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NECESSARY COVARIANCE CONDITIONS FOR A ONE-FIELD LAX PAIR

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We study the covariance with respect to Darboux transformations of polynomial differential and difference operators with coefficients given by functions of one basic field. In the scalar (Abelian) case, the functional dependence is established by equating the Frechèt differential (the first term of the Taylor series on the prolonged space) to the Darboux transform; a Lax pair for the Boussinesq equation is considered. For a pair of generalized Zakharov–Shabat problems (with differential and shift operators) with operator coefficients, we construct a set of integrable nonlinear equations together with explicit dressing formulas. Non-Abelian special functions are fixed as the fields of the covariant pairs. We introduce a difference Lax pair, a combined gauge–Darboux transformation, and solutions of the Nahm equations.

Keywords: Darboux transformation, Lax pair, Boussinesq equation, Zakharov–Shabat problem, shift operator polynomial, Nahm equation

1. Introduction

We consider a kind of form-invariance of differential polynomials, having in mind a link between the operator

$$\sum_{k=0}^{n} a_k \partial^k$$

and the operator of the same order but with new (transformed) coefficients

$$\sum_{k=0}^{n} a_k[1] \partial^k.$$

We call this property the "covariance" of the operator under some kind of transformation. We use a similar term in the case of shift operator polynomials.

The proof of the covariance of the equation

$$\psi_t = \sum_{k=0}^{n} a_k \partial^k \psi \tag{1}$$

with noncommutative coefficients a_k under the classic Darboux transformation (DT) [1]

$$\psi[1] = \psi' - \sigma\psi \tag{2}$$

incorporates the auxiliary relation (we use the short notation $\psi' = \partial\psi = \psi_x$ throughout the paper) [2]

$$\sigma_t = \partial r + [r, \sigma], \qquad r = \sum_{0}^{N} a_n B_n(\sigma), \tag{3}$$

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where B_n are differential Bell polynomials [3]. Relation (3) generalizes the so-called Miura map and becomes an identity when $\sigma = \phi'\phi^{-1}$, where ϕ is a solution of Eq. (1).

The study of jointly covariant combinations of (abstract) derivatives introduces extra constraints on the polynomial coefficients, which can be classified [4], [5]. Briefly, given the general statement about the covariant form of a linear polynomial differential operator that determines transformation formulas for the coefficients, the consistency between such formulas yields constraints. In the scalar case, one-potential constructions for the KdV and Boussinesq equations were studied in [5], [6] (see Sec. 2.1) for the higher KdV and KP equations [7].

Examples of nonlocal equations (with non-Abelian entries) integrable by the DT were considered in [1], [8]. Some of them, reviewed in [9], [10], were recently generalized to a wide class of polynomials of automorphisms on a differential ring [11]. Exploring noncommutative coefficients is more complicated but much more rich and promising. This can already be seen from the standpoint of symmetry classification, starting from the pioneering papers [12] (also see [13], [14] and [15]). Next, the link to the DT covariance approach [16] allows constructing and classifying covariant functions for use in quantum [17] and soliton problems. For example, there is a class of such equations

$$-i\rho_t = \big[H, h(\rho)\big], \tag{4}$$

where $h(\rho)$ is an analytic function. In quantum mechanics, the operators ρ and H play the roles of the density matrix and Hamiltonian. The covariance of (4) under a DT was established in [18]. The cases where $f(\rho) = i\rho^3$ and $f(\rho) = i\rho^{-1}$ were considered as applications in [19]. A further step to essentially non-Abelian functions, e.g.,

$$h(X) = XA + AX, \tag{5}$$

$[A, X] \neq 0$, led to [20], and generalizations were studied in [16]. Further generalizations were demonstrated in [21], where an abundant set of integrable equations was listed. The list partially corresponds with [13], and a link to solutions via the iteration procedures or dressing chains, usual for the DT technique, was shown.

2. One-field Lax pair in the Abelian case

2.1. Covariance equations. We first reproduce the one-field scheme for the scalar commutative fields, generalizing the studies of the example of the Boussinesq equation [5], [22]. We consider Eq. (1) with the coefficients b_k, $k = 0, 1, 2, 3$, reserving a_k for the second operator of the Lax pair. The DT for the third-order operator coefficients (Matveev's generalization of the classic DT [1]) yields

$$b_2[1] = b_2 + b_3', \qquad b_1[1] = b_1 + b_2' + 3b_3\sigma', \tag{6}$$

$$b_0[1] = b_0 + b_1' + \sigma b_2' + 3b_3(\sigma\sigma' + \sigma''), \tag{7}$$

and b_3 does not transform. We now suppose that both operators depend on a single potential function w. The problem for the first operator is formulated as follows: to find restrictions on the coefficients $b_3(x, t)$, $b_2(x, t)$, $b_1 = b(w, t)$, and $b_0 = G(w, t)$ compatible with DT rules (6) and (7) of the potential function w induced by the DT for b_1 or b_0. For simplicity, we set $b_3' = 0$.

The covariance of the corresponding spectral equation

$$b_3\psi_{xxx} + b_2(x, t)\psi_{xx} + b(w, t)\psi_x + G(w, t)\psi = \lambda\psi \tag{8}$$

leads to a relation only between b_1 and b_0. In studying the problem in (8) in the context of the Lax representation for a nonlinear equation, we should take the covariance of the second Lax equation into account from the very beginning. We call this the *principle of joint covariance* [4]. The form of the second Lax equation fixes the place of the field w:

$$\psi_t = a_2(t)\psi_{xx} + a_1(t)\psi_x + w\psi. \tag{9}$$

If we consider (8) and (9) as Lax-pair equations, the DT of w must be compatible with the DT formulas (similar to (6)) for both coefficients in (8) depending only on the variable w. Next, generalized Miura map (3) is [2]

$$\sigma_t = \bigl[a_2(\sigma^2 + \sigma_x) + a_1\sigma + w\bigr]_x \tag{10}$$

for problem (9); for evolution equation (8), it is

$$b_3(\sigma^3 + 3\sigma_x\sigma + \sigma_{xx}) + b_2(\sigma^2 + \sigma_x) + b(w,t)\sigma + G(w) = \text{const}, \tag{11}$$

where ϕ is now a solution of both Lax equations.

We next suppose that the coefficients of the operators are analytic functions of w together with their derivatives (or integrals) with respect to x (such functions are called functions on the prolonged space). For the coefficient $b_1 = b(\partial^{-1}w, w, w_x, \dots, \partial^{-1}w_t, w_t, w_{tx}, \dots)$, the covariance condition is obtained for the Frechèt differential (FD) of the function b on the prolonged space. Equating the expansion to the DT leads to the condition

$$b_2' + 3b_3\sigma' = b_w(a_1' + 2a_2\sigma' + \sigma a_2') + b_{w'}(a_1' + 2a_2\sigma' + \sigma a_2')' + \dots . \tag{12}$$

We call this equation the (first) *joint covariance equation*; it guarantees the consistency between transformations of the coefficients of Lax pair (8), (9). Comparing the two transforms gives the expression for $b(w,t)$ (with arbitrary $\alpha(t)$):

$$b(w,t) = \frac{3b_3w}{2a_2} + \alpha(t). \tag{13}$$

Equating the expansion of $b_0 = G(\dots, w, \dots)$ on the prolonged space,

$$G(w + a_1' + 2a_2\sigma' + \sigma a_2') = G(w) + G_w(a_1' + 2a_2\sigma' + \sigma a_2') + \dots, \tag{14}$$

to DT (7) for the same coefficient, we have

$$\begin{aligned} b_1' + \sigma b_2' + 3b_3\left(\frac{\sigma^2}{2} + \sigma'\right)' &= \\ &= G_{w_x}(a_1' + 2a_2\sigma' + \sigma a_2')' + G_{\partial^{-1}w_t}\bigl[a_{1t} + 2\partial^{-1}(a_2\sigma_t') + \partial^{-1}(\sigma a_2')_t\bigr] + \dots . \end{aligned} \tag{15}$$

This second *joint covariance equation* also simplifies when $a_2' = 0$. We note that relation (10) is used in the left-hand side of (15) and linearizes the FD with respect to σ. Finally,

$$\begin{gathered} b_2 = \frac{3b_3a_1}{2a_2} + \beta(t), \\ G(w,t) = \frac{3b_3w_x}{2a_2} + \frac{3b_3a_1'\partial^{-1}w}{2a_2^2} + \frac{3b_3\partial^{-1}w_t}{2a_2^2}. \end{gathered} \tag{16}$$

Remark 2.1. We truncate the FD formulas at the level that is necessary for the minimal flows. Taking the higher terms into account leads to the entire hierarchy as in [7] for the KdV–KP case.

2.2. Compatibility condition. In the case where $a_2' = 0$, to which we restrict ourself, Lax system (8), (9) produces the compatibility conditions

$$\begin{aligned}
b_{3t} &= 2a_2 b_2' - 3b_3 a_1'', \\
b_{2t} &= a_2 b_2'' + 2a_2 b_1' + a_1 b_2' - 3b_3 a_1'' - 2b_2 a_1' - 3b_3 a_0', \\
b_{1t} &= a_2 b_1'' + a_1 b_1' - b_3 a_1''' - b_2 a_1'' - b_1 a_1' - 3b_3 a_0'' - 2b_2 a_0' + 2a_2 b_0', \\
b_{0t} &= a_1 b_0' + a_2 b_0'' - b_1 a_0' - b_2 a_0'' - b_3 a_0'''.
\end{aligned} \tag{17}$$

Relations (16) and (17) together with the expression for b_{2t} produce

$$\beta_t = -2\beta a_1'. \tag{18}$$

The last two equations (with the constants chosen as $b_3 = 1$, $a_2 = -1$, and $b_2' = a_1' = 0$) give

$$\frac{3b_3(w_t + a_1 w)_t}{4a_2^2} = -\left[\left(\frac{3b_3 w}{2a_2} + \alpha\right)w' - \frac{b_3 w'''}{4} + \frac{3b_3 a_1 w_t}{4a_2^2} + \left(\beta - \frac{3b_3 a_1}{4a_2}\right)w''\right]'. \tag{19}$$

This equation reduces to the Boussinesq equation (see, e.g., [9]) for $b_1 = a_1 = 0$, $b_3 = 1$, and $a_2 = -1$.

3. Non-Abelian case: Zakharov–Shabat (ZS) problem

3.1. Joint covariance conditions for general ZS equations. We change the notation for first-order (n=1) equation (1) with coefficients from a non-Abelian differential ring A (see [2] for details) to

$$\psi_t = (J + u\partial)\psi, \tag{20}$$

where the operator $J \in A$ is independent of x, y, and t and the potential $a_0 \equiv u = u(x, y, t) \in A$ is a function of all the variables. The transformed potential is

$$\tilde{u} = u + [J, \sigma], \tag{21}$$

where again $\sigma = \phi_x \phi^{-1}$ with $\phi \in A$ being a solution of (20).

We suppose that the second operator of the Lax pair has the same form but with different entries,

$$\psi_y = (Y + w\partial)\psi, \tag{22}$$

$Y \in A$ is a constant element, and the potential $w = F(u) \in A$ is a function of the potential of Eq. (20). The principle of joint covariance [4] is then given by

$$\widetilde{w} = w + [Y, \sigma] = F\big(u + [J, \sigma]\big) \tag{23}$$

with the direct corollary

$$F(u) + [Y, \sigma] = F\big(u + [J, \sigma]\big). \tag{24}$$

Hence, Eq. (24) defines the function $F(u)$. We also call this equation the *joint covariance equation.*

3.2. Covariant combinations of symmetric polynomials. The first natural example is the generalized Euler top equation with Hamiltonian (5), mentioned in the introduction. In this case, the covariant Lax pair consists of two equations, Eqs. (20) and (22), the entries of the operators satisfy joint covariance equation (24) and the compatibility condition if $J = H$ and $Y = H^2$ [16]. A more general link $Y = J^n$, $J = H$ leads to the covariance of the function

$$P_n(H,u) = \sum_{p=0}^{n} H^{n-p} u H^p,$$

given in [16]. For a further generalization, we consider combinations of polynomials such as

$$f(H,u) = Hu + uH + S^2u + SuS + uS^2. \tag{25}$$

Substituting (25) as $F(u) = f(H,u)$ in (24) suggests the choice $Y = AB + CDE$, which yields

$$A[B,\sigma] + [A,\sigma]B + CD[E,\sigma] + C[D,\sigma]E + [C,\sigma]DE =$$
$$= H[J,\sigma] + [J,\sigma]H + S^2[J,\sigma] + S[J,\sigma]S + [J,\sigma]S^2.$$

This expression becomes an identity if $A = B = J = H$, $C = D = E = \alpha H$, $S = \beta H$, $[\alpha, H] = 0$, and $[\beta, H] = 0$ with $\alpha^3 = \beta^2$. Continuing this analysis, we obtain the following statement.

Statement. *The joint covariance principle defines a class of homogeneous polynomials $P_n(H,u)$, symmetric under cyclic permutations, as possible Hamiltonians $h(u) = P_n(H,u)$ for Liouville–von Neumann-type evolution (4). A linear combination of such polynomials $\sum_{n=1}^{N} \beta_n P_n(H,u)$ with the coefficients commuting with u and H also yields a covariant pair if $Y = \sum_{n=1}^{N} \alpha_n H^{n+1}$ and $\alpha_1 = \beta_1 = 1$, $\alpha_n^{n+1} = \beta_n^n$, $n \neq 1$, hold.*

The proof can be given by induction based on the homogeneity of P_n and the linearity of the constraints in u. The functions $F_H(u) = \sum_0^\infty a_n P_n(H,u)$ also satisfy the constraints and define a function if the series converges.

4. Nonlocal operators

Let A be an operator ring, and let an automorphism T have the property $T(fg) = T(f)T(g)$ for given elements $f, g \in A$. The general DT formulas for an operator polynomial in T are given in [11]. We call the operator T a *shift operator*, but it could be general as defined above.

4.1. The one-field first-order shift operator evolution. We take the general evolution equation in the case where $N = 1$,

$$\psi_t(x,t) = (U_0 + U_1 T)\psi. \tag{26}$$

There are two types of DT in this case [11], [23], denoted by the superscripts + and −. The DT of the first kind (+) leaves U_0 unchanged. We rewrite the transform of U_1 as

$$U_1^+ = \sigma^+ (TU_1)(T\sigma^+)^{-1}, \tag{27}$$

where $\sigma^+ = \phi(T\phi)^{-1}$ and the superscript + is omitted in what follows.

Nontrivial chain equations occur for the stationary version of (26),

$$(J + UT)\varphi = \varphi\mu, \tag{28}$$

if the constant element μ does not commute with φ and σ.

We derive the identity that links the potential U and σ (cf. [23]) by substituting

$$T(\sigma)T^2(\varphi) = T(\varphi) \tag{29}$$

in shifted equation (28). We have a Miura-like link

$$\sigma T(U)\sigma = U + [J, \sigma], \tag{30}$$

where $T(\sigma) = \sigma^{-1}$ is taken into account. It also simplifies DT formula (27):

$$U^+ = \sigma(TU)\sigma = U + [J, \sigma]. \tag{31}$$

Directly using Eq. (28) to express U in terms of $\tau = \varphi\mu\varphi^{-1}$ and σ gives $U = \tau\sigma - J\sigma$. Substituting the result in (30) yields

$$T(\tau) = \sigma^{-1}\tau\sigma, \tag{32}$$

which defines the element τ on a set of points $T^m x_0$ as an explicit function of σ,

$$\tau = \sigma^{-m}\tau(x_0)\sigma^m, \tag{33}$$

which in turn gives an explicit expression for U. Finally, substituting U in (31), we obtain the dressing chain equation

$$\sigma_n^{-m}\tau(x_0)\sigma_n^m\sigma_n - \sigma_n J = \sigma_{n+1}^{-m}\tau(x_0)\sigma_{n+1}^m\sigma_{n+1} - J\sigma_{n+1} \tag{34}$$

for all n, parameterized by μ via $\tau(x_0)$.

The chain equation is a natural result of expressing the DT in terms of the variable σ via the Miura map and was studied in [24] in connection with the celebrated scalar KdV and the standard Sturm–Liouville problem. A periodic closure of the chain produces integrable bi-Hamiltonian finite-dimensional systems and finite-gap potentials in some special cases [25]. The Boussinesq case was explored in [22] (also see [26]). The chain equations for the classical differential ZS problem and two types of the DT were introduced in [27] (also see [15]).

4.2. The joint covariance equations. We take a replica of Eq. (26),

$$\psi_t(x,t) = \big(V_0 + V_1(U)T\big)\psi, \tag{35}$$

where $U_1 \to V_1 = F(U)$ is changed, thus introducing a one-field Lax pair made of two ZS evolution equations. We mean that the coefficients of both equations depend only on the operator U. Again, the invariance of the constant V_0 is implied as the property of the DT of the first kind. We transform V_1 using (27), considering the result as the same function of U_1^+; this yields the joint covariance equation

$$V_1^+ = F(U^+) = F\big(\sigma^+(TU)(T\sigma^+)^{-1}\big) = \sigma^+\big(TF(U)\big)(T\sigma^+)^{-1}. \tag{36}$$

A similar equation can be derived for the alternative "−" system.

5. Covariance theorems for higher operators

The following theorem establishes the covariance of both equations generating the Lax pair for the Nahm equation [28].

Theorem 5.1. *The equation*

$$\psi_y = uT\psi + v\psi + wT^{-1}\psi \tag{37}$$

is covariant under the DT combined with the gauge transformation

$$\psi[1] = g(y)(T-\sigma)\psi, \tag{38}$$

where $g(y) \in A$, $\sigma = (T\phi)/\phi$, and ϕ is a (different) solution of the same Eq. (37). The transforms of the equation coefficients are

$$\begin{aligned}
u[1] &= gT(u)\big[T(g)\big]^{-1},\\
v[1] &= gT(v)g^{-1} - g\sigma u g^{-1} + gT(u)T(\sigma)g^{-1} + g_y g^{-1},\\
w[1] &= g\sigma w\big[T^{-1}(g\sigma)\big]^{-1}.
\end{aligned} \tag{39}$$

Proof. Substituting (38) in transformed equation (37) gives four equations, while the $T^n\psi$ are independent. Three of them yield transformed potentials (39). The fourth equation, after the transformations are used, becomes

$$\sigma_y = \sigma F - (TF)\sigma, \qquad F = u\sigma + v + w\big[T^{-1}(\sigma)\big]^{-1}. \tag{40}$$

We can verify the condition by substituting the definition of σ and using the equation for ϕ.

Remark 5.1. Theorem 5.1 is naturally applicable to the spectral problem

$$\lambda\psi = uT\psi + v\psi + wT^{-1}\psi, \tag{41}$$

which occurs for stationary solutions of (37) with only the last term absent for the transform $v[1]$. The transformation involves the eigenfunction ϕ with a different eigenvalue μ. Equation (40) becomes a "Miura-transformation" analogue of the function σ,

$$\mu = u\sigma + v + w\big[T^{-1}(\sigma)\big]^{-1}. \tag{42}$$

6. Nahm equation reductions

The Nahm equations can be written in the Lax representation (originally as differential operators [28]) using spectral equation (41) and the evolution equation

$$\psi_t = (q + pT)\psi. \tag{43}$$

The covariance of this equation under DT (38) can be established similarly to Theorem 5.1 with the evolution of the function $\sigma(t)$ taken into account,

$$\sigma_t = T(q)\sigma - \sigma p\sigma + T(p)T(\sigma)\sigma - \sigma q. \tag{44}$$

This proves the transformation formulas for the coefficients in (43):

$$p[1] = gT(p)\big[T(g)\big]^{-1}, \tag{45}$$

$$q[1] = g\big[T(q) - \sigma p + T(p)T(\sigma)\big]g^{-1} + g_y g^{-1}. \tag{46}$$

The choice $p = u + \beta I$, $q = v/2$, $T(\varphi_i) = \varphi_i$,

$$u = \alpha\biggl(-\frac{\imath\varphi_1}{2} - \varphi_3\biggr), \qquad v = \varphi_3, \qquad w = \alpha^{-1}\biggl(-\frac{\imath\varphi_1}{2} + \varphi_3\biggr) \tag{47}$$

produces the Nahm equations

$$\varphi_{i,t} = \imath\epsilon_{ikl}[\varphi_k, \varphi_l]. \tag{48}$$

System (41), (43) is covariant under the combined DT–gauge transformations if the variable $g = e^G$ is chosen as

$$G_t = \frac{\alpha}{2}\biggl[T\biggl(\varphi_3 + \frac{\varphi_1}{2}\biggr)T(\sigma) - \sigma\biggl(\varphi_3 + \frac{\varphi_1}{2}\biggr)\biggr]. \tag{49}$$

Finally, we can formulate the following theorem.

Theorem 6.1. *System* (48) *is invariant under the transformations*

$$\begin{aligned}
\varphi_1[1] &= g\biggl[\biggl(\frac{T(\varphi_1)}{2} - \imath T(\varphi_3)\biggr)T(g)^{-1} + \sigma\biggl(\frac{\varphi_1}{2} + \imath\varphi_3\biggr)\big[T^{-1}(g\sigma)\big]^{-1}\biggr],\\
\varphi_2[1] &= g\biggl[T(\varphi_2) + \alpha\biggl(\frac{\imath\sigma\varphi_1}{2} - \frac{\imath\varphi_1 T(\sigma)}{2} + \sigma\varphi_3 - T(\varphi_3\sigma)\biggr)\biggr]g^{-1},\\
\varphi_3[1] &= g\biggl[\biggl(T\biggl(-\frac{\imath\varphi_1}{2} - \varphi_3\biggr)\biggr)T(g)^{-1} + \sigma\biggl(-\frac{\imath\varphi_1}{2} + \varphi_3\biggr)\big[T^{-1}(g\sigma)\big]^{-1}\biggr]
\end{aligned} \tag{50}$$

with the function $g = e^G$, where G is obtained by integrating (49).

7. Solutions of the Nahm equation

By the construction described in the preceding section, we follow the algorithm for a simple example with T considered as the shift operator $T\psi(x, y) = \psi(x + 1, y)$. As a seed solution of Nahm equations (48), we consider commuting constant matrices $\varphi_i = A_i$, $i = 1, 2, 3$, which implies that u, v, and w are constant in (47). First, we must generate a solution of Lax pair (41), (43); we can find it by

$$\phi = \xi(t)\Phi(x) \tag{51}$$

(all elements are assumed invertible). We obtain the equation for ξ as

$$\xi_t = \biggl[\frac{v}{2} + (u + \beta I)T\biggr]\xi = Z\xi, \tag{52}$$

which is solved by

$$\xi = e^{Zt}\xi_0. \tag{53}$$

Substituting ϕ in (41) yields the difference-shift-operator spectral problem

$$\mu\Phi(x) = \xi^{-1}\big[u\xi\Phi(x+1) + v\xi\Phi + w\xi\Phi(x-1)\big]. \tag{54}$$

Separating the variables again, we construct a class of particular solutions as

$$\Phi = \eta e^{\Sigma x}. \tag{55}$$

We thus obtain a matrix spectral problem for η,

$$\mu\eta = \xi^{-1}\big[u\xi\eta e^{\Sigma} + v\xi\eta + w\xi\eta e^{-\Sigma}\big] \tag{56}$$

with the operator in the right-hand side and hence the spectral parameter μ parameterized by t. Finally, the matrix σ is composed as

$$\sigma = \xi(t)\eta e^{\Sigma}\eta^{-1}\xi^{-1}(t). \tag{57}$$

An appropriate choice of the commutator algebra for A_i, Σ, and η allows obtaining the explicit form of σ and therefore constructing and solving Eq. (49) for G,

$$G_t = \frac{\alpha}{2}\left[\left(\varphi_3 + \frac{\varphi_1}{2}\right)\xi(t)\eta e^{\Sigma}\eta^{-1}\xi^{-1}(t) - \xi(t)\eta e^{\Sigma}\eta^{-1}\xi^{-1}(t)\left(\varphi_3 + \frac{\varphi_1}{2}\right)\right], \tag{58}$$

and then obtaining its exponential, i.e., the matrix g required for dressing formulas (50). We stress that the matrices σ and g are independent of x; hence, the dressed $\varphi[i]$ is also independent of x.

8. Conclusions

We have developed a general method for establishing one-field Darboux-covariant operators. It results in the equations of joint covariance for the operator coefficients as functions on the prolonged space. We found some solutions of the equations via the generalized Taylor expansion of functions [29].

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