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ITERATIVE METHOD FOR SOLVING NONLINEAR INTEGRAL EQUATIONS DESCRIBING ROLLING SOLUTIONS IN STRING THEORY

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We consider a nonlinear integral equation with infinitely many derivatives that appears when a system of interacting open and closed strings is investigated if the nonlocality in the closed string sector is neglected. We investigate the properties of this equation, construct an iterative method for solving it, and prove that the method converges.

Keywords: string theory, nonlinear integral equation, iterative method

Nonlinear equations with infinitely many derivatives have recently become the subject of research in both the standard and p-adic string theories [1]–[8]. The properties of some of them were systematically investigated mathematically in [1]. Here, as a continuation of the investigation of nonlinear equations with infinitely many derivatives, we consider the equation

$$a\Phi^3(t) + (1-a)\Phi(t) = \exp\left(a\frac{d^2}{dt^2}\right)\Phi(t), \tag{1}$$

appearing in string theory, with a constant $a \in (0,1]$.

A precise meaning is given to Eq. (1) in what follows. We note that in the case where $a = 1$, this equation becomes the equation for the p-adic string for $p = 3$, investigated in [1], [2].

Equation (1) appears when a system of interacting open and closed strings is investigated if the nonlocality in the closed string interaction is neglected [3], [4]. The existence of rolling solutions of the corresponding equations of motion is investigated in this model. Interestingly, this equation can be rewritten as

$$\Phi^3(t) - \Phi(t) = \frac{d^2}{dt^2}\Phi(t)$$

in the mechanical approximation [4], [5]. It has the well-known kink solution

$$\Phi(t) = \tanh\left(\frac{t}{\sqrt{2}}\right).$$

We note that a kink usually describes a solution depending on spatial coordinates.

In this paper, we investigate the properties of Eq. (1) and consider boundary value problems for bounded solutions; in particular, we construct rolling solutions that interpolate between two vacuums.

Equation (1) is a pseudodifferential equation with the symbol $e^{-a\xi^2}$, which for positive a can be represented as a nonlinear integral equation (similarly to the equations considered in [1], [6], [7])

$$C_a[\Phi](t) = a\Phi^3(t) + (1-a)\Phi(t), \tag{2}$$

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where $a \in (0,1)$ is a constant and the operator C_a is given by

$$C_a[\psi](t) = \int_{-\infty}^{\infty} \mathcal{C}_a[(t-\tau)^2]\psi(\tau)\, d\tau \tag{3}$$

with the kernel

$$\mathcal{C}_a[(t-\tau)^2] = \frac{1}{\sqrt{4\pi a}} e^{-(t-\tau)^2/(4a)}.$$

We seek solutions of Eq. (2) in the class of real-valued measurable functions.

Theorem 1. *If a solution $\Phi(t)$ of Eq. (2) is bounded, then it satisfies the estimate*

$$|\Phi(t)| \le 1, \quad t \in \mathbb{R}. \tag{4}$$

Proof. Let

$$\sup_t |\Phi(t)| = M, \quad 0 < M < \infty. \tag{5}$$

It follows from (2) and (3) that

$$|a\Phi^3(t) + (1-a)\Phi(t)| = \left| \int_{-\infty}^{+\infty} \Phi(\tau)\mathcal{C}_a[(t-\tau)^2]\, d\tau \right| \le \int_{-\infty}^{+\infty} |\Phi(\tau)|\mathcal{C}_a[(t-\tau)^2]\, d\tau \le$$

$$\le \sup_\tau |\Phi(\tau)| \int_{-\infty}^{+\infty} \mathcal{C}_a[(t-\tau)^2]\, d\tau = M, \tag{6}$$

Hence,

$$\sup_t |a\Phi^3(t) + (1-a)\Phi(t)| = aM^3 + (1-a)M \le M,$$

i.e., $M \le 1$. The theorem is proved.

Remark 1. Theorem 1, as well as Theorems 4 and 5 proved below, are similar to the corresponding theorems in [1], [6] but are proved here using specific features of Eq. (2).

Lemma 1. *If the function $\Phi(t)$ is bounded, then the function $C_a[\Phi](t)$ is continuous in t.*

Proof. As before, let $\sup_t |\Phi(t)| = M$. We consider the chain of inequalities

$$|C_a[\Phi](t+\delta) - C_a[\Phi](t)| \frac{1}{\sqrt{4\pi a}} \left| \int_{-\infty}^{+\infty} (e^{-((t+\delta)-\tau)^2/(4a)} - e^{-(t-\tau)^2/(4a)})\Phi(\tau)\, d\tau \right| \le$$

$$\le \frac{1}{\sqrt{4\pi a}} \int_{-\infty}^{+\infty} \left| e^{-((t+\delta)-\tau)^2/(4a)} - e^{-(t-\tau)^2/(4a)} \right| \cdot |\Phi(\tau)|\, d\tau \le$$

$$\le \sup_\tau |\Phi(\tau)| \frac{1}{\sqrt{4\pi a}} \int_{-\infty}^{+\infty} \left| e^{-((t+\delta)-\tau)^2/(4a)} - e^{-(t-\tau)^2/(4a)} \right| d\tau =$$

$$= \frac{M}{\sqrt{4\pi a}} \int_{-\infty}^{+\infty} \left| e^{-(y+\delta)^2/(4a)} - e^{-y^2/(4a)} \right| dy.$$

Evaluating the absolute value in the integrand, we obtain

$$|C_a[\Phi](t+\delta) - C_a[\Phi](t)| \le$$

$$\le \frac{M}{\sqrt{4\pi a}}\left[\int_{-\infty}^{\delta/2} (e^{-(y+\delta)^2/(4a)} - e^{-y^2/(4a)})\,dy - \int_{\delta/2}^{+\infty} (e^{-(y+\delta)^2/(4a)} - e^{-y^2/(4a)})\,dy\right] =$$

$$= -M\,\mathrm{erf}\left(\frac{\delta}{4\sqrt{a}}\right) + e^{-3\delta/(16a)} M\,\mathrm{erf}\left(\frac{\delta}{2\sqrt{a}}\right) \le$$

$$\le -M\,\mathrm{erf}\left(\frac{\delta}{4\sqrt{a}}\right) + M\,\mathrm{erf}\left(\frac{\delta}{2\sqrt{a}}\right), \tag{7}$$

where

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-\tau^2}\,d\tau \tag{8}$$

is the error function. Estimate (7) with Eq. (8) taken into account implies that the function $C_a[\Phi](t)$ is continuous.

Theorem 2. *All bounded solutions $\Phi(t)$ of Eq. (2) are continuous.*

Proof. By Lemma 1, the function $C_a[\Phi](t)$ is continuous. We prove that the function $\Phi(t)$ is continuous. We suppose the contradictory, that some $\Phi(t)$ is not continuous; it then follows that the functions $a\Phi^3(t)$ and $(1-a)\Phi(t)$ are not continuous. Because $a \in [0,1]$, the function $a\Phi^3(t) + (1-a)\Phi(t)$ is also not continuous, which contradicts the continuity of $\mathcal{C}_a\Phi(t)$ and the fact that the function $\Phi(t)$ satisfies Eq. (2).

Theorem 3. *If a solution $\Phi(t)$ of Eq. (2) is positive and bounded, then the action of the operator C_a is decreasing, i.e., $C_a[\Phi](t) \le \Phi(t)$. If a solution $\Phi(t)$ of Eq. (2) is negative and bounded, then the action of the operator C_a is increasing, i.e., $C_a[\Phi](t) \ge \Phi(t)$.*

Proof. We prove the first statement in the theorem. Because the solution $\Phi(t)$ of (2) is positive and bounded by assumption, it follows from Theorem 1 that $0 \le \Phi(t) \le 1$, and therefore

$$C_a[\Phi](t) = a\Phi^3(t) + (1-a)\Phi(t) \le \Phi(t).$$

The second statement in the theorem is proved similarly.

Theorem 4. *If a solution $\Phi(t)$ of Eq. (2) has a limit as $t \to +\infty$, then it takes one of the possible values -1, 0, or 1.*

Proof. Let $\lim_{t\to+\infty} \Phi(t) = b$. We then find the limit of $C_a[\Phi(t)]$. We have

$$\lim_{t\to+\infty} C_a[\Phi(t)] = \lim_{t\to+\infty} \frac{1}{\sqrt{4\pi a}} \int_{-\infty}^{+\infty} e^{-(t-\tau)^2/(4a)}\Phi(\tau)\,d\tau =$$

$$= \frac{1}{\sqrt{4\pi a}}\left[\lim_{t\to+\infty} \int_{-\infty}^{0} e^{-(t-\tau)^2/(4a)}\Phi(\tau)\,d\tau + \lim_{t\to+\infty} \int_0^{+\infty} e^{-(t-\tau)^2/(4a)}\Phi(\tau)\,d\tau\right].$$

Replacing $\tau \to t - u$ in the first integral and $\tau \to t + u$ in the second, we obtain

$$\lim_{t\to+\infty} C_a[\Phi(t)] = \frac{1}{\sqrt{4\pi a}}\left[\lim_{t\to+\infty}\int_t^{+\infty} e^{-u^2/(4a)}\Phi(t-u)\,du + \lim_{t\to+\infty}\int_{-t}^{+\infty} e^{-u^2/(4a)}\Phi(t+u)\,du\right] =$$

$$= \frac{1}{\sqrt{4\pi a}}\left[\int_{+\infty}^{+\infty} e^{-u^2/(4a)} b\,du + \int_{-\infty}^{+\infty} e^{-u^2/(4a)} b\,du\right] = \frac{b}{\sqrt{4\pi a}}\int_{-\infty}^{+\infty} e^{-u^2/(4a)}\,du,$$

and hence $\lim_{t\to+\infty} C_a[\Phi(t)] = b$.

Taking the termwise limit as $t \to +\infty$ in Eq. (2), we obtain the equation

$$ab^3 + (1-a)b = b,$$

which has the three roots $b = 0$ and $b = \pm 1$.

Theorem 5. *There exists a unique nonnegative bounded continuous solution* $\Phi(t) \equiv 1$ *of Eq.* (2) *that satisfies the boundary conditions*

$$\lim_{t\to-\infty}\Phi(t) = \lim_{t\to+\infty}\Phi(t) = 1. \tag{9}$$

Proof. We note that $\Phi(t) \equiv 1$ is a solution of the boundary value problem in (2) and (9). Let $\Phi^*(t)$, $0 \le \Phi^*(t) \not\equiv 1$, be another bounded continuous solution of this problem. Then $0 \le \Phi^*(t) \le 1$, and by (9), there exists a t_0 such that

$$0 \le \Phi^*(t_0) = \min_t \Phi^*(t) \le 1. \tag{10}$$

From Eq. (2), we then have the estimate

$$a\Phi^{*3}(t_0) + (1-a)\Phi^*(t_0) = \int_{-\infty}^{+\infty} \Phi^*(\tau)\mathcal{C}_a[(t_0-\tau)^2]\,d\tau \ge \Phi^*(t_0). \tag{11}$$

This inequality holds if $\Phi^*(t_0) \ge 1$ or $\Phi^*(t_0) = 0$. Recalling that $|\Phi^*(t)| \le 1$, we obtain $\Phi^*(t_0) = 0$ or $\Phi^*(t_0) = 1$. The value $\Phi^*(t_0) = 0$ does not satisfy boundary conditions (9) because $C_a[\Phi^*](t_0) = 0$ in this case. But because $\Phi^*(t) \ge 0$, we obtain $\Phi^*(t) \equiv 0$. Hence, there exists a unique nonnegative solution $\Phi^*(t) \equiv \Phi(t) \equiv 1$ of the boundary value problem in (2) and (9).

Theorem 6. *There exists a continuous solution of Eq.* (2) *that satisfies the boundary conditions*

$$\lim_{t\to-\infty}\Phi(t) = -1, \qquad \lim_{t\to+\infty}\Phi(t) = 1 \tag{12}$$

and such that the iterative procedure

$$\mathcal{C}_a\Phi_n = a\Phi_{n+1}^3 + (1-a)\Phi_{n+1} \tag{13}$$

converges to this solution.

Proof. Because the equation is invariant under the replacement $\Phi(t) \to -\Phi(-t)$, we seek odd solutions;

for them, the boundary value problem on the positive semiaxis $t \geq 0$ can be rewritten as

$$K_a[\phi](t) = a\phi^3(t) + (1-a)\phi(t), \qquad \lim_{t\to+\infty} \phi(t) = 1, \tag{14}$$

where the operator K_a is given by

$$K_a[\phi](t) = \int_0^\infty \mathcal{K}_a(t,\tau)\phi(\tau)\,d\tau$$

with

$$\mathcal{K}_a(t,\tau) = \frac{1}{\sqrt{4\pi a}}\Big[e^{-(t-\tau)^2/(4a)} - e^{-(t+\tau)^2/(4a)}\Big].$$

A solution $\Phi(t)$ of the original problem follows from $\phi(t)$ via odd continuation: $\Phi(t) = \operatorname{sgn}(t)\phi(|t|)$. We seek a solution of Eq. (14) using an iterative procedure similar to (13), which takes the form

$$K_a\phi_n = a\phi_{n+1}^3 + (1-a)\phi_{n+1} \tag{15}$$

on the semiaxis $t \geq 0$.

Solving this cubic equation for ϕ_{n+1}, we obtain

$$\phi_{n+1} = -v_n(1-a) + \frac{1}{3av_n}, \tag{16}$$

where

$$v_n = \left(\frac{2}{27a^2B_n + \sqrt{108(1-a)^3a^3 + 729a^4B_n^2}}\right)^{1/3}, \tag{17}$$

$$B_n = K_a\phi_n. \tag{18}$$

As the zero iteration, we take the function

$$\phi_0 = \frac{1-e^{-(at)^2}}{2}.$$

Acting with the operator K_a, we have

$$K_a\phi_0 = \frac{1}{2}\Big(1 - e^{-(at)^2/\sqrt{1+a^2}}\Big).$$

Evaluating $a\phi_0^3 + (1-a)\phi_0$, we obtain

$$a\phi_0^3 + (1-a)\phi_0 \leq K_a\phi_0,$$

whence

$$a\phi_0^3 + (1-a)\phi_0 \leq a\phi_1^3 + (1-a)\phi_1,$$

and therefore $\phi_0 \leq \phi_1$. Because the kernel is nonnegative, we obtain $K_a\phi_0 \leq K_a\phi_1$ after integrating, i.e., $B_0 \leq B_1$.

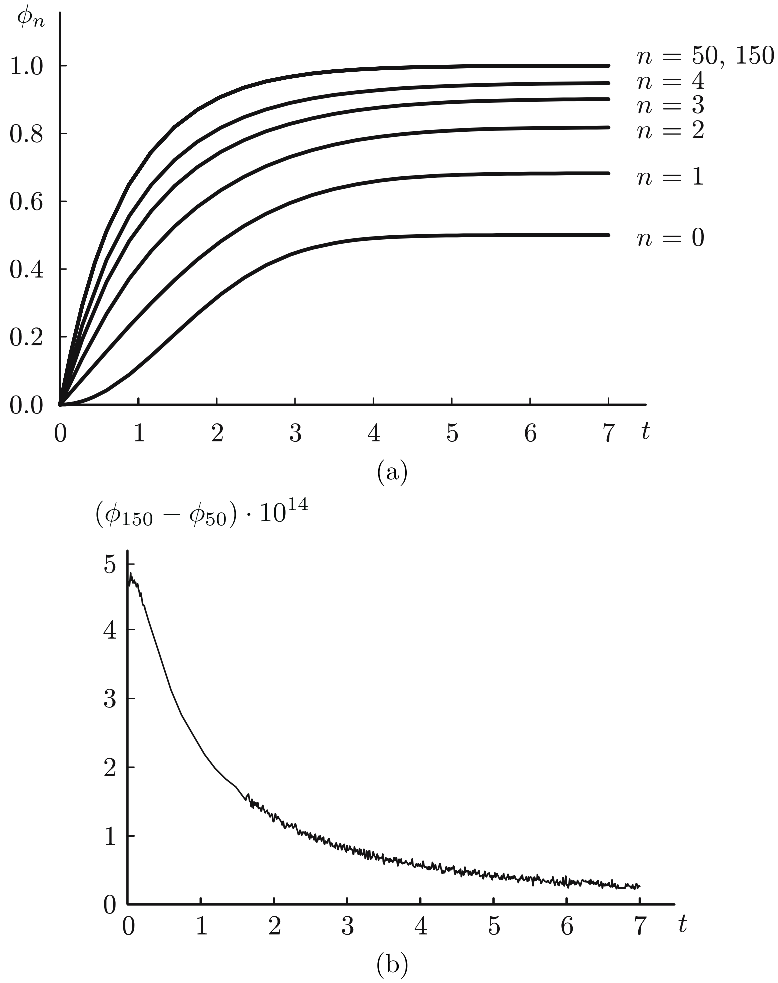

Fig. 1. (a) The graph of the iteration ϕ_n for $n = 0, 1, 2, 3, 4, 50, 150$; the iterations ϕ_{50} and ϕ_{150} are indistinguishable in the graph. (b) The difference $\phi_{150} - \phi_{50}$.

Using explicit expressions (17) and (16) for the functions v_n and ϕ_{n+1}, we conclude that the inequalities $B_0 \le B_1$ imply the inequalities $v_0 \ge v_1$ and $\phi_1 \le \phi_2$. Repeating the above argument $n-1$ times, we obtain

$$\phi_0 \le \phi_1 \le \cdots \le \phi_n \le \phi_{n+1}.$$

The results of calculating the iterations are given in Fig. 1.

We now prove the inequality $\phi_n \le 1$. It is easy to see that the initial iteration ϕ_0 is bounded, $\phi_0 < 1$. By (15), we then have $K_a\phi_0 \le \phi_0 < 1$, and therefore $a\phi_1^3 + (1-a)\phi_1 < 1$. We suppose that there exists $t_1 \ge 0$ such that $\phi_1(t_1) > 1$; then $a\phi_1^3(t_1) + (1-a)\phi_1(t_1) > \phi_1(t_1) > 1$, which contradicts the estimate obtained above, and hence $\phi_1 \le 1$. Repeating this argument n times, we find that all the functions ϕ_{n+1} are bounded, $\phi_{n+1} \le 1$.

We next prove that the functions ϕ_{n+1} are monotonic. We show that $d(K_a[\phi_n](t))/dt \ge 0$ assuming that $\phi_n(t)$ is a nonnegative monotonically increasing function. We have

$$\frac{d}{dt}(K_a[\phi_n](t)) = \int_0^\infty \mathcal{K}_a'(t,\tau)\phi_n(\tau)\,d\tau = \\ = \int_0^\infty (\mathcal{K}_1'(t,\tau) + \mathcal{K}_2'(t,\tau))\phi_n(\tau)\,d\tau,$$

where

$$\mathcal{K}_1'(t,\tau) = \frac{1}{2a\sqrt{4\pi a}}(\tau - t)e^{-(t-\tau)^2/(4a)},$$

$$\mathcal{K}_2'(t,\tau) = \frac{1}{2a\sqrt{4\pi a}}(\tau + t)e^{-(t+\tau)^2/(4a)}.$$

Because for all $t, \tau \geq 0$, the kernel $\mathcal{K}_2'(t,\tau) \geq 0$ and the function $\phi_n(\tau) \geq 0$ and increases, it follows that

$$\frac{d}{dt}(K_a[\phi_n](t)) \geq \int_0^\infty \mathcal{K}_1'(t,\tau)\phi_n(\tau)\,d\tau = \int_0^t \mathcal{K}_1'(t,\tau)\phi_n(\tau)\,d\tau + \int_t^\infty \mathcal{K}_1'(t,\tau)\phi_n(\tau)\,d\tau \geq$$
$$\geq \int_0^t \mathcal{K}_1'(t,\tau)\phi_n(t)\,d\tau + \int_t^\infty \mathcal{K}_1'(t,\tau)\phi_n(t)\,d\tau = \phi_n(t)\int_0^\infty \mathcal{K}_1'(t,\tau)\,d\tau \geq 0.$$

Because $\phi_0(t)$ is a nonnegative monotonically increasing function, we have

$$K_a[\phi_0](t_0) \leq K_a[\phi_0](t_1), \quad t_0 \leq t_1,$$

or

$$a\phi_1^3(t_0) + (1-a)\phi_1(t_0) \leq a\phi_1^3(t_1) + (1-a)\phi_1(t_1).$$

It follows from this inequality that $\phi_1(t_0) \leq \phi_1(t_1)$ for $t_0 \leq t_1$. Repeating the argument n times, we find that $\phi_{n+1}(t_0) \leq \phi_{n+1}(t_1)$ for $t_0 \leq t_1$. We have thus shown that the iterations $\{\phi_{n+1}\}$ are a sequence of monotonically increasing bounded functions; therefore, there is the limit [9]

$$\lim_{n\to\infty} \phi_n(t) = f(t). \tag{19}$$

Taking the limit as $n \to \infty$ in (15) and using the Lebesgue theorem [10], we obtain the equation

$$af^3 + (1-a)f - K_a f = 0, \tag{20}$$

where $f \in L_\infty[0,\infty]$. Therefore, the function f is a solution of Eq. (14). The function f is bounded because $\phi_n \leq 1$; therefore, it is continuous (see Theorem 2). We have thus proved that iterative procedure (15) converges to a continuous solution f of Eq. (14). The function f increases monotonically (because all the ϕ_n increase monotonically) and is bounded, $0 \leq f(t) \leq 1$; therefore, $\lim_{t\to+\infty} f(t)$ exists. Because $\phi_0 \leq f \leq 1$ and $\lim_{t\to+\infty} \phi_0 = 1/2$, it follows from Theorem 4 that $\lim_{t\to+\infty} f(t) = 1$ and that the function $f(t)$ is a solution of problem (14).

We have thus proved that iterative process (13) converges to a continuous solution of the boundary value problem in (2) and (12).

In summary, we have investigated the properties of integral equation (1) with infinitely many derivatives, have constructed an iterative method for solving it, and have proved that this method converges.

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