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EXTERNAL GRAVITATIONAL FIELD OF A NONSTATIC SPHERICALLY SYMMETRIC BODY IN THE INERTIAL FRAME

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We show that the external gravitational field of a nonstatic spherically symmetric source described by a diagonal metric tensor can only be static in the field theory of gravity.

Keywords: relativistic theory of gravity, Birkhoff theorem, graviton mass, nonstatic spherically symmetric source

In general relativity (GR), which relates the gravitational field to the Riemannian space metric tensor, the Birkhoff theorem that claims that the external field of a nonstatic spherically symmetric body can only be static was proved in the class of admissible functions. In the relativistic theory of gravity (RTG) [1], [2], the gravitational field $\phi^{\mu\nu}$ is a physical field evolving in the Minkowski space, its source is the energy–momentum tensor of all physical fields including the gravitational field, and this tensor is conserved in the Minkowski space. This approach results in the effective Riemannian space metric and in a system of equations that differs from the RG system of equations. Therefore, the above problem needs a special investigation.

The complete system of equations of the RTG is

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \frac{m^2}{2}\left[g^{\mu\nu} + \left(g^{\mu\alpha}g^{\nu\beta} - \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\right)\gamma_{\alpha\beta}\right] = 8\pi T^{\mu\nu}, \tag{1}$$

$$D_\nu \tilde{g}^{\nu\mu} = \partial_\nu \tilde{g}^{\nu\mu} + \gamma^\mu_{\alpha\beta}\,\tilde{g}^{\alpha\beta} = 0, \tag{2}$$

where the energy–momentum tensor of matter, following Hilbert, is defined by the equality

$$\sqrt{-g}\,T^{\mu\nu} = -2\frac{\delta L_{\mathrm{M}}}{\delta g_{\mu\nu}},$$

where m is the graviton mass and L_{M} is the density of the matter Lagrangian. The effective metric of the Riemannian space is

$$\tilde{g}^{\mu\nu} = \tilde{\gamma}^{\mu\nu} + \tilde{\phi}^{\mu\nu},$$

$$\tilde{g}^{\mu\nu} = \sqrt{-g}\,g^{\mu\nu}, \qquad \tilde{\gamma}^{\mu\nu} = \sqrt{-\gamma}\,\gamma^{\mu\nu}, \qquad \tilde{\phi}^{\mu\nu} = \sqrt{-\gamma}\,\phi^{\mu\nu}.$$

System of equations (1), (2) is general-covariant w.r.t. arbitrary coordinate transformations and is form invariant w.r.t. the Lorentz transformations. This means that *the relativity principle is universal.* In contrast to GR, this principle is rigorously satisfied in the RTG for all physical phenomena including the gravitational ones. This is why the fundamental laws of the conservation of the energy–momentum and of

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the angular momentum occur in the system. The coordinate system in Eqs. (1) and (2) is governed by the Minkowski space metric tensor $\gamma_{\mu\nu}$. We choose the system of units in which $\hbar = c = G = 1$.

For timelike and isotropic intervals in the effective Riemannian space not to leave the Minkowski space causality cone, the conditions

$$g_{\mu\nu}U^\mu U^\nu \le 0, \qquad \gamma_{\mu\nu}U^\mu U^\nu = 0 \tag{3}$$

must be satisfied.

The effective Riemannian space has a trivial topology in the RTG; the topology in GR is nontrivial in general. Therefore, the field description of gravity in principle cannot result in the RG equations.

Below, we find that an external gravitational field of the form

$$ds^2 = U(t,r)\,dt^2 - V(t,r)\,dr^2 - W^2(t,r)[d\theta^2 + \sin^2\theta\,d\phi^2] \tag{4}$$

created by a *nonstatic* spherically symmetric source in the inertial frame can only be *static*, i.e., its metric coefficients U, V, and W must be independent of the time t.

An inertial frame in the Minkowski space is governed by the interval

$$d\sigma^2 = dt^2 - dr^2 - r^2[d\theta^2 + \sin^2\theta\,d\phi^2]. \tag{5}$$

Starting from Eqs. (1), for the problem defined by Eqs. (4) and (5), we find the equations for the functions U, V, and W:

$$\begin{gathered}
\frac{1}{W^2} - \frac{1}{2V}\frac{\partial}{\partial r}\left(\frac{1}{W^2}\frac{\partial W^2}{\partial r}\right) - \frac{3}{4VW^4}\left(\frac{\partial W^2}{\partial r}\right)^2 - \frac{\partial}{\partial r}\left(\frac{1}{2VW^2}\frac{\partial W^2}{\partial r}\right) + \\
+ \frac{1}{2UW^2}\frac{\partial W^2}{\partial t}\frac{\partial \log(VW)}{\partial t} + \frac{m^2}{2}\left[1 - \frac{r^2}{W^2} + \frac{1}{2}\left(\frac{1}{U} - \frac{1}{V}\right)\right] = 0, \\
\frac{1}{W^2} + \frac{1}{2U}\frac{\partial}{\partial t}\left(\frac{1}{W^2}\frac{\partial W^2}{\partial t}\right) + \frac{3}{4UW^4}\left(\frac{\partial W^2}{\partial t}\right)^2 + \frac{\partial}{\partial t}\left(\frac{1}{2UW^2}\frac{\partial W^2}{\partial t}\right) - \\
- \frac{1}{2VW^2}\frac{\partial W^2}{\partial r}\frac{\partial \log(UW)}{\partial r} + \frac{m^2}{2}\left[1 - \frac{r^2}{W^2} - \frac{1}{2}\left(\frac{1}{U} - \frac{1}{V}\right)\right] = 0, \\
\frac{1}{W^2}\frac{\partial^2 W^2}{\partial t\,\partial r} - \frac{1}{2W^4}\frac{\partial W^2}{\partial r}\frac{\partial W^2}{\partial t} - \frac{1}{2VW^2}\frac{\partial V}{\partial t}\frac{\partial W^2}{\partial r} - \frac{1}{2UW^2}\frac{\partial U}{\partial r}\frac{\partial W^2}{\partial t} = 0.
\end{gathered} \tag{6}$$

Equations (2) for expressions (4) and (5) then become

$$\begin{gathered}
W^2 = \sqrt{\frac{U}{V}}\,q(r), \\
\frac{\partial}{\partial r}\left(W^2\sqrt{\frac{U}{V}}\right) = 2r\sqrt{UV},
\end{gathered} \tag{7}$$

where $q(r)$ is an arbitrary function.

The representation

$$U(t,r) = e^{\mu(t,r)}, \qquad V(t,r) = e^{\nu(t,r)}, \qquad W^2(t,r) = e^{\lambda(t,r)}, \qquad q(r) = e^{\sigma(r)}$$

is convenient in what follows. Equations (6) in the variables μ, ν, λ, and σ take the forms

$$e^{-\lambda} - e^{-\nu}\left(\lambda'' + \frac{3}{4}(\lambda')^2 - \frac{1}{2}\lambda'\nu'\right) + \frac{1}{2}e^{-\mu}\dot{\lambda}\left(\dot{\nu} + \frac{1}{2}\dot{\lambda}\right) + \frac{m^2}{2}\left[1 - r^2 e^{-\lambda} + \frac{1}{2}(e^{-\mu} - e^{-\nu})\right] = 0, \tag{8}$$

$$e^{-\lambda} + e^{-\mu}\left(\ddot{\lambda} + \frac{3}{4}(\dot{\lambda})^2 - \frac{1}{2}\dot{\lambda}\dot{\mu}\right) - \frac{1}{2}e^{-\nu}\lambda'\left(\mu' + \frac{1}{2}\lambda'\right) + \frac{m^2}{2}\left[1 - r^2 e^{-\lambda} - \frac{1}{2}(e^{-\mu} - e^{-\nu})\right] = 0, \tag{9}$$

$$\dot{\lambda}' + \frac{1}{2}\dot{\lambda}\lambda' - \frac{1}{2}\dot{\nu}\lambda' - \frac{1}{2}\dot{\lambda}\mu' = 0, \tag{10}$$

where, for instance, $\dot{\lambda} = \partial\lambda/\partial t$ and $\lambda' = \partial\lambda/\partial r$.

Equations (7) then become

$$\lambda - \frac{1}{2}(\mu - \nu) = \sigma(r), \tag{11}$$

$$\mu' - \nu' + \sigma' = 2re^{\nu-\lambda}. \tag{12}$$

We introduce the notation

$$2\omega = \mu + \nu, \tag{13}$$

$$f = \lambda - \sigma\,(r). \tag{14}$$

In accordance with (11), we have

$$\mu - \nu = 2f. \tag{15}$$

We then find from (13) and (15) that

$$\mu = \omega + f, \qquad \nu = \omega - f. \tag{16}$$

We now express Eq. (12) in terms of the functions ω, f, and σ:

$$2f' + \sigma' = 2re^{\omega - 2f - \sigma}. \tag{17}$$

Differentiating (17) w.r.t. t, we find

$$2\dot{f}' = (2f' + \sigma')(\dot{\omega} - 2\dot{f}).$$

Substituting this expression in Eq. (10) and taking equalities (14) and (16) into account, we obtain the inhomogeneous linear partial differential equation

$$\frac{\partial\omega}{\partial t}f' - \frac{\partial\omega}{\partial r}\dot{f} = 3\dot{f}f'. \tag{18}$$

The system of ordinary differential equations corresponding to Eq. (18) is

$$\frac{dt}{f'} = \frac{dr}{-\dot{f}} = \frac{d\omega}{3\dot{f}f'},$$

whence we find

$$d\omega = 3\dot{f}\,dt, \qquad d\omega = -3f'\,dr.$$

Adding these equalities, we obtain

$$d\omega = \frac{3}{2}\frac{\partial f}{\partial t}\,dt - \frac{3}{2}\frac{\partial f}{\partial r}\,dr.$$

We then find from the condition of the total differential that

$$\frac{\partial^2 f}{\partial t\,\partial r} = 0,$$

but the latter means that the function f is representable in the form

$$f(t,r) = \psi(t) + \varphi(r). \tag{19}$$

The general solution of Eq. (18) is

$$\omega(t,r) = \frac{3}{2}(\psi(t) - \varphi(r)) + F(f), \tag{20}$$

where F is an arbitrary function.

Taking expressions (19) and (20) into account, we can write Eq. (17) as

$$2\varphi' + \sigma' = 2r\exp\left[-\frac{1}{2}\psi(t) - \frac{7}{2}\varphi(r) + F(f)\right].$$

Because the l.h.s. of this equation is independent of t, the r.h.s. must also be independent of t. This is possible only if the function $\psi(t)$ is constant, but this in turn implies that the functions μ, ν, and λ must be time-independent. *In this case, the gravitational field of form* (4) *is static*. But the second case, where $F = f/2$, is also possible. Then,

$$\omega(t,r) = 2\psi(t) - \varphi(r),$$

and the functions μ, ν, and λ, according to (14) and (16), must be

$$\mu(t) = 3\psi(t), \tag{21}$$

$$\nu(t,r) = \psi(t) - 2\varphi(r), \tag{22}$$

$$\lambda(t,r) = \psi(t) + \varphi(r) + \sigma(r) = f + \sigma. \tag{23}$$

To analyze this case, we must return to Eqs. (8) and (9). The difference of these equations is

$$e^{-\nu}\left[-\lambda'' - \frac{1}{2}(\lambda')^2 + \frac{1}{2}\lambda'(\mu' + \nu')\right] + \\ + e^{-\mu}\left[-\ddot{\lambda} - \frac{1}{2}(\dot{\lambda})^2 + \frac{1}{2}\dot{\lambda}(\dot{\mu} + \dot{\nu})\right] + \frac{m^2}{2}\left(e^{-\mu} - e^{-\nu}\right) = 0, \tag{24}$$

and their sum is

$$2e^{-\lambda} - e^{-\nu}\left[\lambda'' + (\lambda')^2 - \frac{1}{2}\lambda'(\mu' - \nu')\right] +$$

$$+ e^{-\mu}\left[\ddot{\lambda} + (\dot{\lambda})^2 - \frac{1}{2}\dot{\lambda}(\dot{\mu} - \dot{\nu})\right] + m^2(1 - r^2 e^{-\lambda}) = 0. \tag{25}$$

Following (11), we have

$$(\dot{\lambda})^2 - \frac{1}{2}\dot{\lambda}(\dot{\mu} - \dot{\nu}) = 0,$$

and Eq. (25) hence becomes slightly simplified,

$$2e^{-\lambda} - e^{-\nu}\left[\lambda'' + (\lambda')^2 - \frac{1}{2}\lambda'(\mu' - \nu')\right] + e^{-\mu}\ddot{\lambda} + m^2(1 - r^2 e^{-\lambda}) = 0. \tag{26}$$

We express Eqs. (24) and (26) in terms of the functions ω, f, and σ:

$$- f'' - \sigma'' - \frac{1}{2}(f' + \sigma')^2 + (f' + \sigma')\omega' - \frac{m^2}{2} + e^{-2f}\left(-\ddot{f} - \frac{1}{2}(\dot{f})^2 + \dot{f}\dot{\omega} + \frac{m^2}{2}\right) = 0, \tag{27}$$

$$2\left(1 - \frac{m^2 r^2}{2}\right)e^{-(\lambda-\nu)} - \left[f'' + \sigma'' + (f' + \sigma')^2 - f'(f' + \sigma')\right] + e^{-2f}(\ddot{f} + m^2 e^{\mu}) = 0. \tag{28}$$

In accordance with (12), we have

$$2re^{\nu-\lambda} = 2f' + \sigma', \tag{29}$$

and after the exponential multiplier is replaced, Eq. (28) hence becomes

$$\frac{1}{r}\left(1 - \frac{m^2 r^2}{2}\right)(2f' + \sigma') - \left[f'' + \sigma'' + (f' + \sigma')^2 - f'(f' + \sigma')\right] + e^{-2f}(\ddot{f} + m^2 e^{\mu}) = 0. \tag{30}$$

Because μ depends only on t and $f = \psi(t) + \varphi(r)$, the variables t and r become separated in Eqs. (27) and (30):

$$- f'' - \sigma'' - \frac{1}{2}(f' + \sigma')^2 + (f' + \sigma')\omega' - \frac{m^2}{2} = ke^{-2\varphi(r)}, \tag{31}$$

$$\ddot{f} + \frac{1}{2}(\dot{f})^2 - \dot{f}\dot{\omega} - \frac{m^2}{2} = ke^{2\psi(t)}, \tag{32}$$

$$f'' + \sigma'' + \sigma'(f' + \sigma') - \frac{1}{r}\left(1 - \frac{m^2 r^2}{2}\right)(2f' + \sigma') = pe^{-2\varphi(r)}, \tag{33}$$

$$\ddot{f} + m^2 e^{\mu} = pe^{2\psi(t)}, \tag{34}$$

where k and p are the separation constants.

We now turn to Eqs. (32) and (34). We introduce a new variable,

$$\psi(t) = \log a^2(t). \tag{35}$$

Equations (32) and (34) then become

$$2a\ddot{a} - 8\dot{a}^2 - \frac{m^2}{2}a^2 = ka^6, \tag{36}$$

$$2a\ddot{a} - 2\dot{a}^2 + m^2a^8 = pa^6, \tag{37}$$

whence we find

$$\dot{a}^2 = -\frac{1}{12}\big[m^2a^2 + 2m^2a^8 + 2(k-p)a^6\big]. \tag{38}$$

Differentiating, we obtain

$$\ddot{a} = -\frac{1}{24}\big[2m^2a + 16m^2a^7 + 12(k-p)a^5\big]. \tag{39}$$

Substituting (38) and (39) in Eq. (36), we obtain the relation $p = -2k$ for the separation constants. We then have

$$\dot{a}^2 = -\frac{1}{12}(m^2a^2 + 2m^2a^8 + 6ka^6). \tag{40}$$

We now pass to analyzing Eqs. (31) and (33). Taking (22) and (23) into account, we write Eq. (29) in the form

$$\varphi' = -\frac{1}{2}\sigma' + re^{-3\varphi-\sigma}. \tag{41}$$

Differentiating this expression, we find

$$\varphi'' = -\frac{1}{2}\sigma'' + \frac{1}{2}r\sigma' e^{-3\varphi-\sigma} + e^{-3\varphi-\sigma} - 3r^2e^{-6\varphi-2\sigma}. \tag{42}$$

Substituting (41) and (42) in Eqs. (31) and (33), we obtain

$$\begin{aligned} &-3r^2e^{-2\sigma}x^2 + 2kx^{2/3} = -x(2+2r\sigma')e^{-\sigma} - \sigma'' + \frac{1}{4}(\sigma')^2 - m^2, \\ &-3r^2e^{-2\sigma}x^2 + 2kx^{2/3} = x\left[2\left(1 - \frac{m^2r^2}{2}\right) - \frac{3}{2}r\sigma' - 1\right]e^{-\sigma} - \frac{1}{2}\sigma'' - \frac{1}{2}(\sigma')^2, \end{aligned} \tag{43}$$

where we introduce $x = e^{-3\varphi}$.

We consider the equations

$$\begin{aligned} &-3r^2e^{-2\sigma}x^2 + 2x(1+r\sigma')e^{-\sigma} + \sigma'' - \frac{1}{4}(\sigma')^2 = 0, \\ &-3r^2e^{-2\sigma}x^2 - x\left(1 - \frac{3}{2}r\sigma'\right)e^{-\sigma} + \frac{1}{2}\sigma'' + \frac{1}{2}(\sigma')^2 = 0. \end{aligned} \tag{44}$$

Subtracting one equation from the other, we obtain

$$x = \frac{(3/4)(\sigma')^2 - (1/2)\sigma''}{3 + (1/2)r\sigma'}e^{\sigma} = \alpha = e^{-3\varphi}. \tag{45}$$

The quantity α is a root of Eqs. (44) at the corresponding σ. Substituting $x = \alpha$ in Eqs. (43), we find

$$2k\alpha^{2/3} = -m^2, \tag{46}$$

$$2k = -\alpha^{1/3} m^2 r^2 e^{-\sigma}. \tag{47}$$

From Eqs. (46) and (47), we then have

$$\alpha = \frac{1}{r^2} e^{\sigma}. \tag{48}$$

Because the quantity α must be constant in accordance with (46), expression (48) implies that

$$\sigma = \log\alpha + \log r^2. \tag{49}$$

From relations (46) and (47), we then have

$$k = -\frac{m^2}{2\alpha^{2/3}}. \tag{50}$$

In accordance with causality principle (3), we have $U \le V$. To satisfy this inequality, it suffices, by virtue of (21) and (22), to impose the conditions

$$1 \le e^{-2\varphi(r)}, \tag{51}$$

$$e^{2\psi(t)} \le 1. \tag{52}$$

Inequality (52) expresses the physical property of the gravitational field to dilate time compared with the inertial time. Taking (45) into account, we can write condition (51) in the form

$$\alpha^{2/3} \ge 1. \tag{53}$$

Substituting the value of k taken from (50) in (40), we obtain

$$\dot{a}^2 = -\frac{m^2 a^8}{12}\left(\frac{1}{a^6} + 2 - \frac{3}{\alpha^{2/3} a^2}\right), \quad a > 0. \tag{54}$$

Because the r.h.s. of (54) with (53) included is strictly negative and the r.h.s. is always positive, the equality is possible only if a is constant and the r.h.s. is zero.

The r.h.s. of equality (54) satisfies the inequality

$$\left(\frac{1}{a^6} + 2 - \frac{3}{\alpha^{2/3} a^2}\right) \ge \frac{2}{a^6}(a^2 - 1)^2\left(a^2 + \frac{1}{2}\right).$$

In order for the l.h.s. of this inequality to vanish, we must set

$$a = 1, \qquad \alpha^{2/3} = 1. \tag{55}$$

Taking (35), (45), and (55) into account, we obtain

$$\psi = 0, \qquad \varphi = 0. \tag{56}$$

By virtue of (49), (55), and (56), we obtain the following metric coefficients in the second case under consideration:

$$U = 1, \qquad V = 1, \qquad W^2 = r^2. \tag{57}$$

These metric coefficients correspond to the Minkowski space metric, which means that the gravitational field is absent.

We thus obtain the general conclusion: for a *nonstatic* spherically symmetric source, the metric coefficients of the interval of an external gravitational field of form (4) in the inertial frame can only be *static*. Because the external gravitational field of form (4) created by a nonstatic spherically symmetric body is static, we conclude based on the results in [3] that the radius of this body is always greater than the Schwarzschild radius because of repulsion forces.

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