2. *for $\alpha=\alpha_{\mathrm{cr}}$, there exist three weakly periodic Gibbs measures μ_0 and $\tilde\mu_1^{\pm}$;*

3. *for $\alpha_{\mathrm{cr}}<\alpha\le\alpha'_{\mathrm{cr}}$, there exists one Gibbs measure μ_0, which corresponds to $h_x=0$; and*

4. *for $\alpha>\alpha'_{\mathrm{cr}}$, there exist five weakly periodic Gibbs measures μ_0, $\mu_{*,1}^{\pm}$, and $\mu_{*,2}^{\pm}$, which correspond to solutions of the form*

$$h_x=0,\qquad \pm\bar h_x^{(1)}=\begin{cases}\pm\bar h_*^{(1)}, & x\in H_3,\quad x_\downarrow\in H_1\\ \mp\bar h_*^{(1)}, & x\in H_1,\quad x_\downarrow\in H_3\\ \mp 4f(\bar h_*^{(1)},\theta), & x\in H_3,\quad x_\downarrow\in H_0\\ \pm 4f(\bar h_*^{(1)},\theta), & x\in H_0,\quad x_\downarrow\in H_3\\ \pm 4f(\bar h_*^{(1)},\theta), & x\in H_1,\quad x_\downarrow\in H_2\\ \mp 4f(\bar h_*^{(1)},\theta), & x\in H_2,\quad x_\downarrow\in H_1\\ \pm\bar h_*^{(1)}, & x\in H_2,\quad x_\downarrow\in H_0\\ \mp\bar h_*^{(1)}, & x\in H_0,\quad x_\downarrow\in H_2\end{cases}$$

on the invariant set I_7 and to solutions of the form

$$\pm\bar{h}_x^{(1)} = \begin{cases} \pm\bar{h}_*^{(1)}, & x \in H_3, \quad x_\downarrow \in H_1 \\ \mp\bar{h}_*^{(1)}, & x \in H_1, \quad x_\downarrow \in H_3 \\ \mp 4f(\bar{h}_*^{(1)}, \theta), & x \in H_3, \quad x_\downarrow \in H_0 \\ \mp 4f(\bar{h}_*^{(1)}, \theta), & x \in H_0, \quad x_\downarrow \in H_3 \\ \pm 4f(\bar{h}_*^{(1)}, \theta), & x \in H_1, \quad x_\downarrow \in H_2 \\ \pm 4f(\bar{h}_*^{(1)}, \theta), & x \in H_2, \quad x_\downarrow \in H_1 \\ \mp\bar{h}_*^{(1)}, & x \in H_2, \quad x_\downarrow \in H_0 \\ \pm\bar{h}_*^{(1)}, & x \in H_0, \quad x_\downarrow \in H_2 \end{cases}$$

on the invariant set I_{10}.

Remark 4. A computer analysis shows that the equation $g(x,\theta,k) = -f\big(kf(x,\theta);\theta\big)+(k-1)f(x,\theta) = x$ has a unique solution $x = 0$ for $k = 1, 2, 3$; if $k \geq 4$, then there exist values of the parameter α such that this equation has five solutions. More precisely, a computer analysis shows that assertion 2 in Theorem 1 and the results in Theorem 2 hold for all $k \geq 4$, i.e., there exist at most five H_A-weakly periodic Gibbs measures regardless of the value of k ($k \geq 4$).

Remark 5. The measures $\tilde{\mu}_i^{\mp}$, $i = 1, 2$, coincide with $H_{\{a_1\}}$-weakly periodic measures in Theorem 1 in the case $A = \{a_1\}$. The measures $\mu_{*,i}^{\mp}$, $i = 1, 2$, are $G_k^{(4)}$ weakly periodic, but they do not coincide with the H_A-weakly periodic measures constructed in Theorem 1.

Remark 6. The new Gibbs measures described in Theorems 1 and 2 allow describing a continuum set of nonperiodic Gibbs measures that differ from the well-known ones.

4. Discussion: Open problems

The results in this paper disclose a new fact for the Ising model. It is well known that there exist at most three periodic Gibbs measures (not translation invariant) for models with a finite interaction radius [4], [5], [7], [10]. In our case, the number of weakly periodic measures turns out to be five.

The functions $h_x^{(i)}$ determined in assertion 1 in Theorem 2 differ significantly from the functions $\bar{h}_x^{(1)}$ determined in assertion 4 in this theorem. The weak (H-weak) periodicity of the function h_x means that the value of h_x depends only on the classes to which x and $x_\downarrow$ belong. Such a dependence was first demonstrated in [3], but weakly periodic solutions were not described there.

We now formulate some problems whose solution turns out to be sufficiently difficult, and they require consideration in the future.

1. Do other invariant sets of operators given by equalities (7) and (14) exist?
2. Do fixed points outside invariant sets exist?
3. How can Remark 4 after Theorem 2 be proved?
4. How can weakly periodic Gibbs measures for normal subgroups of an index greater than four be described?
5. How can such measures be described for other models, for example, for the Potts model or the SOS model?

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