Theoretical and Mathematical Physics, **151**(2): 700–717 (2007)

UNIVERSALLY COUPLED MASSIVE GRAVITY

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We derive Einstein's equations from a linear theory in flat space–time using free-field gauge invariance and universal coupling. The gravitational potential can be either covariant or contravariant and of almost any density weight. We adapt these results to yield universally coupled massive variants of Einstein's equations, yielding two one-parameter families of distinct theories with spin 2 and spin 0. The Freund–Maheshwari–Schonberg theory is therefore not the unique universally coupled massive generalization of Einstein's theory, although it is privileged in some respects. The theories we derive are a subset of those found by Ogievetsky and Polubarinov by other means. The question of positive energy, which continues to be discussed, might be addressed numerically in spherical symmetry. We briefly comment on the issue of causality with two observable metrics and the need for gauge freedom and address some criticisms by Padmanabhan of field derivations of Einstein-like equations along the way.

Keywords: massive gravity, bimetric, ghost, positive mass, causality

1. Introduction

Constructing a relativistic gravitational theory based on principles such as an analogy to Maxwellian electromagnetism, the universal coupling of the gravitational field to a combined gravity–matter energy–momentum complex, and also the requirement that the gravitational field equations alone (without the matter equations, again in analogy with electromagnetic charge conservation) entail energy–momentum conservation was a major part of Einstein's search for an adequate theory of gravity in 1913–1915 (see [1], [2]). Einstein subsequently downplayed these investigations [2], and the above ideas later came to be associated with the non-Einsteinian field theory approach to gravitation. Such a derivation of Einstein's or similar gravitational field equations can use a priori preferred coordinates and a canonical energy–momentum tensor or a flat background metric and variational metric energy–momentum tensor, the difference being basically formal.

Several authors [3]–[14] have discussed the utility of a flat background metric $\eta_{\mu\nu}$ in general relativity or the possibility of deriving that theory, approximately or exactly, from a flat space–time theory.[1] A background metric allows introducing a gravitational energy–momentum tensor [17], not merely a pseudotensor. As a result, gravitational energy and momentum are independent of the coordinates but dependent on the gauge [18]. If we want to regard the background metric seriously as a property of space–time and not just treat it as a useful fiction, then the relation between the effective curved metric's null cone and that of the

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[1]We note that general relativity was also derived from self-interaction on curved backgrounds [10], [15]. The possibility (or otherwise) of massless multigraviton theories analogous to Yang–Mills theories was also investigated [16]. Moreover, it is interesting to consider (A)dS backgrounds, Lorentz-violating theories, and Chern–Simons topologically massive theories. But we consider only the most traditional form of the massive gravity problem here. We work in four space–time dimensions, but the derivations should generalize straightforwardly to any (integral) dimension not less than three.

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 151, No. 2, pp. 311–336, May, 2007. Original article submitted April 11, 2006; revised June 25, 2006.

flat background must be considered. That issue was addressed with some success for the massless case of Einstein's equations [13].

Preparatory to considering massive theories of gravity, we generalize our derivation of Einstein's equations using gauge invariance for the free field and universal coupling [12] to permit almost any density weight and either covariant or contravariant valency for the symmetric rank-two gravitational potential. This generality to some degree parallels that in Kraichnan's classic work [4], but our derivation has several improvements and can be easily adapted to massive theories. The choices of index position and density weight make no difference (after field redefinitions) in the massless theories, but they yield distinct massive theories here. Along the way, we address Padmanabhan's recent objections to field derivations of Einstein's equations [19].

Several authors recently discussed the subject of massive gravity with spin-2 and spin-0 components [20], [21]. While it is permissible simply to postulate the nonlinear features (if any) of a mass term, it seems preferable to find well-motivated theoretical principles to constrain such choices. Some time ago, Ogievetsky and Polubarinov (OP) derived a two-parameter family of massive variants of Einstein's equations [7], which contains both our one-parameter families (quantization was considered briefly in [22]). Their derivation relied on gauge invariance (at least for the massless part of the Lagrangian density) but not universal coupling. Instead, they imposed a spin-limitation principle to exclude some degrees of freedom, many with the wrong sign and thus negative energy, from the full nonlinear interacting theory. The Freund–Maheshwari–Schonberg (FMS) massive theory was originally derived using universal coupling with a canonical, not metric, energy–momentum tensor [23]. While the FMS theory is recovered among our results, it is not the unique universally coupled massive variant of Einstein's equations, contrary to previous claims [23], [24]. Our derivation using the metric energy–momentum tensor seems shorter and cleaner than the FMS derivation using the canonical energy–momentum tensor. In a future work, we will consider universal coupling using a tetrad, not metric, formalism, thus obtaining additional universally coupled theories, and will use the metric formalism to show that all the OP theories are universally coupled. All known universally coupled theories correspond to the OP family of theories; hence, their derivation and the metric universal coupling derivation currently lead to coinciding results. The FMS theory, subsequently adopted by Logunov and collaborators [11], also faces questions regarding positive energy and causality, on which we briefly comment in what follows.

2. Effectively geometric theories from universal coupling and gauge invariance

We previously derived Einstein's theory and other effectively geometric theories using gauge-invariant free-field theories and universal coupling [12], which was significantly based on the work of Kraichnan [4] and Deser [9]. Whereas the gravitational potential was previously taken to be a symmetric rank-two covariant tensor field, we now generalize this derivation to the case of a density of almost any weight and either covariant or contravariant valency. Given the generality of Kraichnan's derivation, it is not surprising that these generalizations again yield only Einstein's and other effectively geometric theories. Nothing especially novel is obtained for massless theories, but the derivations below are adapted to massive gravity for the first time.

2.1. Free-field action for a covariant tensor density potential. For the massless theories, an initial infinitesimal invariance (up to a boundary term) of the free gravitational action is assumed. For the subsequent derivation of massive theories, the gauge freedom is broken by a natural mass term algebraic in the fields, but the derivative terms retain the gauge invariance.

Let S_{f} be the action for a free symmetric tensor density $\tilde{\gamma}_{\mu\nu}$ (of density weight $-l$, where $l \neq 1/2$) in a space–time with a flat metric tensor $\eta_{\mu\nu}$ in arbitrary coordinates. The torsion-free metric-compatible covariant derivative is denoted by ∂_μ; hence, $\partial_\alpha \eta_{\mu\nu} = 0$. It is convenient to use not the flat metric itself but its related densitized metric $\tilde{\eta}_{\mu\nu} = \eta_{\mu\nu}(\sqrt{-\eta})^{-l}$ of weight $-l$. We note that in the forbidden case $l = 1/2$, $\tilde{\eta}_{\mu\nu}$ is noninvertible: $\eta_{\mu\nu}(\sqrt{-\eta})^{-1/2}$ determines only the null cone, not a full metric tensor.[2] The field $\tilde{\gamma}_{\mu\nu}$ turns out to be the gravitational potential. Although it has become customary in work on loop quantum gravity to denote the density weight of fields by placing the corresponding number of tildes above or below a symbol to express its positive or negative density weight, that custom is impossible here. The density weight l ($l \neq 1/2$) can be large, nonintegral, or even irrational, and we therefore merely write a tilde over most densities. All indices are respectively raised and lowered with $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ with two exceptions. For the densitized flat metric $\tilde{\eta}_{\mu\nu}$, the oppositely densitized inverse flat metric is $\tilde{\eta}^{\mu\nu}$. Similarly, the inverse of the densitized curved metric $\tilde{g}_{\mu\nu}$ defined below is $\tilde{g}^{\mu\nu}$. Because we use tensor densities extensively, we recall the forms of their covariant and Lie derivatives. A $(1,1)$-density $\tilde{\phi}^\alpha_\beta$ of weight w is a representative example. The Lie derivative is given by [25]

$$\pounds_\xi \tilde{\phi}^\alpha_\beta = \xi^\mu \tilde{\phi}^\alpha_{\beta},_\mu - \tilde{\phi}^\mu_\beta \xi^\alpha,_\mu + \tilde{\phi}^\alpha_\mu \xi^\mu,_\beta + w \tilde{\phi}^\alpha_\beta \xi^\mu,_\mu\,, \tag{1}$$

and the η-covariant derivative is given by

$$\partial_\mu \tilde{\phi}^\alpha_\beta = \tilde{\phi}^\alpha_{\beta},_\mu + \tilde{\phi}^\sigma_\beta \Gamma^\alpha_{\sigma\mu} - \tilde{\phi}^\alpha_\sigma \Gamma^\sigma_{\beta\mu} - w \tilde{\phi}^\alpha_\beta \Gamma^\sigma_{\sigma\mu}, \tag{2}$$

where $\Gamma^\sigma_{\beta\mu}$ are the Christoffel symbols for $\eta_{\mu\nu}$. After the curved metric $g_{\mu\nu}$ is defined below, the analogous g-covariant derivative ∇ with the Christoffel symbols $\{^{\alpha}_{\sigma\mu}\}$ follows.

The desire to avoid ghosts motivates gauge invariance for linear theories [26]. We require that the free field action S_f change only by a boundary term under the infinitesimal gauge transformation

$$\tilde{\gamma}_{\mu\nu} \to \tilde{\gamma}_{\mu\nu} + \delta\tilde{\gamma}_{\mu\nu}, \qquad \delta\tilde{\gamma}_{\mu\nu} = \partial_\mu \tilde{\xi}_\nu + \partial_\nu \tilde{\xi}_\mu + c\eta_{\mu\nu}\partial^\alpha \tilde{\xi}_\alpha, \tag{3}$$

where $c \neq -1/2$ and $\tilde{\xi}_\nu$ is an arbitrary covector density field of weight $-l$.[3] We can expect the appearance of a connection between l and c. For any S_f invariant in this sense under (3), a certain linear combination of the free field equations is identically divergenceless, as we now show. The action changes by

$$\delta S_\mathrm{f} = \int d^4x \left[\frac{\delta S_\mathrm{f}}{\delta\tilde{\gamma}_{\mu\nu}} (\partial_\nu \tilde{\xi}_\mu + \partial_\mu \tilde{\xi}_\nu + c\eta_{\mu\nu}\partial^\alpha \tilde{\xi}_\alpha) + e^\mu,_\mu \right] = \int d^4x\, f^\mu,_\mu\,. \tag{4}$$

The explicit forms of the boundary terms given by $e^\mu,_\mu$ and $f^\mu,_\mu$ are not needed for our purposes. Integrating by parts, letting $\tilde{\xi}_\mu$ have compact support to annihilate the boundary terms, and using the arbitrariness of $\tilde{\xi}_\mu$, we obtain the identity

$$\partial_\mu \left(\frac{\delta S_\mathrm{f}}{\delta\tilde{\gamma}_{\mu\nu}} + \frac{c}{2} \eta^{\mu\nu} \eta_{\sigma\alpha} \frac{\delta S_\mathrm{f}}{\delta\tilde{\gamma}_{\sigma\alpha}} \right) = 0. \tag{5}$$

This is the generalized Bianchi identity for the free theory, which in the most common case is a linearized version of the original geometric Bianchi identity. A natural choice for S_f is the linearized GR Lagrangian density, but no such detailed assumptions on the form of S_f are used here. Because noninteracting sourceless fields are unobservable [27], the theory is interesting only after an interaction is introduced.

As Kretschmann pointed out long ago in response to Einstein, any theory can be given a generally covariant formulation in the sense that the equations hold in any coordinate system, Cartesian or otherwise [28]–[31]. The resulting formulation might be called weakly or trivially general covariant. Often

[2] This exceptional case was of interest in deriving slightly bimetric theories [12], where $\sqrt{-\eta}$ can appear in the field equations of the interacting theories.

[3] The case $c = -1/2$ merely gives a scalar theory in the somewhat comparable work of OP [7].

achieving weak general covariance involves using absolute objects [29], [30], such as a flat metric tensor, as it does here. The distinctively novel aspect of Einstein's theory of gravity is supposedly its lack of absolute objects or "prior geometry" [29], [32].[4] This property might be called strong or nontrivial general covariance. This distinction between two senses of general covariance was discussed previously [35]. The derivation presented here and in some other works starts with a weakly generally covariant theory and concludes with a strongly generally covariant one without absolute objects.[5] Therefore, the quite misleading claim [19] that flat space–time derivations of Einstein's equations only result in general covariance because they feed it in at the beginning should not be accepted. The two senses of general covariance are very different; Padmanabhan's criticism commits the fallacy of equivocation.

2.2. Metric energy–momentum tensor density. If the energy–momentum tensor is to be the source of the gravitational potential $\tilde{\gamma}_{\mu\nu}$, then consistency requires that the *total* energy–momentum tensor be used including the gravitational energy–momentum, not merely the nongravitational ("matter") energy–momentum, because only the total energy–momentum tensor is divergenceless in the sense of ∂_ν [9] or, equivalently, in the sense of a Cartesian coordinate divergence. To obtain a global conservation law, a vanishing *coordinate* divergence for the four-current is needed.

An expression for the total energy–momentum tensor density can be derived from S using the metric recipe [4], [17], [29], [36] as follows. The action depends on the flat metric density $\tilde{\eta}_{\mu\nu}$, gravitational potential $\tilde{\gamma}_{\mu\nu}$, and matter fields u. Here, u represents an arbitrary collection of dynamical tensor fields of arbitrary rank, index position, and density weight. Using the OP spinor formalism that uses the "square root of the metric" rather than a tetrad or other additional structure [37], we can likely also include spinor fields.

Under an arbitrary infinitesimal coordinate transformation described by a vector field ξ^μ, the action changes by the amount

$$\delta S = \int d^4x \left(\frac{\delta S}{\delta \tilde{\gamma}_{\mu\nu}} \pounds_\xi \tilde{\gamma}_{\mu\nu} + \frac{\delta S}{\delta u} \pounds_\xi u + \frac{\delta S}{\delta \tilde{\eta}_{\mu\nu}} \pounds_\xi \tilde{\eta}_{\mu\nu} + g^\mu{}_{,\mu} \right), \tag{6}$$

with boundary terms from $g^\mu{}_{,\mu}$ vanishing because ξ^μ is compactly supported. But S is a scalar, and hence $\delta S = 0$. Letting the matter and gravitational field equations hold, integrating by parts, discarding vanishing boundary terms, and using the arbitrariness of the vector field ξ^μ gives

$$\partial_\nu \left(\frac{\delta S}{\delta \tilde{\eta}_{\mu\nu}} - \frac{l}{2} \tilde{\eta}_{\alpha\beta} \tilde{\eta}^{\mu\nu} \frac{\delta S}{\delta \tilde{\eta}_{\alpha\beta}} \right) = 0. \tag{7}$$

This quantity is an energy–momentum tensor density for matter and gravitational fields. The flatness of $\eta_{\mu\nu}$ is relaxed in taking the functional derivative $\delta S / \delta \tilde{\eta}_{\mu\nu}$ and is restored later. This move is sometimes criticized [19], but it is unobjectionable even in flat space–time because it is merely a formal trick useful for defining the energy–momentum tensor, not an illicit use of curved space–time. Using the connection between the Rosenfeld energy–momentum tensor and the symmetrized Belinfante canonical energy–momentum tensor [36], [38], we could regard the metric recipe as a mathematical shortcut to the conceptually unimpeachable but mathematically inconvenient Belinfante tensor modified with terms proportional to the equations of motion. The metric energy–momentum tensor is not unique at this stage, because terms proportional

[4] Actually, the matter is more complicated: R. Geroch and Giulini recently noted in effect that g, the determinant of the metric tensor, is an absolute object because any point has a neighborhood with a coordinate system such that the component of g has the value -1 [33], [34].

[5] The fact that g counts as absolute in general relativity in the Anderson–Friedman absolute-object program suggests that general relativity is not strongly covariant after all. But the point remains that two very different notions of general covariance are in play. In addition to the field equations, the topology, boundary conditions, and causality should be examined in seeking absolute objects [13]. We emphasize that the massless cases are considered here; obviously, the massive theories contain the flat metric and are therefore not strongly generally covariant.

to the equations of motion and their derivatives, as well as superpotentials, can be added. Another option would be to use Lagrange multipliers and let the background metric be flat only on-shell [39]. As is shown below, using the superpotential freedom judiciously is important in deriving the field equations.

2.3. Full universally coupled action. We find an action S satisfying the plausible physical postulate that invertible linear combinations of the Euler–Lagrange equations are just invertible linear combinations of the free field equations for S_{f} augmented by the total energy–momentum tensor. A simple way to impose this requirement reduces to the condition

$$\frac{\delta S}{\delta \tilde{\gamma}_{\mu\nu}} = \frac{\delta S_{\mathrm{f}}}{\delta \tilde{\gamma}_{\mu\nu}} - \lambda \frac{\delta S}{\delta \tilde{\eta}_{\mu\nu}}, \tag{8}$$

where $\lambda = -\sqrt{32\pi G}$. (Because the linear combinations of the free Euler–Lagrange equations and of the full Lagrange equations with interaction are invertible in this case, the linear combination can now be applied to the energy–momentum tensor.) It seems prudent to set $c = -l$ to make the generalized Bianchi identity and the energy–momentum tensor density take similar forms.

The basic variables here are the gravitational potential $\tilde{\gamma}_{\mu\nu}$ and the flat metric density $\tilde{\eta}_{\mu\nu}$. But we can freely change the variables in S from $\tilde{\gamma}_{\mu\nu}$ and $\tilde{\eta}_{\mu\nu}$ to the bimetric variables $\tilde{g}_{\mu\nu}$ and $\tilde{\eta}_{\mu\nu}$ [4], where

$$\tilde{g}_{\mu\nu} = \tilde{\eta}_{\mu\nu} - \lambda \tilde{\gamma}_{\mu\nu}. \tag{9}$$

(We can then define the metric $g_{\mu\nu}$ from $\tilde{g}_{\mu\nu}$ using matrix algebra and then define the g-covariant derivative ∇ as usual, but we have little need to use ∇ explicitly.)

Equating coefficients of the variations gives

$$\left.\frac{\delta S}{\delta \tilde{\eta}_{\mu\nu}}\right|\tilde{\gamma} = \left.\frac{\delta S}{\delta \tilde{\eta}_{\mu\nu}}\right|\tilde{g} + \frac{\delta S}{\delta \tilde{g}_{\mu\nu}} \tag{10}$$

for $\delta\tilde{\eta}_{\mu\nu}$ and

$$\frac{\delta S}{\delta \tilde{\gamma}_{\mu\nu}} = -\lambda \frac{\delta S}{\delta \tilde{g}_{\mu\nu}} \tag{11}$$

for $\delta\tilde{\gamma}_{\mu\nu}$. Combining these two results gives

$$\lambda \left.\frac{\delta S}{\delta \tilde{\eta}_{\mu\nu}}\right|\tilde{\gamma} = \lambda \left.\frac{\delta S}{\delta \tilde{\eta}_{\mu\nu}}\right|\tilde{g} - \frac{\delta S}{\delta \tilde{\gamma}_{\mu\nu}}. \tag{12}$$

Equation (12) splits the energy–momentum tensor into two parts: one that vanishes when gravity is on-shell and one that does not. Using this result in universal-coupling postulate (8) gives

$$\lambda \left.\frac{\delta S}{\delta \tilde{\eta}_{\mu\nu}}\right|\tilde{g} = \frac{\delta S_{\mathrm{f}}}{\delta \tilde{\gamma}_{\mu\nu}}. \tag{13}$$

Taking the divergence, recalling free-theory Bianchi identity (5), and using $c = -l$, we derive

$$\partial_\mu \left(\left.\frac{\delta S}{\delta \tilde{\eta}_{\mu\nu}}\right|\tilde{g} - \frac{l}{2} \tilde{\eta}^{\mu\nu} \tilde{\eta}_{\alpha\beta} \left.\frac{\delta S}{\delta \tilde{\eta}_{\alpha\beta}}\right|\tilde{g} \right) = 0. \tag{14}$$

The quantity in parentheses is exactly $(\sqrt{-\eta}\,)^l\, \delta S/\delta\eta_{\mu\nu}|\tilde{g}$. Hence, the part of the energy–momentum tensor not proportional to the gravitational field equations has identically vanishing divergence (on either index), i.e., is a (symmetric) "curl" [29]. The splitting of the energy–momentum tensor ensures that in the massless case, the gravitational field equations *alone*, without separately postulating the matter equations, entail conservation of energy–momentum for the resulting effectively geometric field equations.

Because the quantity $\delta S/\delta\eta_{\mu\nu}|\tilde{g}$ is symmetric and has identically vanishing divergence on either index, it necessarily has the form [40]

$$\frac{\delta S}{\delta\eta_{\mu\nu}}\bigg|\tilde{g} = \frac{1}{2}\partial_\rho\partial_\sigma\big(\mathcal{M}^{[\mu\rho][\sigma\nu]} + \mathcal{M}^{[\nu\rho][\sigma\mu]}\big) + B\sqrt{-\eta}\eta^{\mu\nu}, \tag{15}$$

where $\mathcal{M}^{\mu\rho\sigma\nu}$ is a tensor density of unit weight and B is a constant. This result follows from the converse of the Poincaré lemma in Minkowski space–time. We cannot choose $\mathcal{M}^{\mu\rho\sigma\nu}$ arbitrarily but must choose it such that the term $\delta S_{\mathrm{f}}/\delta\tilde{\gamma}_{\mu\nu}$ is accommodated. The freedom to add an arbitrary curl must therefore be used in a quite definite way. As Huggins, a student of Feynman, showed in his dissertation [41] and Padmanabhan recently emphasized [19], coupling a spin-2 field to the energy–momentum tensor does not lead to a unique theory, because of terms of this curl form. Rather, as Huggins said (p. 39 in [41]): "an additional restriction is necessary. For Feynman this restriction was that the equations of motion be obtained from an action principle; Einstein required that the gravitational field have a geometric interpretation. Feynman showed these two restrictions to be equivalent."

Gathering all dependence on $\eta_{\mu\nu}$ (with $\tilde{g}_{\mu\nu}$ independent) into one term yields $S = S_1[\tilde{g}_{\mu\nu}, u] + S_2[\tilde{g}_{\mu\nu}, \eta_{\mu\nu}, u]$. It is easy to verify that if

$$S_2 = \frac{1}{2}\int d^4x\, R_{\mu\nu\rho\sigma}(\eta)\mathcal{M}^{\mu\nu\rho\sigma}(\eta_{\mu\nu}, \tilde{g}_{\mu\nu}, u) + \int d^4x\, \alpha^\mu{}_{,\mu} + 2B\int d^4x\,\sqrt{-\eta}, \tag{16}$$

then $\delta S_2/\delta\eta_{\mu\nu}|\tilde{g}$ has exactly the desired form, while S_2 does not affect the Euler–Lagrange equations, because $\delta S_2/\delta\tilde{g}_{\mu\nu} = 0$ and $\delta S_2/\delta u = 0$ identically [4].[6] The coefficient B of the four-volume term is naturally chosen to cancel any zeroth-order term (such as from a cosmological constant) in the action such that the action vanishes when there is no gravitational field. The four-divergence $\alpha^\mu{}_{,\mu}$ resolves worries [19] about obtaining terms that are not analytic in the coupling constant λ. It is unclear whether the Hilbert action is best in any event, given its badly behaved conservation laws [42].

Hence, the universally coupled action for the covariant tensor density case is

$$S = S_1[\tilde{g}_{\mu\nu}, u] + \frac{1}{2}\int d^4x\, R_{\mu\nu\rho\sigma}(\eta)\mathcal{M}^{\mu\nu\rho\sigma} + 2B\int d^4x\,\sqrt{-\eta} + \int d^4x\,\alpha^\mu{}_{,\mu}\,. \tag{17}$$

The boundary term is at our disposal; if α^μ is a unit-weight vector density, then S is a coordinate scalar. Using the effective curved metric density $\tilde{g}_{\mu\nu}$, we can define an effective curved metric by $\tilde{g}_{\mu\nu} = g_{\mu\nu}(\sqrt{-g}\,)^{-l}$ and an inverse curved metric density $\tilde{g}^{\mu\nu} = g^{\mu\nu}(\sqrt{-g}\,)^{l}$.

For S_1, we choose the Hilbert action for general relativity plus minimally coupled matter and a cosmological constant:

$$S_1 = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,R(g) - \frac{\Lambda}{8\pi G}\int d^4x\,\sqrt{-g} + S_{\mathrm{matter}}[g_{\mu\nu}, u]. \tag{18}$$

It is well known that the Hilbert action is the simplest (scalar) action that can be constructed using only the metric tensor. If the gravitational field vanishes everywhere, then the gravitational action should also vanish. In the massless case, the result is that $B = \Lambda/(16\pi G)$. For the generalization to the massive case considered below, the gauge-breaking part of the mass term introduces another zeroth-order contribution that also needs to be canceled. It is also possible to couple the matter to the Riemann tensor for $g_{\mu\nu}$ or to allow higher powers of the Riemann tensor in the gravitational action, if desired. In the massless case, we could set $\Lambda = 0$ [23].

[6] If it seems that using this term $(1/2)\int d^4x\, R_{\mu\nu\rho\sigma}(\eta)\mathcal{M}^{\mu\nu\rho\sigma}(\eta_{\mu\nu}, \tilde{g}_{\mu\nu}, u)$ is too clever to be invented without knowing Einstein's theory in advance and thus cheating [19], then a symmetric curl term $\partial_\rho\partial_\sigma(\mathcal{M}^{[\mu\rho][\sigma\nu]} + \mathcal{M}^{[\nu\rho][\sigma\mu]})/2 + B\sqrt{-\eta}\eta^{\mu\nu}$ should be simply added to the metric energy–momentum tensor by hand using the usual underdetermination of the energy–momentum tensor.

3. Massive universally coupled gravity for a covariant tensor density potential

Our goal is to generalize the above derivation to obtain one or more massive finite-range variants of Einstein's equations. Such field equations would be related to Einstein's much as Proca's massive electromagnetic field equations are related to Maxwell's. But a spin-2–spin-0 massive theory would have ghosts at the level of the free linear theory. Good linear behavior is generally required as a guide to good nonlinear behavior. But good linear behavior seems neither necessary nor sufficient for good nonlinear behavior. In the present case, it seems quite possible that the nonlinear form of the Hamiltonian constraint cures the bad behavior of the linear theory. We briefly discuss this matter below.

It can be expected that the mass term for a free field is quadratic in the potential and lacks derivatives. The free-field action S_{f} is now assumed to have two parts: a (mostly kinetic) part S_{f0} that is invariant under the previous gauge transformations as in the massless case above and an algebraic mass term S_{fm} that is quadratic and breaks the gauge symmetry. We seek a full universally coupled theory with an action S that has two corresponding parts: $S = S_0 + S_{\mathrm{ms}}$. They are the familiar part S_0 (yielding the Einstein tensor, the matter action, a cosmological constant, and a zeroth-order four-volume term) and the new gauge-breaking part S_{ms}, which also has another zeroth-order four-volume term. As it turns out, the mass term is constructed from both the algebraic part of S_0 (the cosmological constant and four-volume term) and the purely algebraic term S_{ms}.

Requiring S_{f0} to change only by boundary terms under the variation $\tilde{\gamma}_{\mu\nu} \to \tilde{\gamma}_{\mu\nu} + \partial_\mu \tilde{\xi}_\nu + \partial_\nu \tilde{\xi}_\mu + c\eta_{\mu\nu}\partial^\alpha \tilde{\xi}_\alpha$ for $c \neq -1/2$ implies the identity

$$\partial_\mu \left(\frac{\delta S_{\mathrm{f0}}}{\delta \tilde{\gamma}_{\mu\nu}} - \frac{l}{2} \eta^{\mu\nu} \eta_{\sigma\alpha} \frac{\delta S_{\mathrm{f0}}}{\delta \tilde{\gamma}_{\sigma\alpha}} \right) = 0. \tag{19}$$

We again postulate the universal coupling in the form

$$\frac{\delta S}{\delta \tilde{\gamma}_{\mu\nu}} = \frac{\delta S_{\mathrm{f}}}{\delta \tilde{\gamma}_{\mu\nu}} - \lambda \frac{\delta S}{\delta \tilde{\eta}_{\mu\nu}}. \tag{20}$$

Changing to the bimetric variables $\tilde{g}_{\mu\nu}$ and $\tilde{\eta}_{\mu\nu}$, as before, implies that

$$\frac{\delta S_{\mathrm{f}}}{\delta \tilde{\gamma}_{\mu\nu}} = \lambda \frac{\delta S}{\delta \tilde{\eta}_{\mu\nu}} \bigg| \tilde{g}. \tag{21}$$

We now introduce the quantities S_{fm} and S_{ms}, $S_{\mathrm{f}} = S_{\mathrm{f0}} + S_{\mathrm{fm}}$ and $S = S_0 + S_{\mathrm{ms}}$, in order to treat the pieces that existed in the massless case separately from the new pieces in the massive case. We thus obtain

$$\frac{\delta S_{\mathrm{f0}}}{\delta \tilde{\gamma}_{\mu\nu}} + \frac{\delta S_{\mathrm{fm}}}{\delta \tilde{\gamma}_{\mu\nu}} = \lambda \frac{\delta S_0}{\delta \tilde{\eta}_{\mu\nu}} \bigg| \tilde{g} + \lambda \frac{\delta S_{\mathrm{ms}}}{\delta \tilde{\eta}_{\mu\nu}} \bigg| \tilde{g}. \tag{22}$$

Assuming that the new terms S_{fm} and S_{ms} correspond, we separate this equation into the familiar part $\delta S_{\mathrm{f0}}/\delta \tilde{\gamma}_{\mu\nu} = \lambda \left(\delta S_0/\delta \tilde{\eta}_{\mu\nu}\right)|\tilde{g}$ and the new part $\delta S_{\mathrm{fm}}/\delta \tilde{\gamma}_{\mu\nu} = \lambda \left(\delta S_{\mathrm{ms}}/\delta \tilde{\eta}_{\mu\nu}\right)|\tilde{g}$. Using invariance (19), we derive the form of S_0 as

$$S_0 = S_1[\tilde{g}_{\mu\nu}, u] + S_2, \tag{23}$$

as in the massless case. We again choose the simplest case and obtain the Hilbert action with a cosmological constant, with matter coupled only to the curved metric.

The new part in the massive case is

$$\frac{\delta S_{\mathrm{fm}}}{\delta \tilde{\gamma}_{\mu\nu}} = \lambda \frac{\delta S_{\mathrm{ms}}}{\delta \tilde{\eta}_{\mu\nu}} \bigg| \tilde{g}.$$

Assuming that the free-field mass term is quadratic in the gravitational potential, we find that its variational derivative is

$$\frac{\delta S_{\rm fm}}{\delta\tilde{\gamma}_{\mu\nu}} = a\sqrt{-\eta}\,\tilde{\gamma}_{\alpha\beta}(\tilde{\eta}^{\alpha\mu}\tilde{\eta}^{\beta\nu} + b\tilde{\eta}^{\alpha\beta}\tilde{\eta}^{\mu\nu}).$$

Changing to the bimetric variables gives

$$\frac{a\sqrt{-\eta}}{\lambda}(-\tilde{g}_{\alpha\beta} + \tilde{\eta}_{\alpha\beta})(\tilde{\eta}^{\alpha\mu}\tilde{\eta}^{\beta\nu} + b\tilde{\eta}^{\alpha\beta}\tilde{\eta}^{\mu\nu}) = \lambda\frac{\delta S_{\rm ms}}{\delta\tilde{\eta}_{\mu\nu}}\bigg|\tilde{g}. \tag{24}$$

We take the expression for $S_{\rm ms}$ in the natural form

$$S_{\rm ms} = \int d^4x\,(p\tilde{g}_{\alpha\beta}\tilde{\eta}^{\alpha\beta} + q)\sqrt{-\eta}, \tag{25}$$

where p and q are real numbers to be determined below. We note that the term $\sqrt{-g}$ itself, which gives a cosmological constant, plays no role here and is already included in S_0. Using the relation $\sqrt{-\eta} = (\sqrt{-\tilde{\eta}})^{1/(1-2l)}$, we calculate $(\delta S_{\rm ms}/\delta\tilde{\eta}_{\mu\nu})|\tilde{g}$. Equating $\lambda\,(\delta S_{\rm ms}/\delta\tilde{\eta}_{\mu\nu})|\tilde{g}$ to $\delta S_{\rm fm}/\delta\tilde{\gamma}_{\mu\nu}$, i.e., equating corresponding coefficients, determines several of the constants. Equating the coefficients of the $\sqrt{-\eta}\,\tilde{\eta}_{\mu\nu}$ terms gives $q = (2-4l)a(1+4b)/\lambda^2$, equating the coefficients of the $\sqrt{-\eta}\,\tilde{\eta}^{\mu\nu}\tilde{\eta}^{\alpha\beta}\tilde{g}_{\alpha\beta}$ terms gives $p = -ab(2-4l)/\lambda^2$, and equating the coefficients of the $\sqrt{-\eta}\,\tilde{\eta}^{\alpha\mu}\tilde{\eta}^{\beta\nu}\tilde{g}_{\alpha\beta}$ terms gives $p = a/\lambda^2$. Using the last two results together gives $b = 1/(4l-2)$. Using all three results together gives $q = -2a(2l+1)/\lambda^2$. Combining the algebraic piece of S_0 with $S_{\rm ms}$ gives

$$S_{\rm alg} = -\frac{\Lambda}{8\pi G}\int d^4x\,\sqrt{-g} + 2B\int d^4x\,\sqrt{-\eta} + \frac{a}{\lambda^2}\int d^4x\,(\tilde{g}_{\alpha\beta}\tilde{\eta}^{\alpha\beta} - 4l - 2)\sqrt{-\eta}. \tag{26}$$

When the gravitational potential vanishes, $S_{\rm alg}$ and hence the zeroth-order term should also vanish. Imposing this condition gives

$$B = \frac{\Lambda}{16\pi G} - \frac{a(1-2l)}{\lambda^2}.$$

Because our goal is to find a massive generalization of Einstein's theory, not a theory with an effective cosmological constant, we require that the first-order term in $\tilde{\gamma}_{\mu\nu}$ also vanish. Because $\lambda^2 = 32\pi G$, it follows that $\Lambda = a(1-2l)/2$. Hence, the sign of the *formal* cosmological-constant term depends on the density weight of the initially chosen potential. We also expect the quadratic part of the algebraic component $S_{\rm alg}$ of the action to agree with the free-field mass term $S_{\rm fm}$. After a binomial expansion and some algebra, we see that this is the case.[7] The weak-field expansion of the full massive nonlinear action S allows relating the coefficient a to the mass m of the spin-2 gravitons: $a = -m^2$. For nontachyonic theories, we impose the condition $a < 0$.

Combining all these results gives the total massive action S, which depends on the spin-2 graviton mass m and the parameter l controlling the relative mass of the spin-0 ghost:

$$\begin{aligned} S = {} & \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,R(g) + S_{\rm matter}[\tilde{g}_{\mu\nu}, u] + \\ & + \frac{1}{2}\int d^4x\,R_{\mu\nu\rho\sigma}(\eta)\mathcal{M}^{\mu\nu\rho\sigma}[\tilde{\eta}_{\mu\nu}, \tilde{g}_{\mu\nu}, u] + \int d^4x\,\alpha^{\mu}{}_{,\mu} - \\ & - \frac{m^2}{16\pi G}\int d^4x\left[\sqrt{-g}(2l-1) - \sqrt{-\eta}(2l+1) + \frac{1}{2}\sqrt{-\eta}\,\tilde{g}_{\mu\nu}\tilde{\eta}^{\mu\nu}\right], \end{aligned} \tag{27}$$

[7] In [24], there is a mistake in the binomial expansion for $\sqrt{-g}$ between Eqs. (43) and (44).

where $l \neq 1/2$. These theories are all universally coupled, contrary to the claim that only the FMS theory has this property [23], [24].

The Euler–Lagrange equations are easily found if the metric $g_{\mu\nu}$ is used as the dynamical variable. The result is

$$\frac{\delta S}{\delta g_{\mu\nu}} = -\frac{\sqrt{-g}}{16\pi G} G^{\mu\nu} - \frac{m^2}{16\pi G}\left[\frac{2l-1}{2}\sqrt{-g}\, g^{\mu\nu} + \frac{(\sqrt{-\eta}\,)^{l+1}}{4(\sqrt{-g}\,)^l}(2\eta^{\mu\nu} - l\eta^{\alpha\beta} g_{\alpha\beta} g^{\mu\nu})\right] + \frac{\delta S_{\mathrm{matter}}}{\delta g_{\mu\nu}}. \tag{28}$$

Following Boulware and Deser [24], we can linearize these theories and check whether the spin-0 component is tachyonic. Nontachyonicity entails $-1/2 \leq l < 1/2$. For $l = 0$, the spin-0 ghost has the same mass as the spin-2 degrees of freedom, and this theory (with density weight zero for the potential) is hence the cleanest in the set. (The connection between density "weight" and graviton "mass" is an amusing linguistic accident.) Investigating various masses for the spin-0 degree of freedom might have some empirical consequences. This is especially the case in large-scale and homogeneous situations, for example, in cosmology [21]. The ratio of the spin-0 mass m_0 to the spin-2 mass m_2 is given by

$$\frac{m_0^2}{m_2^2} = \frac{-4l^2+1}{2l^2+1}. \tag{29}$$

Hence, the mass is an even function of l. For $l = -1/2$, the spin-0 degree of freedom is massless, and $\sqrt{-\eta}$ is absent from the mass term. As $l \to 1/2$, the scalar again becomes light, and the coefficient of $\sqrt{-g}$ tends to zero, although the value $l = 1/2$ is forbidden. Between these massless endpoints, the spin-0 degree of freedom becomes heavier. At the midpoint $l = 0$, the scalar has the same mass as the spin-2 field, giving a simple form of the wave equation for the linearized massive Einstein equations. Thus, the spin-0 ghost is never heavier than the spin-2 degrees of freedom, and the $l \neq 0$ theories hence have weaker gravitational attraction at large distances and in homogeneous situations.

4. Derivation of a contravariant tensor density potential: Massless case

We briefly present the contravariant analogue of the above derivation of Einstein's equations. The gravitational potential is now a contravariant symmetric tensor density field $\tilde{\gamma}^{\mu\nu}$ of density weight l, where $l \neq 1/2$. In addition to the obvious moves of some indices, we introduce some sign changes in the contravariant case. The hope for a simple rule relating index moves and sign changes is disappointed because the symmetry between the $(0,2)$ theory of weight $-l$ and the $(2,0)$ theory of weight l is broken: the Lagrangian density is a scalar density of weight 1, not weight 0. The consequences of this imperfect symmetry are more apparent in the massive case below than in the current massless case. It is convenient to use not the inverse flat metric itself but the densitized related metric $\tilde{\eta}^{\mu\nu} = \eta^{\mu\nu}(\sqrt{-\eta}\,)^l$ of weight l, where $l \neq 1/2$.

In the massless theories, we assume an initial infinitesimal invariance (up to a boundary term) of the free gravitational action S_{f} under the infinitesimal gauge transformation $\tilde{\gamma}^{\mu\nu} \to \tilde{\gamma}_{\mu\nu} + \delta\tilde{\gamma}^{\mu\nu}$, where

$$\delta\tilde{\gamma}^{\mu\nu} = \partial^\mu \tilde{\xi}^\nu + \partial^\nu \tilde{\xi}^\mu + c\eta^{\mu\nu}\,\partial_\alpha \tilde{\xi}^\alpha, \tag{30}$$

$c \neq -1/2$, and $\tilde{\xi}^\nu$ is an arbitrary vector density of weight l. It proves expedient to set $c = -l$. For any S_{f} invariant in this sense, a certain linear combination of the free-field equations is identically divergenceless:

$$\partial^\mu\left(\frac{\delta S_{\mathrm{f}}}{\delta\tilde{\gamma}^{\mu\nu}} - \frac{l}{2}\eta_{\mu\nu}\eta^{\sigma\alpha}\frac{\delta S_{\mathrm{f}}}{\delta\tilde{\gamma}^{\sigma\alpha}}\right) = 0. \tag{31}$$

This is the generalized Bianchi identity for the free-field theory. Local energy–momentum conservation, which holds with the use of the Euler–Lagrange equations for the gravity $\tilde{\gamma}^{\mu\nu}$ and matter u, can be written as

$$\partial^{\nu}\left(\frac{\delta S}{\delta\tilde{\eta}^{\mu\nu}} - \frac{l}{2}\tilde{\eta}^{\alpha\beta}\tilde{\eta}_{\mu\nu}\frac{\delta S}{\delta\tilde{\eta}_{\alpha\beta}}\right) = 0. \tag{32}$$

We write the universal-coupling postulate in the form

$$\frac{\delta S}{\delta\tilde{\gamma}^{\mu\nu}} = \frac{\delta S_{\mathrm{f}}}{\delta\tilde{\gamma}^{\mu\nu}} + \lambda\frac{\delta S}{\delta\tilde{\eta}^{\mu\nu}} \tag{33}$$

(the reason for choosing of the sign of the term containing the energy–momentum tensor soon becomes clear). We obtain S, changing from $\tilde{\gamma}^{\mu\nu}$ and $\tilde{\eta}^{\mu\nu}$ to the bimetric variables $\tilde{g}^{\mu\nu}$ and $\tilde{\eta}^{\mu\nu}$, where

$$\tilde{g}^{\mu\nu} = \tilde{\eta}^{\mu\nu} + \lambda\tilde{\gamma}^{\mu\nu}. \tag{34}$$

The coefficient of $\lambda\tilde{\gamma}^{\mu\nu}$ is chosen such that the covariant and contravariant cases, as far as is easily achieved, define the gravitational potential similarly. In particular, to the linear order in the potential in Cartesian coordinates, the traceless part of the gravitational potential γ has the same observable significance, whether it is the density difference between the covariant curved and flat and curved metrics or the density difference between the contravariant flat and curved metrics.[8] Equating coefficients of the variations gives

$$\left.\frac{\delta S}{\delta\tilde{\eta}^{\mu\nu}}\right|\tilde{\gamma} = \left.\frac{\delta S}{\delta\tilde{\eta}^{\mu\nu}}\right|\tilde{g} + \frac{\delta S}{\delta\tilde{g}^{\mu\nu}} \tag{35}$$

and

$$\frac{\delta S}{\delta\tilde{\gamma}^{\mu\nu}} = \lambda\frac{\delta S}{\delta\tilde{g}^{\mu\nu}}, \tag{36}$$

whence we obtain

$$\frac{\delta S_{\mathrm{f}}}{\delta\tilde{\gamma}^{\mu\nu}} = -\lambda\left.\frac{\delta S}{\delta\tilde{\eta}^{\mu\nu}}\right|\tilde{g}. \tag{37}$$

The use of the generalized Bianchi identity implies that S splits into a component $S_1[\tilde{g}^{\mu\nu}, u]$ and a component S_2 that takes form (16). The quantity S_2 contains all the ineliminable dependence on the background metric and does not contribute to the field equations. The simplest choice of S_1 gives the Hilbert action for Einstein's equations and a cosmological constant, as in the covariant case. The specific choice from the allowed values of l makes no difference in the massless case.

5. Derivation for a contravariant tensor density potential: Massive case

The choice of the density weight l makes a difference in the massive generalization of this contravariant derivation, much as in the covariant case. The FMS–Logunov theory turns out to be the l=1 contravariant universally coupled massive theory. While in some clear senses the l=1 theory is the best of the contravariant massive theories, it is not the only such theory. The free-field action S_{f} again has two parts: the part S_{f0} that is mostly kinetic and has a local gauge symmetry and an algebraic mass term S_{fm}. The action S of the full theory again splits into two parts, S_0 and S_{ms}.

[8]If $g_{\mu\nu} = \eta_{\mu\nu} - \lambda\gamma_{\mu\nu}$, then the inverse metric yields an infinite series expansion (at least formally), whose first term has a sign different from what a naive index raising might suggest: $g_{\mu\nu} = \eta_{\mu\nu} + \lambda\gamma_{\mu\nu} + \cdots$ implies that $g^{\mu\nu} - \eta^{\mu\nu} \approx \lambda\gamma^{\mu\nu}$, not $-\lambda\gamma_{\mu\nu}$. If the exact relation $g^{\mu\nu} - \eta^{\mu\nu} = \lambda\psi^{\mu\nu}$ holds (the sign of the λ term is important), then $\psi_{\mu\nu} \approx \gamma_{\mu\nu}$. Thus, the meaning of the gravitational potential is insensitive to a sign change, and it is therefore easier to compare the various massive theories.

Requiring $S_{\rm f0}$ to change only by a boundary term under the infinitesimal variation

$$\delta\tilde{\gamma}^{\mu\nu} = \partial^\mu\tilde{\xi}^\nu + \partial^\nu\tilde{\xi}^\mu - l\eta^{\mu\nu}\,\partial_\alpha\tilde{\xi}^\alpha, \tag{38}$$

where $l \neq 1/2$, implies the identity

$$\partial^\mu\left(\frac{\delta S_{\rm f0}}{\delta\tilde{\gamma}^{\mu\nu}} - \frac{l}{2}\eta_{\mu\nu}\eta^{\sigma\alpha}\frac{\delta S_{\rm f0}}{\delta\tilde{\gamma}^{\sigma\alpha}}\right) = 0. \tag{39}$$

We again postulate the universal coupling in form (33). We change to the bimetric variables $\tilde{g}^{\mu\nu}$ and $\tilde{\eta}^{\mu\nu}$. Letting the new mass terms and the terms previously present agree separately, we find that the mass terms satisfy

$$\frac{\delta S_{\rm fm}}{\delta\tilde{\gamma}^{\mu\nu}} = -\lambda\frac{\delta S_{\rm ms}}{\delta\tilde{\eta}^{\mu\nu}}\bigg|\tilde{g}. \tag{40}$$

The action $S_{\rm fm}$ is chosen to be quadratic in the gravitational potential and to satisfy

$$\frac{\delta S_{\rm fm}}{\delta\tilde{\gamma}^{\mu\nu}} = a\sqrt{-\eta}\,\tilde{\gamma}^{\alpha\beta}(\tilde{\eta}_{\alpha\mu}\tilde{\eta}_{\beta\nu} + b\tilde{\eta}_{\alpha\beta}\tilde{\eta}_{\mu\nu}). \tag{41}$$

The quantity $S_{\rm ms}$ is naturally chosen in the form

$$S_{\rm ms} = \int d^4x\,(p\tilde{g}^{\alpha\beta}\tilde{\eta}_{\alpha\beta} + q)\sqrt{-\eta} \tag{42}$$

for unspecified p and q. Matching the coefficients of several terms gives

$$q = \frac{-(2-4l)a(1+4b)}{\lambda^2}, \qquad p = \frac{ab(2-4l)}{\lambda^2}, \qquad p = \frac{a}{\lambda^2},$$
$$b = -\frac{1}{4l-2}, \qquad q = \frac{2a(2l-3)}{\lambda^2}.$$

Requiring the zeroth-order algebraic term in S to vanish gives

$$B = \frac{\Lambda}{16\pi G} + \frac{a(1-2l)}{\lambda^2}.$$

Requiring the first-order algebraic term to vanish, after some algebra, gives $\Lambda = -a(1-2l)/2$. The second-order term agrees with the free-field mass term, as could be hoped. The spin-2 graviton mass m is given by $a = -m^2$.

Combining all these results, we obtain the total massive action S, which depends on the spin-2 graviton mass m and the parameter l controlling the relative mass of the spin-0 graviton:

$$\begin{aligned} S = {} & \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,R(g) + S_{\rm matter}[\tilde{g}^{\mu\nu}, u] + \\ & + \frac{1}{2}\int d^4x\, R_{\mu\nu\rho\sigma}(\eta)\mathcal{M}^{\mu\nu\rho\sigma}[\tilde{\eta}^{\mu\nu}, \tilde{g}^{\mu\nu}, u] + \int d^4x\,\alpha^\mu{}_{,\mu} - \\ & - \frac{m^2}{16\pi G}\int d^4x\left[-\sqrt{-g}(2l-1) + \sqrt{-\eta}(2l-3) + \frac{1}{2}\sqrt{-\eta}\,\tilde{g}^{\mu\nu}\tilde{\eta}_{\mu\nu}\right], \end{aligned} \tag{43}$$

where $l \neq 1/2$. The empirically doubtful Fierz–Pauli mass term is not among the universally coupled theories considered in this paper. All these theories are contained in the OP 2-parameter family.

While we could use some contravariant tensor (or tensor density) to find the Euler–Lagrange equations, the equations are more easily compared with those previously found if the metric $g_{\mu\nu}$ is used. The resulting equations are

$$\frac{\delta S}{\delta g_{\mu\nu}} = -\frac{\sqrt{-g}}{16\pi G} G^{\mu\nu} - \frac{m^2}{16\pi G}\left[\frac{1-2l}{2}\sqrt{-g}\, g^{\mu\nu} + \frac{(\sqrt{-g}\,)^l \eta_{\alpha\beta}}{4(\sqrt{-\eta}\,)^{l-1}}(lg^{\alpha\beta}g^{\mu\nu} - 2g^{\mu\alpha}g^{\nu\beta})\right] + \frac{\delta S_{\text{matter}}}{\delta g_{\mu\nu}}. \tag{44}$$

Linearizing these theories shows that the spin-0 field is not a tachyon if and only if $1/2 < l \leq 3/2$. For $l = 1$, the spin-0 ghost has the same mass as the spin-2 degrees of freedom, and these field equations, which correspond to the FMS–Logunov theory, are hence the cleanest in the set. At the linear level, the $(0, 2)$ theory of weight $-l$ is identical to the $(2, 0)$ theory of weight $l + 1$. Hence, for the $(2, 0)$ theory of weight $l + 1$, the ratio of the spin-0 mass m_0 to the spin-2 mass m_2 is given by

$$\frac{m_0^2}{m_2^2} = \frac{-4(l-1)^2 + 1}{2(l-1)^2 + 1}.$$

The mass is an even function of $l - 1$. For $l = 3/2$, the spin-0 degree of freedom is massless, and $\sqrt{-\eta}$ is absent from the mass term. As $l \to 1/2$, the scalar again becomes light, and the coefficient of $\sqrt{-g}$ tends to 0, although the value $l = 1/2$ is forbidden. Between these massless endpoints, the spin-0 degree of freedom becomes heavier. At the midpoint $l = 1$ (the FMS–Logunov theory), the spin-0 graviton has the same mass as the spin-2 graviton, giving a simple form of the wave equation for the linearized massive Einstein equations. Hence, the spin-0 ghost is never heavier than the spin-2 degrees of freedom for this family of theories, although such can occur for the larger family of OP massive theories [7].[9]

6. Massive gravities and experiment

In both experimental [43] and theoretical contexts, it is common to speak of *the* mass of *the* graviton, as if all gravitons must have the same mass. While all gravitons do have the same mass in the most famous spin-2–spin-0 massive gravity (developed by FMS and studied by Logunov and collaborators), the existence of the OP theories shows that massive gravity has two mass parameters that should be tested experimentally. It would be worthwhile to ascertain to what degree the tacit assumption of equal spin-2 and spin-0 masses is actually used in finding experimental bounds on massive gravity. Astrophysical tests for changes in the behavior of the degrees of freedom present in massless general relativity primarily bound the spin-2 mass. The empirical bounds on the spin-2 graviton mass are so tight that the spin-2 part of the mass term is empirically negligible except in strong fields or over cosmic distances [20], [21]; these regimes are also those investigated by Logunov and collaborators. The flexibility in the spin-0 mass in the massive theories derived here opens some phenomenological opportunities by increasing the range of the spin-0 repulsion that counterbalances some of the spin-2 attraction. The larger family of OP massive theories permits either longer or shorter range for the spin-0 repulsion compared with the spin-2 attraction. Babak and Grishchuk recently investigated a similar phenomenological flexibility [21]. They noted that their massive spin-2, massless spin-0 special case agreed with general relativity in cosmological contexts because of the high degree of symmetry. Hence, cosmological limits on the graviton mass(es) primarily bound the spin-0 mass. While the OP theories have nonlinear mass terms motivated from first principles in contrast to Babak and Grishchuk's linear mass terms motivated by mathematical simplicity, similar qualitative behaviors of the two kinds of two-parameter massive gravities can be expected outside highly nonlinear regimes.

[9] It is reassuring that in the conformally flat special case $g_{\mu\nu} = \phi\eta_{\mu\nu}$, the $(2, 0)$ theory of weight $l + 1$ and $(0, 2)$ theory of weight $-l$ coincide.

In view of the tight empirical bounds on the graviton masses, the observable consequences of a mass term are rather difficult to detect. But there are two important theoretical issues that arise at the classical level for massive variants of Einstein's equations. The first is the well-known question of stability, positive energy, etc., given the wrong-sign spin 0. The second is the question of causality: massive gravity is a special relativistic field theory with $\eta_{\mu\nu}$ observable, but there is reason to fear that field propagation might violate causality by having the light cone of the effective metric $g_{\mu\nu}$ leak outside that of $\eta_{\mu\nu}$. We now turn to these issues.

7. Positive energy

It has long been argued that massive variants of Einstein's equations pose the unpleasant dilemma [24], [44] that either the mass term is of the Fierz–Pauli form with $5\infty^3$ degrees of freedom (pure spin 2) and is empirically falsified by having a discontinuous massless limit or the theory has $6\infty^3$ degrees of freedom including a wrong-sign spin 0 (a spatial scalar density) and instability arises after linearization. The former problem is the van Dam–Veltman–Zakharov discontinuity [45], about which a large literature has appeared in the last few years after a long period of relative quiet. More relevant for our purposes is whether the massive theories with $6\infty^3$ degrees of freedom (spins 2 and 0) are unstable.

Contrary to widely held views, Visser argued that the massive theories with $6\infty^3$ degrees of freedom might well be stable [20]. More recently, Babak and Grishchuk argued that such theories actually are stable [21]. Concerning the specific case of the FMS theory, the authors of the theory were themselves unconvinced of instability [23], although they did not follow up on the matter after arguments for instability were published. In the middle to late 1980s, Logunov and collaborators (such as Loskutov and Chugreev) adopted the FMS theory as the massive version of the relativistic theory of gravity. They argued that this specific theory might well be stable [11], linearization arguments notwithstanding. More compellingly, Loskutov calculated the gravitational radiation from a bounded source and concluded that it is in fact positive definite [46], even though the theory has a wrong-sign spin-0 component. It is curious that this conclusion has received so little response.

While the question of positive energy (or positive mass, as is often said in a gravitational context) has not been settled with a favorable outcome (in the sense of a general proof that all exact solutions satisfying certain energy conditions have positive mass), neither do the arguments for instability from linearization seem compelling. It is useful to show using a Hamiltonian formalism that linearization is untrustworthy because it essentially changes the form of the Hamiltonian constraint such that instability becomes more plausible than is true for the exact nonlinear theory. Although the wrong-sign spin 0 is present in the nonlinear theory, its kinetic energy is related to that of the right-sign degrees of freedom such that it can easily radiate only together with the positive energy degrees of freedom. Precisely this feature is lost upon linearization. This important feature of the Hamiltonian constraint depends essentially on a cubic term in the Hamiltonian density and hence on a quadratic term in the field equations.

In the case of the FMS–Logunov contravariant weight-1 theory, the field equations impose a lower bound on the Hamiltonian density a little below zero, in contrast to boundary terms (which should be annihilated, at least in static contexts, because of the exponential Yukawa falloff of the gravitational potential of bounded systems). Discarding unimportant terms and factors from action (43), we obtain the Lagrangian density

$$\mathcal{L} = \sqrt{-g}R(g) - m^2\biggl(-\sqrt{-g} - \sqrt{-\eta} + \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\eta_{\mu\nu}\biggr). \tag{45}$$

We use the ADM (3+1)-dimensional split[10] [32] and choose coordinates (Cartesian, spherical, or the like) such that $\eta_{00} = -1$ and $\eta_{0i} = 0$. The curved metric $g_{\mu\nu}$ is then expressed in terms of a lapse function

[10] The ADM split is a noncovariant (3+1)-representation of the metric tensor of Riemannian space first proposed by Arnowitt, Deser, and Misner.

N relating the effective proper time to the coordinate time, a shift vector β^i expressing how the spatial coordinate system appears to shift among the various time slices, and a curved spatial metric h_{ij} with the inverse h^{ij} and determinant h. Letting $g^{\mu\nu}$ be the inverse curved metric as usual, we have $g^{00} = -N^{-2}$ (the inverse metric being most convenient here), $g_{ij} = h_{ij}$, and $g_{0i} = h_{ij}\beta^j$. The indices for three-dimensional quantities are raised and lowered with h_{ij}. Dropping a divergence from the Hilbert-like action above, we have the FMS massive version of the standard (3+1)-dimensional Lagrangian density

$$\mathcal{L} = N\sqrt{h}\left[{}^3R + K_{ab}K^{ab} - K^2 + m^2\left(1 - \frac{h^{ij}\eta_{ij}}{2}\right)\right] + m^2\left[\sqrt{-\eta} + \frac{\sqrt{h}}{2N}(\eta_{ij}\beta^i\beta^j - 1)\right]. \tag{46}$$

Hereafter, we drop the superscript on ${}^3R$.

Defining canonical momenta as usual, we obtain the usual results

$$\pi^{ij} = \frac{\partial\mathcal{L}}{\partial h_{ij,0}} = \sqrt{h}(K^{ij} - h^{ij}K), \qquad P_i = \frac{\partial\mathcal{L}}{\partial\beta^i_{,0}} = 0, \qquad P = \frac{\partial\mathcal{L}}{\partial N_{,0}} = 0. \tag{47}$$

The four vanishing canonical momenta are called primary constraints in the context of constrained dynamics [47].

Performing the generalized Legendre transformation and using the primary constraints gives the canonical Hamiltonian density

$$\mathcal{H} = N\left[\mathcal{H}_0 + m^2\sqrt{h}\left(\frac{1}{2}h^{ij}\eta_{ij} - 1\right)\right] + \beta^i\mathcal{H}_i - m^2\sqrt{-\eta} + \frac{m^2\sqrt{h}}{2N}(1 - \eta_{ij}\beta^i\beta^j), \tag{48}$$

where, as usual,

$$\mathcal{H}_0 = \frac{1}{\sqrt{h}}\left(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2\right) - \sqrt{h}\,R, \qquad \mathcal{H}_i = -2D_j\pi^j_i,$$

and D_j is the three-dimensional torsion-free covariant derivative compatible with h_{ij}. For $m = 0$, we recover the usual form that is purely a sum of constraints, but $m \neq 0$ destroys that form and leads to six, not two, degrees of freedom. We note that we have retained the zeroth-order term $-m^2\sqrt{-\eta}$, and Minkowski space–time hence has zero energy, as it should. Boulware and Deser omitted this term [24]. Varying the lapse N and shift vector β^i, we obtain the secondary constraints, namely, the modified Hamiltonian constraint

$$\frac{\partial\mathcal{H}}{\partial N} = \mathcal{H}_0 + m^2\sqrt{h}\left(-1 + \frac{1}{2}h^{ij}\eta_{ij}\right) - \frac{m^2\sqrt{h}}{2N^2}(1 - \eta_{ij}\beta^i\beta^j) = 0 \tag{49}$$

and the modified momentum constraint

$$\frac{\partial\mathcal{H}}{\partial\beta^i} = \mathcal{H}_i - \frac{m^2\sqrt{h}}{N}\eta_{ij}\beta^j = 0. \tag{50}$$

These constraints are second-class [48]. As Boulware and Deser pointed out, we can use these relations to eliminate the lapse and shift from the Hamiltonian density to obtain a partly on-shell Hamiltonian density purely in terms of the true degrees of freedom and their momenta:

$$\mathcal{H} = \sqrt{2m^2\sqrt{h}\left[\mathcal{H}_0 + m^2\sqrt{h}\left(\frac{h^{ij}\eta_{ij}}{2} - 1\right)\right] + \mathcal{H}_i\mathcal{H}_j\eta^{ij}} - m^2\sqrt{-\eta}.$$

Expressing the lapse in terms of the true degrees of freedom, we obtain

$$N^2 = \frac{hm^4}{2m^2\sqrt{h}\big[\mathcal{H}_0 + m^2\sqrt{h}(h^{ij}\eta_{ij}/2 - 1)\big] + \mathcal{H}_i\mathcal{H}_j\eta^{ij}}. \tag{51}$$

Changing the variables from the lapse N to the recently popular “slicing density” $\alpha = N/\sqrt{h}$ [49] allows writing the on-shell Hamiltonian density as $\mathcal{H} = m^2(1/\alpha - \sqrt{-\eta}\,)$. Boundary terms have been omitted, but they vanish in some cases because of the Yukawa fall-off for localized sources.

We now consider the linearization of the exact Hamiltonian density. The slicing density $\alpha = N/\sqrt{h}$ has the virtue of reducing the number of radicals in the off-shell Hamiltonian density and also giving the on-shell Hamiltonian density a simple form. Before linearization, we have

$$\mathcal{H} = \alpha\left[\pi^{ij}\pi^{kl}\left(h_{ik}h_{jl} - \frac{1}{2}h_{ij}h_{kl}\right) - hR + m^2 h\left(-1 + \frac{1}{2}h^{ij}\eta_{ij}\right)\right] + \\ + \beta^i\mathcal{H}_i - m^2\sqrt{-\eta} + \frac{m^2}{2\alpha}(1 - \eta_{ij}\beta^i\beta^j). \tag{52}$$

We now let $\eta_{ij} = \delta_{ij}$, $h_{ij} = \delta_{ij} + \phi_{ij}$, and $\alpha = 1 + a$. We are only interested in the kinetic term. The quintic and quartic terms containing $a\Pi^2\phi^2$, $\Pi^2\phi^2$, and $a\Pi^2\phi$ (indices suppressed) are dispensable, but dropping the cubic term $a(\pi^{ij}\pi^{ij} - \pi^{ii}\pi^{jj}/2)$ creates serious problems connected with the term $-\pi^{ii}\pi^{jj}/2$ that did not arise in the exact theory. The kinetic term containing the worrisome wrong-sign scalar $-\pi^2/2$ is located in the Hamiltonian constraint of the exact theory, where its ability to do damage is mitigated, but after linearization, the $-\pi^2/2$ term leads to troubles, such as by radiating arbitrarily much negative energy away or by permitting the radiation of arbitrarily much positive energy, leading to instability. If we had the cubic terms such as $a(\pi^{ij}\pi^{ij} - \pi^{ii}\pi^{jj}/2)$ and perhaps its spatial derivative analogue, then the approximate Hamiltonian constraint would still substantially resemble the exact form and might behave better.

Boulware and Deser [24], noting the on-shell square root form of $\mathcal{H}$ above (and lacking the zeroth-order term), commented that “the Hamiltonian form (in terms of 6 degrees of freedom)… appears to be positive definite. Since in addition, the linearized approximation here corresponds to a scalar-ghost admixture, and so gives the linearized Einstein interaction in the weak-field limit, it would seem that this model has at least two improvements over [the empirically doubtful Fierz–Pauli theory and generalizations thereof]: Its energy is positive and it has correct linearized behavior. However, it is unacceptable: The vacuum is not a local minimum, but only a saddle point, as may be seen by considering equilibrium (static) configurations, or simply expanding H to quadratic order, where it is found to agree with the linearized (ghost) version H. That for appropriate excitations the quadratic part of H can be negative may seem irrelevant in view of the apparent positivity of the complete H…. Unfortunately, the argument of the square root is not intrinsically positive… even though its positivity is required for the theory to make sense, i.e., for N^2 to be positive… (otherwise, the effective Riemannian metric ‘seen’ by matter will become pathological). Therefore one would have to *impose* that the excitations respect this requirement, i.e., cut off arbitrarily those modes which take H below its vanishing vacuum ($g = \eta$) state value. Instability near vacuum ($g \simeq \eta$) is the reason for rejecting this and other models whose quadratic mass is not of Pauli–Fierz form.”

This is a puzzling argument because $N \leq 0$ gives a singularity; there is hence no need to “impose” $N > 0$ by hand. A remaining question is whether the theory hits $N = 0$ so often that singularities form in mundane contexts. No argument to that effect has been given, while the fact that $N \to 0$ implies gravitational time dilation suggests that N has little tendency to vanish [50]. There seems to be no need for $\mathcal{H}$ to be positive everywhere as long as $H = \int \mathcal{H}\, d^3x$ is positive (or nonnegative) for some suitable boundary conditions. The restoration of the negative zeroth-order term to $\mathcal{H}$ implies that $\mathcal{H}$ is bounded from below but not by zero.

There are currently (to our knowledge) no known solutions of the nonlinear field equations, exact or numerical, of FMS–Logunov or any other “ghost” theory of the families considered here that have negative total energy. The same is true for solutions that indicate instability by radiating negative net energy. Given the need for a nonperturbative treatment, the question of stability might best be resolved with the help of numerical relativists. It suffices to work in spherical symmetry, where the wrong-sign field can radiate but most of the right-sign fields cannot.

8. Causality

Given that massive gravities are considered in Minkowski space–time with an observable background metric $\eta_{\mu\nu}$, a further issue worth considering is whether the null cone of the flat background metric is a bound of the effective curved metric $g_{\mu\nu}$. The flat metric is observable, and violation of the null cone of $\eta_{\mu\nu}$ hence implies backwards causation in some Lorentz frames, which is usually rejected. Causality for higher-spin theories has already caused trouble in the case of spin-3/2 fields. Velo and Zwanziger [51] concluded that the "main lesson to be drawn from our analysis is that special relativity is not automatically satisfied by writing equations that transform covariantly. In addition, the solutions must not propagate faster than light."

The argument has been made that massive gravity leads to causality violation in the sense of special relativity (this is relevant because of the observable flat metric [13], [52]). As Chugreev rightly notes, the static field of sources, if any, must be taken into account; for cosmological models, the presence of matter everywhere might suffice to preserve causality [53]. But surely it is a contingent rather than necessary truth that the universe is filled with matter everywhere.[11] Gravitational radiation decays as $1/r$, and the static field due to localized sources decays as e^{-mr}. Therefore, the radiation eventually wins and threatens the proper relation between the two null cones. Because we wish to regard many solutions without matter everywhere and with gravitational radiation as physically meaningful (although apparently not corresponding to the actual world), some additional strategy for ensuring the correct relation between the null cones of the two metrics is needed. The only option that comes to mind is to install artificial gauge freedom, perhaps using parameterization along the lines of [52], [54], and then to use the same strategy that we used to impose η-causality in the massless case [13]. The resulting gauged massive gravity has a gauge transformation groupoid, not a group. It is noteworthy that parameterization yields results that for the lowest-order terms resemble Stueckelberg's trick for introducing gauge freedom into massive Proca electromagnetism. Stueckelberg's trick has sometimes been used in the lowest order in gravity [55], but it remained unclear what the generalization to nonlinear field equations might be. It seems plausible that other methods for installing artificial gauge freedom, such as the BFT procedure [56] or gauge unfixing [57], ought to give similar results, although we have not investigated those questions carefully.

Acknowledgments. One of the authors (J. B. P.) thanks Katherine Brading for the assistance with the history of Einstein's use of energy conservation and related principles in his quest for gravitational field equations. The correspondence with Yu. V. Chugreev, Stanley Deser, A. A. Logunov, Thanu Padmanabhan, and Matt Visser, not all of whom agree with everything said here, is gratefully acknowledged.

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[11] Indeed, that claim is better regarded as a convention and not a fact, even in cosmology intending to describe the actual world [6].

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