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COMMUTATOR IDENTITIES ON ASSOCIATIVE ALGEBRAS AND THE INTEGRABILITY OF NONLINEAR EVOLUTION EQUATIONS

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We show that commutator identities on associative algebras generate solutions of the linearized versions of integrable equations. In addition, we introduce a special dressing procedure in a class of integral operators that allows deriving both the nonlinear integrable equation itself and its Lax pair from such a commutator identity. The problem of constructing new integrable nonlinear evolution equations thus reduces to the problem of constructing commutator identities on associative algebras.

Keywords: nonlinear evolution equation, Lax pair

1. Introduction

In [1], we mentioned that any two arbitrary elements A and B of an arbitrary associative algebra satisfy the commutator identity

$$\big[A^3,[A,B]\big]-\frac{3}{4}\big[A^2,[A^2,B]\big]-\frac{1}{4}\big[A,\big[A,\big[A,[A,B]\big]\big]\big]=0. \tag{1.1}$$

Defining the adjoint action of a power of an element A on the associative algebra as

$$\operatorname{ad}_n B=[A^n,B], \tag{1.2}$$

we can rewrite identity (1.1) as a relation between these adjoint actions:

$$\operatorname{ad}_3\operatorname{ad}_1-\frac{3}{4}(\operatorname{ad}_2)^2-\frac{1}{4}(\operatorname{ad}_1)^4=0. \tag{1.3}$$

Identity (1.1), being a trivial consequence of the associativity property, readily proves that the function

$$B(t_1,t_2,t_3)=e^{t_1A+t_2A^2+t_3A^3}Be^{-t_1A-t_2A^2-t_3A^3} \tag{1.4}$$

satisfies the linearized Kadomtsev–Petviashvili (KP) equation [2] in the variables t_j, i.e.,

$$\frac{\partial^2B(t)}{\partial t_1\partial t_3}-\frac{3}{4}\frac{\partial^2B(t)}{\partial t_2^2}-\frac{1}{4}\frac{\partial^4B(t)}{\partial t_1^4}=0, \tag{1.5}$$

more exactly, the KPII equation. It was noted in [1] that analogous identities also exist for the higher powers. Indeed, it is easy to see that the equalities

$$\big[A^n,\underbrace{\big[A,\ldots,[A,}_{n-2}B]\ldots\big]=$$

$$=\frac{1}{2^n}\sum_{m=1}^{n}\frac{n!\,\big(1-(-1)^m\big)}{m!\,(n-m)!}\underbrace{\big[A^2,\ldots,\big[A^2,}_{n-m}\underbrace{\big[A,\ldots,A,}_{2(m-1)}B\big]\ldots\big],\quad n\geq 2, \tag{1.6}$$

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are satisfied identically, and we thus obtain the higher linearized equations of the KP hierarchy,

$$\frac{\partial^{n-1}B(t)}{\partial t_1^{n-2}\partial t_n}=\frac{1}{2^n}\sum_{m=1}^{n}\frac{n!\,\bigl(1-(-1)^m\bigr)}{m!\,(n-m)!}\,\frac{\partial^{n+m-2}B(t)}{\partial t_1^{2(m-1)}\partial t_2^{n-m}}, \tag{1.7}$$

where now

$$B(t_1,t_2,t_n)=e^{t_1A+t_2A^2+t_nA^n}Be^{-t_1A-t_2A^2-t_nA^n}. \tag{1.8}$$

It is obvious by construction that all these flows are in involution.

In terms of definition (1.2), identities (1.1) and (1.6) show that the adjoint action of A^n (more exactly, $\mathrm{ad}_n(\mathrm{ad}_1)^{n-2}$) is given in terms of the adjoint actions of the lowest powers, ad_1 and ad_2, which, in this sense, generate the commutative algebra of ad_n, $n=1,2,\dots$. Relation (1.1), corresponding to the case $n=3$ in (1.6), is the lowest in this hierarchy. Indeed, in the case $n=2$, we obtain $\mathrm{ad}_2=\mathrm{ad}_2$, and in the case $n=1$ (first commuting (1.6) with A), we obtain $\mathrm{ad}_1=\mathrm{ad}_1$. We also mention that all the identities are homogeneous in A and B and are linear in B.

Equality (1.4) was derived in [1] in the framework of the resolvent approach (see [3]–[7]) as a representation describing the time evolution of (operator) scattering data. It is easy to see that any integrable equation, more exactly, any equation solvable by the inverse scattering method, can be reduced to some commutator identity using the procedure described in [1]. Different examples of such commutator identities and linearized versions of the corresponding integrable equations are given in Sec. 2. But our main aim here is to demonstrate that a special version of the dressing procedure developed in [1], [3]–[10] allows solving the inverse problem: to derive both the *nonlinear* integrable evolution equation and its Lax pair from a given commutator identity. This construction is performed in Sec. 3, where we give an explicit realization of the elements A and B as integral operators.

2. Examples of commutator identities on associative algebras

Again let A and B be arbitrary elements of an associative algebra, and let the element A be invertible. A direct calculation then shows that the identity

$$\bigl[A^2,\bigl[A^2,[A^{-1},B]\bigr]\bigr]-\bigl[A,\bigl[A,\bigl[A,\bigl[A,[A^{-1},B]\bigr]\bigr]\bigr]\bigr]+4\bigl[A,\bigl[A,[A,B]\bigr]\bigr]=0 \tag{2.1}$$

holds. Now introducing the time dependence, for example, as

$$B(t_1,t_2,t_3)=e^{t_1A^{-1}+t_2A+t_3A^2}Be^{-t_1A^{-1}-t_2A-t_3A^2}, \tag{2.2}$$

we find that this function $B(t)$ satisfies the differential equation

$$\frac{\partial^3B(t)}{\partial t_3^2\,\partial t_1}-\frac{\partial^5B(t)}{\partial t_2^4\,\partial t_1}+4\frac{\partial^3B(t)}{\partial t_3^3}=0, \tag{2.3}$$

which is a linearized version of the Boiti–Leon–Pempinelli (BLP) equation [8] (also see [9], [10]).

Identity (2.1) also belongs to its hierarchy, which is infinite in both directions in this case. The generic identity of this hierarchy is quite complicated, and we omit it here, only mentioning that the identities for the odd powers are simpler than those for the even powers, for example,

$$\bigl[A^3,[A^{-1},B]\bigr]-\bigl[A,\bigl[A,\bigl[A,[A^{-1},B]\bigr]\bigr]\bigr]+3\bigl[A,[A,B]\bigr]=0. \tag{2.4}$$

Setting

$$B(t_1,t_2,t_3)=e^{t_1A^{-1}+t_2A+t_3A^3}Be^{-t_1A^{-1}-t_2A-t_3A^3} \tag{2.5}$$

in analogy with (2.2), we then find that by (2.4), this function satisfies the differential equation

$$\frac{\partial^2 B(t)}{\partial t_3\,\partial t_1} - \frac{\partial^4 B(t)}{\partial t_2^3\,\partial t_1} + 3\frac{\partial^2 B(t)}{\partial t_2^2} = 0, \tag{2.6}$$

which is a linearized version of the nonlinear equation proposed in [10]. In this case, the commutative algebra of commuting flows given by ad_n is generated by the two adjoint elements ad_1 and ad_{-1}. To obtain identities that, for example, give the commutator with the square in terms of the commutator with the first power, we need at least three elements (A, B_1, and B_2) of an associative algebra. We set

$$B = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix} \tag{2.7}$$

and introduce projection operators

$$I_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.8}$$

It is then easy to verify that we have the identities

$$\sigma_3\bigl[(AI_1)^2, B\bigr] = \bigl[AI_1, [AI_1, B]\bigr], \tag{2.9}$$

$$\sigma_3\bigl[(AI_2)^2, B\bigr] = -\bigl[AI_2, [AI_2, B]\bigr], \tag{2.10}$$

where $\sigma_3 = I_1 - I_2$ is the standard Pauli matrix. Moreover, if A is invertible, then

$$\bigl[AI_j, [A^{-1}I_j, B]\bigr] = B, \quad j = 1, 2. \tag{2.11}$$

Now introducing

$$B_j(t) = e^{I_j(t_1A + t_2A^2)} B e^{-I_j(t_1A + t_2A^2)}, \tag{2.12}$$

$$B'_j(t) = e^{I_j(t_1A + t_{-1}A^{-1})} B e^{-I_j(t_1A + t_{-1}A^{-1})}, \tag{2.13}$$

we see that these functions satisfy the differential equations

$$\sigma_3 \partial_{t_2} B_j(t) = (-1)^{j+1} \partial_{t_1}^2 B_j(t), \tag{2.14}$$

$$\partial_{t_1}\partial_{t_{-1}} B'_j(t) = B'_j(t), \tag{2.15}$$

i.e., linearized versions of the nonlinear Schrödinger and sine-Gordon equations.

It is obvious that relations (2.9) and (2.10) are the lowest members of the corresponding hierarchies of commuting flows, for example,

$$\sigma_3^n [A^n I_1, B] = \bigl[\underbrace{AI_1, \dots [AI_1}_{n}, B] \dots\bigr], \quad n = 1, 2, \dots. \tag{2.16}$$

Relations (2.9) and (2.10) can be used to construct more complicated identities. Let h be a constant block-diagonal matrix,

$$h = \mathrm{diag}\{h_1, h_2\} \equiv h_1 I_1 + h_2 I_2, \tag{2.17}$$

and let B be given by (2.7). From (2.9) and (2.10), we then derive the matrix commutator identities

$$\sigma_3[hA^2, B] = h_1\big[AI_1, [AI_1, B]\big] - h_2\big[AI_2, [AI_2, B]\big], \tag{2.18}$$

$$[A^3, B] + 3\big[A, \big[AI_1, [AI_2, B]\big]\big] - \big[A, \big[A, [A, B]\big]\big] = 0, \tag{2.19}$$

where

$$A = AI_1 + AI_2. \tag{2.20}$$

Hence, if we introduce

$$B(t_1, t_2, t_3) = e^{t_1 AI_1 + t_2 AI_2 + t_3 hA^2} B e^{-t_1 AI_1 - t_2 AI_2 - t_3 hA^2}, \tag{2.21}$$

$$B'(t_1, t_2, t_3) = e^{t_1 AI_1 + t_2 AI_2 + t_3 A^3} B e^{-t_1 AI_1 - t_2 AI_2 - t_3 A^3}, \tag{2.22}$$

then by (2.18), we have

$$\sigma_3 \frac{\partial B(t)}{\partial t_3} = h_1 \frac{\partial^2 B(t)}{\partial t_1^2} - h_2 \frac{\partial^2 B(t)}{\partial t_2^2}, \tag{2.23}$$

which is a linearized version of the Davey–Stewartson (DS) equations [11]. By (2.19), we have

$$\frac{\partial B'(t)}{\partial t_3} - \frac{\partial^3 B'(t)}{\partial t_1^3} - \frac{\partial^3 B'(t)}{\partial t_2^3} = 0, \tag{2.24}$$

i.e., a linearized version of the Veselov–Novikov (VN) equation [12].

3. Reconstruction of nonlinear integrable equations and Lax pairs

3.1. Space of operators. We showed that the commutator identities on associative algebras considered above generate solutions of linearized versions of integrable equations. We also mentioned that these (operator) linearized equations appear naturally in the framework of the resolvent approach to the inverse scattering transform as equations on the (operator) scattering data (see [1] for (1.1)). In this section, we introduce a kind of dressing procedure that allows reconstructing the corresponding nonlinear equation itself and its Lax pair for a given commutator identity. For this, we should present some elements of the resolvent approach (see [3]–[7]).

We work in the space of linear integral operators $F(q)$, $G(q)$, etc., with the corresponding kernels $F(x, x'; q)$, $G(x, x'; q)$, etc., where $x = (x_1, x_2)$, $x' = (x'_1, x'_2)$, $q = (q_1, q_2)$, and all x_j, x'_j, and q_j are real variables. We assume that all these kernels belong to the space of distributions $\mathbb{S}'$ with respect to all their six real variables. Hence, there exists the "shifted" Fourier transformation

$$F(p; \mathbf{q}) = \frac{1}{(2\pi)^2} \int dx \int dx'\, e^{i(p+\mathbf{q}_\Re)x - i\mathbf{q}_\Re x'} F(x, x'; \mathbf{q}_\Im), \tag{3.1}$$

$$F(x, x'; \mathbf{q}_\Im) = \frac{1}{(2\pi)^2} \int dp \int d\mathbf{q}_\Re\, e^{-i(p+\mathbf{q}_\Re)x + i\mathbf{q}_\Re x'} F(p; \mathbf{q}), \tag{3.2}$$

where p and $\mathbf{q} = \mathbf{q}_\Re + i\mathbf{q}_\Im$ are respectively real and complex two-dimensional vectors. The vector $q = \mathbf{q}_\Im$ plays the role of a parameter and is not affected by the composition of such operators

$$(FG)(x, x'; q) = \int dy\, F(x, y; q) G(y, x'; q), \tag{3.3}$$

defined for pairs of operators for which the integral in the r.h.s. exists in the sense of distributions. In the $(p;\mathbf{q})$ space, this composition takes the form

$$(FG)(p;\mathbf{q}) = \int dp'\, F(p-p';\mathbf{q}+p')G(p';\mathbf{q}). \tag{3.4}$$

Differential operators form a special subclass of this class of operators. With each differential operator $\mathcal{L}(x,\partial_x)$ with the kernel

$$\mathcal{L}(x,x') = \mathcal{L}(x,\partial_x)\delta(x_1-x_1')\delta(x_2-x_2'), \tag{3.5}$$

we associate the operator $L(q)$ (we call it the *extension* of $\mathcal{L}$) with the kernel

$$L(x,x';q) = e^{-q(x-x')}\mathcal{L}(x,x') \equiv \mathcal{L}(x,\partial_x+q)\delta(x-x'), \tag{3.6}$$

where $qx = q_1x_1 + q_2x_2$. The kernel $L(p;\mathbf{q})$ of the extended differential operator (as defined in (3.1)) depends on the variables $\mathbf{q}$ polynomially. In particular, if $D_j(q)$, $j = 1,2$, denotes the extension of the differential operator $i\partial_{x_j}$, i.e.,

$$D_j(x,x';q) = i(\partial_{x_j}+q_j)\delta(x-x'), \tag{3.7}$$

then the kernel of this operator in the $(p,\mathbf{q})$ space is

$$D_j(p;\mathbf{q}) = \mathbf{q}_j\delta(p). \tag{3.8}$$

The polynomial dependence of kernels of differential operators in the $(p,\mathbf{q})$ space on the variables $\mathbf{q}$ suggests introducing the operation of $\bar{\partial}$-differentiation with respect to these variables:

$$(\bar{\partial}_jF)(p;\mathbf{q}) = \frac{\partial F(p;\mathbf{q})}{\partial\bar{\mathbf{q}}_j}, \quad j = 1,2. \tag{3.9}$$

In the class of operators under consideration, the differential operators are then selected by the condition

$$\bar{\partial}_jL = 0, \quad j = 1,2. \tag{3.10}$$

We note that in the case of a differential operator with constant coefficients, by (3.6), we have

$$L(p;\mathbf{q}) = l(\mathbf{q})\delta(p), \tag{3.11}$$

where $l(\mathbf{q})$ is a polynomial function of its arguments. For the commutator of the differential operator L with an arbitrary operator F of the considered class, by (3.4), we then obtain

$$[L,F](p;\mathbf{q}) = \big(l(p+\mathbf{q}) - l(\mathbf{q})\big)F(p;\mathbf{q}), \tag{3.12}$$

which also holds if $l(\mathbf{q})$ in (3.11) is a meromorphic function.

We here skip additional conditions that allow defining inverse operators uniquely (if they exist) and only mention for future use that the operator inverse to D_j is defined as (cf. (3.8))

$$D_j^{-1}(p;\mathbf{q}) = \frac{\delta(p)}{\mathbf{q}_j}, \tag{3.13}$$

and its kernel in the x space can be found from (3.2). This construction is used in Sec. 3.3 in the case where the real variables x, x', and q are one-dimensional. The corresponding simplifications of the above formulas are obvious, and we omit them here.

3.2. (2+1)-dimensional integrable equations. We now realize elements of the associative algebra as operators $A(q)$, $B(q)$, etc., in the sense of the definitions in Sec. 3.1, and we impose specific conditions on this realization that allow deriving the Lax pairs and nonlinear integrable equations. The examples given in Secs. 1 and 2 show that the number of generating elements of the commutator identities is equal to the number of space variables of the associated linear differential equations. Correspondingly, we choose the kernels $A(x,x';q)$ and $B(x,x';q)$ to be dependent on vectors x and x' of the same dimension as the number of generating elements. In this section, we consider identities (1.1), (1.6), (2.1), (2.4), (2.18) (in the case $h_1h_2 \neq 0$), and (2.19), i.e., those identities that are generated by two adjoint actions of powers of A (or its matrix functions as in (2.18) and (2.19)). Hence, the real vectors x and x' (and then also q) are two-dimensional here. In what follows, we assume the following conditions.

Condition 1. The time dependence is introduced using one of relations (1.4), (1.8), (2.2), (2.6), (2.21), or (2.22), and this gives operators $B(t,q)$ with kernels $B(x,x';t,q)$ belonging to the same space of operators.

Taking into account that the time variables corresponding to two generators of these identities are denoted by t_1 and t_2, we next impose the following condition.

Condition 2. The time dependence of $B(t,q)$ on the variables t_1 and t_2 reduces to a shift of the space variables of the kernel, i.e.,

$$B(x,x';t_1,t_2,q) = B(x_1+t_1, x_2+t_2, x'_1+t_1, x'_2+t_2; q), \tag{3.14}$$

where the dependence on the other variables t_m is omitted.

In terms of the $(p,\mathbf{q})$ kernels defined by (3.1), this means that

$$B(p;t,\mathbf{q}) = e^{-it_1p_1-it_2p_2}B(p;\mathbf{q}). \tag{3.15}$$

In the differential form, we have

$$\partial_{t_j}B(t,x,x';q) = (\partial_{x_j}+\partial_{x'_j})B(t,x'x';q), \tag{3.16}$$

$$\partial_{t_j}B(t,p;\mathbf{q}) = -ip_jB(t,p;\mathbf{q}), \quad j=1,2, \tag{3.17}$$

which by (3.7) or (3.8) and (3.12) can be written as

$$i\partial_{t_j}B(t,q) = \bigl[D_j(q), B(t,q)\bigr]. \tag{3.18}$$

To explain Condition 2, we note that in the study in [1] in terms of the inverse scattering transform of the algebraic schemes developed in [13]–[15], relation (3.14) was derived as the evolution equation of the operator scattering data.

Condition 3. If the dependence of the kernel $A(p;\mathbf{q})$ on the $\mathbf{q}$ variables reduces to only one linear combination of $\mathbf{q}_1$ and $\mathbf{q}_2$ with constant coefficients, then the $\mathbf{q}$ dependence of the kernel $B(p;\mathbf{q})$ is the same.

Formally, this condition means that the operator B is a function of the space variables and the operator A. Below, we consider consequences of these conditions in the operator realization of the commutator identities listed in Secs. 1 and 2.

3.2.1. The KPII equation. We illustrate the procedure for reconstructing the Lax pair and nonlinear equation using identity (1.1) as an example. By (1.4), we obtain

$$\frac{\partial B(t)}{\partial t_1} = \bigl[A, B(t)\bigr], \qquad \frac{\partial B(t)}{\partial t_2} = \bigl[A^2, B(t)\bigr]. \tag{3.19}$$

The first equality here and (3.18) show that we must set

$$A(p;\mathbf{q}) = -i\mathbf{q}_1\delta(p), \qquad \text{i.e.,} \qquad A = -iD_1. \tag{3.20}$$

It now follows from the second equality in (3.19) that because of (3.12) and (3.17), the kernel of the operator $B(q)$ in the $(p,\mathbf{q})$ space satisfies

$$\bigl[ip_2 + (p_1+\mathbf{q}_1)^2 - \mathbf{q}_1^2\bigr]B(p;\mathbf{q}) = 0. \tag{3.21}$$

Taking (3.12) into account, we can write this as the equality

$$\bigl[L_0(q), B(q)\bigr] = 0, \tag{3.22}$$

where we introduce the operator L_0 with the kernel

$$L_0(p;\mathbf{q}) = (i\mathbf{q}_2 + \mathbf{q}_1^2)\delta(p). \tag{3.23}$$

Taking definitions (3.8) into account, we can write this operator as

$$L_0 = iD_2 + D_1^2. \tag{3.24}$$

By (3.5)–(3.7), this operator is an extension of the operator

$$\mathcal{L}_0 = -\partial_{x_2} - \partial_{x_1}^2, \tag{3.25}$$

which is the differential part (the part corresponding to a zero potential) of the Lax operator associated with the KPII equation (see [16], [17]):

$$\mathcal{L} = -\partial_{x_2} - \partial_{x_1}^2 - u(x). \tag{3.26}$$

Condition 3 means that the kernel $B(p;\mathbf{q})$ is independent of $\mathbf{q}_2$, and by (3.21), it therefore has the form

$$B(p;t,\mathbf{q}) = \delta\bigl(ip_2 + p_1(p_1+2\mathbf{q}_1)\bigr)b(p,t) \tag{3.27}$$

(we here skip consideration of more complicated solutions of (3.21) that are given by $\partial/\partial\bar{\mathbf{q}}_1$ derivatives of (3.27)), where we use the notation for the δ-function of a complex argument, $\delta(z) = \delta(z_{\Re})\delta(z_{\Im})$. In (3.27), the function $b(p,t)$ is

$$b(p,t_1,t_2,t_3) = \exp\biggl(-ip_1t_1 - ip_2t_2 + it_3\frac{p_1^4 - 3p_2^2}{4p_1}\biggr)b(p), \tag{3.28}$$

where the dependence on t_1, t_2, and t_3 is given by (2.2) and $b(p)$ is independent of t (and $\mathbf{q}$).

We now introduce the operator ν with the kernel $\nu(p;\mathbf{q})$ depending on the same linear combination of $\mathbf{q}_1$ and $\mathbf{q}_2$ that was mentioned in Condition 3. In this case, because of (3.20) and (3.27), this means that $\nu(p;\mathbf{q}) = \nu(p;\mathbf{q}_1)$ and

$$\bar{\partial}_1 \nu = \nu B \tag{3.29}$$

(see (3.9)). To define the operator ν uniquely, we normalize it to be the unity operator at the singularity point of the kernel $L_0(p;\mathbf{q})$ with respect to $\mathbf{q}_1$. By (3.20), this here means that

$$\nu(p;\mathbf{q}_1) = \delta(p) + O(\mathbf{q}_1^{-1}), \quad \mathbf{q}_1 \to \infty. \tag{3.30}$$

Because of (3.10), it then follows from (3.22) that $[L_0, \nu]$ satisfies

$$\bar{\partial}_1 [L_0, \nu] = [L_0, \nu] B, \tag{3.31}$$

i.e., the same Eq. (3.29). By (3.12) and (3.30), the kernel $[L_0, \nu](p;\mathbf{q}_1)$ has some constant asymptotic behavior (with respect to $\mathbf{q}_1$) at infinity. Letting $-u(p,t)$ denote this asymptotic form and assuming the unique solvability of (3.29), we find that $[L_0, \nu] = u\nu$, i.e.,

$$L\nu = \nu L_0, \tag{3.32}$$

where L is the extension (in the sense of (3.6)) of differential operator (3.26). Equation (3.32) means that ν is a dressing (transformation) operator. It was shown in [6] that by substituting (3.27) in (3.29), we obtain the standard equation for the inverse problem for the Jost solution (see [18]) and that the second operator of the Lax pair and also the differential equation for $u(x,t)$ (the Fourier transform of $u(p,t)$) follow from (3.28) and (3.29). We do not reproduce these details here.

3.2.2. The KPI equation. We derived the KPII equation above under the assumption that all the times in (1.4) are real. Substituting $t_j \to -it_j$ in (1.4) and (1.6), instead of (3.19), we obtain

$$i\frac{\partial B(t)}{\partial t_1} = \bigl[A, B(t)\bigr], \qquad i\frac{\partial B(t)}{\partial t_2} = \bigl[A^2, B(t)\bigr], \tag{3.33}$$

and so on. Therefore, the same as in (3.20), we now obtain

$$A = D_1. \tag{3.34}$$

It follows from (3.12), (3.17), and the second equality in (3.33) that the kernel of the operator $B(q)$ in the $(p,\mathbf{q})$ space in this case satisfies

$$\bigl[p_2 - (p_1 + \mathbf{q}_1)^2 + \mathbf{q}_1^2\bigr] B(p;\mathbf{q}) = 0. \tag{3.35}$$

This means that the operator $B(q)$ satisfies the same commutator equality (3.22), while now

$$L_0 = D_2 - D_1^2. \tag{3.36}$$

In terms of the discussion in Sec. 3.1, this is an extension of the differential operator

$$\mathcal{L}_0 = i\partial_{x_2} + \partial_{x_1}^2, \tag{3.37}$$

i.e., of the differential part of the linear operator associated with the KPI equation (see [16], [17]):

$$\mathcal{L} = i\partial_{x_2} + \partial_{x_1}^2 - u(x). \tag{3.38}$$

Because of Condition 3 and in analogy with (3.27) and (3.28), we now obtain

$$B(t,p;\mathbf{q}_1) = \delta\bigl(p_2 - p_1(p_1 + 2\mathbf{q}_1)\bigr)b(t,p), \tag{3.39}$$

where

$$b(t_1,t_2,t_3,p) = \exp\biggl(-ip_1t_1 - ip_2t_2 - it_3\frac{p_1^4 + 3p_2^2}{4p_1}\biggr)b(p). \tag{3.40}$$

Again, we can define the transformation operator ν using (3.29) and (3.30). But in this case, the argument of the δ-function in the r.h.s. of (3.39) is proportional to $\mathbf{q}_{1\Im} = q_1$, and instead of $\bar{\partial}$-problem (3.29), we therefore obtain a nonlocal Riemann–Hilbert problem, i.e., we encounter a well-known difference between the inverse problems for KPII and KPI equations (compare [18] with [19] and [20]). Again, $L_0(p;\mathbf{q})$ here behaves polynomially at infinity, and we therefore impose the same normalization condition (3.30). Further, using (3.22), we prove relation (3.32), where L is now the extension in the sense of (3.6) of differential operator (3.38). Referring to [5], we can derive the second equation of the Lax pair and prove that $u(x,t)$ satisfies the KPI equation when the dependence on the three times t_1, t_2, and t_3 is introduced. In the same way, we can use identities (1.6) to derive higher equations of the KP hierarchy (see [1]). By construction, it is clear that all of them have the same associated linear operator $\mathcal{L}$.

3.2.3. The BLP equation. Now let the operator $B(t)$ be defined by (2.2)). Hence, by (2.1), it satisfies (2.3). Instead of (3.19), we then obtain

$$\frac{\partial B(t)}{\partial t_1} = \bigl[A, B(t)\bigr], \qquad \frac{\partial B(t)}{\partial t_2} = \bigl[A^{-1}, B(t)\bigr]. \tag{3.41}$$

Using Condition 2 or one of its differential forms (3.16)–(3.18), we find that the operator A is the same as in (3.20). Analogously to the derivation of (3.21), it then follows from the second equality in (3.41), (3.12), and (3.13) for $j = 2$ that the kernel of the operator B satisfies

$$\biggl(p_2 + \frac{1}{p_1 + \mathbf{q}_1} - \frac{1}{\mathbf{q}_1}\biggr)B(p;\mathbf{q}) = 0. \tag{3.42}$$

Hence, $B(q)$ again satisfies (3.22), where by (3.12),

$$L_0 = D_2 + D_1^{-1}. \tag{3.43}$$

Condition 3 yields $B(p;\mathbf{q}) = B(p;\mathbf{q}_1)$, and relation (3.42) means that the kernel $B(p;\mathbf{q})$ has the form (cf. (3.27))

$$B(t,p;\mathbf{q}) = \delta\biggl(p_2 + \frac{1}{p_1 + \mathbf{q}_1} - \frac{1}{\mathbf{q}_1}\biggr)b(t,p), \tag{3.44}$$

where the function $b(t,p)$ is

$$b(t_1,t_2,t_3,p) = \exp\biggl(-ip_1t_1 - ip_2t_2 - it_3p_1^2\sqrt{1 - \frac{4}{p_1p_2}}\biggr)b(p), \tag{3.45}$$

where the dependence on t_1, t_2, and t_3 corresponds to (2.2).

The kernel of the operator L_0 is equal to $L_0(p;\mathbf{q}) = (\mathbf{q}_2 + 1/\mathbf{q}_1)\delta(p)$; therefore, in contrast to the KP case, it is only a meromorphic function of $\mathbf{q}_1$. Correspondingly, the kernel $\nu(p;\mathbf{q}_1)$ of the transformation operator is defined as the solution of the same $\bar{\partial}$-problem (3.29). But $L_0(p;\mathbf{q})$ is now singular at $\mathbf{q}_1 = 0$, the normalization condition is consequently given by

$$\nu(p;\mathbf{q}_1)\big|_{\mathbf{q}_1=0} = \delta(p), \tag{3.46}$$

and we seek solutions bounded as $\mathbf{q}_1 \to \infty$. As a result, the $\bar{\partial}$-equation for the commutator $[L_0,\nu]$ has an additional inhomogeneous term (see (3.31)). Nevertheless, simple calculations prove that (3.32) also holds in this case with the dressed operator

$$L = D_2 + D_1^{-1} + D_1^{-1}\alpha + \beta, \tag{3.47}$$

where α and β are multiplication operators, i.e., they have the kernel $\alpha(x,x';q) = \alpha(x)\delta(x-x')$ and the analogous kernel for β in the x space. Operator (3.47) is not a differential operator. A differential operator is given by the product $\tilde{L} = D_1 L$, which by (3.6) is the extension of the operator

$$\tilde{\mathcal{L}} = \partial_{x_1}\partial_{x_2} + 1 + \alpha_x + \partial_{x_1}\cdot\beta(x), \tag{3.48}$$

i.e., the auxiliary linear operator of the BLP equation (see [8]). It was noted in [9] that studying the direct and inverse problems for this operator in fact reduces to studying these problems for operator (3.47). It was proved in [10] that by (3.45), the time evolution (with respect to t_3) of the spectral data B leads to the nonlinear differential BLP equation for α and β.

The exponent in (3.45) is not purely imaginary. Correspondingly, the time evolution for the BLP equation is unstable (see [10]). It was also proved in [10] that if the dependence on t_3 is given by (2.5), then the resulting nonlinear differential equation for α and β does not have this defect. Here, we omit further details and only mention that the dependence on t_1 and t_2 in (2.2) and (2.5) coincides; hence, in the case of identity (2.6), using the above procedure, we obtain the same L-operator (3.47).

3.2.4. The DS and VN equations. We now briefly consider the consequences of identities (2.18) and (2.19), where B is the matrix given by (2.7) and the matrices I_j are defined by (2.8). Let B_1 and B_2 be operators belonging to the space in Sec. 3.1 with the kernels $B_j(x,x';q)$ in the x space (with the kernels $B_j(p;\mathbf{q})$ in the $(p,\mathbf{q})$ space). Under the action of evolutions (2.21) and (2.22), we obtain the corresponding operators with the kernels $B(t,x,x';q)$ and $B'(t,x,x';q)$. We consider equality (2.18). Imposing Condition 2 on the operator $B(t,q)$ and taking (3.17) and (2.8) into account, we obtain

$$i(AB_1)(p;\mathbf{q}) = p_1 B_1(p;\mathbf{q}), \qquad -i(B_2A)(p;\mathbf{q}) = p_1 B_2(p;\mathbf{q}), \tag{3.49}$$

$$-i(B_1A)(p;\mathbf{q}) = p_2 B_1(p;\mathbf{q}), \qquad i(AB_2)(p;\mathbf{q}) = p_2 B_2(p;\mathbf{q}). \tag{3.50}$$

Summing the equations in (3.49) and (3.50) pairwise by columns, we obtain $i[A,B_j](p;\mathbf{q}) = (p_1+p_2)B_j(p;\mathbf{q})$, $j = 1,2$. Therefore, by (3.12), we must here set

$$A(p;\mathbf{q}) = -i(\mathbf{q}_1 + \mathbf{q}_2)\delta(p), \tag{3.51}$$

which, because of (3.8), can be written in the operator form as

$$A = -i(D_1 + D_2). \tag{3.52}$$

It now remains to satisfy the equations, for example, in (3.49), giving the conditions $(\mathbf{q}_1+\mathbf{q}_2+p_2)B_1(p;\mathbf{q}) = 0$ and $(\mathbf{q}_1+\mathbf{q}_2+p_1)B_2(p;\mathbf{q}) = 0$. To satisfy these equalities, it suffices to choose the kernel of the operator B in the form

$$B(p;t,\mathbf{q}) = \begin{pmatrix} 0 & \delta(\mathbf{q}_1+\mathbf{q}_2+p_2)b_1(p,t) \\ \delta(\mathbf{q}_1+\mathbf{q}_2+p_1)b_2(p,t) & 0 \end{pmatrix}, \tag{3.53}$$

where we use the fact that the kernel $B(t,p;\mathbf{q})$ is independent of the difference $\mathbf{q}_1-\mathbf{q}_2$ by Condition 3 and (3.51). The time dependence of the operator B in the case where all three times are introduced by (2.21) is given using the explicit relation

$$B(p;t,\mathbf{q}) = e^{-it_1p_1-it_2p_2-t_3\sigma_3(h_1p_1^2-h_2p_2^2)} \times \\ \times \begin{pmatrix} 0 & \delta(\mathbf{q}_1+\mathbf{q}_2+p_2)b_1(p) \\ \delta(\mathbf{q}_1+\mathbf{q}_2+p_1)b_2(p) & 0 \end{pmatrix}. \tag{3.54}$$

Consequently, we should change t_3 to it_3 to avoid instability.

We must now construct a nontrivial operator L_0 with constant coefficients that satisfies Eq. (3.22). Taking the δ-functions in (3.53) and the off-diagonal matrix structure of the operator B into account, we see that $L_0(p;\mathbf{q}) = \operatorname{diag}\{P_1(\mathbf{q}),P_2(\mathbf{q})\}\delta(p)$, where $P_j(\mathbf{q})$ are polynomials. Under the condition that these polynomials are nontrivial and have the lowest possible degree, we easily find that

$$L_0 = \begin{pmatrix} D_2 & 0 \\ 0 & -D_1 \end{pmatrix}. \tag{3.55}$$

Because of our definitions (3.5)–(3.8), this operator is an extension of the operator

$$\mathcal{L}_0 = \begin{pmatrix} \partial_{x_2} & 0 \\ 0 & -\partial_{x_1} \end{pmatrix}, \tag{3.56}$$

which is the differential part of the two-dimensional Zakharov–Shabat problem, known to be the auxiliary linear problem for the DS equation [21].

The transformation operator ν with the kernel $\nu(p;\mathbf{q}_1+\mathbf{q}_2)$ is defined using $\bar\partial$-problem (3.29) and is normalized to be the unit matrix operator (cf. (3.30)) at the singularity point of the kernel $L_0(p;\mathbf{q})$, i.e., as $\mathbf{q}_1+\mathbf{q}_2\to\infty$. We can then prove that (3.31) holds because of (3.22) and (3.55). Further, taking the asymptotic behavior of the kernel $[L_0,B](p;\mathbf{q})$ as $\mathbf{q}\to\infty$ into account, we derive (3.32), where L is now the extension of the linear operator of the Zakharov–Shabat problem. The consideration of identity (2.19) goes along the same lines because the dependences on t_1 and t_2 in (2.21) and (2.22) coincide; we consequently obtain the VN equation.

3.3. (1+1)-dimensional equations. We have seen that without loss of generality, the operator A can always be chosen as a linear combination of the operators D_1 and D_2 with constant (matrix) coefficients (see (3.7) and (3.8)). To obtain integrable (1+1)-dimensional equations, we can perform a reduction, imposing the condition that the operator B commutes with an additional linear combination of D_1 and D_2. As follows from (3.12), the function $b(p)$ in (3.28), (3.40), and (3.45) is then proportional to $\delta(p_2)$, and the functions $b_j(p)$ in (3.53) are proportional to $\delta(p_1+p_2)$. It is easy to verify that we thus indeed obtain the corresponding (1+1)-dimensional nonlinear evolution equations. On the other hand, we can directly apply the procedure described above to identities (2.9)–(2.11) and the corresponding Eqs. (2.14) and (2.15).

As an example, we consider (2.9). Here, we have only one generator of the commutator algebra, the commutator with AI_1. Correspondingly, we realize the operator B as an integral operator in the one-dimensional space, the operator with the kernel $B(x_1, x_1'; q_1)$. Using obvious one-dimensional versions of the formulas given in Sec. 3.1, we here impose the condition

$$B(x_1, x_1'; t_1, q_1) = B(x_1 + t_1, x_1' + t_1, ; q_1) \tag{3.57}$$

instead of Condition 2 (see (3.14)), and we then have

$$\partial_{t_1} B(p_1; t_1, \mathbf{q}_1) = -ip_1 B(p_1; t_1, \mathbf{q}_1) \tag{3.58}$$

by (3.17) for $j = 1$. On the other hand, by (2.12), we have $\partial_{t_1} B(t) = \big[AI_1, B(t)\big]$. We choose A as a differential operator with constant coefficients; because of (3.11), we then have the representation $A(p_1, \mathbf{q}_1) = a(\mathbf{q}_1)\delta(p_1)$ for its kernel, where $a(\mathbf{q}_1)$ is a polynomial. Using (2.7), (2.8), and property (3.12), we obtain

$$\big(a(p_1 + \mathbf{q}_1) + ip_1\big) B_1(p_1, \mathbf{q}_1) = 0, \qquad \big(a(\mathbf{q}_1) - ip_1\big) B_2(p_1, \mathbf{q}_1) = 0. \tag{3.59}$$

In this case, we have no variables p_2 and $\mathbf{q}_2$, and the zeroes of the expressions in parentheses must occur for the same values of $\mathbf{q}_1$ and p_1. This condition defines $a(\mathbf{q}_1)$ as a first-order polynomial equal to $-2i\mathbf{q}_1$ (up to a shift by a constant), i.e.,

$$A(p_1; \mathbf{q}_1) = -2i\mathbf{q}_1 \delta(p_1) \tag{3.60}$$

or by (3.8),

$$A = -2iD_1. \tag{3.61}$$

Equalities (3.59) then reduce to $(p_1 + 2\mathbf{q}_1)B_j(p_1, \mathbf{q}_1) = 0$; by (2.7), we hence have the representation

$$B(t, p_1; \mathbf{q}_1) = \begin{pmatrix} 0 & b_1(t, p_1) \\ b_2(t, p_1) & 0 \end{pmatrix} \delta(p_1 + 2\mathbf{q}_1) \tag{3.62}$$

for the kernel of the operator B. As above, it is easy to prove that the operator L_0 satisfying relation (3.22) is given by

$$L_0 = \sigma_3 D_1, \tag{3.63}$$

and we thus obtain the differential part of the one-dimensional Zakharov–Shabat problem. The transformation operator ν and the potential u can be constructed along the same lines as above using $\bar{\partial}$-problem (3.29), which here, as in the KPI and DS cases, reduces to the Riemann–Hilbert problem because of the specific argument of the δ-function in (3.62).

4. Concluding remarks

We have demonstrated that the existence of commutator identities leads to integrable nonlinear evolution equations. The proposed scheme for reconstructing such equations together with their Lax pairs is general and model independent, although the examples considered here correspond only to the known integrable equations. The problem of describing all commutator identities on associative algebras thus arises. Another open question is the existence of commutator identities with three generators. Following the lines of the procedure proposed here, we find it reasonable to expect that such identities would lead to (3+1)-dimensional integrable equations. The commutator identities described here hold on arbitrary associative algebras, while the realization of the operators A and B that allowed obtaining nonlinear evolution equations was rather specific. It is worthwhile to assume that the same identities are related to some other integrable systems under some other realization of the elements of the associative algebra.

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