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MIXED-SYMMETRY MASSLESS GAUGE FIELDS IN $\mathbf{AdS_5}$

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Using the $su(2,2)$ spinor language, we describe free mixed-symmetry massless bosonic and fermionic gauge fields of arbitrary spins in the AdS_5 space. We construct manifestly covariant action functionals and derive field equations.

Keywords: higher spins, mixed-type symmetry, anti-de Sitter space, Young tableau

1. Introduction

We continue the study of the five-dimensional higher-spin gauge theory [1]–[5]. Our goal is to formulate manifestly covariant gauge-invariant actions describing the free-field dynamics of massless mixed-symmetry bosonic and fermionic gauge fields of arbitrary spins $\mathbf{s} = (s_1, s_2)$. Our treatment of AdS_5 mixed-symmetry gauge fields is based on the framelike approach proposed in [6], [7]. But instead of the (spinor)–tensor language used in [6], [7], we use a spinorial description of AdS_5 higher-spin fields based on the well-known fact that the AdS_5 algebra $o(4,2)$ is isomorphic to $su(2,2)$. Therefore, $o(4,2)$ (spinor)–tensor fields can be described equivalently as $su(2,2)$ multispinors. The main advantage of the spinorial description is that bosonic and fermionic fields of any spin $\mathbf{s} = (s_1, s_2)$ can be considered uniformly.

Although the d-dimensional analysis in [6], [7] certainly includes the AdS_5 case, reformulating these results in the spinor language may be interesting in several aspects. In general, such a reformulation is motivated by the desire to take a step toward a supersymmetric nonlinear higher-spin gauge theory, which is interesting in the context of a higher-spin version of the AdS_5/CFT_4 correspondence (see [8] for a review). The nonlinear equations of motion for AdS_d totally symmetric bosonic fields and the underlying higher-spin gauge algebras are now constructed [9], but extending them to general mixed-symmetry fields remains an open problem. On the other hand, in the case of the AdS_5 higher-spin dynamics, there is a real possibility to construct a nonlinear theory of higher-spin fields of all symmetry types. Indeed, in five dimensions, one benefits from using the isomorphism $o(4,2) \sim su(2,2)$. In particular, in the spinor language, (supersymmetric) higher-spin algebras were identified as certain star-product algebras with $su(2,2)$ spinor generating elements [10], [11]. There also exist manifestly covariant formulations of free AdS_5 higher-spin dynamics[1] [1]–[4] and $\mathcal{N}$=0, 1 (supersymmetric) action functionals that describe cubic interactions of totally symmetric AdS_5 fields [2], [5].

This paper is organized as follows. In Sec. 2, we describe the background AdS_5 geometry in the spinor notation [2]. In Sec. 3, we consider AdS_5 higher-spin gauge fields of mixed-symmetry type and describe their multispinor and (spinor)–tensor forms. In Sec. 4, we introduce higher-rank tensors as functions of auxiliary spinor variables [2] and construct gauge transformations and linearized higher-spin curvatures.

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[1] The light-cone equations of motion for AdS_d massless fields of arbitrary spins were constructed in [12], and the light-cone actions for general AdS_5 mixed-symmetry massless fields were considered in [13]. Also, there are various manifestly covariant Lagrangian formulations for particular examples of AdS_d mixed-symmetry gauge fields [14].

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In Sec. 5, we construct manifestly gauge-invariant higher-spin actions. In Sec. 6, we derive equations of motions and discuss constraints for extra fields. We make concluding remarks in Sec. 7.

2. AdS_5 background geometry in the spinor notation

Gravitational fields in AdS_5 are identified with a 1-form connection that takes values in the AdS_5 algebra $su(2,2)$,

$$\Omega(x) = dx^{\underline{n}}\,\Omega_{\underline{n}}{}^{\alpha}{}_{\beta} t_{\alpha}{}^{\beta}, \tag{2.1}$$

where $t_{\alpha}{}^{\beta}$ are basis elements of $su(2,2)$ and $\Omega^{\alpha}{}_{\alpha} = 0$.[2] The $su(2,2)$ gauge field in (2.1) decomposes into a frame field and a Lorentz spin connection. This splitting can be performed in a manifestly $su(2,2)$-covariant manner by introducing a compensator field [2], [15], an antisymmetric bispinor $V^{\alpha\beta} = -V^{\beta\alpha}$. The compensator is normalized such that $V_{\alpha\gamma}V^{\beta\gamma} = \delta_{\alpha}{}^{\beta}$ and $V_{\alpha\beta} = (1/2)\varepsilon_{\alpha\beta\gamma\rho}V^{\gamma\rho}$. The Lorentz subalgebra is identified with the stability algebra of the compensator. This allows defining the frame field $E^{\alpha\beta}$ and Lorentz spin connection $\omega^{\alpha(2)}$ as [2]

$$\begin{aligned} &E^{\alpha\beta} = DV^{\alpha\beta} \equiv dV^{\alpha\beta} + \Omega^{\alpha}{}_{\gamma}V^{\gamma\beta} + \Omega^{\beta}{}_{\gamma}V^{\alpha\gamma}, \qquad E^{\alpha\beta}V_{\alpha\beta} = 0,\\ &\omega^{\alpha}{}_{\beta} = \Omega^{\alpha}{}_{\beta} + \frac{\lambda}{2}E^{\alpha\gamma}V_{\gamma\beta}, \end{aligned} \tag{2.2}$$

where λ is a cosmological parameter, $\lambda^2 > 0$. We note that because the compensator is Lorentz-invariant, we regard $V^{\alpha\beta}$ as a symplectic metric that allows raising and lowering spinor indices in a Lorentz-covariant way: $X^{\alpha} = V^{\alpha\beta}X_{\beta}$, $Y_{\alpha} = Y^{\beta}V_{\beta\alpha}$.

The AdS_5 field strength corresponding to gauge field (2.1) has the form

$$R_{\alpha}{}^{\beta} = d\Omega_{\alpha}{}^{\beta} + \Omega_{\alpha}{}^{\gamma} \wedge \Omega_{\gamma}{}^{\beta}, \tag{2.3}$$

and the background AdS_5 space is described by the 1-form field $\Omega_{0}^{\alpha}{}_{\beta} = (h^{\alpha\beta}, \omega_0^{\alpha(2)})$, which satisfies the zero-curvature condition [16]

$$R_{\alpha}{}^{\beta}(\Omega_0) = 0. \tag{2.4}$$

3. Higher-spin fields

To describe a spin-(s_1, s_2) gauge field, we introduce a pair of mutually conjugate $su(2,2)$ traceless multispinors symmetric in the lower and upper indices [1]–[6],[3]

$$\Omega^{\alpha(m)}{}_{\beta(n)}(x) \oplus \overline{\Omega}^{\beta(n)}{}_{\alpha(m)}(x), \qquad \Omega^{\alpha(m-1)\gamma}{}_{\beta(n-1)\rho}(x)\,\delta^{\rho}_{\gamma} = 0, \tag{3.1}$$

which are 1-forms

$$\Omega^{\alpha(m)}{}_{\beta(n)}(x) = dx^{\underline{n}}\,\Omega_{\underline{n}}{}^{\alpha(m)}{}_{\beta(n)}(x) \tag{3.2}$$

with

$$m = s_1 + s_2 - 1, \qquad n = s_1 - s_2 - 1. \tag{3.3}$$

[2]Throughout the paper, we work within the "mostly minus" signature $(+-\ldots-)$ and use $\alpha, \beta, \gamma = 1,2,3,4$ for $su(2,2)$ spinor indices, $\underline{m}, \underline{n} = 0,1,2,3,4$ for world indices, $a, b, c = 0,1,2,3,4$ for tangent Lorentz $so(4,1)$ vector indices, and $A, B, C = 0,1,2,3,4,5$ for tangent $so(4,2)$ vector indices. We also use condensed notation for a set of symmetric spinor indices $\alpha(k) \equiv \alpha_1 \ldots \alpha_k$. Indices denoted by the same letter are assumed to be symmetrized as $X^{\alpha}Y^{\alpha} \equiv X^{\alpha_1}Y^{\alpha_2} + X^{\alpha_2}Y^{\alpha_1}$.

[3]The complex conjugation operation is defined by the rule $\overline{X}_{\alpha} = X^{\beta}C_{\beta\alpha}$, $\overline{Y}^{\alpha} = C^{\alpha\beta}Y_{\beta}$, where the bar denotes complex conjugation and $C^{\alpha\beta} = -C^{\beta\alpha}$ and $C_{\alpha\beta} = -C_{\beta\alpha}$ are some real matrices such that $C_{\alpha\gamma}C^{\beta\gamma} = \delta_{\alpha}{}^{\beta}$ [2].

To decompose representations (3.1) of the AdS$_5$ algebra $su(2,2)$ into representations of its Lorentz subalgebra, we use the compensator $V^{\alpha\beta}$. The result of the reduction is given by

$$\Omega^{\alpha(s_1+s_2-1)}{}_{\beta(s_1-s_2-1)}(x) = \sum_{t=0}^{s_1-s_2-1} \omega^{\alpha(s_1+s_2-1)\gamma(t),\gamma(s_1-s_2-t-1)}(x) V_{\beta(s_1-s_2-1),\gamma(s_1-s_2-1)}, \tag{3.4}$$

where the condensed notation $V_{\alpha(k),\beta(k)} \equiv V_{\alpha_1\beta_1}V_{\alpha_2\beta_2}\cdots V_{\alpha_k\beta_k}$ is introduced. The Lorentz algebra irreducible components

$$\omega^{\alpha(s_1+s_2+t-1),\beta(s_1-s_2-t-1)}(x), \quad 0 \le t \le s_1-s_2-1, \tag{3.5}$$

satisfy the Young symmetry condition

$$\omega^{\alpha(s_1+s_2+t-1),\alpha\beta(s_1-s_2-t-1)}(x) = 0, \tag{3.6}$$

and contractions with $V_{\alpha\beta}$ are zero,

$$\omega^{\alpha(s_1+s_2+t-1),\beta(s_1-s_2-t-1)}(x) V_{\alpha\beta} = 0. \tag{3.7}$$

According to the analysis in [1], [2], multispinors with $|m-n|=0$ correspond to totally symmetric spin-s_1 bosonic fields and are self-conjugate. Other fields with $|m-n| \ge 1$ are described by a pair of mutually conjugate multispinors and correspond either to totally symmetric fermionic spin-s_1 fields, $|m-n| = 1$ [3], [4], or to mixed-symmetry bosonic and fermionic fields, $|m-n| \ge 2$ [2], [4]. We note that mixed-symmetry gauge fields necessarily occur in the spectrum of $\mathcal{N}\geq 2$-extended five-dimensional higher-spin gauge superalgebras, while $\mathcal{N}\leq 1$ (super)algebras describe only totally symmetric fields [11].

To relate the spinor and (spinor)–tensor forms of mixed-symmetry field dynamics, we examine the $o(4,1)$ (spinor)–tensor cousins of multispinor fields (3.5)–(3.6) at $s_2 \neq 0$. The result is that a collection of $o(4,1)$ gauge fields is represented by complex-valued (spinor)–tensor fields of the form

$$\omega^{a(s_1-1),b(s_2+t)} = dx^{\underline{n}}\, \omega_{\underline{n}}{}^{a(s_1-1),b(s_2+t)}, \quad 0 \le t \le s_1-s_2-1, \tag{3.8}$$

for bosonic mixed-symmetry fields and

$$w^{\alpha|a(s_1-1),b(s_2+t)} = dx^{\underline{n}}\, w_{\underline{n}}{}^{\alpha|a(s_1-1),b(s_2+t)}, \quad \alpha = 1,2,3,4, \quad 0 \le t \le s_1-s_2-1, \tag{3.9}$$

for fermionic mixed-symmetry fields (α denotes a five-dimensional Dirac spinor index). In both cases, fields (3.8) and (3.9) have the Young symmetry property and are traceless (bosons) or gamma-transverse (fermions).

In accordance with the nomenclature in [6], fields (3.8) and (3.9) with the parameter $t \ge 1$ are called extra fields. Fermionic field (3.9) at $t=0$ is called the physical field. To classify the bosonic field in (3.8) with $t=0$, we decompose it into real and imaginary parts as

$$\omega^{a(s_1-1),b(s_2)} = \omega_1{}^{a(s_1-1),b(s_2)} + i\omega_2{}^{a(s_1-1),b(s_2)}. \tag{3.10}$$

Using the five-dimensional Levi-Civita symbol, one of the fields, ω_1 or ω_2, can be dualized into a field with one index in the third row, for example,

$$\omega^{a(s_1-1),b(s_2)} = \omega_1{}^{a(s_1-1),b(s_2)} + i\epsilon^{abcde}\omega_2{}^{a(s_1-2)}{}_{c,}{}^{b(s_2-1)}{}_{d,e}, \tag{3.11}$$

where the dual three-row field $\omega_2^{a(s_1-1),b(s_2),e}$ is traceless and has the Young symmetry property. We call the real part $\operatorname{Re}\omega^{a(s_1-1),b(s_2)}$ of field (3.11) the physical field. The imaginary part $\operatorname{Im}\omega^{a(s_1-1),b(s_2)}$ of this field is called an auxiliary field. In fact, any bosonic field (3.8) can be represented as a pair of real fields with one of them having one index in a third row. The resulting collection of real Lorentz-covariant fields is described by three-row $o(4,1)$ Young tableaux arising as a decomposition of a certain $o(4,2)$ three-row Young tableau [2], [6].

4. Higher-spin linearized curvatures

The analysis of linearized curvatures in this section is close to the analysis in the previous papers on totally symmetric fields [2], [3]. It turns out that the general form of gauge transformations for mixed-symmetry fields remains intact except for an additional dependence on the spin s_2 and the appearance of a nonzero operator $\mathcal{T}_0$ for bosonic nonsymmetric fields (see (4.18) and (4.20)). This last feature is not typical for bosonic systems and is a reflection of an implicit presence of the Levi-Civita symbol in the definition of real bosonic components of complex-valued fields (3.8). The calculation of the bosonic operator $\mathcal{T}_0$ is the main result in this section.

We introduce auxiliary commuting variables a_α and b^β transforming under the fundamental and the conjugate fundamental representations of $su(2,2)$. It is convenient to represent higher-spin fields (3.2) as functions of the auxiliary variables,

$$\Omega(a,b|x) = \Omega^{\alpha(s_1+s_2-1)}{}_{\beta(s_1-s_2-1)}(x) a_{\alpha(s_1+s_2-1)} b^{\beta(s_1-s_2-1)}, \tag{4.1}$$

where

$$a_{\alpha(m)} = a_{\alpha_1}\cdots a_{\alpha_m}, \qquad b^{\beta(n)} = b^{\beta_1}\cdots b^{\beta_n}. \tag{4.2}$$

The corresponding five-dimensional linearized higher-spin curvature is given by

$$R(a,b|x) = d\Omega(a,b|x) + \Omega_0{}^\alpha{}_\beta\left(b^\beta\frac{\partial}{\partial b^\alpha} - a_\alpha\frac{\partial}{\partial a_\beta}\right)\wedge\Omega(a,b|x), \tag{4.3}$$

where the background 1-form connection $\Omega_0{}^\alpha{}_\beta$ satisfies zero-curvature condition (2.4). The linearized (Abelian) higher-spin transformations are

$$\delta\Omega(a,b|x) = D_0\xi(a,b|x), \tag{4.4}$$

where the background covariant derivative is given by

$$D_0 = d + \Omega_0{}^\alpha{}_\beta\left(b^\beta\frac{\partial}{\partial b^\alpha} - a_\alpha\frac{\partial}{\partial a_\beta}\right). \tag{4.5}$$

Condition (2.4) implies that $\delta R(a,b|x) = 0$. The Bianchi identities have the form

$$D_0 R(a,b|x) = 0. \tag{4.6}$$

In the subsequent analysis, we use two sets of differential operators in the auxiliary variables [2],

$$S^- = a_\alpha\frac{\partial}{\partial b^\beta}V^{\alpha\beta}, \qquad S^+ = b^\alpha\frac{\partial}{\partial a_\beta}V_{\alpha\beta}, \qquad S^0 = N_b - N_a \tag{4.7}$$

and

$$T^- = \frac{1}{4}\frac{\partial^2}{\partial a_\alpha \partial b^\alpha}, \qquad T^+ = a_\alpha b^\alpha, \qquad T^0 = \frac{1}{4}(N_a + N_b + 4), \tag{4.8}$$

where

$$N_a = a_\alpha\frac{\partial}{\partial a_\alpha}, \qquad N_b = b^\alpha\frac{\partial}{\partial b^\alpha}. \tag{4.9}$$

With (4.7) and (4.8), the irreducibility conditions for $\Omega(a,b)$ are reformulated as

$$T^-\Omega(a,b) = 0, \qquad (S^0 + 2s_2)\Omega(a,b) = 0. \tag{4.10}$$

As demonstrated in Sec. 3, the higher-spin gauge field Ω decomposes into Lorentz subalgebra representations in accordance with formula (3.4). In terms of operators (4.7) and (4.8), formula (3.4) is rewritten as

$$\Omega(a,b|x) = \sum_{t=0}^{s_1+s_2-1} (S^+)^t \omega^t(a,b|x), \tag{4.11}$$

where

$$\omega^t(a,b|x) = \omega^{\alpha(s_1+s_2+t-1),\beta(s_1-s_2-t-1)}(x) a_{\alpha(s_1+s_2+t-1)} b_{\beta(s_1-s_2-t-1)} \tag{4.12}$$

are Lorentz-covariant gauge fields (3.5). The irreducibility conditions in (3.6) and (3.7) become

$$S^-\omega^t(a,b) = 0, \qquad T^-\omega^t(a,b) = 0. \tag{4.13}$$

Higher-spin gauge symmetry (4.4) requires the bosonic and fermionic Lorentz-covariant higher-spin curvatures r^t and gauge transformations to be given by

$$r^t = \mathcal{D}\omega^t + \mathcal{T}^-\omega^{t+1} + \lambda\mathcal{T}^0\omega^t + \lambda^2\mathcal{T}^+\omega^{t-1}, \tag{4.14}$$

$$\delta\omega^t = \mathcal{D}\xi^t + \mathcal{T}^-\xi^{t+1} + \lambda\mathcal{T}^0\xi^t + \lambda^2\mathcal{T}^+\xi^{t-1}, \tag{4.15}$$

where the 0-forms ξ^t are Lorentz-covariant gauge parameters and $\mathcal{D}$ is the background Lorentz-covariant derivative

$$\mathcal{D} = d + w_0{}^\alpha{}_\beta \left(a_\alpha \frac{\partial}{\partial a_\beta} + b_\alpha \frac{\partial}{\partial b_\beta} \right). \tag{4.16}$$

The operators $\mathcal{T}^-$, $\mathcal{T}^+$, and $\mathcal{T}^0$ have the forms

$$\mathcal{T}^+ = \left(1 - \frac{\Delta^2}{(S^0)^2}\right) h^\alpha{}_\beta\, a_\alpha \frac{\partial}{\partial b_\beta}, \tag{4.17}$$

$$\mathcal{T}^0 = -\frac{\Delta}{S^0} h^\alpha{}_\beta \left(b_\alpha \frac{\partial}{\partial b_\beta} - a_\alpha \frac{\partial}{\partial a_\beta} + \frac{2}{S^0-2} \left(b_\gamma \frac{\partial}{\partial a_\gamma} \right) a_\alpha \frac{\partial}{\partial b_\beta} \right), \tag{4.18}$$

$$\begin{aligned} \mathcal{T}^- = \frac{1}{1-S^0} h^\alpha{}_\beta \Bigg(&(2-S^0) b_\alpha \frac{\partial}{\partial a_\beta} + b_\gamma \frac{\partial}{\partial a_\gamma} \left(b_\alpha \frac{\partial}{\partial b_\beta} - a_\alpha \frac{\partial}{\partial a_\beta} \right) + \\ &+ \frac{1}{S^0-3} \left(b_\gamma \frac{\partial}{\partial a_\gamma} \right)^2 a_\alpha \frac{\partial}{\partial b_\beta} \Bigg), \end{aligned} \tag{4.19}$$

where the parameter Δ takes the values

$$\Delta = \begin{cases} 2s_2 & \text{for bosons,} \\ 2s_2+1 & \text{for fermions,} \end{cases} \tag{4.20}$$

and satisfy the relations

$$\begin{aligned} &\{\mathcal{T}^0, \mathcal{T}^-\} = \{\mathcal{T}^0, \mathcal{T}^+\} = 0, \\ &(\mathcal{T}^-)^2 = 0, \qquad (\mathcal{T}^+)^2 = 0, \\ &\mathcal{D}^2 + \lambda^2\{\mathcal{T}^-, \mathcal{T}^+\} + \lambda^2(\mathcal{T}^0)^2 = 0. \end{aligned} \tag{4.21}$$

We note that the coefficients in (4.17)–(4.19) can be changed by field redefinitions of the form $\widetilde{\omega}^t = C(t,s)\omega^t$ with $C \neq 0$.

5. Higher-spin action

Before considering actions for arbitrary mixed-symmetry gauge fields, we examine the case of the simplest nonsymmetric bosonic field of spin $(2, 1)$ described by a 1-form $\Omega^{\alpha(2)}(x)$. Up to total derivative terms, the action functional has the unique form

$$\mathcal{S}_2^{(2,1)} = \int_{\mathcal{M}^5} h^{\alpha}{}_{\beta} \wedge R^{\beta\gamma} \wedge \overline{R}_{\gamma\alpha}, \tag{5.1}$$

where $h^{\alpha}{}_{\beta}$ is the background AdS_5 frame field and the curvature is

$$R^{\alpha(2)} = D_0\Omega^{\alpha(2)} \equiv \mathcal{D}\Omega^{\alpha(2)} + \lambda h^{\alpha}{}_{\gamma} \wedge \Omega^{\gamma\alpha}. \tag{5.2}$$

The equations of motion resulting from action (5.1) are

$$H_2{}^{\alpha}{}_{\gamma} \wedge R^{\gamma\beta} + H_2{}^{\beta}{}_{\gamma} \wedge R^{\gamma\alpha} = 0 \tag{5.3}$$

plus the complex-conjugate equations for $\overline{\Omega}_{\alpha\beta}$. We note that these bosonic equations are of the first order, which makes them similar to fermionic equations. But as discussed in Sec. 3, the real and imaginary parts of the complex-valued field $\Omega^{\alpha(2)}(x)$ are regarded as physical and auxiliary fields (3.11), with the auxiliary field being expressed by virtue of its equation of motion in terms of first derivatives of the physical field. To describe this mechanism in more detail, we consider the tensor form of action (5.1). According to (3.10), the $o(4, 1)$ field isomorphic to $\Omega^{\alpha(2)}(x)$ is

$$\omega^{[ab]} = \omega_1^{ab} + i\omega_2^{ab}. \tag{5.4}$$

The corresponding linearized curvature and gauge transformations have the forms

$$R^{ab} = \mathcal{D}\omega^{ab} - \frac{i\lambda}{2}\epsilon^{abcde}h_c \wedge \omega_{de}, \qquad \delta\omega^{ab} = \mathcal{D}\xi^{ab} - \frac{i\lambda}{2}\epsilon^{abcde}h_c\xi_{de}, \tag{5.5}$$

where $\mathcal{D}$ is the background Lorentz-covariant derivative, ξ^{ab} is a 0-form complex gauge parameter, and h^a is the background frame field. We note that the terms in (5.5) involving the Levi-Civita symbol are in fact the operator $\mathcal{T}^0$ expressed in the spinor notation by formula (4.18). The Bianchi identities are

$$\mathcal{D}R^{ab} - \frac{i\lambda}{2}\epsilon^{abcde}h_c \wedge R_{de} = 0. \tag{5.6}$$

The action has a form analogous to (5.5),

$$\mathcal{S}_2^{(2,1)} = \int_{\mathcal{M}^5} \epsilon_{abcde}h^e \wedge R^{ab} \wedge \overline{R}^{cd}, \tag{5.7}$$

where $\overline{R}^{cd}$ is complex-conjugate curvature (5.5). The equations of motion are

$$H_a{}^c \wedge R_{cb} - H_b{}^c \wedge R_{ca} = 0, \qquad H_{ab} \overset{\mathrm{def}}{=} h_a \wedge h_b \tag{5.8}$$

plus the complex-conjugate equations.

To clarify the dynamical content of these equations, we regard the real or imaginary part of the field ω^{ab} given by (5.4) as a dualized auxiliary field, for example,

$$\omega_1^{ab} = \omega_1^{ab}, \qquad \omega_2^{ab} = \frac{1}{\lambda}\epsilon^{abcde}\omega_{2\,cde}, \tag{5.9}$$

where ω_1^{ab} and ω_2^{abc} are the physical and auxiliary fields with antisymmetric indices and the factor λ^{-1} is introduced to express the fact that the mass dimensions of physical and auxiliary fields are different. These fields can be unified into a single $o(4,2)$ field $\Omega^{[ABC]}$ [6]. It can be shown that action (5.7) can be rewritten as

$$\mathcal{S}_2^{(2,1)} = \frac{1}{\lambda^2}\int_{\mathcal{M}^d} \epsilon_{ABCDEF} h^E V^F \wedge R^{ABM} \wedge R^{CDN} V_M V_N. \tag{5.10}$$

In this form, the action coincides with the action for the AdS_d "hook" field explicitly studied in [6]. We note that the flat limit of action (5.7) (or, equivalently, (5.10)) yields a dual description of the spin-2 field, which precisely corresponds to Curtright's action [17]. Another comment is that the described procedure for unifying dynamical and auxiliary fields into a single complex-valued field was used to study the so-called odd-dimensional self-duality for massive antisymmetric tensor fields in Minkowski space [18].

Action for nonsymmetric $\mathbf{AdS_5}$ gauge fields. In what follows, we construct free actions describing mixed-symmetry bosonic and fermionic gauge fields in AdS_5. The case of totally symmetric fields was considered in [2], [3].

As in [2], [3], [6], we seek mixed-symmetry field action functionals in the form

$$\mathcal{S}_2^{(s_1,s_2)} = \int_{\mathcal{M}^5} \widehat{H} \wedge R^{s_1,s_2}(a_1,b_1) \wedge \overline{R}^{s_1,s_2}(a_2,b_2)|_{a_i=b_i=0}, \tag{5.11}$$

where R^{s_1,s_2} is linearized higher-spin curvature (4.3) and $\widehat{H}$ is the 1-form differential operator

$$\widehat{H} = \left(\alpha(p,q) h_{\alpha\beta} \frac{\partial^2}{\partial a_{1\alpha}\partial a_{2\beta}} \hat{b}_{12} + \beta(p,q) h^{\alpha\beta} \frac{\partial^2}{\partial b_1^\alpha \partial b_2^\beta} \hat{a}_{12} + \right.$$
$$\left. + \gamma(p,q) h_\alpha{}^\beta \frac{\partial^2}{\partial a_{2\alpha}\partial b_1^\beta} \hat{c}_{12} + \zeta(p,q) h_\alpha{}^\beta \frac{\partial^2}{\partial a_{1\alpha}\partial b_2^\beta} \hat{c}_{21}\right)(\hat{c}_{12})^{2s_2}. \tag{5.12}$$

Here, $h_\alpha{}^\beta$ is the background frame field, and the coefficients α, β, γ, and ζ are functions of the operators

$$p = \hat{a}_{12}\hat{b}_{12}, \qquad q = \hat{c}_{12}\hat{c}_{21}, \tag{5.13}$$

where

$$\hat{a}_{12} = V_{\alpha\beta} \frac{\partial^2}{\partial a_{1\alpha}\partial a_{2\beta}}, \qquad \hat{b}_{12} = V^{\alpha\beta} \frac{\partial^2}{\partial b_1^\alpha \partial b_2^\beta},$$
$$\hat{c}_{12} = \frac{\partial^2}{\partial a_{1\alpha}\partial b_2^\alpha}, \qquad \hat{c}_{21} = \frac{\partial^2}{\partial a_{2\alpha}\partial b_1^\alpha}. \tag{5.14}$$

These functions are responsible for various types of index contractions between the frame field and the curvatures. The action is invariant under complex conjugation $\overline{\mathcal{S}}_2 = \mathcal{S}_2$ when the coefficients α, β, γ, and ζ are real.

Because the general variation of the linearized curvatures is $\delta R = D_0 \delta\Omega$ and because the action is formulated in an AdS_5 covariant way, integrating by parts yields the variation

$$\delta\mathcal{S}_2^{(s_1,s_2)} = \int_{\mathcal{M}^5} D_0 \widehat{H} \wedge \delta\Omega(a_1,b_1) \wedge \overline{R}(a_2,b_2)|_{a_i=b_i=0} + \text{c.c.} \tag{5.15}$$

The derivative D_0 produces the frame field each time it hits the compensator $D_0 V^{\alpha\beta} = h^{\alpha\beta}$. Taking $D_0 h^{\alpha\beta} = 0$, $h_\alpha{}^\beta = h_{\alpha\gamma} V^{\beta\gamma}$ into account and using the notation $H^{\alpha\beta} = H^{\beta\alpha} = h^\alpha{}_\gamma \wedge h^{\beta\gamma}$, we find

$$D_0\widehat{H} = \left(\rho_1 H_\alpha{}^\beta \frac{\partial^2}{\partial a_{2\alpha}\partial b_1{}^\beta} \hat{c}_{12} + \rho_2 H_\alpha{}^\beta \frac{\partial^2}{\partial a_{1\alpha}\partial b_2{}^\beta} \hat{c}_{21} + \right.$$
$$\left. + \rho_3 H_{\alpha\beta} \frac{\partial^2}{\partial a_{1\alpha}\partial a_{2\beta}} \hat{b}_{12} + \rho_3 H^{\alpha\beta} \frac{\partial^2}{\partial b_1^\alpha \partial b_2^\beta} \hat{a}_{12}\right)(\hat{c}_{12})^{2s_2}, \tag{5.16}$$

where

$$\rho_1 = \frac{1}{2}\left(1 + p\frac{\partial}{\partial p}\right)(-2\gamma(p,q) + (\alpha+\beta)(p,q)),$$
$$\rho_2 = \frac{1}{2}\left(1 + p\frac{\partial}{\partial p}\right)(-2\zeta(p,q) + (\alpha+\beta)(p,q)), \tag{5.17}$$
$$\rho_3 = \frac{1}{2}q\frac{\partial}{\partial p}(\zeta(p,q) - \gamma(p,q)).$$

For the trivial solution $\rho_i = 0$, the covariant derivative of $\widehat{H}$ vanishes, $D_0\widehat{H} = 0$, and the corresponding action functional is a total derivative. It follows from (5.17) that $\rho_i = 0$ whenever

$$(\alpha+\beta)(p,q) = 2\gamma(p,q), \qquad \zeta(p,q) = \gamma(p,q). \tag{5.18}$$

Clearly, by adding total derivatives with the coefficients satisfying (5.18), we can always set $\gamma = 0$ and $\beta = 0$ in action (5.11), (5.12).

Generally, action (5.11) does not describe massless higher-spin fields, because there are too many nonphysical dynamical variables associated with the extra fields. To eliminate the corresponding degrees of freedom, we must fix the operator $\widehat{H}$ in an appropriate form by virtue of the *decoupling condition* [6], [19]. It requires the variation of the quadratic action with respect to the extra fields to be identically zero,

$$\frac{\delta\mathcal{S}_2^{(s_1,s_2)}}{\delta\omega^{t>0}} \equiv 0. \tag{5.19}$$

To analyze the extra field decoupling condition, we observe that all gauge fields of the extra type can be combined into a single irreducible $su(2,2)$ tensor $\xi(a,b)$ satisfying $(N_a - N_b - 2s_2 - 2)\xi(a,b) = 0$. Then the variation of the extra fields becomes

$$\delta\Omega^{\mathrm{extra}}(a,b) = S^+\xi(a,b), \tag{5.20}$$

and the extra field decoupling condition (5.19) amounts to

$$\left(\frac{\partial}{\partial p} - \frac{\partial}{\partial q}\right)(q\rho_2) + \rho_3 = 0,$$
$$\left(\frac{\partial}{\partial p} - \frac{\partial}{\partial q}\right)\rho_3 = 0, \tag{5.21}$$
$$\rho_1 + \rho_3 = 0.$$

Modulo total derivative contributions (5.18), the general solution of system (5.21) is

$$\gamma(p,q) = 0, \qquad \beta(p,q) = 0, \qquad \zeta(p,q) = \zeta^{(0)}\frac{(p+q)^{s_1-s_2-1}}{q},$$
$$\alpha(p,q) = -\zeta^{(0)}(s_1 - s_2 - 1)\int_0^1 d\tau\,(p\tau + q)^{s_1-s_2-2} = \tag{5.22}$$
$$= -\zeta^{(0)}\sum_{k=0}^{s_1-s_2-2}\frac{(s_1-s_2-1)!}{(k+1)!(s_1-s_2-k-2)!}p^k q^{s_1-s_2-k-2}.$$

The factor q^{-1} appearing in $\zeta(p,q)$ can be removed by redefining $\zeta(p,q) \to q\zeta(p,q)$. This operation does remove the singularity because the last term in the operator $\widehat{H}$ in (5.12) contains the combination $\hat{c}_{21}(\hat{c}_{12})^{2s_2}$, which is always $q(\hat{c}_{12})^{2s_2-1}$ by the definition of q in (5.14). An overall factor $\zeta^{(0)}$ in front of the action of a given spin (s_1, s_2) cannot be fixed from the analysis of the free action and represents the residual ambiguity in the coefficients.

6. Equations of motion and constraints

To obtain equations of motion, we rewrite the nontrivial part of variation (5.15) as

$$\delta\mathcal{S}_2^{(s_1,s_2)} = -\frac{\zeta^{(0)}(s_1-s_2-1)}{2}\int(p+q)^{s_1-s_2-2}\biggl(\frac{(s_1-s_2+1)q+2(s_1-s_2)p}{s_1-s_2-1}\times$$

$$\times H_\alpha{}^\beta\frac{\partial^2}{\partial a_{1\alpha}\partial b_2{}^\beta}+H_\alpha{}^\beta\frac{\partial^2}{\partial a_{2\alpha}\partial b_1{}^\beta}(\hat{c}_{12})^2-H_{\alpha\beta}\frac{\partial^2}{\partial a_{1\alpha}\partial a_{2\beta}}\hat{b}_{12}\hat{c}_{12}-$$

$$-H^{\alpha\beta}\frac{\partial^2}{\partial b_1^\alpha\partial b_2^\beta}\hat{a}_{12}\hat{c}_{12}\biggr)(\hat{c}_{12})^{2s_2-1}\wedge r^0(a_1,b_1)\wedge\delta\overline{\omega}^0(a_2,b_2)+\text{c.c. part.} \tag{6.1}$$

Substituting the fields

$$\begin{aligned} r^0(a_1,b_1) &= r^{0\,\alpha(s_1+s_2-1),}{}_{\beta(s_1-s_2-1)}a_{1\,\alpha(s_1+s_2-1)}b_1^{\beta(s_1-s_2-1)},\\ \overline{\omega}^0(a_2,b_2) &= \overline{\omega}^0{}_{\gamma(s_1+s_2-1),}{}^{\rho(s_1-s_2-1)}a_{2\,\rho(s_1-s_2-1)}b_2^{\gamma(s_1+s_2-1)} \end{aligned} \tag{6.2}$$

in variation (6.1) and using their Young symmetry properties

$$\begin{aligned} &Y_1\equiv S_1^-: \quad Y_1 r^0(a_1,b_1)=0,\\ &Y_2\equiv S_2^+: \quad Y_2\overline{\omega}^0(a_2,b_2)=0, \end{aligned} \tag{6.3}$$

we obtain equations of motion that can be conveniently written as

$$\widehat{E}\wedge r^0(a,b)=0, \tag{6.4}$$

where $\widehat{E}$ is a 2-form differential operator given by

$$\widehat{E}=H^\alpha{}_\beta\biggl(a_\alpha\frac{\partial}{\partial a_\beta}+\kappa_2 b_\alpha\frac{\partial}{\partial b_\beta}+\kappa_3 S^+a_\alpha\frac{\partial}{\partial b_\beta}+\kappa_4 T^+\frac{\partial^2}{\partial a^\alpha\partial b_\beta}+\kappa_5 T^+S^+\frac{\partial^2}{\partial b^\alpha\partial b_\beta}\biggr) \tag{6.5}$$

with the coefficients

$$\kappa_2=\frac{1+(s_1+s_2-1)(s_2+1)}{1-(s_1-s_2+1)(s_2+1)}, \qquad \kappa_3=-\kappa_4=\frac{1-\kappa_2}{2(s_2+1)}, \qquad \kappa_5=\frac{\kappa_2-1}{4s_1(s_2+1)}. \tag{6.6}$$

Analogous equations hold for the complex-conjugate physical field $\overline{\omega}^0$. The operator $\widehat{E}$ satisfies the conditions

$$[S^-,\widehat{E}]=0, \qquad [T^-,\widehat{E}]=0, \tag{6.7}$$

i.e., preserves the Young symmetry and V-transversality properties of the physical curvature r^0. By construction, this operator also satisfies the extra field decoupling condition, which means that the term $\mathcal{T}^-\omega^1$ containing the extra field ω^1 in the curvature $r^0=\mathcal{D}\omega^0+\mathcal{T}^0\omega^0+\mathcal{T}^-\omega^1$ does not contribute to the equations of motion, i.e., $\widehat{E}\wedge\mathcal{T}^-\omega^1=0$.

As in the papers on totally symmetric fields [19], we assume that the constraints for extra fields have the form

$$\Upsilon_2^+\wedge r^t(a,b)=0, \quad 0\le t<s_1-s_2-1, \tag{6.8}$$

where Υ_2^+ is the 2-form operator that increases t and satisfies the condition

$$\mathcal{T}^+ \wedge \Upsilon_2^+ = 0. \tag{6.9}$$

The operator Υ_2^+ is required to have property (6.9) because it ensures that the number of independent algebraic relations imposed on the curvature r^t coincides with the number of components of extra fields $\omega^{t>0}$ modulo pure gauge components of the form $\delta\omega^{t+1} = \mathcal{T}^-\epsilon^{t+2}$. It can be shown that the operator Υ_2^+ is uniquely fixed in the form

$$\Upsilon_2^+ = \mathcal{T}^0 \wedge \mathcal{T}^+. \tag{6.10}$$

By virtue of constraints (6.8), the field ω^{t+1} can be expressed via derivatives of ω^t for any $t > 0$. Finally, we can obtain the fields ω^t expressed in terms of the derivatives of ω^0 with the highest derivative order equal to t.

7. Conclusion

We have constructed a manifestly covariant Lagrangian formulation for AdS_5 mixed-symmetry massless gauge fields in the framework of the $su(2,2)$ spinor formalism. The approach we used is based on the frame-like formulation of mixed-symmetry fields elaborated in [6], [7]. Our results can be regarded as the final step in the study of the manifestly covariant Lagrangian formulation of AdS_5 higher-spin gauge fields in the $su(2,2)$ formalism. An important problem for further research is to develop the unfolded form of free mixed-symmetry field dynamics based on the Weyl tensors following from the equations of motion and constraints for extra fields analyzed in Sec. 6. This will allow formulating the central on-mass-shell theorem similarly to the case of totally symmetric gauge fields [2], [19] and establishing a relation with the unfolded formulation of mixed-symmetry fields developed in [4]. Also, the constructed Lagrangian formulation allows studying $\mathcal{N}$-extended supersymmetric cubic interactions of AdS_5 gauge fields at the level of action functionals, thus generalizing the $\mathcal{N}=0,1$ results in [2], [5].

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