Theoretical and Mathematical Physics, **156**(2): 1209–1217 (2008)

RADIATION BEYOND FOUR SPACE–TIME DIMENSIONS

A. D. Mironov*† and A. Yu. Morozov*

We present a set of formulas describing classical radiation of a rank-s tensor field from an accelerated pointlike source in a flat space–time of an arbitrary even dimension d. These formulas allow straightforwardly and algorithmically evaluating the total intensity and radiated momentum for any s and d by hand or using a computer. The practical application of the obtained results is limited for $s > 1$ because the energy–momentum tensor for the pointlike source is not conserved. This usually means that contributions to the radiation from tensions of the forces causing the acceleration of the radiation source cannot be neglected.

Keywords: classical radiation, higher dimensions, string theory

Radiation processes were traditionally considered subjects of direct physical application and were therefore only deeply investigated in at most d=4 space–time dimensions [1]. Even for $d = 2$ and $d = 3$, where there are obvious applications to sound waves in media, for example, the theory remains poorly represented in the literature. Only recently, after string-inspired multidimensional models [2] attracted increasing attention [3], some papers on multidimensional radiation began to appear [4]–[8]. Of course, they are still too few to cover the field exhaustively, as for the literature on four-dimensional radiation. In this paper, we take a step that we think is necessary for studying physical effects *systematically*. We present general formulas describing *classical* radiation for an arbitrary dimension d and for an arbitrary rank s of the *radiated* fields. This should help to clarify the physical and mathematical structures underlying radiation in higher dimensions. In particular, the radiation damping force in higher dimensions can be immediately obtained from our results, for example, by the method in [4].

To find the radiation intensity, the following chain of calculations must be performed.

Step 1. Solve the wave equation for a pointlike source of the rank-s field, moving along a world line $z^\mu(\tau)$,

$$\Box A_{\mu_1\ldots\mu_s}(x) = \oint u_{\mu_1}\cdots u_{\mu_s}\delta^{(d)}(x-z(\tau))\,d\tau, \tag{1}$$

and select the contribution that decreases most slowly at large distances. For even d, it is given by a simple formula for the retarded Liénard–Wiechert potential:

$$\begin{aligned} A^{\mathrm{rad}}_{\mu_1\ldots\mu_s} &= \left(\frac{1}{(Ru)}\,\partial_\tau\right)^{(d-4)/2}\frac{u_{\mu_1}\cdots u_{\mu_s}}{Ru}, \\ \partial_\mu A^{\mathrm{rad}}_{\mu_1\ldots\mu_s} &= R_\mu\left(\frac{1}{(Ru)}\,\partial_\tau\right)^{(d-2)/2}\frac{u_{\mu_1}\cdots u_{\mu_s}}{Ru}, \end{aligned} \tag{2}$$

*Institute for Theoretical and Experimental Physics, Moscow, Russia, e-mail: mironov@itep.ru, morozov@itep.ru.

†Lebedev Physical Institute, RAS, Moscow, Russia, e-mail: mironov@lpi.ru.

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 156, No. 2, pp. 282–291, August, 2008. Original article submitted April 23, 2007.

where $u_\mu = \partial_\tau z_\mu$ is the d-velocity of the source ($u^2 = 1$) evaluated at the radiation instant $t' = z^0(\tau)$ determined by the condition $R^2 = 0$, where R_μ is the d-vector with the components $R^\mu \equiv x^\mu - z^\mu(\tau)$. Here, τ is the proper time of the source. In particular,

$$\frac{\partial\tau}{\partial x^\mu} = \frac{R_\mu}{(Ru)}. \tag{3}$$

We also introduce the notation $n^\mu \equiv (1, -\vec{n}) = R^\mu/R$, where R is the length of the spatial part of R^μ, $R \equiv \sqrt{(R^i)^2}$ (in contrast to R^μ, n^μ is not a d-vector, and we define it in the laboratory frame).

It is convenient to rewrite (2) in condensed notation,

$$\begin{aligned} A_{\text{rad}} &= \frac{1}{R^{d-2}}\left(\frac{1}{U}\,\partial_\tau\right)^{p-1}\frac{S}{U} = \frac{1}{R^{d-2}}\sum_{q=0}^{p-1}(\partial_\tau^q S)A_q^{p-1}, \\ \partial_\mu A_{\text{rad}} &= \frac{n_\mu}{R^{d-2}}\sum_{q=0}^{p}(\partial_\tau^q S)A_q^p, \end{aligned} \tag{4}$$

where $U = (nu)$, $S = u_{\mu_1}\cdots u_{\mu_s}$, and $p = (d-2)/2$, and to find A_q^p from the recurrence relations

$$A_q^{p+1} = \frac{1}{U}(\partial_\tau A_q^p + A_{q-1}^p). \tag{5}$$

Step 2. Check the transversality. In the leading order in $1/R$ when derivatives of R can be neglected,

$$\partial^{\mu_s}A^{\text{rad}}_{\mu_1\mu_2\ldots\mu_s} = \left(\frac{1}{(Ru)}\,\partial_\tau\right)^{(d-2)/2} u_{\mu_1}\cdots u_{\mu_{s-1}} \tag{6}$$

does not vanish for $s \geq 2$, and this is not cured by subtracting traces. The physical reason for the nontransversality is the neglect of radiation from tensions of the forces that cause the source acceleration: only taking all the tensions into account makes the radiation problem well defined for $s \geq 2$. For $s \geq 2$, the formulas in this paper provide only part of the total answer.

Step 3. Develop the energy–momentum tensor in the wave zone. It is equal to

$$T^{(s)}_{\mu\nu} = R_\mu R_\nu\left\{\left(\frac{1}{(Ru)}\,\partial_\tau\right)^{(d-2)/2}\frac{u_{\mu_1}\cdots u_{\mu_s}}{Ru}\right\}^2 = \frac{R_\mu R_\nu}{R^d}\left\{\sum_{q=0}^{p}A_q^p\partial_\tau^q S\right\}^2. \tag{7}$$

Certain linear combinations of such stress tensors arise in applications, for instance, for the scalar waves

$$T^{(\text{scalar})}_{\mu\nu} = -T^{(0)}_{\mu\nu}. \tag{8}$$

This is because only the spatial components of all nonzero spin fields have any physical meaning (e.g., survive in physical gauges), thus yielding the overall minus sign of the kinetic part of the energy–momentum tensor compared with the scalar fields. For gravitational waves, we have

$$T^{(\text{grav})}_{\mu\nu} = T^{(2)}_{\mu\nu} - \frac{1}{2}T^{(0)}_{\mu\nu}. \tag{9}$$

Similar redefinitions are also needed for higher spins $s > 2$. The results for such linear transformations can be easily obtained using the formulas for $T^{(s)}_{\mu\nu}$.

Step 4. Evaluate the radiated momentum flux through the sphere of radius R. It is equal to

$$d\mathcal{P}^\mu_{d|s} = -\oint \{T^{\mu i} n_i \, dx^0\} \, dS, \tag{10}$$

where dS is an infinitesimal element of this sphere (i.e., the integration is over the sphere). Thus (using $dx^0 = U\,d\tau = (nu)\,d\tau$), we finally obtain

$$\begin{aligned}
\frac{\partial \mathcal{P}^\mu_{d|s}}{d\tau} &= -\oint T^{\mu i} n_i R^{d-2} U \, d\Omega_{d-2} = \\
&= \sum_{q',q''=0}^{p} \left(\partial_\tau^{q'}(u_{\mu_1}\cdots u_{\mu_s})\partial_\tau^{q''}(u^{\mu_1}\cdots u^{\mu_s})\right) \int n^\mu U A^p_{q'} A^p_{q''} \, d\Omega_{d-2}(n) = \\
&= \sum_{k=0}^{(d-4)/2} \left(\partial_\tau^k u^\mu\right) P_k^{(d|s)}(\kappa).
\end{aligned} \tag{11}$$

where $d\Omega_{d-2}$ is the solid angle in the $d-1$ space and $P_k^{(d|s)}$ are functions of various scalar products $(\partial_\tau^l u_\nu \partial_\tau^m u^\nu)$ with $l + m \le d - 4 - k$. In particular, the radiated energy loss is ($dt' = u^0\,d\tau = \gamma\,d\tau$)

$$\frac{d\mathcal{P}^0_{d|s}}{dt'} = \frac{1}{\gamma} \sum_{k=0}^{(d-4)/2} (\partial_\tau^k \gamma) P_k^{(d|s)}(\kappa) \quad \overset{\gamma=(1-v^2)^{-1/2}=\text{const}}{\longrightarrow} \quad P_0^{(d|s)}(\kappa). \tag{12}$$

We once again emphasize that because of effects like (6) and (8), (9), this quantity represents only part of the total answer for $s \ge 2$. Moreover, it can even be negative(!): for example, for fields with an asymptotically large rank s, the leading contribution to the radiated momentum $d\mathcal{P}^\mu_{d|s}/d\tau$ behaves as $-(s\dot{u}^2)^{d/2-1} \sim (-1)^{d/2}$ with $\dot{u}^2 \le 0$ taken into account. For $s = 0$, the negativeness is corrected by changing the overall sign (see (8)), but for $s \ge 3$, the procedure must be more sophisticated.

Contractions $I_{d|s} = u^\mu \, d\mathcal{P}^\mu_{d|s}/d\tau$ are also sometimes considered [6], but they do not have any direct physical meaning unless $d\mathcal{P}^\mu_{d|s}/d\tau \sim u_\mu$, as happens for $d = 4$. In this case, $I_{d|s} = d\mathcal{P}^0_{d|s}/dt'$.

Step 5. Calculate angular integrals over isotropic (lightlike) unit vectors n^μ in (11) (although n^μ is not a d-vector, these integrals are Lorentz invariant because (11) is Lorentz invariant),

$$\int \frac{n^{\mu_1}\cdots n^{\mu_m}}{(nu)^{d+m-2}} \, d\Omega_{d-2}(n) \sim \mathrm{Pr}_d(u^{\mu_1}\cdots u^{\mu_m}). \tag{13}$$

These integrals are expressed in terms of the trace-eliminating projection operator Pr_d (because $n^2 = 0$ and contraction of any pair of μ-indices in the left-hand side of (13) should give zero):

$$\begin{aligned}
&\text{spin } s=2: \quad \mathrm{Pr}_d(u_\mu u_\nu) = u_\mu u_\nu - \frac{1}{d}\eta_{\mu\nu}, \\
&\text{spin } s=3: \quad \mathrm{Pr}_d(u_{\mu_1} u_{\mu_2} u_{\mu_3}) = u_{\mu_1} u_{\mu_2} u_{\mu_3} - \frac{1}{d+2}(\eta_{\mu_1\mu_2} u_{\mu_3} + \eta_{\mu_1\mu_3} u_{\mu_2} + \eta_{\mu_2\mu_3} u_{\mu_1}), \\
&\text{spin } s=4: \quad \mathrm{Pr}_d(u_{\mu_1} u_{\mu_2} u_{\mu_3} u_{\mu_4}) = u_{\mu_1} u_{\mu_2} u_{\mu_3} u_{\mu_4} - \frac{1}{d+4}(\eta_{\mu_1\mu_2} u_{\mu_3} u_{\mu_4} + 5 \text{ permutations}) + \\
&\qquad\qquad + \frac{1}{(d+4)(d+2)}(\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4} + \eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4} + \eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}), \\
&\quad\vdots \qquad\qquad \vdots
\end{aligned}$$

For arbitrary m, we have

$$\mathrm{Pr}_d(u_{\mu_1}\cdots u_{\mu_m}) = \sum_{k=0}^{[m/2]}(-1)^k\frac{(d+2m-4-2k)!!}{(d+2m-4)!!}\times$$
$$\times\left(\overbrace{u_{\mu_1}\cdots u_{\mu_{m-2k}}\left(\underbrace{\eta_{\mu_{m-2k+1}\mu_{m-2k+2}}\cdots\eta_{\mu_{m-1}\mu_m}+\cdots}_{(2k-1)!!\ \text{permutations}}\right)+\cdots}^{C_{2k}^m\ \text{permutations}}\right)$$

($[m/2]$ means the integer part of a number $m/2$). It is convenient to rewrite this projection operator in terms of the generating function: after contraction with m copies of a d-vector x, we obtain

$$\mathrm{Pr}_d((xu)^m) = \sum_{k=0}^{[m/2]}(-1)^k(2k-1)!!C_{2k}^m\frac{(d+2m-4-2k)!!}{(d+2m-4)!!}(xu)^{m-2k}(x^2)^k =$$
$$= \sum_{k=0}^{[m/2]}(-1)^k\frac{(d+2m-4-2k)!!}{(d+2m-4)!!}\frac{m!(xu)^{m-2k}(x^2)^k}{2^k k!(m-2k)!}. \tag{14}$$

To determine the normalization factor, we must know the sum of the coefficients in series (14),

$$c_d(m) = \sum_{k=0}^{[m/2]}(-1)^k C_{2k}^m(2k-1)!!\frac{(d+2m-4-2k)!!}{(d+2m-4)!!} =$$
$$= \frac{(d+2[m/2]-3)!!(d+2(m-[m/2])-4)!!}{(d-3)!!(d+2m-4)!!}. \tag{15}$$

Rewritten in terms of the generating function with the normalization factor restored, angular integral (13) is

$$\int\frac{(nx)^m\,d\Omega_{d-2}(n)}{(nu)^{d+m-2}} = \frac{S_{n-2}}{c_d(s)}\,\mathrm{Pr}_d((ux)^m) =$$
$$= \frac{2^{d/2}\pi^{(d-2)/2}}{(d+2[m/2]-3)!!}\times$$
$$\times\sum_{j=0}^{[m/2]}(-1)^j\frac{(d+2m-4-2j)!!}{(d+2m-2[m/2]-4)!!}\frac{m!}{2^j j!(m-2j)!}(ux)^{m-2j}(x^2)^j. \tag{16}$$

It remains to replace the vector x with $tx_0+\sum_{k=0}^{(d-2)/2}a_k\partial_\tau^k u$ and select the coefficient of the relevant combination of t and a_k. We note that this calculation uses the generation function, which provides an additional combinatorial coefficient that must be taken into account explicitly. For example, to obtain the correct coefficients of the term $\partial_\tau^{k_1}u\cdots\partial_\tau^{k_n}u$ with different values of $k_1,\ldots,k_n$, the generating function must be divided by $n!$.

The final results can be equivalently expressed both in terms of the scalar products $U_{ij}\equiv\partial_\tau^l\vec{u}_\nu\partial_\tau^m\vec{u}^\nu$ and in terms of the Frenet curvatures κ_m and their derivatives $\partial_\tau^l\kappa_m$. Hence, we have the last step.

Step 6. Express U_{ij} in terms of the Frenet curvatures [9] parameterizing the moving orthonormal basis associated with the world line $x^\mu(\tau)$. It is formed from d-vectors $\vec{N}^{(\mu)}$, $\mu = 0, \ldots, d-1$,

$$\vec{N}^{(\mu)}\vec{N}^{(\nu)} = \eta^{\mu\nu} \tag{17}$$

with $\vec{N}^{(0)} = \vec{u}$ (i.e., $N^{(0)}_\mu = u_\mu$) and other vectors given by the recurrence relations

$$\partial_\tau \vec{N}^{(\mu)} = -k_{\mu+1}\vec{N}^{(\mu+1)} + \sum_{\nu=1}^{\mu} \beta_{\mu\nu}\vec{N}^{(\nu)}.$$

Differentiating the orthonormality condition with respect to τ, we obtain

$$(\partial_\tau \vec{N}^{(\mu)})\vec{N}^{(\nu)} + \vec{N}^{(\mu)}(\partial_\tau \vec{N}^{(\nu)}) = 0.$$

It hence follows that for $\nu \le \mu$,

$$\beta_{\mu\nu} = (\partial_\tau \vec{N}^{(\mu)})\vec{N}^{(\nu)} = -\vec{N}^{(\mu)}(\partial_\tau \vec{N}^{(\nu)}) = \kappa_\mu \delta_{\nu,\mu-1}\eta^{\mu\mu}$$

and

$$\partial_\tau \vec{N}^{(\mu)} = -k_{\mu+1}\vec{N}^{(\mu+1)} + k_\mu \vec{N}^{\mu-1}. \tag{18}$$

The parameters k_μ (they are not d-vectors!) depend on the shape of the world line $x^\mu(\tau)$ in the infinitesimal vicinity of its point and are called Frenet curvatures.

We summarize. All the indicated steps are easily performed using MAPLE or Mathematica.[1] In the appendix, we obtain explicit formulas for the radiated d-momentum $d\mathcal{P}^\mu_{d|s}$ and also somewhat simpler formulas for $P_0^{(d|s)}$, the radiated intensity at a constant γ ($\kappa_2^2 = \gamma^2\kappa_1^2/(\gamma^2-1) = \text{const}$, $\kappa_i = 0$, $i > 2$; see (12)) and for the contractions $I_{d|s} = u^\mu\, d\mathcal{P}^\mu_{d|s}/d\tau$ [6] for the "realistic" values $d = 4, 6, 8, 10$ and an arbitrary rank s. First, we list results for $d = 4, 6, 8$, where the formulas are relatively simple and their general structure can be understood. The much more involved formulas in the most interesting case $d = 10$ are placed in a separate section.

Appendix

Formulas for $d\mathcal{P}^\mu_{d|s}$ for $d = 4, 6, 8$.

$$\frac{d\mathcal{P}^\mu_{4|s}}{d\tau} = \frac{4-12s}{3}\pi\dot{u}^2 u^\mu,$$

$$\frac{d\mathcal{P}^\mu_{6|s}}{d\tau} = \pi^2\left[\frac{19}{3} - (2s-3)^2\right]\dot{u}^4 u^\mu + \frac{8\pi^2}{15}(1-5s)\ddot{u}^2 u^\mu +$$

$$+ \frac{16\pi^2}{35}(2-7s)(\dot{u}\ddot{u})\dot{u}^\mu + \frac{16\pi^2}{105}(7s-4)\dot{u}^2\ddot{u}^\mu,$$

$$\frac{d\mathcal{P}^\mu_{8|s}}{d\tau} = \left\{-\frac{16\pi^3}{9}\left(\frac{7s^3}{10} - \frac{135}{14}s^2 + \frac{838}{35}s - 5\right)\dot{u}^6 + \frac{32\pi^3}{3}\left(-s^2 + \frac{101}{35}s - \frac{41}{63}\right)(\dot{u}\ddot{u})^2 +\right.$$

$$+ \frac{16\pi^3}{3}\left(-\frac{s^2}{5} + \frac{59}{35}s - \frac{20}{63}\right)\dot{u}^2\ddot{u}^2 + 32\pi^3\left(\frac{s^2}{15} - \frac{2s}{7} + \frac{11}{189}\right)\dot{u}^2(\dot{u}\dddot{u}) -$$

[1] A program for the calculations can be found, for example, at `http://thesaurus.itep.ru/project/0703mm/intens9.zip`.

$$- \frac{16\pi^3}{15}\left(s - \frac{1}{7}\right)\ddot{u}^2\bigg\}u^\mu +$$

$$+ \left\{\frac{32\pi^3}{7}\left(-\frac{6s^2}{5} + \frac{19}{5}s - \frac{283}{297}\right)\dot{u}^2(\dot{u}\ddot{u}) + \frac{64\pi^3}{7}\left(\frac{1}{27} - \frac{s}{5}\right)\ddot{u}\dddot{u}\right\}\dot{u}^\mu +$$

$$+ \left\{\frac{64\pi^3}{21}\left(\frac{2s^2}{5} - 2s + \frac{65}{99}\right)\dot{u}^4 + \frac{64\pi^3}{21}\left(\frac{1}{9} - \frac{2s}{5}\right)\dot{u}\dddot{u}\right\}\ddot{u}^\mu +$$

$$+ \frac{32\pi^3}{35}\left(s - \frac{17}{27}\right)(\dot{u}\ddot{u})\dddot{u}^\mu.$$

Formulas for $P_0^{(d|s)}$ for $d = 4, 6, 8$.

$$P_0^{(4|s)} = \frac{12s-4}{3}\pi\kappa_1^2,$$

$$P_0^{(6|s)} = -\pi^2\left(4s^2 - \frac{28}{3}s + \frac{32}{15}\right)4\kappa_1^4 + \frac{8\pi^2}{15}(5s-1)\kappa_1^2\kappa_2^2,$$

$$P_0^{(8|s)} = \frac{16\pi^3}{105}(7s-1)(\kappa_1^2\kappa_2^2\kappa_3^2 + \kappa_1^2\kappa_2^4) +$$

$$+ \frac{8\pi^3}{315}(49s^3 - 549s^2 + 1004s - 216)\kappa_1^6 - 16\pi^3\left(\frac{s^2}{5} - s + \frac{64}{315}\right)\kappa_1^4\kappa_2^2.$$

Formulas for $I_{d|s}$ for $d = 4, 6, 8$.

$$I_{4|s} = \frac{12s-4}{3}\pi\kappa_1^2,$$

$$I_{6|s} = -\pi^2\left(4s^2 - \frac{124}{15}s + \frac{32}{21}\right)\kappa_1^4 + \frac{8\pi^2}{15}(5s-1)(\dot{\kappa}_1^2 + \kappa_1^2\kappa_2^2),$$

$$I_{8|s} = \frac{16\pi^3}{105}\Bigg[\left(\frac{49}{6}s^3 - \frac{167}{2}s^2 + \frac{406}{3}s - \frac{276}{11}\right)\kappa_1^6 -$$

$$- \left(21s^2 - 97s + \frac{172}{9}\right)\kappa_1^4\kappa_2^2 - \left(77s^2 - 180s + \frac{109}{3}\right)\kappa_1^2\dot{\kappa}_1^2 +$$

$$+ 2(7s^2 - 9s + 4)\kappa_1^3\ddot{\kappa}_1 + (7s-1)\Big\{\kappa_1^2\kappa_2^2\kappa_3^2 + (\kappa_1\kappa_2^2 - \ddot{\kappa}_1)^2 + (2\kappa_2\dot{\kappa}_1 + \kappa_1\dot{\kappa}_2)^2\Big\}\Bigg].$$

Results for the most interesting case $d = 10$.

$$\frac{d\mathcal{P}^\mu_{10|s}}{d\tau} = \left\{\frac{4\pi^4}{315}\left(19s^4 - \frac{6466}{9}s^3 + \frac{236237}{33}s^2 - \frac{1609666}{99}s + \frac{9800}{3}\right)\dot{u}^8 -\right.$$

$$- \frac{32\pi^4}{7}\left(s^3 - \frac{200}{9}s^2 + \frac{2147}{33}s - \frac{6230}{429}\right)\dot{u}^2(\dot{u}\ddot{u})^2 +$$

$$+ \frac{16\pi^4}{35}\left(s^3 - \frac{61}{27}s^2 - \frac{8316}{297}s + \frac{11690}{1287}\right)\dot{u}^4\ddot{u}^2 +$$

$$+ \frac{64\pi^4}{35}\left(s^3 - \frac{1037}{54}s^2 + \frac{32653}{594}s - \frac{4865}{429}\right)\dot{u}^4(\dot{u}\dddot{u}) -$$

$$
\begin{aligned}
&-\frac{16\pi^4}{105}\left(19s^2-\frac{611}{9}s+\frac{512}{33}\right)\ddot{u}^4-\frac{32\pi^4}{15}\left(\frac{17}{7}s^2-151s+\frac{34}{21}\right)(\dot{u}\dddot{u})^2-\\
&-\frac{64\pi^4}{105}\left(s^2-\frac{53}{3}s+\frac{100}{33}\right)(\dot{u}\ddot{u})(\ddot{u}\dddot{u})-\frac{256\pi^4}{35}\left(s^2-\frac{263}{108}s+\frac{119}{198}\right)\ddot{u}^2(\dot{u}\dddot{u})-\\
&-\frac{32\pi^4}{105}\left(s^2-\frac{134}{9}s+\frac{25}{11}\right)\dot{u}^2\dddot{u}^2+\frac{64\pi^4}{105}\left(s^2-\frac{74}{9}s+\frac{15}{11}\right)\dot{u}^2(\ddot{u}\ddddot{u})+\\
&+\frac{64\pi^4}{315}(9s^2-49s+10)(\dot{u}\ddot{u})(\dot{u}\ddddot{u})+\frac{32\pi^4}{105}\left(\frac{1}{9}-s\right)\ddddot{u}^4\Bigg\}u^\mu-\\
&-\Bigg\{\frac{32\pi^4}{63}\left(\frac{8s^3}{3}-\frac{593}{11}s^2+\frac{72173}{429}s-\frac{5670}{143}\right)\dot{u}^4(\dot{u}\ddot{u})+\\
&+\frac{64\pi^4}{189}\left(20s^2-\frac{829}{11}s+\frac{2664}{143}\right)\ddot{u}^2(\dot{u}\ddot{u})+\\
&+\frac{256\pi^4}{189}\left(5s^2-\frac{133}{11}s+\frac{453}{143}\right)(\dot{u}\dddot{u})(\dot{u}\ddot{u})-\frac{128\pi^4}{189}\left(s^2-\frac{109}{22}s+\frac{147}{143}\right)\dot{u}^2(\dot{u}\ddddot{u})+\\
&+\frac{320\pi^4}{819}\left(1-\frac{208}{33}s\right)\dot{u}^2(\ddot{u}\dddot{u})+\frac{64\pi^4}{189}\left(2s-\frac{3}{11}\right)(\dddot{u}\ddddot{u})\Bigg\}\dot{u}^\mu+\\
&+\Bigg\{\frac{128\pi^4}{189}\left(s^3-\frac{785}{44}s^2+\frac{30843}{572}s-\frac{2065}{143}\right)\dot{u}^6+\\
&+\frac{320\pi^4}{189}\left(s^2-\frac{53s}{11}+\frac{216}{143}\right)(\dot{u}\ddot{u})^2-\frac{640\pi^4}{189}\left(s^2-4s+\frac{1569}{1430}\right)\dot{u}^2(\dot{u}\dddot{u})-\\
&-\frac{64\pi^4}{189}\left(4s^2+s+\frac{123}{143}\right)\dot{u}^2\ddot{u}^2+\frac{128\pi^4}{189}\left(\frac{2}{11}-s\right)(\ddot{u}\ddddot{u})\Bigg\}\ddot{u}^\mu+\\
&+\Bigg\{\frac{256\pi^4}{4158}\left(22s^2-167s+\frac{795}{13}\right)\dot{u}^2(\dot{u}\ddot{u})+\frac{64\pi^4}{189}\left(\frac{3}{11}-s\right)(\dot{u}\ddddot{u})\Bigg\}\dddot{u}^\mu-\\
&-\Bigg\{\frac{32\pi^4}{945}\left(s^2-\frac{141}{11}s+\frac{1020}{143}\right)\dot{u}^4-\frac{64\pi^4}{315}\left(s-\frac{23}{33}\right)\ddot{u}^2-\\
&-\frac{256\pi^4}{945}\left(s-\frac{29}{44}\right)(\dot{u}\dddot{u})\Bigg\}\ddddot{u}^\mu,
\end{aligned}
$$

$$
\begin{aligned}
P_0^{(10|s)}=\frac{32\pi^4}{315}\Bigg[&\left(-\frac{19}{8}s^4+\frac{2747}{36}s^3-\frac{152021}{264}s^2+\frac{369827}{396}-192\right)\kappa_1^8+\\
&+\left(\frac{9}{2}s^3-605s^2+933s-188\right)\kappa_1^6\kappa_2^2-\\
&-\left(\frac{33}{2}s^2-325s+28\right)\kappa_1^4\kappa_2^4-\left(9s^2-\frac{88}{3}s+\frac{43}{3}\right)\kappa_1^4\kappa_2^2\kappa_3^2+\\
&+\left(3s-\frac{1}{3}\right)(\kappa_1^2\kappa_2^6+\kappa_1^2\kappa_2^2\kappa_3^2\kappa_4^2+2\kappa_1^2\kappa_2^4\kappa_3^2+\kappa_1^2\kappa_2^2\kappa_3^4)\Bigg],
\end{aligned}
$$

$$
I_{10|s}=\frac{16\pi^4}{945}\Bigg[\left(81s^3-1693s^2+\frac{53244}{11}s-\frac{133048}{143}\right)\kappa_1^6\kappa_2^2+
$$

$$
\begin{aligned}
&+\left(243s^3-4785s^2+\frac{106818}{11}s-\frac{300280}{143}\right)\kappa_1^4\dot{\kappa}_1^2-\\
&-\left(99s^2-931s+\frac{1744}{11}\right)\kappa_1^4\kappa_2^4+\left(36s^2-220s+\frac{416}{11}\right)\kappa_1^4\kappa_2\ddot{\kappa}_2+\\
&+\left(324s^2-1780s+\frac{3392}{11}\right)\kappa_1^3\ddot{\kappa}_1\kappa_2^2-\left(468s^2-1160s+\frac{2812}{11}\right)\kappa_1\ddot{\kappa}_1\dot{\kappa}_1^2+\\
&+\left(144s^2-568s+\frac{1336}{11}\right)\kappa_1^2\dot{\kappa}_1\dddot{\kappa}_1-\left(171s^2-413s+\frac{1062}{11}\right)\dot{\kappa}_1^4-\\
&-\left(54s^2-488s+\frac{866}{11}\right)\kappa_1^4\kappa_2^2\kappa_3^2-\\
&-\left(\frac{57}{4}s^4-\frac{2507}{6}s^3+\frac{125989}{44}s^2-\frac{3644975}{858}s+\frac{111744}{143}\right)\kappa_1^8-\\
&-\left(108s^3-1506s^2+\frac{31794}{11}s-\frac{83808}{143}\right)\kappa_1^5\ddot{\kappa}_1-\\
&-\left(324s^2-822s+\frac{1878}{11}\right)\kappa_1^2\ddot{\kappa}_1^2-\left(432s^2-2752s+\frac{5760}{11}\right)\kappa_1^3\dot{\kappa}_1\kappa_2\dot{\kappa}_2-\\
&-\left(18s^2-268s+\frac{450}{11}\right)\kappa_1^4\dot{\kappa}_2^2-\left(450s^2-3714s+\frac{7520}{11}\right)\kappa_1^2\dot{\kappa}_1^2\kappa_2^2+\\
&+2(9s-1)[(\dddot{\kappa}_1-3\dot{\kappa}_1\kappa_2^2-3\kappa_1\kappa_2\dot{\kappa}_2)^2+\\
&+(3\ddot{\kappa}_1\kappa_2+3\dot{\kappa}_1\dot{\kappa}_2+\kappa_1\ddot{\kappa}_2-\kappa_1\kappa_2^3)^2+(2\kappa_1\dot{\kappa}_2\kappa_3+\kappa_1\kappa_2\dot{\kappa}_3+3\dot{\kappa}_1\kappa_2\kappa_3)^2-\\
&-\kappa_1^2\kappa_2^2\kappa_3^2(2\kappa_2^2+\kappa_3^2+\kappa_4^2)-2\kappa_1\kappa_2\kappa_3^2(3\ddot{\kappa}_1\kappa_2+\kappa_1\ddot{\kappa}_2+3\dot{\kappa}_1\dot{\kappa}_2)]\Big].
\end{aligned}
$$

We note that in the expressions for the radiated momentum, the coefficients of terms of the degree k in U_{ij} are polynomials of the degree k in the rank s. The relative simplicity of the coefficients of these polynomials implies that they may have a general formula for an arbitrary dimension d. But we once again emphasize that using these expressions for higher ranks is limited because contributions from other energy–momentum tensors must be taken into account. Moreover, to separate the contribution of the given *spin* s from the expression for the *rank* s, proper combinations of energy–momentum tensors for lower rank fields must be added. This can be done straightforwardly using the formulas obtained here.

Acknowledgments. The authors thank P. Kazinsky and S. Lyakhovich for the informative discussions.

This work is supported in part by a program of the Federal Nuclear Energy Agency, the Russian Foundation for Basic Research (Grant Nos. 07-02-00878-a, A. D. M.; 07-02-00645, A. Yu. M.; and 06-01-92059-CE), the Program for Supporting Leading Scientific Schools (Grant No. NSh-8004.2006.2), the NWO (Project No. 047.011.2004.026), INTAS (Grant No. 05-1000008-7865), and the ANR (Project NO. ANR-05-BLAN-0029-01).

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