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HIGHER-SPIN CONFORMAL CURRENTS IN MINKOWSKI SPACE

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Using the unfolded formulation of equations for massless free fields of all spins, we obtain an explicit form of the gauge-invariant higher-spin conformal conserved charges bilinear in four-dimensional massless fields of arbitrary spins.

Keywords: higher-spin gauge theory, conformal current

1. Introduction

In this paper, we construct an explicit form of gauge-invariant higher-spin (HS) conserved currents constructed from four-dimensional massless fields of all spins. To the best of our knowledge, a complete realization of HS conformal currents constructed from massless fields of arbitrary spins in the four-dimensional Minkowski space has not yet been available in the literature, although some particular examples of HS conformal currents constructed from massless fields of lower spins $s \leq 1$ were considered. In particular, x-independent HS conformal currents constructed from massless scalar, spinor, and Maxwell fields were found in [1], and x-dependent HS currents constructed from massless scalar and spinor fields were found in [2]. We extend these results in two directions: the currents are constructed, first, from fields of arbitrary spin and, second, with explicit dependence on the space–time coordinates.

The analysis of HS currents has several applications. Most notably, constructing conserved currents is the first step toward a nonlinear HS theory because they characterize the so-called Noether cubic interactions of gauge fields of all spins. At the same time, charges constructed from the currents serve as generators of the respective lower and higher symmetries. Hence, the construction of HS currents sheds some light on the structure of HS symmetries and interactions.

Our construction is based on the unfolded formulation of dynamical equations in the form of zero-curvature equations [3] and is analogous to the construction of HS currents [4] in the generalized space–time with matrix coordinates [5]–[8].

2. Unfolded massless field equations in four-dimensional Minkowski space

It was shown in [3], [7] that the equations for field strengths of massless fields of all spins in Minkowski space can be concisely formulated in the unfolded form

$$\frac{\partial}{\partial x^{a\dot{b}}} C(w,\overline{w}\,|\,x) + \frac{\partial^2}{\partial w^a \partial \overline{w}^{\dot{b}}} C(w,\overline{w}\,|\,x) = 0, \tag{1}$$

where w^a and $\overline{w}^{\dot{b}}$ are auxiliary commuting conjugate two-component spinor coordinates ($a, b = 1, 2$ and $\dot{a}, \dot{b} = \dot{1}, \dot{2}$) and $x^{a\dot{b}}$ are Minkowski coordinates in the two-component spinor notation. The two-component

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indices are raised and lowered as

$$A^a = \varepsilon^{ab} A_b, \qquad A_a = \varepsilon_{ba} A^b, \qquad \varepsilon_{ab} = -\varepsilon_{ba}, \quad \varepsilon_{12} = 1, \tag{2}$$

and analogously for dotted indices. The relation to the tensor notation is given by

$$V^{a\dot{b}} = A^\nu \sigma_\nu^{a\dot{b}}, \tag{3}$$

where $\sigma_\nu^{a\dot{b}}$, $\nu = 0, 1, 2, 3$, are four Hermitian 2×2 matrices.

The meaning of Eqs. (1) is as follows. The fields $C(w, \overline{w}\,|\,x)$ are assumed to be expandable in power series in w^a and $\overline{w}^{\dot{a}}$,

$$C(w, \overline{w}\,|\,x) = \sum_{m,n=0}^{\infty} C_{a_1 \ldots a_n \dot{a}_1 \ldots \dot{a}_m}(x) w^{a_1} \cdots w^{a_n} \overline{w}^{\dot{a}_1} \cdots \overline{w}^{\dot{a}_m}. \tag{4}$$

The operator

$$N_{w,\overline{w}} = w^a \frac{\partial}{\partial w^a} - \overline{w}^{\dot{a}} \frac{\partial}{\partial \overline{w}^{\dot{a}}}$$

commutes with $\partial^2/(\partial w^a\, \partial \overline{w}^{\dot{a}})$. Solutions of Eqs. (1) with fixed eigenvalues of $N_{w,\overline{w}}$ form an invariant subspace describing fields of different helicities h,

$$N_{w,\overline{w}} C(w, \overline{w}\,|\,x) = 2h C(w, \overline{w}\,|\,x). \tag{5}$$

The holomorphic fields

$$C(w, 0\,|\,x) = \sum_{2s=0}^{\infty} C_{a_1 \ldots a_{2s}}(x) w^{a_1} \cdots w^{a_{2s}}$$

and their complex conjugates (w^a is complex conjugate to $\overline{w}^{\dot{a}}$)

$$C(0, \overline{w}\,|\,x) = \sum_{2s=0}^{\infty} C_{\dot{a}_1 \ldots \dot{a}_{2s}}(x) \overline{w}^{\dot{a}_1} \cdots \overline{w}^{\dot{a}_{2s}}$$

respectively describe self-dual (positive helicity) and anti-self-dual (negative helicity) gauge-invariant on-shell nontrivial (on the equations of motion) combinations of derivatives of massless gauge fields of all spins $s = 0, 1/2, 1, \ldots, \infty$, where

$$w^a \frac{\partial}{\partial w^a} C(w, 0\,|\,x) = 2s C(w, 0\,|\,x)$$

and

$$\overline{w}^{\dot{a}} \frac{\partial}{\partial \overline{w}^{\dot{a}}} C(0, \overline{w}\,|\,x) = 2s C(0, \overline{w}\,|\,x).$$

These include the scalar field ($s = 0$)

$$c(x) = C(0, 0\,|\,x),$$

the spinor field ($s = 1/2$)

$$c_a(x) = \frac{\partial}{\partial w^a} C(w, 0\,|\,x)\bigg|_{w=0}, \qquad \bar{c}_{\dot{a}}(x) = \frac{\partial}{\partial \overline{w}^{\dot{a}}} C(0, \overline{w}\,|\,x)\bigg|_{\overline{w}=0},$$

the Maxwell tensor ($s = 1$)

$$c_{ab}(x) = \frac{1}{2} \frac{\partial^2}{\partial w^a\, \partial w^b} C(w, 0\,|\,x)\bigg|_{w=0}, \qquad \bar{c}_{\dot{a}\dot{b}}(x) = \frac{1}{2} \frac{\partial^2}{\partial \overline{w}^{\dot{a}}\, \partial \overline{w}^{\dot{b}}} C(0, \overline{w}\,|\,x)\bigg|_{\overline{w}=0},$$

the Rarita–Schwinger field strength ($s = 3/2$)

$$c_{abc}(x) = \frac{1}{3!}\frac{\partial^3}{\partial w^a\,\partial w^b\,\partial w^c} C(w,0\,|\,x)\bigg|_{w=0}, \qquad \bar{c}_{\dot{a}\dot{b}\dot{c}}(x)\frac{1}{3!}\frac{\partial^3}{\partial \overline{w}^{\dot{a}}\,\partial \overline{w}^{\dot{b}}\,\partial \overline{w}^{\dot{c}}} C(0,\overline{w}\,|\,x)\bigg|_{\overline{w}=0},$$

the Weyl tensor ($s = 2$)

$$c_{abcd}(x) = \frac{1}{4!}\frac{\partial^4}{\partial w^a\,\partial w^b\,\partial w^c\,\partial w^d} C(w,0\,|\,x)\bigg|_{w=0},$$

$$\bar{c}_{\dot{a}\dot{b}\dot{c}\dot{d}}(x) = \frac{1}{4!}\frac{\partial^3}{\partial \overline{w}^{\dot{a}}\,\partial \overline{w}^{\dot{b}}\,\partial \overline{w}^{\dot{c}}\,\partial \overline{w}^{\dot{d}}} C(0,\overline{w}\,|\,x)\bigg|_{\overline{w}=0},$$

and so on [7].

The *primary* fields are those contained in $C(w,0\,|\,x)$ and their complex conjugates $C(0,\overline{w}\,|\,x)$. They describe gauge-invariant combinations of lowest-order derivatives with respect to x of massless gauge fields, which were considered by many authors (see, e.g., [9]). We note that the order of derivatives of a spin s field is equal to $[s]$.

The descendants are described by the components of $C(w,\overline{w}\,|\,x)$ that depend on both w and $\overline{w}$ and are therefore expressed in terms of derivatives of the primary fields by (1).

The dynamical HS field equations are consequences of (1):

$$\frac{\partial}{\partial x^{a\dot{b}}}\frac{\partial}{\partial w_a} C(w,0\,|\,x) = 0, \qquad \frac{\partial}{\partial x^{a\dot{b}}}\frac{\partial}{\partial \overline{w}_{\dot{b}}} C(0,\overline{w}\,|\,x) = 0, \tag{6}$$

for fields of spin $s \neq 0$ and the massless Klein–Gordon equation

$$\frac{\partial^2}{\partial x^{a\dot{b}}\,\partial x_{a\dot{b}}} c(x) = 0 \tag{7}$$

for scalar fields (for $s > 0$, (7) is a consequence of (6)).

A given function $C(w,\overline{w}\,|\,0)$ of the spinors w^a and $\overline{w}^{\dot{a}}$ uniquely reconstructs a solution of Eqs. (1) by

$$C(w,\overline{w}\,|\,x) = \exp\left(-x^{a\dot{b}}\frac{\partial^2}{\partial w^a\,\partial \overline{w}^{\dot{b}}}\right) C(w,\overline{w}\,|\,0).$$

Conversely, a given solution of Eqs. (1) reconstructs the full dependence on w and $\overline{w}$ as follows. The Taylor expansion gives

$$C(w,\overline{w}\,|\,x) = \exp\left(w^a\frac{\partial}{\partial v^a} + \overline{w}^{\dot{a}}\frac{\partial}{\partial \bar{v}^{\dot{a}}}\right) C(v,\bar{v}\,|\,x)\bigg|_{v=\bar{v}=0}.$$

For a given helicity $h \geq 0$, we obtain

$$C(w,\overline{w}\,|\,x) = \frac{1}{(2h)!}\left(w^b\frac{\partial}{\partial v^b}\right)^{2h} F_h\left(w^a\frac{\partial}{\partial v^a}\overline{w}^{\dot{a}}\frac{\partial}{\partial \bar{v}^{\dot{a}}}\right) C(v,\bar{v}\,|\,x)\bigg|_{v=\bar{v}=0},$$

where the function

$$F_h(r) = \sum_{n=0}^{\infty}\frac{(2h)!}{n!\,(2h+n)!} r^n$$

is related to the regular Bessel functions $I_k(x)$ (see, e.g., [10]) by

$$\frac{r^h}{(2h)!} F_h(r) = I_{2h}(2r^{1/2}).$$

Again using Eqs. (1), we now obtain

$$C^h(w,\overline{w}\,|\,x) = F_h(-w^a\overline{w}^{\dot b}\partial_{a\dot b})C^h(w,0\,|\,x) \tag{8}$$

for a field with a positive helicity h, where $\partial_{a\dot a} = \partial/\partial x^{a\dot a}$. Analogously, we obtain

$$C^h(w,\overline{w}\,|\,x) = F_{|h|}(-w^a\overline{w}^{\dot b}\partial_{a\dot b})C^h(0,\overline{w}\,|\,x) \tag{9}$$

for negative helicities $h < 0$. This reconstructs the dependence on w and $\overline{w}$.

3. Higher-spin conformal currents

It follows from Eqs. (1) that the field equations for massless fields of all spins are $sp(8)$ symmetric with $sp(8)$ being a maximal finite-dimensional subalgebra of the infinite-dimensional HS symmetry [7]. This symmetry is conformal because $sp(8)$ contains the four-dimensional conformal algebra $su(2,2)$ as a subalgebra. The infinite set of conformal HS symmetries suggests the existence of conserved HS currents.

The HS charges in Minkowski space should have the form

$$Q(\eta) = \int_{\Sigma^3} \Omega^3(\eta), \tag{10}$$

where η denotes the HS symmetry parameters, $\Omega^3(\eta)$ is an on-shell closed 3-form (on the equations of motion) dual to the conserved current, and Σ^3 is an arbitrary three-dimensional surface in the Minkowski space–time usually identified with the space surface $\mathbb{R}^3$, i.e., the Cauchy surface.

Using the unfolded form of the massless field equations, we can easily write explicit formulas for the conserved HS charges in the four-dimensional Minkowski space. We consider the 3-form in the Minkowski space M^4

$$\begin{aligned}\Omega^3(\eta) = dx_{a\dot a}\wedge dx^{a\dot c}\wedge dx^{c\dot a}\eta_{b_1\dots b_l\dot b_{l+1}\dots\dot b_t}{}^{\alpha_1\dots\alpha_s}\times\\ \times\, x^{b_1\dot e_1}\dots x^{b_l\dot e_l}x^{e_{l+1}\dot b_{l+1}}\cdots x^{e_t\dot b_t}T_{c\dot c\alpha_1\dots\alpha_s e_{l+1}\cdots e_t\dot e_1\dots\dot e_l},\end{aligned} \tag{11}$$

where $\eta_{\beta_1\dots\beta_t}{}^{\alpha_1\dots\alpha_s}$ are HS symmetry parameters symmetric in the upper and lower indices and the generalized stress tensor $T_{\alpha_1\dots\alpha_n}$ is also symmetric. We use notation with four-component Greek indices being equivalent to a pair of undotted and dotted two-component indices, e.g., $\alpha = a, \dot a$.

Form (11) is closed,

$$d\Omega^3(\eta) = 0, \tag{12}$$

if the generalized stress tensor $T_{\alpha_1\dots\alpha_n}(x)$ satisfies the conservation condition

$$\frac{\partial}{\partial x^{b\dot b}}T^{b\dot b}{}_{\alpha_1\dots\alpha_{n-2}}(x) = 0. \tag{13}$$

Indeed, because $T_{\alpha_1\dots\alpha_n}(x)$ is symmetric, (12) is a simple consequence of the identity

$$dx_{a\dot a}\wedge dx^a{}_{\dot c}\wedge dx_c{}^{\dot a}\wedge dx^{b\dot b}\frac{\partial}{\partial x^{b\dot b}} = \frac{1}{4}dx_{a\dot a}\wedge dx^a{}_{\dot b}\wedge dx_b{}^{\dot a}\wedge dx^{b\dot b}\frac{\partial}{\partial x^{c\dot c}}.$$

For the case with an equal number of dotted and undotted indices among the indices α in (13), we obtain the usual conservation condition for traceless symmetric tensors, which is well known to be related to conformal

HS symmetries [1]. But it follows from Eq. (13) that all irreducible tensors of the four-dimensional Lorentz algebra in the general case can arise as generalized conserved stress tensors except those containing only indices of one type: dotted (i.e., self-dual) or undotted (i.e., anti-self-dual) components. We note that in the tensor language, the Lorentz algebra of generalized stress tensors of integer spins is described by all possible traceless Lorentz tensors that have symmetry properties of Young tableaux with at most two rows. The components of $T_{\alpha_1\dots\alpha_n}(x)$ that do not contribute to the conserved charge are described by all possible two-row rectangular Young tableaux.

The key observation is that the stress tensor

$$T^{kl}_{\alpha_1\dots\alpha_n}(x) = \frac{\partial}{\partial y^{\alpha_1}}\cdots\frac{\partial}{\partial y^{\alpha_n}}\big(C^k(y\,|\,x)C^l(iy\,|\,x)\big)\Big|_{y=0}, \tag{14}$$

where $y^\alpha = (w^a, \overline{w}^{\dot a})$, satisfies conservation condition (13) if the field $C^k(y\,|\,x^{a\dot a})$ satisfies four-dimensional unfolded equation (1). Indeed, it follows from (1) that

$$\frac{\partial}{\partial x^{b\dot b}}\frac{\partial}{\partial w_b}\frac{\partial}{\partial \overline{w}_{\dot b}}\big(C^k(y\,|\,x)C^l(iy\,|\,x)\big) =$$

$$= -\frac{\partial}{\partial w_b}\frac{\partial}{\partial \overline{w}_{\dot b}}\left(C^k(y\,|\,x)\frac{\partial}{\partial w^b}\frac{\partial}{\partial \overline{w}^{\dot b}}C^l(iy\,|\,x) - \left(\frac{\partial}{\partial w^b}\frac{\partial}{\partial \overline{w}^{\dot b}}C^k(y\,|\,x)\right)C^l(iy\,|\,x)\right) = 0.$$

We note that the conserved currents constructed from HS fields according to (14) contain higher derivatives. This completely agrees with the analysis in [11] and also with the general property of HS theories that their interactions contain higher derivatives [12], [13].

4. Examples

In this section, we consider some examples of conserved currents resulting from the general construction. In terms of two-component fields, dynamical equations (6) and (7) on the (anti-)self-dual components $c_{a_1a_2\dots a_{2s}}(x)$ and $\bar c_{\dot a_1\dot a_2\dots\dot a_{2s}}(x)$ are rewritten as

$$\partial^{a_1\dot a_1}c_{a_1a_2\dots a_{2s}} = 0, \qquad \partial^{a_1\dot a_1}\bar c_{\dot a_1\dot a_2\dots\dot a_{2s}} = 0. \tag{15}$$

These equations imply that space–time derivatives of the field strengths are separately symmetric in the dotted and the undotted indices.

Straightforwardly substituting (8) and (9) in (14) gives the generalized stress tensor that contains p derivatives acting on spin-s self-dual and spin-s' anti-self-dual fields,

$$T^{s,s',p\,|\,k,l}_{a(2s+p),\dot a(2s'+p)} = \sum_{j=0}^{p}(-1)^j\frac{(2s)!\,(2s+p)!\,(2s')!\,(2s'+p)!}{(2s+j)!\,(p-j)!\,(2s'+p-j)!\,j!}\partial_{a\dot a(j)}c^k_{a(2s)}\partial_{a\dot a(p-j)}\bar c^l_{\dot a(2s')}, \tag{16}$$

where the notation $\partial_{a\dot a(k)} \equiv \partial^k/(\partial x^{a\dot a}\cdots\partial x^{a\dot a})$ is used and indices denoted by the same letter are assumed to be symmetrized (with the convention that the symmetrization is a projection operator, i.e., symmetrizing twice leaves a symmetrized tensor unchanged). Analogously, we can construct self-dual–self-dual and anti-self-dual–anti-self-dual generalized stress tensors $T^{s,s',p\,|\,k,l}_{a(2s+2s'+p),\dot a(p)}$ and $T^{s,s',p\,|\,k,l}_{a(p),\dot a(2s+2s'+p)}$.

The generalized irreducible angular momentum tensors obtained from (11) have the form

$$M^{s,s',p\,|\,m,m',n\,|\,k,l}_{a(2s+p-m+m'-n),\dot a(2s'm'-n)} =$$

$$= T^{s,s',p\,|\,k,l}_{a(2s+p-m-n)b(m)c(n),\dot a(2s'+p-m'-n)\dot b(m')\dot c(n)}x^{b(m)}{}_{\dot a(m)}\,x^{\dot b(m')}{}_{a(m')}\,x^{c\dot c(n)}, \tag{17}$$

where $x^{b(m)}{}_{\dot{a}(m)} \equiv \overbrace{x^b{}_{\dot{a}} \cdots x^b{}_{\dot{a}}}^{m}$.

In particular, for fields of equal spins, we obtain the generalized stress tensors

$$T^{s,s,p\,|\,k,l}_{a(2s+p),\dot{a}(2s+p)} = \sum_{j=0}^{p} \frac{(-1)^j \big((2s)!\big)^2 \big((2s+p)!\big)^2}{(2s+j)!\,(p-j)!\,(2s+p-j)!\,j!} \partial_{a\dot{a}(j)} c^k_{a(2s)}(x) \partial_{a\dot{a}(p-j)} \bar{c}^l_{\dot{a}(2s)} \tag{18}$$

and $T^{s,s,p\,|\,k,l}_{a(4s+p),\dot{a}(p)}$ and $T^{s,s,p\,|\,k,l}_{a(p),\dot{a}(4s+p)}$ corresponding to symmetric traceless tensors $T^{s,s,p\,|\,k,l}_{\mu(2s+p)}$ of rank $2s+p$ in the tensor notation of the Lorentz algebra.

We consider some lower-spin examples. A spin-0 massless scalar field $c^k(x)$ satisfies the Klein–Gordon equation

$$\partial^\mu \partial_\mu c^k(x) = 0, \tag{19}$$

which is equivalent to (7). The HS totally symmetric conserved currents constructed from higher derivatives of the scalar field [1], [2] have the form

$$T^{0,0,p\,|\,k,l}_{a(p),\dot{a}(p)} = \sum_{j=0}^{p} (-1)^j \left\{ \frac{p!}{(p-j)!\,j!} \right\}^2 \partial_{a\dot{a}(j)} c^k(x) \partial_{a\dot{a}(p-j)} c^l(x). \tag{20}$$

For the particular cases of $p=1$ and $p=2$, we obtain the electric current

$$J^{k,l}_{a,\dot{a}} = \partial_{a\dot{a}} c^l c^k - \partial_{a\dot{a}} c^k c^l \tag{21}$$

and the improved energy–momentum tensor

$$T^{0,0,2\,|\,k,l}_{aa,\dot{a}\dot{a}} = c^k \partial_{a\dot{a}} \partial_{a\dot{a}} c^l - 4 \partial_{a\dot{a}} c^k \partial_{a\dot{a}} c^l + \partial_{a\dot{a}} \partial_{a\dot{a}} c^k c^l, \tag{22}$$

which is symmetric in the color indices. In the tensor notation, these currents have the forms

$$\begin{aligned} J^{k,l}_\mu &= c^k \partial_\mu c^l - c^l \partial_\mu c^k, \\ T_{\mu\nu} &= \partial_\mu c \partial_\nu c - \frac{1}{2} c \partial_{\mu\nu} c - \frac{1}{4} \eta_{\mu\nu} \partial_\lambda c \partial^\lambda c. \end{aligned} \tag{23}$$

A spin-1/2 field $\psi(x)$ satisfying the massless Dirac equation $\gamma_\mu \partial^\mu \psi(x) = 0$ is described by $c_a(x)$ and $\bar{c}_{\dot{a}}(x)$ that satisfy (15). With the color indices omitted, the electric current $T^{1/2,1/2,0}$ and the energy–momentum tensor $T^{1/2,1/2,1}$ are

$$\begin{aligned} T^{1/2,1/2,0}_{a,\dot{a}} &= c_a \bar{c}_{\dot{a}}, \\ T^{1/2,1/2,1}_{a(2),\dot{a}(2)} &= 2(c_a \partial_{a\dot{a}} \bar{c}_{\dot{a}} - \partial_{a\dot{a}} c_a \bar{c}_{\dot{a}}). \end{aligned} \tag{24}$$

A supercurrent mixing spin-0 and spin-1/2 fields is given by

$$T^{1/2,0,1}_{aa,\dot{a}} = 2 c_a \partial_{a\dot{a}} c - \partial_{a\dot{a}} c_a c \tag{25}$$

and its complex conjugate.

A massless spin-1 field can be described by a gauge-invariant field strength satisfying the Maxwell equations

$$\partial^\mu F_{\mu\nu} = 0, \qquad \partial_{[\rho} F_{\mu\nu]} = 0. \tag{26}$$

In terms of two-component spinors, $F_{\mu\nu}$ is described by c_{aa} and $\bar{c}_{\dot{a}\dot{a}}$, while the Maxwell equations have form (15).

The energy–momentum tensor

$$T_{\mu\nu} = -F_{\mu}{}^{\sigma} F_{\sigma\nu} + \frac{1}{4}\eta_{\mu\nu} F^2 \tag{27}$$

corresponds to $T^{1,1,0}$ and has the form

$$T^{1,1,0}_{aa,\dot{a}\dot{a}} = 4 c_{aa} \bar{c}_{\dot{a}\dot{a}}. \tag{28}$$

As in the scalar field case, there exist totally symmetric HS conserved currents constructed from higher derivatives of the spin-1 field strength [1] corresponding to the sequence $T^{1,1,p}$.

Using the energy–momentum tensor $T_{\mu\nu}$, we can construct an angular momentum tensor

$$M_{\mu,\nu\lambda} = T_{\mu\nu} x_\lambda - T_{\mu\lambda} x_\nu. \tag{29}$$

In the case of spin 0, it corresponds to the spinor tensors $M^{0,0,2\,|\,1,0,0}_{a,\dot{a}(3)}$, $M^{0,0,2\,|\,0,1,0}_{a(3),\dot{a}}$, and $M^{0,0,2\,|\,0,0,1}_{a,\dot{a}}$.

A massless spin-2 field describes the linearized gravity. The linearized gauge-invariant combinations of derivatives of a linearized metric tensor are given by the linearized Riemann tensor, whose trace part is zero by virtue of the Einstein equations. The nonzero traceless part is called the Weyl tensor $H_{\mu\nu\lambda\rho}$. As a consequence of the Einstein equations, the Weyl tensor satisfies differential restrictions, the Bianchi identities. In terms of two-component spinors, the Weyl tensor is described by the self-dual component $c_{abcd}(x)$ and anti-self-dual component $\bar{c}_{\dot{a}\dot{b}\dot{c}\dot{d}}(x)$. The consequences of the Einstein equations have form (15).

It is well known that there is a conserved current bilinear in the Weyl tensor, called the Bel–Robinson tensor [14], [15]. In the tensor notation, it has the form

$$T_{\mu\nu\lambda\rho} = H_{\mu\sigma\nu\eta} H_{\lambda}{}^{\sigma}{}_{\rho}{}^{\eta} + {}^{*}H_{\mu\sigma\nu\eta}{}^{*}H_{\lambda}{}^{\sigma}{}_{\rho}{}^{\eta}, \tag{30}$$

where the Hodge star $*$ denotes dualization by the Levi-Civita tensor $\varepsilon_{\mu\nu\lambda\rho}$. In terms of two-component spinors, the Bel–Robinson tensor corresponds to $T^{2,2,0}$ and has the simple form

$$T^{2,2,0}_{a(4),\dot{a}(4)} = (4!)^2 c_{a(4)} \bar{c}_{\dot{a}(4)}. \tag{31}$$

Other generalized stress tensors constructed from the Weyl tensor are given by formula (18) with $s = 2$.

5. Conclusion

Although the obtained list of conserved currents is infinite, it does not contain some of the expected symmetry generators and is thus incomplete. This is not surprising, because even ordinary conserved currents like the energy–momentum tensor and the electric charge for HS fields are not in the class of gauge-invariant currents. Indeed, it is well known that the energy–momentum conservation in the theory of gravity is described in terms of a gauge-noninvariant pseudotensor [16], which nevertheless yields gauge-invariant total energy and momentum conservation laws in the free-field approximation. The same happens for all HS fields [11], [17], [18]. The reason is that for a spin-s gauge field, the minimum degree in the derivatives of the gauge-invariant tensors $C(w, \overline{w}\,|\,x)$ is equal to s; therefore, conserved tensors must themselves have sufficiently high spins.

The system of all HS fields is $sp(8)$ invariant [5], [7]. The symmetry parameters and corresponding conserved currents in the two-component spinor notation have the following forms: generalized translations correspond to the symmetry parameters $\eta^{a\dot{a}}$, η^{aa}, and $\eta^{\dot{a}\dot{a}}$ and the conserved currents $T^{k,l}_{a,\dot{a}\dot{a}\dot{a}}$, $T^{k,l}_{aaa,\dot{a}}$, and $T^{k,l}_{aa,\dot{a}\dot{a}}$; generalized Lorentz boosts and dilatations correspond to the symmetry parameters $\eta_b{}^a$, $\eta_{\dot{b}}{}^a$, $\eta_b{}^{\dot{a}}$, and $\eta_{\dot{b}}{}^{\dot{a}}$ and the conserved currents $T^{k,l}_{aa,\dot{a}\dot{b}}x^{b\dot{b}}$, $T^{k,l}_{aab,\dot{a}}x^{b\dot{b}}$, $T^{k,l}_{a,\dot{a}\dot{a}\dot{b}}x^{b\dot{b}}$, and $T^{k,l}_{ab,\dot{a}\dot{a}}x^{b\dot{b}}$; generalized special conformal transformations correspond to the symmetry parameters η_{bb}, $\eta_{b\dot{b}}$, and $\eta_{\dot{b}\dot{b}}$ and the conserved currents $T^{k,l}_{a,\dot{a}\dot{b}\dot{b}}x^{b\dot{b}}x^{b\dot{b}}$, $T^{k,l}_{ab,\dot{a}\dot{b}}x^{b\dot{b}}x^{b\dot{b}}$, and $T^{k,l}_{abb,\dot{a}}x^{b\dot{b}}x^{b\dot{b}}$.

The list of generators of this type (which includes the generators of the usual conformal algebra $su(2,2) \subset sp(8)$) that can be constructed in terms of invariant HS tensors is quite short,

$$\begin{aligned}
&T^{k,l}_{aa,\dot{a}\dot{a}} = c^k\partial_{a\dot{a}}\partial_{a\dot{a}}c^l - 4\partial_{a\dot{a}}c^k\partial_{a\dot{a}}c^l + \partial_{a\dot{a}}\partial_{a\dot{a}}c^kc^l,\\
&4c_{aa}\bar{c}_{\dot{a}\dot{a}} \qquad 2(c_a\partial_{a\dot{a}}\bar{c}_{\dot{a}} - \partial_{a\dot{a}}c_a\bar{c}_{\dot{a}}),\\
&T^{k,l}_{aaa,\dot{a}} = 2(c_a\partial_{a\dot{a}}\bar{c}_{\dot{a}} - \partial_{a\dot{a}}c_a\bar{c}_{\dot{a}}), \qquad 6c^k_{aaa}\bar{c}^l_{\dot{a}},\\
&T^{k,l}_{a,\dot{a}\dot{a}\dot{a}} = 6\bar{c}^l_{\dot{a}\dot{a}\dot{a}}c^k_a, \qquad 6\bar{c}^l_{\dot{a}\dot{a}}\partial_{a\dot{a}}c^k - 2\partial_{a\dot{a}}\bar{c}^l_{\dot{a}\dot{a}}c^k,
\end{aligned} \tag{32}$$

and obviously incomplete because the $sp(8)$ symmetry mixes fields of all spins while generators (32) do not contain HS fields at all. It is still unknown whether the list of HS conserved currents presented here can be supplemented with the HS pseudotensors that may not be gauge invariant but allow constructing invariant conserved charges.

In conclusion, we note that the formula for HS conformal currents presented here is analogous to the formula in [4], [19] for HS conserved currents in the ten-dimensional space–time $\mathcal{M}_4$ suggested for describing four-dimensional massless HS fields in [5], [7], [8]. In fact, expression (14) for $T^{kl}_{\alpha_1\ldots\alpha_n}(y\,|\,x^{a\dot{a}})$ is the reduction to the Minkowski space of the generalized energy–momentum tensor [4] $T^{kl}_{\alpha_1\ldots\alpha_n}(y\,|\,X^{\alpha\beta})$, where $X^{\alpha\beta}$ are symmetric matrix coordinates in $\mathcal{M}_4$. Conservation condition (13) is also the reduction of the conservation condition in $\mathcal{M}_4$. But the explicit relation between the two constructions, which requires an appropriate integration over a noncontractible cycle in $\mathcal{M}_4$, remains undeveloped.

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