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ANALYTIC PROPERTIES OF THE *S*-MATRIX FOR INTERACTIONS WITH YUKAWA-POTENTIAL TAILS

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We give an explicit analytic expression for the S-matrix in the case of an arbitrary central interaction inside a sphere of finite radius with a Yukawa-potential tail at large distances. The method uses the completeness of the wave functions outside the finite sphere and also the unitarity and the symmetry conditions for the S-matrix.

Keywords: S-matrix, centrally symmetric potential, Yukawa potential

1. Introduction

The analytic structure of the nonrelativistic one-channel S-matrix is now known rather completely for a wide class of spherical interactions (see, e.g., [1]–[5]). It seems that the case of interactions with Yukawa-potential tails, for which the explicit analytic representations of the S-matrix is unknown, is the only essential exception. Indeed, this case is characteristic for nuclear physics, both for nucleon–nucleon and nucleon–nucleus interactions. Our goal in this paper is to fill this gap to some extent based on the method developed in [3]–[5].

2. Basic formulas and general *S*-matrix properties

We assume that the interaction between two colliding particles outside a sphere of finite radius a is described by a central Yukawa potential

$$V = V_0\big[(br)^{-1}e^{-br}\big], \quad V_0 < 0, \quad b^{-1} \sim a, \tag{1}$$

while it has a given arbitrary form inside the sphere.

The radial scattering wave function $R_l^{(+)}$ with outgoing-wave boundary conditions for the partial wave l, the wave number k, and $r > a$ can be written as

$$R_l^{(+)}(k,r) = \frac{i}{2kr}\big[f_{l-}(k,r)e^{il\pi/2} - S_l(k)f_{l+}(k,r)e^{-il\pi/2}\big]. \tag{2}$$

The radial wave function for a bound state R_l^n in the same region can be written in the form

$$R_l^n(k_{nl},r) = (2\pi)^{-1/2}\frac{B_l(k_{nl})f_{l+}(k_{nl},r)}{r}, \tag{3}$$

where k_{nl} is the corresponding eigenvalue and B_l is the so-called bound-state constant. The S-matrix or, more precisely, the function $S_l(k)$ in Eq. (2) satisfies the (extended) unitarity condition

$$S_l(k)S_l^*(k^*) = 1 \tag{4}$$

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and the symmetry condition

$$S_l(k)S_l(-k) = 1 \tag{5}$$

(see, e.g., [1]–[5]). The definitions and some properties of the functions $f_{l\pm}(k,r)$ are given in [1]. All the information on the interaction for $r < a$ is contained in the functions $S_l(k)$, constants $B_l(k_{nl})$, and numbers k_{nl}.

We now consider the analytic properties of $f_{l\pm}(k,r)$ for Yukawa potentials (1). First, according to [1], for the case $l = 0$, we have

$$f_{0\pm}(k,r) = \left[1 + \int_b^\infty S_\pm(b',k)e^{-b'r}\,db'\right]e^{\pm ikr}, \tag{6}$$

where $S_\pm(b',k)$ is the solution of the equation

$$b(b \mp 2ik)S_\pm(b,k) = \rho + \int_b^\infty \rho S_\pm(b',k)\,db', \tag{7}$$

where $\rho = 2\mu V_0/(\hbar^2 b^3) < 0$ and μ is the reduced mass. The solution of Eq. (7) can be written as

$$S_\pm(b,k) = \rho\left\{b(b \mp 2ik)\left[1 \mp \frac{i\rho}{2k}\log\left(1 \mp \frac{2ik}{b}\right)\right]^{-1}\right\}, \tag{8}$$

and we obtain

$$f_{0\pm}(k,r) = \left\{1 + \rho\left[1 \mp \frac{i\rho}{2k}\log\left(1 \mp \frac{2ik}{b}\right)\right]^{-1}\int_b^\infty \frac{e^{-b'r}\,db'}{b'(b' \mp 2ik)}\right\}e^{\pm ikr} \tag{6a}$$

from Eq. (6).

In Eq. (8), there is a logarithmic singularity of $f_{0\pm}(k,r)$ at the point $k = k_\gamma = ib/2$. We consider the analytic properties of the factor

$$A_- = \left[1 + \frac{i\rho}{2k}\log\left(1 + \frac{2ik}{b}\right)\right]^{-1}. \tag{9}$$

It is easy to verify that for complex values of k with both $\mathrm{Re}\,k$ and $\mathrm{Im}\,k$ nonzero, the factor A_-^{-1} has no zeros, and A_- therefore has no poles. The same result is obtained for real k. Further, we set $k = ix$, where x is real, and rewrite Eq. (9) in the form

$$A_- = \left[1 + \frac{\rho}{2x}\log\left(1 - \frac{2x}{b}\right)\right]^{-1}.$$

In the case $2x/b > 1$, the factor A_-^{-1} has no zeros because the logarithm is complex, and A_- therefore has no poles. In the case $0 \le 2x/b < 1$ for $\rho < 0$, as in the case of the long-range part of the nuclear forces, A_- again has no poles. Finally, in the case $2x/b \le 0$, poles can exist, but they must be located in the lower half of the complex k plane.

In conclusion, we note that the factor A_- contains no additional singularities in the upper half of the complex k plane except a branch point at $k_\gamma = ib/2$.

The treatment can be extended to higher angular momenta $l > 0$. The same logarithmic singularity at the point $k = k_\gamma = ib/2$ also appears in $f_{l-}(k,r)$. To show this, the following integral equation, which allows computing $f_{l-}(k,r)$ from $f_{0-}(k,r)$, can be used:

$$f_{l-}(k,r) = f_{0-}(k,r) + l(l+1)\int_r^\infty G(k;r,r')(r')^{-2}f_{l-}(k,r')\,dr', \tag{10}$$

where $r > a$ and the Green's function $G(k;r,r')$ has the form [1]

$$G(k;r,r') = (2ik)^{-1}\big[f_{0-}(k,r)f_{0+}(k,r') - f_{0-}(k,r')f_{0+}(k,r)\big] = \\ = \big[f_{0+}(k,0)\big]^{-1}\big[\Phi(k,r)f(k,r') - \Phi(k,r')f(k,r)\big]. \tag{11}$$

Because the function $\Phi(k,r)$ is regular everywhere, the Green's function in Eq. (11) has no singularity at the point k. The solution $f_{l-}(k,r)$ of Eq. (10) for any value of l contains the same logarithmic singularity as the function $f_{0-}(k,r)$ at the point $k_\gamma = ib/2$.

After these preliminary considerations, we investigate the analytic structure of the S-matrix $S_l(k)$. For this, as in [3], [5], we use the completeness condition for the wave function in the range $r \ge a$. We assume that in this external region, the colliding particle can be described using the Schrödinger equation with a single Yukawa potential (with the possible addition of a potential decreasing faster than any exponential). Therefore, we can write

$$\frac{2}{\pi}\int_0^\infty k^2 R_l^{(+)}(k,r)R_l^{(+)*}(k,r')\,dk + \sum_n R_l^{(n)}(k_{nl},r)R_l^{(n)*}(k_{nl},r') = r^{-2}\delta(r-r') \tag{12}$$

for $r, r' > a$. Using the well-known property [1]

$$f_{l+}^*(k,r) = f_{l-}(k,r), \qquad f_{l-}^*(k,r) = f_{l+}(k,r), \tag{13}$$

valid for real k, and conditions (4) and (5), we can write Eq. (12) in the form

$$\frac{1}{2r'}\int_C f_{l-}(k,r)f_{l+}(k,r')\,dk - \frac{(-1)^l}{rr'}\int_C S_l(k)f_{l+}(k,r)f_{l+}(k,r')\,dk + \\ + \frac{1}{rr'}\sum_n \big(B_l(k_{nl})\big)^2 f_{l+}(k_{nl},r)f_{l+}(k_{nl},r') = \frac{2\pi\delta(r-r')}{r^2}. \tag{14}$$

The integration path C in Eq. (14) goes along the real axis from $-\infty$ to ∞ except near the origin $k = 0$, where it follows a semicircle of infinitely small radius in the upper half-plane D^+. At the point $k = 0$, the functions $f_{l\pm}(k,r)$ have poles of order l.

Because for the chosen potential in the external region ($r > a$), the functions $f_{l\pm}(k,r)$ have the limits $e^{\pm ikr}$ for $|k| \to \infty$ in all directions in the complex plane, we can deform the integration path inside D^+ to another path composed of two pieces: an arbitrarily large semicircle above the real axis and a set of contours inside D^+ going around all the existing singularities, either cuts or poles, as close to them as is convenient. After such a deformation, using the identities

$$\begin{aligned} &\int_{\Gamma^+} f_{l-}(k,r)f_{l+}(k,r')\,dk = \int_{\Gamma^+} e^{ik(r'-r)}\,dk = \int_{-\infty}^{\infty} e^{ik(r'-r)}\,dk = 2\pi\delta(r-r'), \\ &\int_{\Gamma^+} S_l(k)f_{l+}(k,r)f_{l+}(k,r')\,dk = \int_{\Gamma^+} S_l(k)e^{ik(r'+r)}\,dk, \end{aligned} \tag{15}$$

we obtain [5]

$$\sum_m \oint_{k_m} f_{l-}(k,r)f_{l+}(k,r')\,dk + \sum_p \oint_{\gamma_p} f_{l-}(k,r)f_{l+}(k,r')\,dk - \\ - (-1)^l \sum_s \oint_{k_s} S_l(k)f_{l+}(k,r)f_{l+}(k,r')\,dk - (-1)^l \sum_q \oint_{\gamma_q} S_l(k)f_{l+}(k,r)f_{l+}(k,r')\,dk - \\ - (-1)^l \int_{\Gamma^+} S_l(k)e^{ik(r'+r)}\,dk + \sum_n \big(B_l(k_{nl})\big)^2 f_{l+}(k_{nl},r)f_{l+}(k_{nl},r') = 0. \tag{16}$$

Here, Γ^+ is an infinitely large semicircle above the real axis, k_m and k_s denote isolated singularities, and γ_p and γ_q are contours encircling nonisolated singularities or going along the edges of a cut in the case of branch points. Because the functions $f_{l+}(k,r)$ and e^{ikr} are independent as functions of r at different points k_{nl}, k_m, k_s, γ_p, and γ_q (singularities for $f_{l-}(k,r)$ and $S_l(k)$) in D^+, Eq. (16) is equivalent to the set of equations

$$-(-1)^l \oint_{k_{nl}} S_l(k) f_{l+}(k,r) f_{l+}(k,r')\,dk = \big(B_l(k_{nl})\big)^2 f_{l+}(k_{nl},r) f_{l+}(k_{nl},r'), \tag{17}$$

$$(-1)^l \oint_{k_m} S_l(k) f_{l+}(k,r) f_{l+}(k,r')\,dk = \oint_{k_m} f_{l-}(k,r) f_{l+}(k,r')\,dk, \tag{18}$$

$$(-1)^l \oint_{\gamma_p} S_l(k) f_{l+}(k,r) f_{l+}(k,r')\,dk = \oint_{\gamma_p} f_{l-}(k,r) f_{l+}(k,r')\,dk, \tag{19}$$

$$\int_{\Gamma^+} S_l(k) e^{ik(r'+r)}\,dk = 0, \tag{20}$$

where $r, r' \geq a$. According to the residue theorem, we can conclude from Eq. (17) that in D^+, $S_l(k)$ can have first-order poles corresponding to the bound states and having residues equal to $(-1)^{l+1} i (2\pi)^{-1} (B_{nl})^2$. Equation (18) can be rewritten as

$$\oint_{k_m} f_{l+}(k,r) f_{l+}(k,r') \left[\frac{f_{l-}(k,r')}{f_{l+}(k,r')} - (-1)^l S_l(k) \right] dk = 0,$$

whence we can conclude that in D^+, $S_l(k)$ can have additional isolated singularities coinciding with the isolated singularities of $f_{l-}(k,r)$ in D^+, near which

$$\lim_{k\to k_m} f_{l-}(k,r) = \lim_{k\to k_m} D_m(k) f_{l+}(k,r). \tag{21}$$

The function $D_m(k)$ is independent of r and has an isolated singularity at the point k_m. These results were obtained in [1] using a different method. In Appendix 1, we present the English translation of a paper by Olkhovsky and Tsekhmistrenko [6] (where similar results were presented concerning poles for interactions that are more general than those in [1]); it was previously published in an Ukrainian edition, which had not been translated into Russian and English and was hence previously unknown to readers not knowing Ukrainian.

We now consider Eq. (19) near the logarithmic singularities of $f_{l-}(k,r)$ in D^+. The contour γ_q can be chosen in the form shown in Fig. 1. It consists of the almost closed circle γ_{acc} around $k_\gamma = ib/2$ with the small radius $\varepsilon \equiv (\varepsilon/b)b$ and the two infinite lines γ_{edge} along the edges of the cut with a much smaller distance between them given by $(\varepsilon/b)^\delta b$, $\delta > 2$, i.e., $\gamma_q = \gamma_{\text{acc}} + \gamma_{\text{edge}}$. We let γ_{12} denote the segment joining the lowest points 1 and 2 of the two lines (see Fig. 1). Letting γ_{c} denote the closed contour formed by the almost closed circle and the segment, we can write the identity

$$\int_{\gamma_{\text{acc}}} = \oint_{\gamma_{\text{c}}} - \int_{\gamma_{12}} \quad \underset{\varepsilon\to 0}{\longrightarrow} \quad \oint_{\gamma_{\text{c}}} \tag{22}$$

for the integrals in Eq. (19). Because the length of γ_{12} is $(\varepsilon/b)^\delta b$, the integrals over γ_{12} and over γ_{edge} vanish as $O(\varepsilon^{\delta-2})$ as $\varepsilon \to 0$. Therefore, only the contour integral over the closed circle γ_{c} centered at the point k_γ remains.

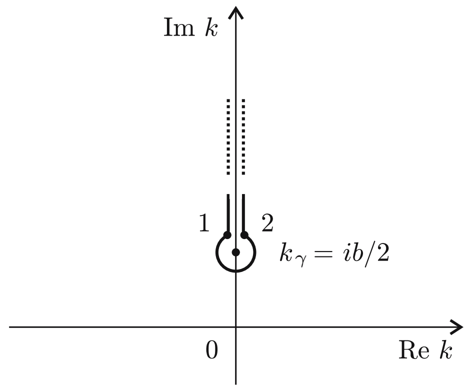

Fig. 1. A form of the contour γ_q.

We consider the integral over γ_c in detail. The value of the integral is determined by the behavior of the integrand as the radius of the circle tends to zero. Therefore, we consider the limit

$$\lim_{k\to k_\gamma}\left[1+\frac{i\rho}{2k}\log\left(1+\frac{2ik}{b}\right)\right]^{-1}\int_b^\infty \frac{e^{(ik-b')r}}{b'(b'+2ik)}\,db'. \tag{23}$$

It is easy to show that the integral over the variable b' in this expression has a logarithmic divergence at k_γ that cancels when the corresponding factor vanishes, and the function $f_{0-}(k,r)$ therefore has no pole at k_γ. Explicitly evaluating limit (23) shows that in the vicinity of k_γ, the function $f_{0-}(k,r)$ can be written as (see Appendix 2)

$$f_{0-}(k,r)\to W(k,r)+\left[1+\frac{i\rho}{2k}\log\left(1+\frac{2ik}{b}\right)\right]^{-1}U(k,r), \tag{24}$$

where the functions W and U are analytic functions of k at the point k_γ and inside the small closed circle γ_c. Equation (19) can therefore be rewritten as

$$\begin{aligned}\oint_{\gamma_c} S_0(k)f_{0+}(k,r)f_{0+}(k,r')\,dk &= \oint_{\gamma_c} f_{0+}(k,r)f_{0-}(k,r')\,dk = \\ &= \oint_{\gamma_c} W(k,r)f_{0+}(k,r')\,dk+ \\ &+\left[1+\frac{i\rho}{2k}\log\left(1+\frac{2ik}{b}\right)\right]^{-1}\oint_{\gamma_c} U(k,r)f_{0+}(k,r')\,dk.\end{aligned} \tag{25}$$

Because each integral in the right-hand side vanishes, we can conclude that

$$\oint_{\gamma_c} S_0(k)f_{0+}(k,r)f_{0+}(k,r')\,dk = 0. \tag{26}$$

It hence follows that $S_0(k)$ can contain at most a singular factor of the type

$$F=\left[1-\frac{i\rho}{2k}\log\left(1-\frac{2ik}{b}\right)\right]\left[1+\frac{i\rho}{2k}\log\left(1+\frac{2ik}{b}\right)\right]^{-1} \tag{27}$$

connected with the analogous logarithmic branch points at $k=k_\gamma$. Of course, there may be no such factor, or $S_0(k)$ may contain other factors that have a logarithmic branch point but vanish at k_γ.

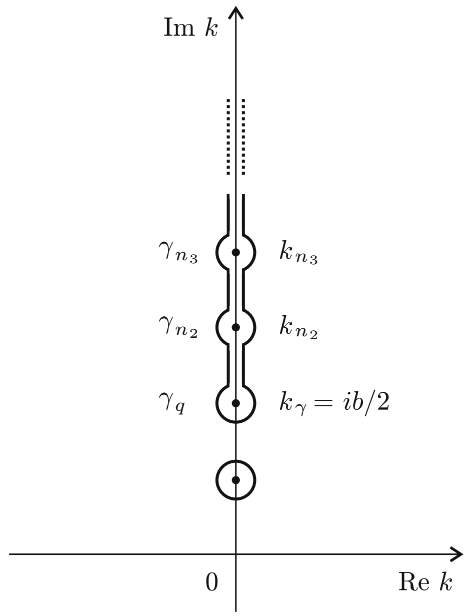

Fig. 2. A disposition of the poles and the cut.

We consider a few special cases where this factor actually occurs. If the interaction inside the sphere $r \leq a$ is such that the scattering wave function in the external region ($r > a$) can be written in the form

$$\Psi_{\rm ext} = f_{0-}(kr) - S_0(k) f_{0+}(kr) \tag{28}$$

and vanishes at some point $r = r_0 > a$, then

$$S_0(k) = \frac{f_{0-}(k, r_0)}{f_{0+}(k, r_0)}. \tag{29}$$

It follows that $S_0(k)$ *must contain* a factor F given by Eq. (27).

Another possibility occurs for a wide class of potentials [1], namely, when the interaction inside the sphere $r \leq a$ is such that the continuity relations

$$\begin{aligned} &\Psi_{\rm int} \equiv {\rm const} \cdot \Phi(k,a) = f_{0-}(k,a) - S_0(k) f_{0+}(k,a), \\ &\left.\frac{d\Psi_{\rm int}}{dr}\right|_{r=a} \equiv {\rm const} \cdot \left.\frac{d\Phi(k,r)}{dr}\right|_{r=a} = \left.\frac{df_{0-}(kr)}{dr}\right|_{r=a} - S_0 \left.\frac{df_{0+}(k,r)}{dr}\right|_{r=a} \end{aligned} \tag{30}$$

are satisfied. Here, the function $\Phi(k,r)$ is the regular solution of the radial Schrödinger equation inside the sphere $r \leq a$ with the boundary condition $\Phi(k,0) = 0$. This function is determined only by the interaction inside the sphere $r \leq a$. Equations (30) determine the constant and the corresponding S-matrix

$$S_0(k) = \frac{\varphi(k,a)\, df_{0-}(k,a)/da - f_{0-}(k,a)\, d\varphi(k,a)/da}{f_{0+}(k,a)\, d\varphi(k,a)/da - \varphi(k,a)\, df_{0+}(k,a)/da}, \tag{31}$$

i.e., $S_0(k)$ also must then contain factor (27). According to (10) and (11), the same result is also obtained for $S_l(k)$ with $l > 0$.

Using the same approach, we can study a more general case where there is a covering of the cut (see Fig. 2) by the poles (corresponding to bound states and/or "redundant" poles that appear when the potentials decrease exponentially). In this case, it suffices to use equalities analogous to (22),

$$\int_{\gamma_{\mathrm{acc}}(k_n)} = \oint_{k_n} - \int_{\gamma_{12}} \quad \underset{\varepsilon\to 0}{\longrightarrow} \quad \oint_{k_n}$$

and simply repeat the reasoning preceding (24). It is then easy to prove that for all these singularities, Eq. (21) continues to hold, and the results obtained previously concerning the singularities of $S_l(k)$ in D^+ also continue to hold.

Finally, the analytic continuation of the functions $S_l(k)$ to the lower half-plane D^- can be found as usual based on symmetry condition (5) and the known general theorem on analytic continuation.

3. The explicit analytic *S*-matrix representation for the chosen class of interactions

We now try to find the explicit analytic representation of $S_l(k)$. The factor F defined by (27) contains logarithmic singularities analogous to those of (8) at the points $k_b = ib/2$ and $-k_b$, satisfies symmetry condition (5), has the absolute value equal to unity on the real axis k, and, finally, has no other singularities except possibly a zero of the denominator in (27) and no zeros except possibly a zero of the numerator in (27).

Taking the behavior of (25) and equality (20) into account, we can easily see that for any value of l, the functions $\widetilde{S}_l^{(F)}(k) = F^{-1}\widetilde{S}_l(k)$ with $\widetilde{S}_l(k) = S_l(k)e^{2ika}$ contain no logarithmic singularities, no zeros, and no poles connected with the presence of logarithmic factor (27). They are regular and bounded everywhere in the upper half-plane (including the real axis) except the points k_{nl}.[1] And for $\widetilde{S}_l^{(F)}(k)$, we can find the product expansion following an approach that was outlined schematically in [5] and received further development here.

First, we consider some intermediate products. The product

$$\prod_n \frac{k + k_{nl}}{-k + k_{nl}} \tag{32}$$

contains all the poles of $S_l(k)$ in D^+, has no other singularities or zeros, satisfies conditions (4) and (5) and has the absolute value equal to unity on the real axis k.

The product

$$\prod_v \frac{-k + k_{vl}}{k + k_{vl}} \tag{33}$$

contains all the zeros of $S_l(k)$ on the positive imaginary axis (except, of course, zeros of F), contains all the poles of $S_l(k)$ on the negative imaginary axis (in accordance with conditions (4) and (5), which are satisfied for (33)), has the absolute value equal to unity on the real axis k, and is regular in D^+ in the case of convergence.

The product

$$\prod_s \frac{(k_{sl} - k)(k_{sl}^* + k)}{(k_{sl} + k)(k_{sl}^* - k)} \tag{34}$$

contains all the zeros of $S_l(k)$ in D^+ (except the imaginary axis and zeros of F), satisfies conditions (4) and (5), is regular in D^+ in the case of convergence, and has the absolute value equal to unity on the real axis k.

[1] The possible zero of the numerator in (25) is excluded in $\widetilde{S}_l^{(F)}(k)$.

The function

$$J_{lN}(k) = \widetilde{S}_l^{(F)}(k)\Bigg[\prod_n \frac{k+k_{nl}}{-k+k_{nl}} \prod_v^{N_1} \frac{-k+k_{vl}}{k+k_{vl}} \prod_s^{N_2} \frac{(k_{sl}-k)(k_{sl}^*+k)}{(k_{sl}+k)(k_{sl}^*-k)}\Bigg]^{-1} \tag{35}$$

for finite numbers $N = N_1 + N_2$ is regular and bounded on the real axis k. If the limit

$$J_l(k) = \lim_{N\to\infty} J_{lN}(k)$$

exists, then it has the same properties, and then

$$S_l(k) = e^{-2ika} F(k) \prod_n \frac{k+k_{nl}}{-k+k_{nl}} \prod_v \frac{-k+k_{vl}}{k+k_{vl}} \prod_s \frac{(k_{sl}-k)(k_{sl}^*+k)}{(k_{sl}+k)(k_{sl}^*-k)}. \tag{36}$$

To be certain of the validity (correctness) of (36), it is necessary to show that two infinite products in (36) converge. The condition for their absolute convergence is the convergence of the sum

$$\sum_s \Bigg[\bigg|\frac{k_{sl}-k}{k_{sl}+k} - 1\bigg| + \bigg|\frac{k_{sl}^*+k}{k_{sl}^*-k} - 1\bigg|\Bigg] + \sum_v \bigg|\frac{k_{vl}-k}{k_{vl}+k} - 1\bigg| =$$

$$= 2|k|\Bigg\{\sum_s \frac{1}{|k+k_{sl}|} + \sum_s \frac{1}{|k_{sl}^*-k|} + \sum_v \frac{1}{|k+k_{vl}|}\Bigg\}. \tag{37}$$

In turn, the convergence of (37) is determined by the convergence of the sum

$$2\sum_s \frac{1}{|k_{sl}|} + \sum_v \frac{1}{|k_{vl}|} \tag{38}$$

because $|k_{sl}| \to \infty$ as $s \to \infty$ and $|k_{vl}| \to \infty$ as $v \to \infty$. It is easy to see that sum (38) converges if the analyticity of the function

$$\tilde{J}_l(k) = \frac{\widetilde{S}_l^{(F)}(k)}{\displaystyle\prod_n \frac{k+k_{nl}}{-k+k_{nl}}} \tag{39}$$

in D^+ is taken into account together with the absence of its zeros above the real axis k and if the following theorem is used.

Theorem [7]**.** *Let a function $f(z)$ be bounded and analytic for $\operatorname{Re} z \geq 0$, and let its zeros in the right half-plane z be $r_1 e^{\theta_1}, r_2 e^{\theta_2}, \ldots$. Then the series $\sum_{n=1}^{\infty} r_n^{-1} \cos\theta_n$ converges.*

Because $\cos\theta_n = |\cos\theta_n| \geq \varepsilon$, where $\varepsilon \neq 0$, for $\tilde{J}_l(\rho)$ with $\rho = ik$, we have

$$\varepsilon \sum_{n=1}^{\infty} r_n^{-1} < \sum_{n=1}^{\infty} r_n^{-1} \cos\theta_n < \infty,$$

which proves that sum (38) converges. Hence, the infinite products in (36) converge uniformly and give a meromorphic function with the poles $-k_{sl}$ and $-k_{vl}$.

We note that for all known interactions in the field of nuclear and elementary particle physics, the "resonance" zeros of the S-matrix (in the single-channel limit) are always located at increasing distances from the real axis k (and hence at increasing distances from the imaginary axis ρ for $\tilde{J}_l(\rho)$) with increasing values of k. Therefore, the lower bound ε of $\cos\theta_n$ as $n \to \infty$ can be not very small, and the sum $\sum_{n=1}^{\infty} r_n^{-1}$ is hence not only finite but also not very large, being at least proportional to ε^{-1}.

Moreover, for all interactions for which the Levinson theorem is applicable, sum (37) must converge as a direct consequence of the nonpositive (negative or zero) integral $\int_0^\infty \Delta\tau(E)\,dE$, where $\Delta\tau(E) = (\hbar/2)(\partial \arg S_l/\partial E)E$ is the time delay, $\hbar = h/(2\pi)$, and h is the Planck constant, over the whole positive half-axis of kinetic energies $E = \hbar^2 k^2/(2\mu)$ for the relative motion of colliding particles [8], [9] because the contribution of the "resonance" zeros k_s and "virtual-state" zeros k_v to $\int_0^\infty \Delta\tau(E)\,dE$ is positive and moreover is less than the absolute value of the contribution of the *finite* number of poles corresponding to bound states and of the terms connected with reflection (from the interaction boundary).

Returning to the function $J_l(k)$, we note that it is not only analytic in D^+ but also, being an entire function without zeros, can be written in the form e^{u+iv}, where $u + iv$ is an entire function (see, e.g., the relevant theorem in [7]). The real function $u(k)$ must be negative in D^+ because of equality (20) and positive in D^- because of conditions (4) and (5). Therefore, according to the Cauchy–Riemann equations, the condition

$$0 = \frac{\partial u}{\partial\,\mathrm{Im}\,k} = -\frac{\partial v}{\partial k}, \qquad \mathrm{Im}\,k = 0, \tag{40}$$

must be satisfied on the real axis k. It hence follows that the function $v(k)$ increases monotonically and takes any real value not more than once. Then the function $u + iv$ takes any imaginary value not more than once and consequently must be a linear function of k:

$$u + iv = 2i\alpha_1 k + \alpha_2. \tag{41}$$

Obviously, $\alpha_1 = 0$, and because of the equality $S_l(0) = 1$, we have $\alpha_2 = 0$. Thus, we finally obtain

$$S_l(k) = e^{-2ik\alpha} F(k) \prod_n \frac{k + k_{nl}}{-k + k_{nl}} \prod_v \frac{-k + k_{vl}}{k + k_{vl}} \prod_s \frac{(k_{sl} - k)(k_{sl}^* + k)}{(k_{sl} + k)(k_{sl}^* - k)}, \tag{42}$$

where $\alpha = a - \alpha_1 \le a$ and the factor F is defined by (27).

4. Conclusions

Formula (42) is obtained for the first time and is a direct generalization of the results in [3]–[5] to interactions with the Yukawa-potential tail. In turn, it can be easily generalized further to noncentral parity-violating interactions and interactions with absorption using the methods presented here and in [5]. A separate work will be devoted to investigating many-channel scattering for interactions with the Yukawa-potential tail.

We note that in all kinds of dispersion relations, it is necessary to take residues not only at the poles k_{nl} but also at the "redundant" poles for the potential decreasing exponentially and also to take the integrals over the contours γ_q around logarithmic singularities into account.

Appendix 1:[2] Necessary and sufficient conditions for the existence of the "redundant" poles $(m/2)ib$ with $b > 0$, $m = 1, 2, \ldots$, in $f_0(k, r)$ and hence also in $S_0(k)$

Using the Jost equation

$$f_0(k, r) = e^{ikr} + k^{-1} \int_r^\infty \sin\bigl[k(r' - r)\bigr] V(r') f_0(k, r')\,dr' \tag{A.1}$$

[2]This translation of a paper from the Ukrainian [6] is dedicated to the memory of Yu. V. Tsekhmistrenko.

and solving it formally by the method of successive approximations, we obtain the sum

$$f_0(k,r) = \sum_{v=0}^{\infty} f_{0v}(k,r), \tag{A.2}$$

where

$$\begin{aligned} f_{00}(k,r) &= e^{ikr}, \\ f_{0v}(k,r) &= e^{ikr}(2ikr)^v \int_r^{\infty} \bigl(e^{2ik(r_1-r)} - 1\bigr) V(r_1) \times \cdots \\ &\cdots \times \int_{r_{v-1}}^{\infty} e^{-2ik(r_v - r_{v-1})} V(r_v)\, dr_v \cdots dr_1, \quad v = 1, 2, \dots, \end{aligned} \tag{A.3}$$

for the cases where $|V| < M/r^{2+\delta}$, $M < \infty$, and $\delta > 0$ [10].

Further, we use the following theorems.

Theorem 1 [7]**.** *Let $F(k,r)$ be a function of the complex variables k and r that is definite and continuous for all values of k in some domain D and for all values of r on the contour C. Then the function*

$$\Phi(k) = \int_C F(k,r)\, dr$$

is an analytic function of k in the domain D. If the contour C is infinite, then uniform convergence of the integral is also necessary.

Theorem 2 [7]**.** *Let all the functions in the series $u_1(z), u_2(z), \dots$ be analytic functions of z in the domain D and the sum $\sum_{n=1}^{\infty} u_n(z)$ be uniformly convergent in every domain D' inside D. Then the function $u(z) = \sum_{n=1}^{\infty} u_n(z)$ is an analytic function of z inside D.*

It follows from (A.3) and Theorem 1 that $f_{0v}(k,r)$, $v = 0, 1, 2, \dots$, are analytic functions of k in the upper half-plane. If Born series (A.2) is uniformly convergent, then by Theorem 2, the function $f_0(k,r)$ is analytic in the upper half-plane. Analogously, $f_0(-k,r)$ is analytic in the lower half-plane. Moreover, if $f_{0v}(k,r)$, $v \geq 1$, is analytic in the lower half-plane, then all successive terms are also analytic.

We now show that the following theorem holds.

Theorem 3. *If $f_{0v}(k,r)$, $v > 1$, has singularities, then $f_{01}(k,r)$ must also have them.*

Indeed, let $f_{01}(k,r)$ be analytic everywhere. Then all successive terms are also analytic everywhere. This contradiction proves Theorem 3.

We use the analytic structure of $f_{01}(k,r)$ in the upper half-plane. Obviously, the problem can be reduced to studying the analytic structure of the integral

$$I = \int_r^{\infty} e^{-2ikr'} V(r')\, dr'$$

because all other terms give functions that are analytic in the whole plane.

As shown in [10], [11], in the case of the potential

$$V(r) = P_n(r) e^{-br}, \tag{A.4}$$

where $P_n(r)$ is an nth-order polynomial and $b > 0$, the function $f_0(-k,r)$ has poles of an order not higher than $n+1$ at the points $ib/2$, ib, $3ib/2$, …, and it is analytic at all other points of the complex plane.

We now show that if $f_{01}(-k,r)$ has a pole of an order not higher than $n+1$ at the point $ib/2$, then the potential must have a term of type (A.4). It follows from the hypothesis that the integral

$$I_1 = \int_r^\infty e^{-2ikr'} V_1(r')\,dr',$$

where $V_1 = V - V_2$ (the term V_2 does not give poles), can be represented on the real k axis in the form

$$I_1 \equiv \sum_{\mu=0}^{n} \frac{\varphi_\mu(-2ik,r)}{(2ik+b)^\mu}, \tag{A.5}$$

where $\varphi_\mu(-2ik,r)$ is analytic at all the poles and is nonzero at the point $ib/2$. Further, rewriting I_1 in the form

$$I_1 = \int_r^\infty e^{-(2ik+b)r'} V_1(r') e^{br'}\,dr'$$

and successively integrating by parts, we can transform the right-hand side of identity (A.5) into the series

$$-e^{-2ikr}V_1 - \frac{1}{2ik+b}e^{-2ikr}\left(bV_1 + \frac{dV_1}{dr}\right) - \cdots - \frac{1}{(2ik+b)^n}e^{-2ikr} \times$$
$$\times \left(b^n V_1 + nb^{n-1}\frac{dV_1}{dr} + \cdots + \frac{d^n V_1}{dr^n}\right) - \cdots \equiv \sum_{\mu=0}^{n} \frac{\varphi_\mu(-2ik,r)}{(2ik+b)^\mu}. \tag{A.6}$$

Comparing the coefficients of equal powers of $2ik+b$, we obtain a successive system of the corresponding identities. Because there is no term containing $(2ik+b)^{-m}$, $m > n$, in the right-hand side of (A.6), we obtain

$$b^{n+1}V_1 + (n+1)b^n\frac{dV_1}{dr} + \frac{(n+1)n}{2} - b^{n-1}\frac{d^2V_1}{dr^2} + \cdots + \frac{d^{n+1}V_1}{dr^{n+1}} = 0,$$

whence follows

$$V = P_n(r)e^{-br} + V_2,$$

where V_2 is an arbitrary function that cannot be brought to the form $P_s(r)e^{-b_s r}$.

The following general theorem can be proved analogously.

Theorem 4. *For $f_0(-k,r)$ to have poles of an order not higher than n_1+1 at the points $ib_1/2$, ib_1, $3ib_1/2$, …, not higher than n_2+1 at the points $ib_2/2$, ib_2, $3ib_2/2$, …, and not higher than n_m+1 at the points $ib_m/2$, ib_m, $3ib_m/2$, …, it is necessary and sufficient that the corresponding potential have a term $\sum_{m,n_m} P_{n_m}(r)e^{-b_m r}$.*

Obviously, to have *essentially singular points*, it is necessary and sufficient that the corresponding potential contain a term $X(r)e^{-br}$, where $X(r)$ is an uniformly convergent infinite series of the type $\sum_{n=0}^\infty \alpha_n r^n$ not equal to $e^{\mathrm{const}\cdot r^\alpha}$, $0 < \alpha < \infty$.

Investigating the behavior of I on the axis $k = ib/2$ in the case where $V(r) = v(r)e^{-br}$, where $v(r)$ is an arbitrary function not having a factor $e^{\mathrm{const}\cdot r^\alpha}$, $\alpha \geq 1$, we can easily conclude that branch points can appear on that axis. One of the simplest cases is the potential $[e^{-br}\sin(cr)]/r^q$. For various $q > 0$, this potential can give branch points of different types at $k = ib/2 \pm c/2$.

If $I(r) = v(r)e^{-br^\alpha}$ and $\alpha > 1$ is an integer, then I and hence $f_0(k,r)$ are analytic functions in the whole plane.

Thus, the presence of the factor e^{-br} leads to the function $f_0(-k,r)$ becoming nonanalytic in the upper half-plane.

Appendix 2: Derivation of formula (24)

We rewrite expression (6a) for $f_{0-}(k,r)$ inside the circle γ_c around the point $k_\gamma = ib/2$ in the form

$$f_{0-}(k,r) \to \lim_{k\to k_\gamma} \left\{1+\rho\left[1+\frac{i\rho}{2k}\log\left(1+\frac{2ik}{b}\right)\right]^{-1}\int_b^\infty \frac{e^{-b'r}}{b'(b'\mp 2ik)}\,db'\right\}e^{\pm ikr} =$$

$$= \lim_{\eta\to 0} \left\{1+\rho\left[1+\frac{i\rho}{2k}\log\left(1+\frac{2ik}{b}\right)\right]^{-1}\int_0^\infty \frac{e^{-(\tilde b+b)r}}{(\tilde b+\eta)(\tilde b+b)}\,d\tilde b\right\}e^{-ikr}, \tag{A.7}$$

where we introduce the variables $\tilde b = b' - b$ and $\eta = 2ik + b$. We then make the following simple transformations of the right-hand side of (A.7):

$$f_{0-}(k,r) \to \lim_{\eta\to 0} \left\{1+\rho\left[1+\frac{i\rho}{2k}\log\left(1+\frac{2ik}{b}\right)\right]^{-1}\int_0^\infty \frac{e^{-\tilde b r}e^{2ikr}\,d\tilde b}{(\tilde b+\eta)(\tilde b-2ik+\eta)}\right\}e^{-ikr} =$$

$$= e^{-ikr} + \rho\left[1+\frac{i\rho}{2k}\log\left(1+\frac{2ik}{b}\right)\right]^{-1}\int_0^\infty \frac{e^{-\tilde b r}\,d\tilde b}{\tilde b(\tilde b-2ik)}e^{ikr} +$$

$$+ \rho\left[1+\frac{i\rho}{2k}\log\left(1+\frac{2ik}{b}\right)\right]^{-1}\lim_{\eta\to 0}\int_0^b \frac{e^{-\tilde b r}\,d\tilde b}{(\tilde b+\eta)(\tilde b-2ik+\eta)}e^{ikr} =$$

$$= e^{-ikr} + \frac{A_-}{A_+}\left[f_{0+}(k,r) - e^{ikr}\right] +$$

$$+ \rho\left[1+\frac{i\rho}{2k}\log\left(1+\frac{2ik}{b}\right)\right]^{-1}\lim_{\eta\to 0}\int_0^b \frac{e^{-\tilde b r}\,d\tilde b}{(\tilde b+\eta)(\tilde b-2ik+\eta)}e^{ikr}, \tag{A.8}$$

where

$$A_\mp = \left[1 \pm \frac{i\rho}{2k}\log\left(1\pm\frac{2ik}{b}\right)\right]^{-1}$$

and we use definition (6a) for $f_{0+}(k,r)$.

We now analyze the last integral in the right-hand side of (A.8), using formulas (3.352.1) and (8.214.1) in [12]:

$$J = \lim_{\eta\to 0}\int_0^b \frac{e^{-\tilde b r}\,d\tilde b}{(\tilde b+\eta)(\tilde b-2ik+\eta)} = \lim_{\eta\to 0}\frac{1}{b-\eta}\int_0^b e^{-\tilde b r}\left[\frac{1}{\tilde b+\eta} - \frac{1}{\tilde b+b}\right]d\tilde b =$$

$$= \lim_{\eta\to 0}\left\{\frac{1}{b-\eta}e^{\eta r}\left[\mathrm{Ei}(-br-\eta r) - \mathrm{Ei}(-\eta r)\right] - \frac{1}{b-\eta}e^{br}\left[\mathrm{Ei}(-2br) - \mathrm{Ei}(-br)\right]\right\} =$$

$$= \lim_{\eta\to 0}\frac{1}{b}\left[-\log(\eta r) + X(\eta,r)\right], \tag{A.9}$$

where Ei is the Airy function and

$$X(\eta,r) = \sum_{k=1}^\infty \frac{(-br+\eta r)^k}{k\cdot k!} - \sum_{k=1}^\infty \frac{(-\eta r)^k}{k\cdot k!} - e^{br}\left[\sum_{k=1}^\infty \frac{(-2br)^k}{k\cdot k!} - \sum_{k=1}^\infty \frac{(-br)^k}{k\cdot k!}\right]$$

is an analytic function of $\eta = b + 2ik$ at the point $\eta = 0$ and in a small circle $|\eta| < b$. Further, we can obviously rewrite (A.9) as

$$J = \frac{1}{b}\left\{\left[1+\frac{i\rho}{2k}\log\left(1+\frac{2ik}{b}\right)\right]\frac{2ik}{\rho} + Z(k,r)\right\}, \tag{A.9a}$$

where $Z(k,r) = X(\eta,r) - \log(br) - 2ik/\rho$ is an analytic function of k at the point $k = k_\gamma = ib/2$ and in a small circle $|k| < b/2$.

Using (A.9a), we continue the transformations of (A.8):

$$f_{0-}(k,r) \underset{k\to k_\gamma}{\longrightarrow} e^{-ikr} + \frac{A_-}{A_+}\left[f_{0+}(kr) - e^{ikr}\right] + \rho\left[1 + \frac{i\rho}{2k}\log\left(1 + \frac{2ik}{b}\right)\right]^{-1} \times$$
$$\times \lim_{\eta\to 0} e^{ikr}\frac{1}{b}\left\{\left[1 + \frac{i\rho}{2k}\log\left(1 + \frac{2ik}{b}\right)\right]\frac{2ik}{\rho} + Z(k,r)\right\} =$$
$$= W(k,r) + \left[1 + \frac{i\rho}{2k}\log\left(1 + \frac{2ik}{b}\right)\right]^{-1} U(k,r),$$

where

$$W(k,r) = e^{-ikr} + \frac{2ik}{b}e^{ikr},$$
$$U(k,r) = \left[1 - \frac{i\rho}{2k}\log\left(1 - \frac{2ik}{b}\right)\right]^{-1} f_{0+}(k,r) + e^{ikr}\left\{\frac{\rho}{b}Z(k,r) - \left[1 - \frac{i\rho}{2k}\log\left(1 - \frac{2ik}{b}\right)\right]\right\}$$

are analytic functions of k at the point $k = k_\gamma = ib/2$ and in a small circle $|k| < b/2$.

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