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TRAVELING-WAVE SOLUTIONS OF THE CALOGERO–DEGASPERIS–FOKAS EQUATION IN 2+1 DIMENSIONS

M. L. Gandarias* **and S. Saez***

Soliton solutions are among the more interesting solutions of the (2+1)*-dimensional integrable Calogero–Degasperis–Fokas* (*CDF*) *equation. We previously derived a complete group classification for the CDF equation in* 2+1 *dimensions. Using classical Lie symmetries, we now consider traveling-wave reductions with a variable velocity depending on an arbitrary function. The corresponding solutions of the* (2+1)*-dimensional equation involve up to three arbitrary smooth functions. The solutions consequently exhibit a rich variety of qualitative behaviors. Choosing the arbitrary functions appropriately, we exhibit solitary waves and bound states.*

Keywords: Lie symmetries, partial differential equations, solitary waves

1. Introduction

We discuss the (2+1)-dimensional integrable generalization of the Calogero–Degasperis–Fokas (CDF) equation

$$u_t + \frac{1}{4}u_{xxz} - \frac{1}{2}\frac{u_x u_{xz}}{u} - \frac{1}{4}\frac{u_{xx}u_z}{u} + \frac{1}{2}\frac{u_x^2 u_z}{u^2} - \\ - \frac{1}{8}u_x\partial_x^{-1}\left(\frac{u_x^2}{u^2}\right)_z + \frac{1}{4}a^2u^2u_z + \frac{1}{8}a^2u_x\partial_x^{-1}(u^2)_z + \\ + \frac{1}{2}abu_z + \frac{1}{4}abu_x + \frac{1}{4}b^2\frac{u_z}{u^2} + \frac{1}{8}b^2u_x\partial_x^{-1}\left(\frac{1}{u^2}\right)_z = 0, \tag{1}$$

where $\partial_x^{-1}u = \int u\,dx$. This equation was derived by Toda and Yu [1].

A wide class of differential equations with interesting properties is integrable by the inverse spectral transform. One of these equations is the CDF equation in 1+1 dimensions. The CDF equation is a (1+1)-dimensional nonlinear equation having the form

$$u_t + \frac{1}{4}u_{xxx} - \frac{3}{4}\frac{u_x u_{xx}}{u} + \frac{3}{8}\frac{u_x^3}{u^2} + \frac{3}{8}\frac{u_x}{u^2}(au^2+b)^2 = 0, \tag{2}$$

where a and b are arbitrary constants. Equation (2) was introduced by Calogero and Degasperis [2], investigating equations solvable by a matrix variant of the inverse transformation, and independently by Fokas [3], investigating KdV-type equations with certain Lie–Bäcklund symmetries. Exact multisoliton solutions of (2) were obtained from its bilinear form (when $a > 0$) [4]. The CDF equation was also studied by others [5], [6]. In [7], an extended Dym equation was generated by the purely binormal motion of an

*Departamento de Matemáticas, Universidad de Cádiz, PO Box 40, 11510, Puerto Real, Cádiz, Spain, e-mail: marialuz.gandarias@uca.es.

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inextensible curve of constant curvature. This extended Dym equation is readily established by a reciprocal link to the the CDF equation. Moreover, it is well known that the CDF equation reduces to the Calogero–Korteweg–de Vries (CKdV) equation [8] when $a = 0$ and $b = \pm 1$, the Chen equation [9] when $a = -b = 1$ after the transformation $u = exp(kw)$ and $w \to \pm iw$, and the Schwartzian KdV (SKdV) equation when $a = b = 0$ and $u = \phi_x$, which is a potential transformation of u.

The study of higher-dimensional integrable systems is one of the main themes in integrability theory. Toda and Yu [10] developed several models integrable in the context of (2+1)-dimensional equations, i.e., equations with two spatial variables and one temporal variable. These equations were recently derived using a method proposed by Calogero, i.e., by modifying one of the operators of the Lax pair for 1+1 dimensions. Equation (1) was thus obtained from the CDF equation. Although this equation arises in a nonlocal form, it can be written as

$$\begin{aligned} 8\frac{u_{tx}}{u_x} - 8\frac{u_t u_{xx}}{u_x^2} + 2\frac{u_{xxxz}}{u_x} - 2\frac{u_{xx}u_{xxz}}{u_x^2} - 4\frac{u_{xxz}}{u} + \\ + 4\frac{u_x u_{xz}}{u^2} - 2\frac{u_z u_{xxx}}{u u_x} - 2\frac{u_{xx}u_{xz}}{u u_x} + 2\frac{u_z u_{xx}^2}{u u_x^2} + 6\frac{u_z u_{xx}}{u^2} + \\ + 2\frac{u_x u_{xz}}{u^2} - 6\frac{u_z u_x^2}{u^3} + 6u u_z a^2 + 2a^2\frac{u^2 u_{xz}}{u_x} - 2a^2\frac{u^2 u_z u_{xx}}{u_x^2} + \\ + 4ab\frac{u_{xz}}{u_x} - 4ab\frac{u_z u_{xx}}{u_x^2} + 2b^2\frac{u_{xz}}{u^2 u_x} - 2b^2\frac{u_z u_{xx}}{u^2 u_x^2} - 6b^2\frac{u_z}{u^3} = 0. \end{aligned} \tag{3}$$

Although there exist different tools for investigating the properties of the integrable (2+1)-dimensional equations, we choose the Lie symmetry analysis. The invariance properties of some of the physically important nonlinear evolution equations, such as the Kadomtsev-Petviashvili equation and the Davey–Stewartson equation, have been studied using Lie symmetry analysis [11], [12]. In most cases, the corresponding Lie algebra has a Kac–Moody–Virasoro-type subalgebra, but some integrable (2+1)-dimensional equations do not admit a Virasoro-type subalgebra. Examples of such equations are a breaking soliton equation introduced by Bogoyavlenskii, a (2+1)-dimensional generalization of the nonlinear Schrödinger equation [13], and the SKdV equation [14].

The classical method for finding symmetry reductions of partial differential equations (PDEs) is the Lie group method of infinitesimal transformations. Using this method, we develop the previously unknown invariance and similarity properties that reduce Eq. (3) to (1+1)-dimensional PDEs.

In this paper, we derive traveling-wave reductions with a variable velocity depending on the form of an arbitrary function. We first obtain a point-transformation group that leaves Eq. (3) invariant. We consider the reductions derived from the translation groups and from the infinite-dimensional groups. An interesting feature of our study is that this integrable equation in 2+1 dimensions admits infinite-dimensional Lie point symmetry groups, but it does not admit Virasoro-type subalgebras.

The invariance study of these reduced (1+1)-dimensional equations and further reductions lead to second-order integrable ODEs. The solutions of all these ODEs are expressible in terms of known functions; some solutions can be expressed in terms of the second and third Painlevé transcendents. We also derive exact solutions for the (2+1)-dimensional integrable generalization of the CDF equation. Some of these solutions are soliton solutions, localized on a curve and decaying exponentially away from that curve.

2. Lie symmetries

To apply the classical method to (2+1)-dimensional PDE (3), we consider the one-parameter Lie group of infinitesimal transformations in (x, t, z, u). The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\mathbf{v} = \xi \frac{\partial}{\partial x} + \zeta \frac{\partial}{\partial z} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}. \tag{4}$$

We then require that this transformation leave the set of solutions of (3) invariant. This yields an overdetermined linear system of equations for the infinitesimals $\xi(x,z,t,u)$, $\zeta(x,z,t,u)$, $\tau(x,z,t,u)$, and $\eta(x,z,t,u)$. Having determined the infinitesimals, we find the symmetry variables by solving the invariant surface condition

$$\Phi_1 \equiv \xi \frac{\partial u}{\partial x} + \zeta \frac{\partial u}{\partial z} + \tau \frac{\partial u}{\partial t} - \eta = 0. \tag{5}$$

When we apply the classical method to PDE (3), the corresponding Lie symmetry algebra depends on the constants a and b, and we can distinguish the following cases:

1. If $a \neq 0$ and $b \neq 0$, then we obtain the generators

$$\mathbf{v}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{v}_2 = \frac{\partial}{\partial z}, \qquad \mathbf{v}_3 = t\frac{\partial}{\partial t} + z\frac{\partial}{\partial z}, \qquad \mathbf{v}_f = f(t)\frac{\partial}{\partial x}, \tag{6}$$

   where $f(t)$ is an arbitrary function of t.

2. If $a \neq 0$ and $b = 0$, we obtain generators (6) and

$$\mathbf{v}_4^1 = x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z} - u\frac{\partial}{\partial u}.$$

3. If $a = 0$ and $b \neq 0$, we obtain generators (6) and

$$\mathbf{v}_4^2 = x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}.$$

4. If $a = 0$ and $b = 0$, we obtain generators (6) and

$$\mathbf{v}_4^3 = x\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial z}, \qquad \mathbf{v}_k = k(z)u\frac{\partial}{\partial u},$$

   where $k(z)$ is an arbitrary function of z.

We remark that Eq. (3) does not admit a Virasoro-type subalgebra. In a previous paper, we listed the similarity variables and similarity solutions as well as the systems of PDEs obtained when (2+1)-dimensional equation (3) is reduced using the generators obtained by adding the infinite-dimensional generators to the generators of the optimal system.

Our aim in this paper is to use the theory of symmetry reductions to find traveling-wave solutions for the (2+1)-dimensional CDF equation. To obtain these solutions, we consider the following reductions arising from translations and the infinite-dimensional vector field, i.e., from $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_f$, and $\mathbf{v}_k$.

Reduction 1. Using the generator $\mathbf{v}_1 + \lambda\mathbf{v}_2 + \mathbf{v}_f$, we obtain the similarity variables and similarity solution

$$z_1 = x - \int f(t)\,dt, \qquad z_2 = z - \lambda t, \qquad u = h(z_1, z_2) \tag{7}$$

and the PDE

$$\begin{aligned}
&8\,h^3 h_{z_1z_1} h_{z_2}\lambda - 8\,h^3 h_{z_1} h_{z_1z_2}\lambda - 2\,h^2 h_{z_1} h_{z_1z_1z_1} h_{z_2} + 2\,h^2 h_{z_1z_1}^2 h_{z_2} + \\
&\quad + 6\,h h_{z_1}^2 h_{z_1z_1} h_{z_2} - 2\,a^2\,h^5 h_{z_1z_1} h_{z_2} - 4\,a\,b\,h^3 h_{z_1z_1} h_{z_2} - 2\,b^2\,h h_{z_1z_1} h_{z_2} - \\
&\quad - 6 h_{z_1}^4 h_{z_2} + 6\,a^2 h^4 h_{z_1}^2 h_{z_2} - 6\,b^2 h_{z_1}^2 h_{z_2} + 2\,h^3 h_{z_1} h_{z_1z_1z_1z_2} - \\
&\quad - 2\,h^3 h_{z_1z_1} h_{z_1z_1z_2} - 4\,h^2 h_{z_1}^2 h_{z_1z_1z_2} - 2\,h^2 h_{z_1} h_{z_1z_1} h_{z_1z_2} + 6\,h h_{z_1}^3 h_{z_1z_2} + \\
&\qquad + 2\,a^2\,h^5 h_{z_1} h_{z_1z_2} + 4\,a\,b\,h^3 h_{z_1} h_{z_1z_2} + 2\,b^2 h_{z_1} h_{z_1z_2} = 0.
\end{aligned} \tag{8}$$

When $a = 0$ and $b = 0$, we obtain the generator $\mathbf{v}_1 + \lambda\mathbf{v}_2 + \mathbf{v}_f + \mathbf{v}_k$ and the similarity solution

$$z_1 = x - \int f(t)\,dt, \qquad z_2 = z - \lambda t, \qquad u = h(z_1, z_2)\exp\left\{\frac{1}{\lambda}\int k(z)\,dz\right\}. \tag{9}$$

Reduction 2. Using the generator $\mathbf{v}_2 + \mathbf{v}_f$, we obtain the similarity variables and similarity solution

$$z_1 = x - zf(t), \qquad z_2 = t, \qquad u = h(z_1, z_2) \tag{10}$$

and the PDE

$$\begin{aligned} &-8\,h^3\,h_{z_1z_1}h_{z_2} - 2\,f(z_2)\,h^3\,h_{z_1}\,h_{z_1z_1z_1z_1} + 2\,f(z_2)\,h^3\,h_{z_1z_1}\,h_{z_1z_1z_1} + \\ &\quad + 6\,f(z_2)\,h^2\,h_{z_1}^2\,h_{z_1z_1z_1} - 12\,f(z_2)\,h\,h_{z_1}^3\,h_{z_1z_1} + 8\,h^3\,h_{z_1}\,h_{z_1z_2} + \\ &\quad\quad + 6\,f(z_2)h_{z_1}^5 - 6\,a^2\,f(z_2)\,h^4\,h_{z_1}^3 + 6\,b^2\,f(z_2)\,h_{z_1}^3 = 0. \end{aligned} \tag{11}$$

When $a = 0$ and $b = 0$, we obtain the generator $\mathbf{v}_2 + \mathbf{v}_f + \mathbf{v}_k$, the similarity solution

$$z_1 = x - zf(t), \qquad z_2 = t, \qquad u = h(z_1, z_2)\exp\left\{\int k(z)\,dz\right\} \tag{12}$$

and PDE (11) with $a = 0$ and $b = 0$.

3. Symmetry reductions to ODEs

The reduced PDEs in 1+1 variables admit symmetries that lead to further reductions to ODEs, and we again use the techniques of Lie group theory.

Equation (8) admits the symmetries

$$\mathbf{v}_{11} = \frac{\partial}{\partial z_1}, \qquad \mathbf{v}_\alpha = \alpha(z_2)\frac{\partial}{\partial z_2}, \tag{13}$$

where $\alpha(z_2)$ is an arbitrary function of z_2. Using $\mathbf{v}_{11} + \mathbf{v}_\alpha$, we obtain the similarity variable and similarity solutions

$$w = z_1 - \int \frac{1}{\alpha(z_2)}\,dz_2, \qquad h = g(w) \tag{14}$$

and the autonomous ODE

$$\begin{aligned} &-g^3g'g'''' + g^3g''g''' + 3g^2(g')^2g''' - 6g(g')^3g'' + \\ &\quad + 3(g')^5 - 3a^2g^4(g')^3 + 3b^2(g')^3 = 0. \end{aligned} \tag{15}$$

Dividing by $g^2(g')^2$, integrating once over w, and then multiplying by $g^3(g')^2$, we can reduce Eq. (15) to the second-order autonomous ODE

$$g'' = \frac{3}{2}\frac{(g')^2}{g} - \frac{a^2}{2}g^3 + \frac{3b^2}{2}\frac{1}{g} + k_1g + k_2. \tag{16}$$

Multiplying by $g^{-3}g'$ and integrating once over w, we obtain

$$(g')^2 = -a^2g^4 + 2k_1g^2 + k_2g + b^2 + 2k_3g^3. \tag{17}$$

The integration can be completed in terms of elliptic functions.

For $b = 0$ and $\lambda \neq 0$, Eq. (8) admits symmetries (13). For $\lambda = 0$, in addition to the previous ones, it admits the generator

$$\mathbf{v}_{12} = z_1 \frac{\partial}{\partial z_1} - h \frac{\partial}{\partial h}. \tag{18}$$

Using $\mathbf{v}_{11} + \mathbf{v}_\alpha$, we obtain the similarity variable and similarity solutions (14) and ODE (15) with $b = 0$.

Using $\mathbf{v}_{12} + \mathbf{v}_\alpha$, we obtain the similarity variable and similarity solutions

$$w = z_1 \exp\left\{-\int \frac{1}{\alpha(z_2)}\, dz_2\right\}, \qquad h = g(w) \exp\left\{-\int \frac{1}{\alpha(z_2)}\, dz_2\right\} \tag{19}$$

and the ODE

$$\begin{aligned} & - g^3 g' g'''' w + g^3 g'' g''' w + 3g^2 (g')^2 g''' w - 6g(g')^3 g'' w + \\ & \quad + 3(g')^5 w - 3a^2 g^4 (g')^3 w - 3g^3 g' g''' + 2g^3 (g'')^2 + \\ & \quad\quad + 5g^2 (g')^2 g'' + a^2 g^6 g'' - 3g(g')^4 - 5a^2 g^5 (g')^2 = 0. \end{aligned} \tag{20}$$

Dividing (20) by $g^2(g')^2$, integrating once over w, then multiplying by $g^{-3}(g')$, and integrating again over w, we obtain the Painlevé III equation

$$g'' = \frac{(g')^2}{g} - \frac{g'}{w} - a^2 g^3 + \frac{k_1}{2w} - \frac{k_2 g^2}{w}. \tag{21}$$

Equation (11) admits the symmetries

$$\mathbf{v}_\zeta = \zeta(z_2) \frac{\partial}{\partial z_1}, \qquad \mathbf{v}_\beta = \frac{1}{f(z_2)} \frac{\partial}{\partial z_2}, \quad f(z_2) \neq 0, \tag{22}$$

where $\zeta(z_2)$ and $f(z_2)$ are arbitrary functions of z_2. Using $\mathbf{v}_\zeta + \mathbf{v}_\beta$, we obtain the similarity variable and similarity solutions

$$w = z_1 - \int \zeta(z_2) f(z_2)\, dz_2, \qquad h = g(w) \tag{23}$$

and ODE (15), which can be integrated in terms of elliptic functions.

Equation (11) with $b = 0$ admits symmetries (23) and

$$\mathbf{v}_\gamma = z_1 \frac{\partial}{\partial z_1} + \frac{3}{f(z_2)} \int f(z_2)\, dz_2 \frac{\partial}{\partial z_2} - h \frac{\partial}{\partial h}, \quad f(z_2) \neq 0. \tag{24}$$

Using $\mathbf{v}_\zeta + \mathbf{v}_\beta + \mathbf{v}_\gamma$, we obtain the similarity variable and similarity solutions

$$\begin{aligned} w &= z_1 \left(1 + 3 \int f(z_2)\, dz_2\right)^{1/3} - \int \frac{f(z_2)\alpha(z_2)\, dz_2}{\left(1 + 3 \int f(z_2)\, dz_2\right)^{4/3}}, \\ h &= g(w) \left(1 + 3 \int f(z_2)\, dz_2\right)^{-1/3} \end{aligned} \tag{25}$$

and the ODE

$$\begin{aligned} & - g^3 g' g'''' + g^3 g'' g''' + 3g^2 (g')^2 g''' - 6g(g')^3 g'' + \\ & \quad + 4g^4 g'' + 3(g')^5 - 3a^2 g^4 (g')^3 - 8g^3 (g')^2 = 0. \end{aligned} \tag{26}$$

Dividing (26) by $g^3(g')^2$, integrating once over w, multiplying by g', and integrating again over w, we obtain the second-order ODE

$$g'' = \frac{3}{2}\frac{(g')^2}{g} - \frac{a^2}{2}g^3 + k_2 g - 4wg + k_1. \tag{27}$$

The change of variables $g = y^{-1}$ leads to

$$y'' = \frac{1}{2}\frac{(y')^2}{g} - \frac{a^2}{2y} + k_2 y^2 + 4wy + k_1 y. \tag{28}$$

The change of variables $y = \alpha V(Z)$ with $Z = \beta w$ leads to the equation

$$V'' = \frac{1}{2}\frac{(V')^2}{V} + 4cV^2 - ZV - \frac{1}{2}\frac{1}{V}, \tag{29}$$

whose solutions can be written in terms of the Painlevé II equation (see [15]).

4. Some traveling-wave solutions

We now present some explicit solutions of the second-order ODEs as well as the corresponding traveling-wave solutions of the (2+1)-dimensional CDF equation. Equation (17) can be integrated in terms of elliptic functions.

Setting $b = 0$ and $k_2 = -4$ in (17), we obtain an exact solution in terms of the Weierstrass function $\wp$. Clearly, any of the rational, hyperbolic, or trigonometric degenerations of the function $\wp$ also gives a solution.

In particular, setting $k_1 = -2c_2^2$, $k_2 = 0$, and $k_3 = (2c_1c_2 - a)(2c_1c_2 + a)/c_1$, we obtain the solitary-wave result

$$g = \frac{1}{c_1 - \big(c_1 + a^2/(4c_1c_2^2)\big)\cosh^2(c_2 w)}.$$

Considering the corresponding symmetry reductions (7) and (14), we find that a "curve" soliton solution for the (2+1)-dimensional CDF equation can be written as

$$u = \frac{1}{c_1 - \big(c_1 + a^2/(4c_1c_2^2)\big)\cosh^2\big(c_2 x - \varphi(t) - \delta(z - \lambda t)\big)} \tag{30}$$

with

$$\varphi(t) = c_2\int f(t)\,dt, \qquad \delta(z_2) = c_2\int\frac{dz_2}{\alpha(z_2)}, \qquad z_2 = z - \lambda t. \tag{31}$$

In Fig. 1, we can see solution (30) with $c_1 = -1$, $c_2 = 1$, $a = 4$, $\delta(z - \lambda t) = \sin(z - t)$, and $\varphi(t) = -t$ for $t = 1$.

Considering the corresponding symmetry reductions (10) and (23), we find that a soliton solution for the (2+1)-dimensional CDF equation can be written as

$$u = \frac{1}{c_1 - \big(c_1 + a^2/(4c_1c_2^2)\big)\cosh^2\big(c_2 x - zf(t) - \psi(t)\big)} \tag{32}$$

with

$$\psi(t) = c_2\int\delta(t)f(t)\,dt. \tag{33}$$

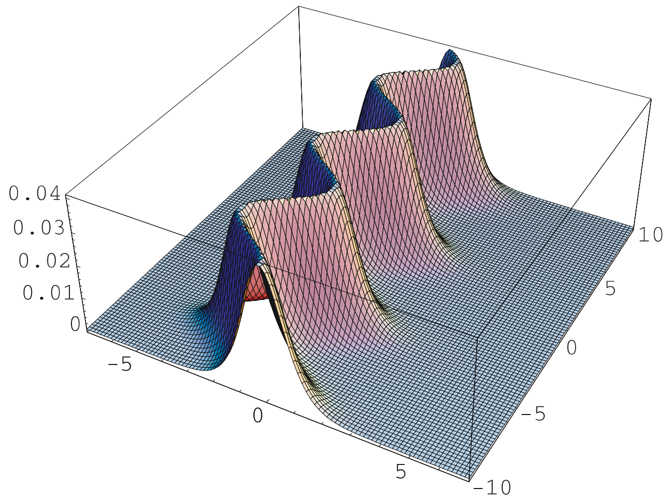
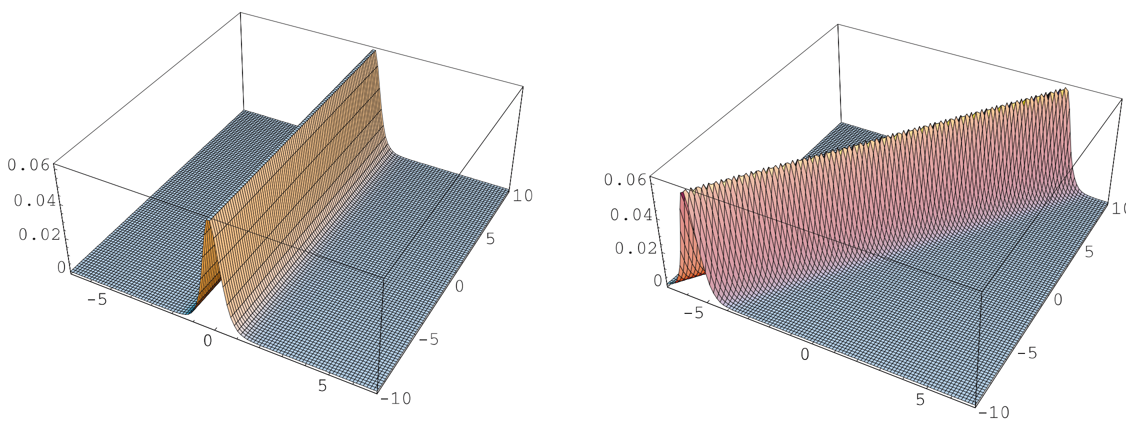


Fig. 1. Curve soliton, $t = 1$.



a

b

Fig. 2. Evolution of the rotating soliton.

In Fig. 2, we can see (32) with $c_1 = 1$, $c_2 = 1$, $a = 4$, $f(t) = t^4$, and $\psi(t) = t$ for $t = 0$ (Fig. 2a) and $t = 1$ (Fig. 2b). We observe that this soliton solution is "rotating."

Setting $a = 0$, $b = 0$, $k_1 = -2c_1$, $k_2 = 0$, and $k_3 = 4c_2$ in (17), we obtain the solution

$$g = -\frac{c_1}{c_2 \cosh^2 \sqrt{c_1} w}.$$

Considering the corresponding symmetry reductions (7) and (14), we find that a solution of the (2+1)-dimensional CDF equation can be written as

$$u = -\frac{c_1 \rho(z)}{c_2 \cosh^2\bigl(\sqrt{c_1}(x - \varphi(t) - \delta(z - \lambda t))\bigr)}. \tag{34}$$

Considering the corresponding symmetry reductions (10) and (23), we find that a solution of the (2+1)-dimensional CDF equation can be written as

$$u = \frac{c_1 \rho(z)}{c_2 \cosh^2\bigl(\sqrt{c_1}(x - z f(t) - \psi(t))\bigr)} \tag{35}$$

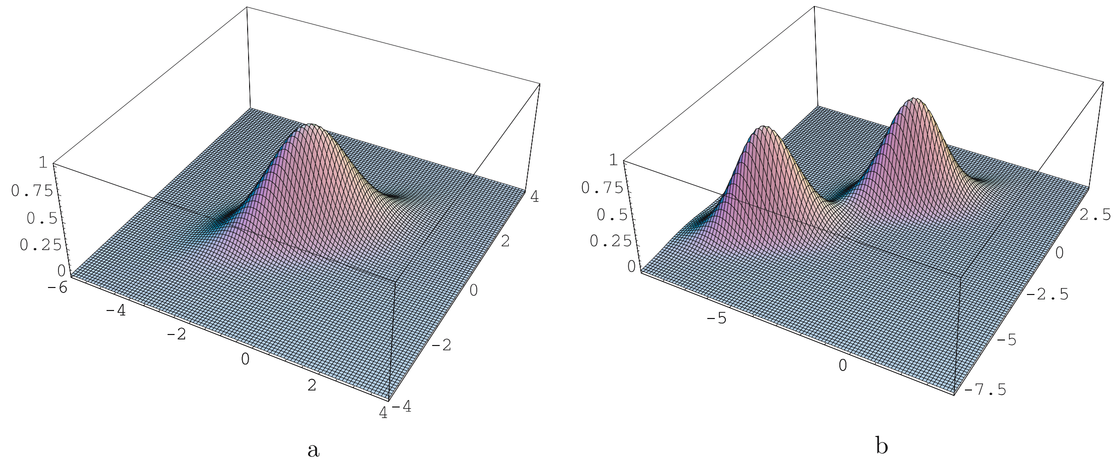


Fig. 3. Dromion and coherent structure, $t = 1$.

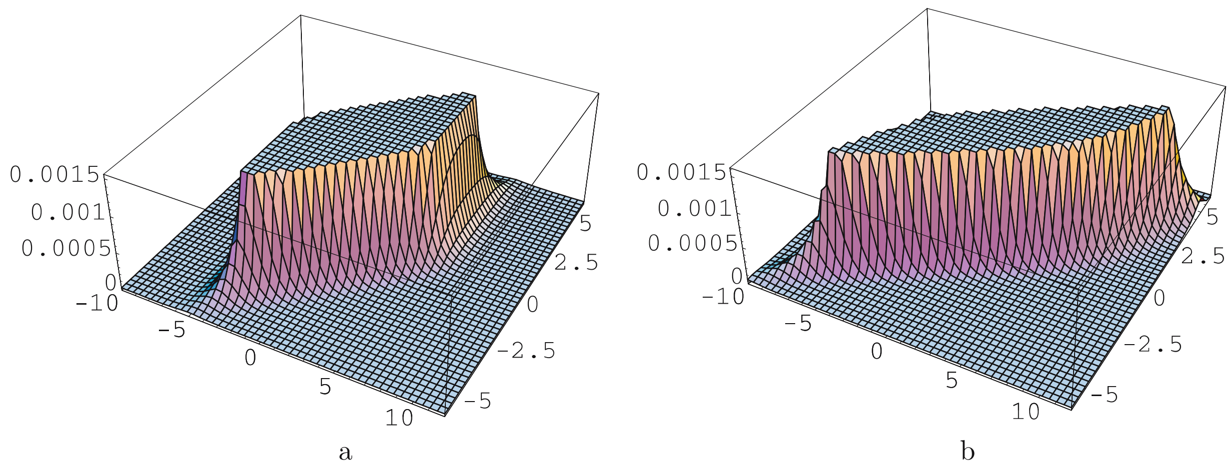


Fig. 4. Evolution of solution (35).

with

$$\psi(t) = c_2 \int \delta(t) f(t)\, dt. \tag{36}$$

In Fig. 3, we can see solution (34) with $\varphi(t) = 0$ and $\delta(z - \lambda t) = z - t$ with $-c_1\rho(z) = \cosh^{-2}(z)$ (Fig. 3a) and with $-c_1\rho(z) = \cosh^{-2}(z) + \cosh^{-2}(z + 4)$ (Fig. 3b) for $t = 1$. We observe that these dromions and coherent structures are localized in all directions.

In Fig. 4, we can see solution (35) with $f(t) = t$, $\psi(t) = t$, and $-c_1\rho(z) = \cosh^{-2}(z)$ for $t = 1$ (Fig. 4a) and $t = 2$(Fig. 4b). We observe that the solution is localized in all directions and evolves by "rotating" and changing its shape.

Setting $a = 0$ in (17), we obtain

$$g = \frac{\sqrt{k_2^2 - 8k_1 b^2}}{4k_1} \sin(\sqrt{2k_1}(w + c) + k_2), \qquad g = \frac{\sqrt{k_2^2 - 8k_1 b^2}}{4k_1} \cos(\sqrt{2k_1}(w + c) - k_2),$$

$$g = \frac{\sqrt{-k_2^2 + 8k_1 b^2}}{4k_1} \sinh(\sqrt{-2k_1}(w + c) + k_2), \quad g = \frac{\sqrt{k_2^2 - 8k_1 b^2}}{4k_1} \cosh(\sqrt{-2k_1}(w + c) + k_2).$$

Considering the corresponding symmetry reductions (7) and (14), we find that some exact solutions for the (2+1)-dimensional CDF equation can be written as

$$u = -\frac{\sqrt{k_2^2 - 8k_1 b^2}}{4k_1} \sin\bigl(\sqrt{2k_1}(x - \varphi(t) - \delta(z - \lambda t) + k_2)\bigr), \tag{37}$$

$$u = -\frac{\sqrt{k_2^2 - 8k_1 b^2}}{4k_1} \cos\bigl(\sqrt{2k_1}(x - \varphi(t) - \delta(z - \lambda t) - k_2)\bigr), \tag{38}$$

$$u = -\frac{\sqrt{-k_2^2 + 8k_1 b^2}}{4k_1} \sinh\bigl(\sqrt{-2k_1}(x - \varphi(t) - \delta(z - \lambda t) - k_2)\bigr), \tag{39}$$

$$u = \frac{\sqrt{k_2^2 - 8k_1 b^2}}{4k_1} \cosh\bigl(\sqrt{-2k_1}(x - \varphi(t) - \delta(z - \lambda t) - k_2)\bigr). \tag{40}$$

Setting $a = 1$ and $b = 0$ in (17), we obtain

$$g = -\frac{2\tan\bigl((\sqrt{3}/2)w\bigr)}{\tan\bigl((\sqrt{3})/2w\bigr) + 3}.$$

Setting $a = \sqrt{2}i/\sqrt{3}$, $k_1 = 0$, $k_2 = 8/3$, and $b = 0$ in (17), we obtain

$$g = \frac{2\operatorname{sech}^2 w - 2}{2\operatorname{sech}^2 w + 1}.$$

Setting $b = 0$, $k_1 = k_2 = 0$, and $\lambda = k_3/a^2$ in (17), we obtain

$$g = \frac{2\lambda}{1 + a^2\lambda^2 w^2}.$$

Setting $a = 0$, $b = 0$, and $k_2 = k_3 = 0$ in (17), we obtain

$$g = \rho(z)e^{\sqrt{-2k_1}w}.$$

Considering the corresponding symmetry reductions (7) and (14), we find that some exact solutions for the (2+1)-dimensional CDF equation can be written as

$$u = -\frac{2\tan\bigl((\sqrt{3}/2)(x - \varphi(t) - \delta(z - \lambda t))\bigr)}{\tan(\sqrt{3}/2)(x - \varphi(t) - \delta(z - \lambda t)) + 3}, \tag{41}$$

$$u = \frac{2\operatorname{sech}^2(x - \varphi(t) - \delta(z - \lambda t)) - 2}{2\operatorname{sech}^2(x - \varphi(t) - \delta(z - \lambda t)) + 1}, \tag{42}$$

$$u = \frac{2\lambda}{1 + a^2\lambda^2(x - \varphi(t) - \delta(z - \lambda t))^2}, \tag{43}$$

$$u = \rho(z)\exp\bigl\{\sqrt{-2k_1}(x - \varphi(t) - \delta(z - \lambda t))\bigr\} \tag{44}$$

with

$$\varphi(t) = c_2 \int f(t)\,dt, \qquad \delta(z_2) = c_2 \int \frac{dz_2}{\alpha(z_2)}, \qquad z_2 = z - \lambda t. \tag{45}$$

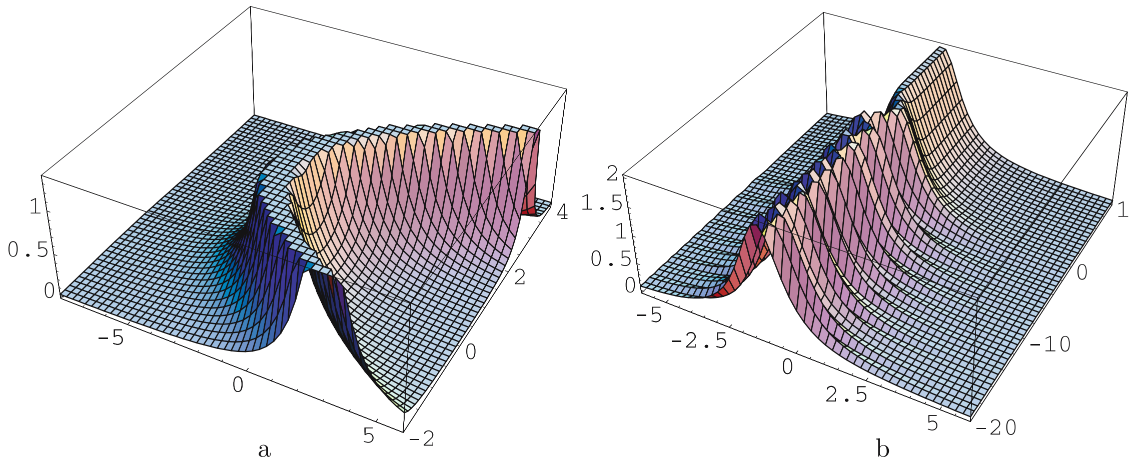


Fig. 5. Curve solitons (43), $t = 1$.

In Fig. 5, we can see "curve" soliton solution (43) for $a = 1$, $c = 1$, and $\varphi(t) = t$ with $\delta(z_2) = (z - \lambda t)^2$ (Fig. 5a) and with $\delta(z_2) = \mathrm{Ai}(z - \lambda t)$ (Fig. 5b) for $t = 1$.

Setting $b = 0$ and $k_2 = 0$ in (17), we obtain

$$g = \frac{4k_1}{\sqrt{k_4^2 - 8k_1a^2}\cosh\bigl(\sqrt{-2k_1}(w+c) - k_4\bigr)}, \qquad g = \frac{4k_1}{\sqrt{k_4^2 - 8k_1a^2}\cos\bigl(\sqrt{2k_1}(w+c) - k_4\bigr)}.$$

Considering the corresponding symmetry reductions (7) and (14), we find that some exact solutions of the (2+1)-dimensional CDF equation can be written as

$$u = \frac{4k_1}{\sqrt{k_4^2 - 8k_1a^2}\cosh\bigl(\sqrt{-2k_1}(x - \varphi(t) - \delta(z - \lambda t) + c) - k_4\bigr)}, \tag{46}$$

$$u = \frac{4k_1}{\sqrt{k_4^2 - 8k_1a^2}\cos\bigl(\sqrt{2k_1}(x - \varphi(t) - \delta(z - \lambda t) + c) - k_4\bigr)}. \tag{47}$$

The most interesting solutions are the soliton solutions. The entrance of the arbitrary functions $\rho(z)$, $\varphi(t)$, and $\delta(z - \lambda t)$ allows a wide variety of qualitative and physical behaviors of these solutions.

5. Conclusions

We have discussed symmetry reductions and exact solutions of the (2+1)-dimensional integrable generalization of the CDF equation. Using classical Lie symmetries, we have considered traveling-wave reductions for this (2+1)-dimensional integrable equation. It is interesting feature that this (2+1)-dimensional integrable equation does not admit Virasoro-type subalgebras. Using the classical Lie method, we obtained PDEs in 1+1 dimensions and systems of ODEs and, by further reductions, second-order integrable ODEs whose solutions are all expressible in terms of known functions, some of them expressible in terms of the second and third Painlevé transcendents. For the (2+1)-dimensional CDF equation, we obtained families of solutions with a rich variety of qualitative behaviors because of the freedom in choosing the arbitrary functions $\varphi(t)$, $\rho(z)$, and $\delta(z - \lambda t)$.

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