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INTEGRABLE MODEL OF INTERACTING ELLIPTIC TOPS

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We suggest a method for constructing a system of interacting elliptic tops. It is integrable and symplectomorphic to the Calogero–Moser model by construction.

Keywords: integrable systems, algebraic geometry, symplectic geometry

1. Gaudin models over elliptic curves

Let Σ_τ be an elliptic curve with the periods $(1,\tau)$ and marked points $\{z_i\}$, $i=1,2,\dots,m$, and let V be a holomorphic vector bundle of rank N and degree k over it. By analogy with the rational case, we define the Lax matrix $L^{m|k|N}(z)$ of the elliptic Gaudin model $EG(m|k|N)$ over Σ to be a meromorphic section of a bundle $\operatorname{End} V$ with simple poles at $\{z_i\}$ and with the fixed residues $S^i\in sl^*(N,\mathbb{C})$. In the Hitchin approach to integrable systems [1], [2], the corresponding 1-form $L^{m|k|N}(z)\,dz$ describes the reduced Higgs field. Choosing the vector bundle V fixes the corresponding quasiperiodic boundary conditions on the lattice $\langle 1,\tau\rangle$:

$$L^{m|k|N}(z+1)=g_1L^{m|k|N}(z)g_1^{-1},\qquad L^{m|k|N}(z+\tau)=g_\tau L^{m|k|N}(z)g_\tau^{-1}.$$

Conceptually, the bundles are distinguished by their degrees. The dimension of the moduli space of a fixed-degree bundle, which is associated with a principal $SL(N,\mathbb{C})$ bundle, is $\operatorname{GCD}(N,k)-1$ [3]. Its maximum is at $k=0\ (\operatorname{mod} N)$ and its minimum is at $k=1\ (\operatorname{mod} N)$. We consider these two cases in more detail.

The case $\deg V=0$ was first considered in [4]. The bundles are then described by N parameters $\{u_i\}$, $i=1,2,\dots,N$: $u_1+\dots+u_N=0$. The corresponding multipliers are

$$g_1=\operatorname{Id}_N,\qquad g_\tau=\mathbf{e}(-\mathbf{u})=\operatorname{diag}(\mathbf{e}(-u_1),\dots,\mathbf{e}(-u_N)),\tag{1}$$

where $\mathbf{e}(x)=e^{2\pi\sqrt{-1}x}$. These conditions define the Lax matrix for $EG(m|0|N)$,

$$L^{m|0|N}_{ij}(z)=\delta_{ij}v_i+\delta_{ij}\sum_{l=1}^m S^l_{ii}E_1(z-z_l)+(1-\delta_{ij})\sum_{l=1}^m S^l_{ij}\phi(z-z_l,u_i-u_j),\tag{2}$$

up to a conjugation by an element from the Cartan subgroup H of $SL(N,\mathbb{C})$. The functions $E_1(z)$ and $\phi(x,y)$ are given by Eqs. (A.2) and (A.3). The Hamiltonian reduction of the direct product of the orbits of the coadjoint action $\{\mathcal{O}^1\times\dots\times\mathcal{O}^m\}//\,\mathrm{H}$ and this symmetry describes the "spin" part of the phase space of the Gaudin model [5]. The moment map corresponding to the action has the form

$$\mu=\sum_{l=1}^m S^l_{ii}.\tag{3}$$

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Equation $\mu = 0$ together with fixing the action of H provides the final answer for $L^{m|0|N}(z)$. The spinless part of the phase space is a cotangent bundle to the moduli space of holomorphic bundles of degree zero and corresponds to a dynamics of N interacting particles with the momenta v_i and coordinates u_i in the center-of-mass frame.

An important particular case is the Gaudin model corresponding to a single marked point and a coadjoint orbit of minimal dimension at this point. This model coincides with the elliptic $sl(N,\mathbb{C})$ Calogero–Moser [6] model after the reduction described above,

$$L_{ij}^{1|0|N}(z) = \delta_{ij} v_i + (1-\delta_{ij})\sqrt{-1}\nu\phi(z, u_i - u_j), \tag{4}$$

where ν is the interaction constant. The quadratic Hamiltonian has the form

$$H = \sum_{i=1}^{N} \frac{1}{2} v_i^2 + \sum_{i\neq j} \nu^2 \wp(u_i - u_j). \tag{5}$$

The case $\deg V = 1$ was first considered in [7]. The multipliers of the bundle V have the forms

$$g_1 = Q^{-1}, \qquad g_\tau = -\mathbf{e}\left(\frac{\tau}{2N} + \frac{z}{N}\right)\Lambda^{-1}, \tag{6}$$

where Q and Λ are matrices defining a standard representation of the finite Heisenberg group (see the appendix). We write the Lax matrix in a special basis $\{E_\alpha\}$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ of the $sl(N,\mathbb{C})$ Lie algebra:

$$\begin{aligned} &L^{m|1|N}(z) = \sum_{l=1}^{m} \sum_{\alpha\neq 0} E_\alpha S_\alpha^l \varphi_\alpha(z - z_l), \qquad \varphi_\alpha(z) = \mathbf{e}(z\partial_\tau \omega_\alpha)\phi(z, \omega_\alpha), \\ &\omega_\alpha = \frac{\alpha_1 + \alpha_2\tau}{N}. \end{aligned} \tag{7}$$

The phase space $EG(1|1|N)$ here is a direct product of the coadjoint orbits $\mathcal{O}^1 \times \cdots \times \mathcal{O}^m$:

$$\{S_\alpha^i, S_\beta^j\} = 2\sqrt{-1}\sin\left[\frac{\pi}{N}(\alpha_2\beta_1 - \alpha_1\beta_2)\right]\delta^{ij} S_{\alpha+\beta}^i. \tag{8}$$

Poisson brackets (8) can be written in the R-matrix form

$$\{L_1^{m|1|N}(z), L_2^{m|1|N}(w)\} = [L_1^{m|1|N}(z) + L_2^{m|1|N}(w), r(z,w)] \tag{9}$$

using the Belavin–Drinfeld r-matrix [8]

$$r(z,w) = \sum_{\alpha\neq 0} E_\alpha \otimes E_{-\alpha}\varphi_\alpha(z-w). \tag{10}$$

Correspondence between the Calogero–Moser model and the elliptic top. The modification was defined in [9] as a procedure changing the degree of a bundle V by one. It acts on the sections of the bundle $\operatorname{End} V$ as a gauge transformation degenerated at a fixed point. In [10], the transformation between $L^{1|0|N}$ and $L^{1|1|N}$ was constructed; in other words, the gauge equivalence between the Calogero–Moser model and the elliptic top was proved. In $sl(2,\mathbb{C})$ case, it is easy to find an explicit change of variables,

$$\begin{aligned} &\{v, u\} = 1, \qquad \{S_\alpha, S_\beta\} = 2\sqrt{-1}\varepsilon_{\alpha\beta\gamma}S_\gamma, \\ &L^{1|0|2} = \begin{pmatrix} v & \nu\phi(2u,z) \\ \nu\phi(-2u,z) & -v \end{pmatrix}, \\ &L^{1|1|2} = \begin{pmatrix} S_3\varphi_3(z) & S_1\varphi_1(z) - iS_2\varphi_2(z) \\ S_1\varphi_1(z) + iS_2\varphi_2(z) & -S_3\varphi_3(z) \end{pmatrix}, \end{aligned} \tag{11}$$

where the indices agree with the Pauli matrix enumeration: $(1,2,3) = (01,11,01)$. The equivalence of the models means that there exists a gauge transformation $\Xi(z)$ such that

$$L^{1|1|2}(z) = \Xi(z) L^{1|0|2}(z) \Xi^{-1}(z). \tag{12}$$

It was shown in [10] that in the $sl(2,\mathbb{C})$ case, the transformation has the form

$$\Xi(z) = \begin{pmatrix} \theta_{00}(z-2u, 2\tau) & -\theta_{00}(z+2u, 2\tau) \\ -\theta_{10}(z-2u, 2\tau) & \theta_{10}(z+2u, 2\tau) \end{pmatrix}. \tag{13}$$

It then follows from (12) that

$$\begin{aligned} S_{01} &= -v \frac{\theta_{01}(0)}{\vartheta'(0)} \frac{\theta_{01}(2u)}{\vartheta(2u)} + \nu \frac{\theta_{01}^2(0)}{\theta_{00}(0)\theta_{10}(0)} \frac{\theta_{00}(2u)\theta_{10}(2u)}{\vartheta^2(2u)}, \\ -\sqrt{-1} S_{11} &= -v \frac{\theta_{00}(0)}{\vartheta'(0)} \frac{\theta_{00}(2u)}{\vartheta(2u)} + \nu \frac{\theta_{00}^2(0)}{\theta_{10}(0)\theta_{01}(0)} \frac{\theta_{10}(2u)\theta_{01}(2u)}{\vartheta^2(2u)}, \\ S_{10} &= -v \frac{\theta_{10}(0)}{\vartheta'(0)} \frac{\theta_{10}(2u)}{\vartheta(2u)} + \nu \frac{\theta_{10}^2(0)}{\theta_{00}(0)\theta_{01}(0)} \frac{\theta_{00}(2u)\theta_{01}(2u)}{\vartheta^2(2u)}. \end{aligned} \tag{14}$$

2. The model of interacting tops

We consider an elliptic top corresponding to a bundle of degree n and having the rank $N = np$, $N > n$. This means that its Lax matrix has the quasiperiodic boundary conditions

$$\begin{aligned} L(z+1) &= Q L(z) Q^{-1}, \\ L(z+\tau) &= \Lambda^n L(z) \Lambda^{-n}. \end{aligned} \tag{15}$$

But only a degenerate $L(z)$ matrix can satisfy these conditions because there exists a diagonal matrix with n different eigenvalues $\bar{A} = \operatorname{diag}\{u_1, \ldots, u_n, p, u_1, \ldots, u_n\}$ simultaneously commuting with Q and Λ^n,

$$Q\bar{A} = \bar{A}Q, \qquad \Lambda^n \bar{A} = \bar{A}\Lambda^n.$$

To fix this freedom, we change conditions (15) to

$$\begin{aligned} L(z+1) &= Q L(z) Q^{-1}, \\ L(z+\tau) &= \mathbf{e}(-\bar{A}) \Lambda^n L(z) \Lambda^{-n} \mathbf{e}(\bar{A}). \end{aligned} \tag{16}$$

These boundary conditions demonstrate the existence of an $(n-1)$-dimensional moduli space.

Proposition. *There exists a numerical matrix M such that*

$$\begin{aligned} M\bar{A}M^{-1} &= \bigoplus_{J=1}^{n} u_J \operatorname{Id}_{p\times p}, \\ MQM^{-1} &= \bigoplus_{J=1}^{n} \mathbf{e}\left(\frac{J-p}{N}\right) Q_{p\times p}, \\ M\Lambda^n M^{-1} &= \bigoplus_{J=1}^{n} \Lambda_{p\times p}. \end{aligned} \tag{17}$$

The last equation indicates that Λ^n can be transformed to a block-diagonal form with n blocks where each block represents a $p{\times}p$ Λ-matrix.

Proof. Let $m = (\alpha - 1)n + \beta$, where $\alpha = 1, 2, \ldots, p$ and $\beta = 1, 2, \ldots, n$. We define a permutation operation

$$\psi_{n,p}(m) = (\beta - 1)p + \alpha.$$

We claim that the desired matrix has the form

$$M_{ij} = \delta(\psi_{p,n}(i), j).$$

We prove this. We note that we have

$$(M^{-1})_{kl} = \delta(\psi_{n,p}(k), l)$$

for the inverse matrix. At this stage, we have

$$M_{ij}\bar{A}_{jk}M^{-1}_{kl} = \delta(\psi_{p,n}(i), j)\delta(j, k)\bar{A}_{jj}\delta(\psi_{n,p}(k), l) = \delta(\psi_{n,p}(i), l)\bar{A}_{ii}$$

(here we assume summation over repeated indices). We then have

$$M_{ik}Q_{kl}M^{-1}_{li} = \delta(i, j)\mathbf{e}\biggl(\frac{\psi_{p,n}(i)}{N}\biggr),$$

$$M_{ij}\Lambda^n_{jk}M^{-1}_{kl} = \delta(\psi_{p,n}(i), j)\delta(\mathrm{mod}_N(j + n), k)\delta(\psi_{n,p}(k), l) =$$

$$= \delta(\psi_{n,p}(\mathrm{mod}_N(\psi_{p,n}(i) + n)), l).$$

Setting $i = (\beta - 1)p + \alpha$, we have

a. if $\alpha < p$, then $\mathrm{mod}_N(\psi_{p,n}(i)+n) = \psi_{p,n}(i)+n = \alpha n+\beta$, and $\psi_{n,p}(\alpha n+\beta) = (\beta-1)p+\alpha+1 = i+1$,

b. if $\alpha = p$, then $\mathrm{mod}_N((p - 1)n + \beta + n) = \beta$, and $\psi_{n,p}(\beta) = (\beta - 1)p + 1$.

This completes the proof.

In what follows, we use capital Latin letters for indices taking values from 1 to n and small letters for indices taking values from 1 to p. We also use the notation $\sum_{m,n} = \sum_{m,n=0,\, m^2+n^2\neq 0}^{p-1}$.

We now use the proved proposition to rewrite the Lax matrix in the twisted basis. For the $p{\times}p$ blocks, we then have

$$\begin{aligned} L_{IJ}(z + 1) &= \mathbf{e}\biggl(\frac{I - J}{N}\biggr)Q_{p\times p}L_{IJ}(z)Q^{-1}_{p\times p}, \\ L_{IJ}(z + \tau) &= \mathbf{e}(-u_I)\Lambda_{p\times p}L_{IJ}(z)\Lambda^{-1}_{p\times p}\mathbf{e}(u_J). \end{aligned} \tag{18}$$

The factor $\mathbf{e}((I - J)/N)$ can be canceled by the change of variables

$$L_{IJ}(z) \to L_{IJ}(z)\mathbf{e}\biggl(-z\frac{I - J}{N}\biggr), \qquad u_I \to u_I - I\frac{\tau}{N}.$$

Finally, the boundary conditions are

$$\begin{aligned} L_{IJ}(z + 1) &= Q_{p\times p}L_{IJ}(z)Q^{-1}_{p\times p}, \\ L_{IJ}(z + \tau) &= \mathbf{e}(-u_I)\Lambda_{p\times p}L_{IJ}(z)\Lambda^{-1}_{p\times p}\mathbf{e}(u_J). \end{aligned} \tag{19}$$

It is easy to find an operator L satisfying conditions (19) with a fixed residue,

$$\begin{aligned} L_{IJ}(z) &= \frac{1}{p}\delta_{IJ}v_I + \sum_{m,n}(S_{IJ})_{mn}\phi_{mn}(z,u_{IJ})E_{mn}, \\ \phi_{mn}(z,u_{IJ}) &= \mathbf{e}\left(\frac{-nz}{N}\right)\phi\left(u_{IJ}-\frac{m+n\tau}{N},z\right). \end{aligned} \tag{20}$$

We introduce the factor $1/p$ to ensure that the brackets $\{v_I,u_J\}=\delta_{IJ}$ are canonical.

The Poisson brackets for the matrix elements of S are the Poisson–Lie brackets corresponding to the structure constants of $gl(N,\mathbb{C})$:

$$\{(S_{IJ})_{ab},(S_{KL})_{cd}\} = 2\sqrt{-1}\sin\left[\frac{\pi}{p}(bc-ad)\right](\delta_{KJ}(S_{IL})_{a+c,b+d}-\delta_{IL}(S_{KJ})_{a+c,b+d}). \tag{21}$$

Here, we use two different bases: the standard one for the $p\times p$ blocks ($I,J=1,2,\ldots,n$) and sine-algebra basis (A.13)–(A.16) for the elements of these blocks.

The quadratic Hamiltonian has the form

$$H=\frac{1}{2}\sum_{I=1}^{n}v_I^2-\frac{1}{2}\sum_{I,J}\sum_{m,n}\operatorname{Tr}(S_{IJ}E_{-m,-n})\operatorname{Tr}(S_{JI}E_{mn})E_2\left(u_{IJ}-\frac{m+n\tau}{N}\right) \tag{22}$$

or

$$\begin{aligned} H=\frac{1}{2}\sum_{I=1}^{n}v_I^2-\frac{1}{2}\sum_{I}\sum_{m,n}\operatorname{Tr}(S_{II}E_{-m,-n})\operatorname{Tr}(S_{II}E_{mn})E_2\left(\frac{m+n\tau}{N}\right)- \\ -\frac{1}{2}\sum_{I\neq J}\sum_{m,n}\operatorname{Tr}(S_{IJ}E_{-m,-n})\operatorname{Tr}(S_{JI}E_{mn})E_2\left(u_{IJ}-\frac{m+n\tau}{N}\right). \end{aligned} \tag{23}$$

The first and second terms represent the Hamiltonians of n $p\times p$ tops with the momenta v_I. We note that the analogue of the reduction by the Cartan subgroup in the $\deg V=0$ case here requires $\operatorname{Tr}(S_{II})=\mathrm{const}\,\forall I$. It would be interesting to interpret the last term as the potential energy of pairwise interaction. We can do this in the case where S is the coadjoint orbit of minimal dimension, i.e., $rkS=1$. Indeed, we use the known parameterization [11] of such orbits. For simplicity, let $S\in gl(N,\mathbb{C})$. Then $S=\xi\times\eta$, where ξ and η are a column and a row of length N and $\{\xi_a,\eta_b\}=\delta_{ab}$. Thus,

$$\operatorname{Tr}(S_{IJ}E_{-m,-n})\operatorname{Tr}(S_{JI}E_{mn})=\operatorname{Tr}(S_{II}E_{mn}S_{JJ}E_{-m,-n}).$$

The condition $\operatorname{Tr}(S_{II})=\mathrm{const}$ means that the matrices S_{II} describe the coadjoint orbits of minimal dimension equal to $2p-2$. In this case, the Hamiltonian has a simple physical interpretation:

$$\begin{aligned} H=\frac{1}{2}\sum_{I=1}^{n}v_I^2-\frac{1}{2}\sum_{I}\sum_{m,n}\operatorname{Tr}(S_{II}E_{-m,-n})\operatorname{Tr}(S_{II}E_{mn})E_2\left(\frac{m+n\tau}{N}\right)- \\ -\frac{1}{2}\sum_{I\neq J}\sum_{m,n}\operatorname{Tr}(S_{II}E_{m,n}S_{JJ}E_{-m,-n})E_2\left(u_{IJ}-\frac{m+n\tau}{N}\right). \end{aligned} \tag{24}$$

The first terms describe the energy of n $p\times p$ tops, and the last term describes the interaction for each pair. As shown in [10], all the systems of type $EG(1|k|N)$, $k=1,2,\ldots,N$ are symplectomorphic to one another. Therefore, the obtained model of interacting tops is symplectomorphic to the N-particle Calogero–Moser model.

Appendix 1: Elliptic functions

In this appendix, we collect basic definitions and relations for elliptic functions needed for proving the results in this paper. The majority of formulas are borrowed from [12] and [13]. We introduce $q = e^{2\pi i\tau}$, where τ is the modular parameter of the elliptic curve E_τ. The basic element is the theta function

$$\vartheta(z|\tau) = q^{1/8}\sum_{n\in\mathbb{Z}}(-1)^n e^{\pi i(n(n+1)\tau+2nz)} =$$

$$= q^{1/8}e^{-i\pi/4}(e^{i\pi z}-e^{-i\pi z})\prod_{n=1}^{\infty}(1-q^n)(1-q^n e^{2i\pi z})(1-q^n e^{-2i\pi z}). \tag{A.1}$$

The Eisenstein functions are

$$E_1(z|\tau) = \partial_z \log\vartheta(z|\tau), \qquad E_1(z|\tau) \sim \frac{1}{z} - 2\eta_1 z, \tag{A.2}$$

where

$$\eta_1(\tau) = \zeta\left(\frac{1}{2}\right) = \frac{3}{\pi^2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty\,\prime}\frac{1}{(m\tau+n)^2} = \frac{24}{2\pi i}\frac{\eta'(\tau)}{\eta(\tau)}$$

and

$$\eta(\tau) = q^{1/24}\prod_{n>0}(1-q^n)$$

is the Dedekind function. The second Eisenstein function has the form

$$E_2(z|\tau) = -\partial_z E_1(z|\tau) = \partial_z^2 \log\vartheta(z|\tau), \qquad E_2(z|\tau) \sim \frac{1}{z^2} + 2\eta_1.$$

The next important function is

$$\phi(u,z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}, \qquad \varphi_\alpha(z,\alpha+u) = \mathbf{e}(z\partial_\tau\alpha)\phi(z,\alpha+u). \tag{A.3}$$

It has a pole at $z=0$ and admits the decomposition

$$\phi(u,z) = \frac{1}{z} + E_1(u) + \frac{z}{2}(E_1^2(u) - \wp(u)) + \dots. \tag{A.4}$$

Its derivative is

$$\phi(u,z)^{-1}\partial_u\phi(u,z) = E_1(u+z) - E_1(u).$$

It is related to the Weierstrass functions,

$$\zeta(z|\tau) = E_1(z|\tau) + 2\eta_1(\tau)z, \qquad \wp(z|\tau) = E_2(z|\tau) - 2\eta_1(\tau),$$

$$\phi(u,z) = e^{-2\eta_1 uz}\frac{\sigma(u+z)}{\sigma(u)\sigma(z)}, \tag{A.5}$$

$$\phi(u,z)\phi(-u,z) = \wp(z) - \wp(u) = E_2(z) - E_2(u).$$

The series representations are

$$
\begin{aligned}
E_1(z|\tau) &= -2\pi i\left(\frac{1}{2}+\sum_{n\neq 0}\frac{e^{2\pi iz}}{1-q^n}\right)= \\
&= -2\pi i\left(\sum_{n<0}\frac{1}{1-q^n e^{2\pi iz}}+\sum_{n\geq 0}\frac{q^n e^{2\pi iz}}{1-q^n e^{2\pi iz}}+\frac{1}{2}\right), \\
E_2(z|\tau) &= -4\pi^2\sum_{n\in\mathbb{Z}}\frac{q^n e^{2\pi iz}}{(1-q^n e^{2\pi iz})^2}, \\
\phi(u,z) &= 2\pi i\sum_{n\in\mathbb{Z}}\frac{e^{-2\pi inz}}{1-q^n e^{-2\pi iu}}.
\end{aligned}
\tag{A.6}
$$

The parity properties are

$$
\begin{aligned}
&\vartheta(-z)=-\vartheta(z), \qquad E_1(-z)=-E_1(z), \qquad E_2(-z)=E_2(z), \\
&\phi(u,z)=\phi(z,u)=-\phi(-u,-z).
\end{aligned}
\tag{A.7}
$$

The behavior on the lattice is

$$
\begin{aligned}
&\vartheta(z+1)=-\vartheta(z), \qquad \vartheta(z+\tau)=-q^{-1/2}e^{-2\pi\sqrt{-1}z}\vartheta(z), \\
&E_1(z+2\omega_\alpha)=E_1(z)-4\pi\sqrt{-1}\partial_\tau\omega_\alpha, \\
&E_1(z+1)=E_1(z), \qquad E_1(z+\tau)=E_1(z)-2\pi\sqrt{-1}, \\
&E_2(z+2\omega_\alpha)=E_2(z), \qquad E_2(z+1)=E_2(z), \qquad E_2(z+\tau)=E_2(z), \\
&\phi(u+1,z)=\phi(z,u), \qquad \phi(u+\tau,z)=e^{-2\pi\sqrt{-1}z}\phi(z,u).
\end{aligned}
\tag{A.8}
$$

We also need the addition formulas

$$
\phi(u,z)\partial_v\phi(v,z)-\phi(v,z)\partial_u\phi(u,z)=(E_2(v)-E_2(u))\phi(u+v,z) \tag{A.9}
$$

or

$$
\phi(u,z)\partial_v\phi(v,z)-\phi(v,z)\partial_u\phi(u,z)=(\wp(v)-\wp(u))\phi(u+v,z). \tag{A.10}
$$

The proof of (A.9) is based on (A.4), (A.7), and (A.8) for the function $\phi(u,z)$. In fact, $\phi(u,z)$ satisfies a more general relation that follows from the Fay trisecant formula

$$
\phi(u_1,z_1)\phi(u_2,z_2)-\phi(u_1+u_2,z_1)\phi(u_2,z_2-z_1)-\phi(u_1+u_2,z_2)\phi(u_1,z_1-z_2)=0. \tag{A.11}
$$

A particular case of this formula is

$$
\phi(u_1,z)\phi(u_2,z)-\phi(u_1+u_2,z)(E_1(u_1)+E_1(u_2))+\partial_z\phi(u_1+u_2,z)=0. \tag{A.12}
$$

Appendix 2: Sine algebra

The generators of sine algebra E_{mn} are defined using the generators Q and Λ of the finite Heisenberg group:

$$E_{mn} = \mathbf{e}\left(\frac{mn}{2N}\right) Q^m \Lambda^n, \quad m = 0, 1, \ldots, N-1,$$
$$n = 0, 1, \ldots, N-1, \quad m^2 + n^2 \neq 0 \pmod N, \tag{A.13}$$

in the basis $sl(N, \mathbb{C})$, where

$$\mathbf{e}(z) = e^{2\pi\sqrt{-1}z}, \qquad Q = \mathrm{diag}(\mathbf{e}(1/N), \ldots, \mathbf{e}(m/N), \ldots, 1),$$
$$\Lambda = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{A.14}$$

The commutators are

$$[E_{sk}, E_{nj}] = 2\sqrt{-1}\sin\left[\frac{\pi}{N}(kn - sj)\right] E_{s+n,k+j}, \tag{A.15}$$

$$\mathrm{Tr}(E_{sk}E_{nj}) = \delta_{s,-n}\delta_{k,-j}N. \tag{A.16}$$

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REFERENCES

1. N. Hitchin, *Duke Math. J.*, **54**, 91–114 (1987).
2. A. Levin and M. Olshanetsky, *Comm. Math. Phys.*, **188**, 449–466 (1997).
3. M. F. Atiyah, *Proc. London Math. Soc.*, **7**, 414–452 (1957).
4. N. Nekrasov, *Comm. Math. Phys.*, **180**, 587–604 (1996); hep-th/9503157 (1995).
5. I. Krichever, O. Babelon, E. Billey, and M. Talon, *Am. Math. Soc. Transl.*, **170**, 83–119 (1995).
6. F. Calogero, *J. Math. Phys.*, **12**, 419 (1971); "Erratum," **37**, 4636 (1996); J. Moser, *Adv. Math.*, **16**, 197–220 (1975).
7. A. G. Reiman amd M. A. Semenov-Tyan-Shanskii, *J. Sov. Math.*, **46**, 1631–1640 (1989).
8. A. A. Belavin, *Nucl. Phys.* [FS2], **180**, 189–200 (1981).
9. D. Arinkin and S. Lysenko, *Math. Res. Lett.*, **4**, 181–190 (1997).
10. A. Levin, M. Olshanetsky, and A. Zotov, *Comm. Math. Phys.*, **236**, 93–133 (2003); *Math. Ann.*, **248**, 205–239 (1980).
11. J. Avan, O. Babelon, and M. Talon, "Construction of the classical R-matrices for the Toda and Calogero models," hep-th/9306102 (1993).
12. A. Weyl, *Elliptic Functions According to Eisenstein and Kronecker*, Springer, Berlin (1976).
13. D. Mumford, *Tata Lectures on Theta*, Vols. 1 and 2 (Progr. Math., Vols. 28 and 43), Birkhäuser, Boston (1983, 1984).