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ORIGINAL RESEARCH

Potential rationality in collective decision-making

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Abstract
This study investigates Suzumura consistency as a condition for the rationality of social preferences. A preference is said to be Suzumura-consistent when all preference cycles include only indifference relations. This condition is equivalent to transitivity in the presence of completeness, but, in general, it is substantially weaker than transitivity when preference is incomplete. Notably, Suzumura consistency is especially significant for a preference because it is necessary and sufficient for the existence of an ordering (transitive and complete preference) that is compatible with the original preference. This coherency property can be regarded as a requirement for potential rationality. In this study, we examine the implications of shifting from actual rationality to potential rationality in collective decision-making. We introduce the concept of an alternative-dependent coherent collection in order to obtain a representation of a class of Suzumura-consistent collective choice rules that satisfy the axioms imposed in Arrow's impossibility theorem. This demonstrates that the power structure to determine social choice can be alternative-dependent.

Keywords Rationality · Suzumura consistency · Transitivity · Arrow's impossibility theorem · Theory choice · Alternative dependency

1 Introduction

Many decision theories and moral arguments assume transitivity, which is a property of rationality for preference and betterness. The concept of transitivity posits that the choice x is better than the choice y if there exists a third choice z, under the suppositions that x is better than z and z is better than y. Although this requirement sounds quite reasonable, many scholars have criticized it for being too demanding in actuality.[1] Specifically, transitivity can be problematic when it is combined with completeness,

[1] See Temkin (1996) and Rachels (1998) for famous criticism of transitivity. See also Handfield (2014), Handfield and Rabinowicz (2018), and Nebel (2018) for related arguments.

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which requires perfect comparability of alternatives.[2] Indeed, it is known that aggregating multiple values becomes difficult if both transitivity and completeness are satisfied. This point is an immediate consequence of Arrow's impossibility theorem, which is the central tenet of the social choice theory; the theorem states that individual preferences cannot be aggregated to a social preference by a rational method/algorithm (Arrow, 1951, 1963, 2012). Although there are various escape routes for this negative result, transitivity and completeness are indispensable in deriving Arrow's theorem.

Does the theory of collective decision-making need to impose completeness and transitivity on social preferences? How can we escape the impossibility result when both are relaxed? These questions cannot be answered unless a plausible approach is identified to weaken the two conditions. This study considers the concept of Suzumura consistency; this property for a preference (or, more generally, for a binary relation) was introduced by Suzumura (1976) under the name of "consistency." To be precise, a preference is said to be Suzumura-consistent if, for any preference cycle, all its included alternatives are indifferent.[3] Notably, this is a natural weakening of transitivity. That is, a preference satisfies this condition as long as it satisfies transitivity. On the other hand, there is a Suzumura consistent but intransitive preference. We argue that transitivity can be viewed as a condition for actual rationality, whereas Suzumura consistency can be viewed as a condition for potential rationality. This condition is plausible in both individual and social decision-making processes; see Sect. 2.

This study develops an approach to characterizing Suzumura-consistent collective choice rules within Arrow's framework. To this end, we introduce the concept of an *alternative-dependent coherent collection* that identifies a pair of disjoint coalitions for each pair of alternatives. We prove that a Suzumura-consistent collective choice rule can be represented by an alternative-dependent coherent collection if it satisfies Arrow's axioms.[4]This result substantially generalizes the existing theorems of Arrovian social choice.

The rest of this paper is organized as follows. Section 2 explains the meanings and implications of Suzumura consistency as a coherent property of preferences by introducing the formal framework. Section 3 presents the concept of an alternative-dependent coherent collection. Section 4 presents our representation theorem. In Section 5, two anonymity axioms are considered; the results in this section are applications of the main theorem. Finally, Section 6 concludes the paper.

2 Preliminaries

2.1 Binary relation

We assume that X is the set of social alternatives; X contains at least three alternatives. Let $R \subseteq X \times X$ be a binary relation on X. Throughout this paper, R represents a weak

[2] Sen (2004) provides a fundamental criticism of completeness.

[3] Bossert and Suzumura (2010) provide a synthesis of the analysis of Suzumura consistency.

[4] Note that alternative dependency is compatible with independence of irrelevant alternatives. The latter requires social preferences to not be dependent on the individual rankings over "irrelevant" alternatives, but it does not restrict use of information about individual rankings over "relevant" alternatives.

preference. That is, $(x, y) \in R$ means "x is at least as good as y." The symmetric and asymmetric parts of R are denoted by $I(R)$ and $P(R)$, respectively.[5] In other words, $(x, y) \in I(R)$ means "x is indifferent to y," while $(x, y) \in P(R)$ means "x is strictly better than y." Let $\mathcal{B}$ be the set of all binary relations R on X.

Now, we mention three well-known properties of a binary relation.[6] The first one states that x is at least as good as x for each alternative x.

Reflexivity: For all $x \in X$, $(x, x) \in R$.

The second one is completeness, which requires that alternatives are comparable. Note that completeness implies reflexivity.

Completeness: For all $x, y \in X$, $(x, y) \in R$ or $(y, x) \in R$.

The above two conditions are about the comparability of alternatives. The following condition is associated with the coherency or consistency of preferences; it is the key concept of rationality for decision-making. We will refer to this rationality by the term "actual rationality."

Transitivity: For all $x, y, z \in X$,

$$[(x, y) \in R \text{ and } (y, z) \in R] \Rightarrow (x, z) \in R.$$

A reflexive and transitive binary relation on X is called a quasi-ordering. An ordering R on X is a reflexive, complete, and transitive binary relation on X. Let $\mathcal{R}$ be the set of all orderings R defined on X.

Two properties that are logically weaker than transitivity have been extensively examined; see Sen (1969, 1970) and Cato (2016). The following requires that the asymmetric part of R is transitive.

Quasi-transitivity: For all $x, y, z \in X$,

$$[(x, y) \in P(R) \text{ and } (y, z) \in P(R)] \Rightarrow (x, z) \in P(R).$$

The next property states that there is no cycle of strict preferences.

Acyclicity: For all $M \geq 3$, and for all $x^1, \ldots, x^M \in X$,

$$[(x^{m-1}, x^m) \in P(R) \text{ for all } m \in \{2, \ldots, M\}] \Rightarrow (x^M, x^1) \notin P(R).$$

Let us introduce Suzumura consistency. This property requires the following: if there is a chain (or sequence) of alternatives, $x^1, x^2, \ldots, x^{M-1}, x^M$, such that x^{m-1} is at least as good as x^m for all $m = 2, \ldots, M$, then x^M cannot be strictly better than x^1. Formally, Suzumura consistency is defined as follows.

[5] That is, $I(R) := \{(x, y) \in X \times X | (x, y) \in R \text{ and } (y, x) \in R\}$ and $P(R) := \{(x, y) \in X \times X | (x, y) \in R \text{ and } (y, x) \notin R\}$.

[6] See Cato (2016) for explanations of these properties.

Suzumura consistency: For all $M \geq 3$, and for all $x^1, \ldots, x^M \in X$,

$$[(x^{m-1}, x^m) \in R \text{ for all } m \in \{2, \ldots, M\}] \Rightarrow (x^M, x^1) \notin P(R).$$

We note that if transitivity is satisfied, Suzumura consistency is satisfied. In other words, Suzumura consistency is logically weaker than (implied by) transitivity. This is because x^1 is at least as good as x^K as long as there is a chain stated in this condition. Therefore, it must be false that x^K is strictly better than x^1. Moreover, it is easy to find an example of a preference that is Suzumura-consistent, but not transitive.[7] If completeness is satisfied, then Suzumura consistency is equivalent to transitivity. To confirm this, suppose that x is at least as good as y and y is at least as good as z. Because this is a chain of the three alternatives, Suzumura consistency requires that z not be strictly better than x. Completeness implies that x is at least as good as z; thus, transitivity is satisfied. In summary, there is no logical difference between Suzumura consistency and transitivity in the presence of completeness. Therefore, Suzumura consistency has its significance only when completeness is dropped or relaxed. Indeed, this is a natural weakening of transitivity; see, for instance, Bossert and Suzumura (2010) and Bradley (2015).

It is easy to see that Suzumura consistency implies acyclicity, but not vice versa. However, quasi-transitivity and Suzumura consistency are independent. As claimed by Bossert and Suzumura (2010), Suzumura consistency is a plausible weakening of transitivity. The reason is related to the concept of an extension. A binary relation R' is said to be an *extension* of a binary relation R if $R \subseteq R'$ and $P(R) \subseteq P(R')$. That is, when R' is an extension of R, x is at least as good as y with respect to R' if x is at least as good as y with respect to R, and x is better than y with respect to R' if x is better than y with respect to R. If an extension R' of R is an ordering, then it is referred to as an *ordering extension* of R. Notably, an ordering extension R' is an ordering that is perfectly compatible with the original preference R. Any preference has an extension because it is an extension in itself. However, preferences do not always have ordering extensions. For instance, the following preference does not have an ordering extension:

x is better than y, y is better than z, and z is better than x.

Then, when does a preference have an ordering extension? Regarding this query, it is known that any quasi-ordering has an ordering extension; see Szpilrajn (1930), Arrow (1951, 1963, 2012), and Hansson (1968). That is, the conjunction of reflexivity and transitivity is sufficient for the existence of an ordering that is compatible with the original binary relation. However, it is not a necessary condition. Indeed, Suzumura consistency is a necessary and sufficient condition for the existence of an ordering extension. We now formally state this as a lemma.

Lemma 1 (Suzumura, 1976) *A binary relation R on X has an ordering extension if and only if it is Suzumura-consistent.*

[7] Assuming that there are only three alternatives, x, y, and z, let a preference be such that x is better than y, y is indifferent to z, and x and z are incomparable. This preference is Suzumura consistent but not transitive.

This lemma shows the reason why we regard Suzumura consistency as "potential rationality." That is, this essentially implies that a binary relation R on X has a *transitive* extension if and only if it is Suzumura-consistent. The 'if' part is obvious from the lemma because an ordering extension is a transitive extension. The 'only-if' part is proved as follows. Assume that R is not Suzumura-consistent. Then, there exist $M \geq 3$ and $x^1, \ldots, x^M \in X$ such that

$$(x^{m-1}, x^m) \in R \text{ for all } m \in \{2, \ldots, M\}] \text{ and } (x^M, x^1) \in P(R).$$

For any extension R' of R, it holds that

$$(x^{m-1}, x^m) \in R' \text{ for all } m \in \{2, \ldots, M\}] \text{ and } (x^M, x^1) \in P(R').$$

We note that R' cannot be transitive. Thus, R on X has no transitive extension. Thus, the claim is proved. We observe that Suzumura-consistency is a necessary and sufficient condition for extending it to a transitive binary relation.

2.2 Collective choice rule

We assume that N is the (finite or countably infinite) set of individuals.[8] Each individual $i \in N$ has a preference ordering $R_i \in \mathcal{R}$ on X. A preference profile $\mathbf{R} = (R_i)_{i \in N} \in \mathcal{R}^N$ is a list of individual preference orderings on X. Note that each individual's preference is assumed to be reflexive, complete, and transitive. A *collective choice rule* is a function $f : \mathcal{R}^N \to \mathcal{B}$ that maps each profile $\mathbf{R} \in \mathcal{R}^N$ to a unique social preference $f(\mathbf{R}) \in \mathcal{B}$. That is, $(x, y) \in f(\mathbf{R})$ means that x is socially at least as good as y under a preference profile $\mathbf{R}$.[9] A collective choice rule f is said to be reflexive, complete, transitive, or Suzumura-consistent, if, for every $\mathbf{R} \in \mathcal{R}^N$, $f(\mathbf{R})$ is reflexive, complete, transitive, or Suzumura-consistent, respectively. Arrow (1951, 1963, 2012) focused on the case where f is complete and transitive (i.e., $f(\mathbf{R})$ is an ordering for any profile $\mathbf{R} \in \mathcal{R}^N$). He referred to a collective choice rule with range $\mathcal{R}$ as a social welfare function. In the remaining part of this paper, we examine a more general class of collective choice rules.

We now introduce the main axioms. The first axiom states that if an alternative is better than another alternative for all individuals, then the former is socially better than the latter.

Weak Pareto: For all $\mathbf{R} \in \mathcal{R}^N$, and for all $x, y \in X$,

$$(x, y) \in \bigcap_{i \in N} P(R_i) \Rightarrow (x, y) \in P(f(\mathbf{R})).$$

[8] For simplicity, we exclude the case where N is uncountably infinite. When we impose anonymity axioms (in Section 4), the cardinality measure becomes crucial if N is allowed to be uncountably infinite. With an appropriate measure, our analysis is valid for such a case. Our main result is robust independently of this cardinality problem.

[9] The concept of a collective choice rule is introduced by Sen (1970) in order to investigate the case where social preferences are not necessarily complete or transitive.

The second axiom requires that the ranking between two alternatives is dependent only on the individual rankings between the two.

Independence of Irrelevant Alternatives: For all $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^N$, and for all $x, y \in X$, if, for all $i \in N$,

$$(x, y) \in R_i \Leftrightarrow (x, y) \in R_i' \text{ and } (y, x) \in R_i \Leftrightarrow (y, x) \in R_i',$$

then $f(\mathbf{R})$ and $f(\mathbf{R}')$ agree on $\{x, y\}$.

Throughout this paper, we impose these two axioms on f. These axioms were introduced by Arrow (1951, 1963, 2012), who showed that a dictator exists if the two of them are simultaneously imposed on a social welfare function. A collective choice rule that satisfies the two axioms is called Arrovian. The formal statement of the impossibility theorem by Arrow (1951, 1963, 2012) is as follows: If a social welfare function satisfies weak Pareto and independence of irrelevant alternatives, there is an individual d such that, for all $x, y \in X$, x is socially better than y as long as x is personally better than y for individual d, that is, there is a dictator.

Arrow's theorem has been extended and applied to various fields beyond the voting problem and welfare economics. For example, Arrow's theorem has been generalized in the field of judgment aggregations. A preference (or a binary relation) is regarded as a special type of judgment. From this perspective, transitivity is considered a certain requirement for logical coherency. It is very natural to examine a general class of aggregation rules of logical judgments (List & Pettit, 2002; List, 2012). Indeed, an appropriate generalization of Arrow's theorem holds true for this problem; see Dietrich and List (2007a). Arrow's theorem has also been applied to the theory choice problem in the field of the philosophy of science (Okasha, 2011; Stegenga, 2013). According to Kuhn (1998), there are multiple criteria for choosing a plausible theory from among various competing ones. He notes that the procedure must be rational in a certain way. According to Okasha (2011), Arrow's theorem has negative implications for theory choice because theories are considered alternatives, and multiple criteria over theories are regarded as individual values. Thus, there is no rational method for theory choice. In this context, the relevance of Arrow's theorem has been discussed extensively of late.

This paper imposes Suzumura consistency instead of transitivity as a requisite property for social preference. Arguably, Suzumura consistency is relevant as a condition for social preference. From Lemma 1, if a social preference satisfies Suzumura consistency, there is an ordering extension. Moreover, it is easy to see that if a social preference is Suzumura-consistent, there exists a transitive (and not necessarily complete) extension. Thus, transitivity is *potentially* established under any Suzumura-consistent social preference. This may be referred to as "potential rationality." We note that the converse is also true. That is, if the aforementioned extension is possible (i.e. if there exists some transitive relation that can reflect all judgments of a social preference), then the original social preference must be Suzumura-consistent. Arguably, social preferences must be potentially rational but not necessarily actually rational.

3 Alternative-dependent coherent collection

3.1 Decisiveness and the field expansion lemma

The core concept of Arrow's theory is the *decisiveness* of coalitions (Sen, 1970, 1979). Notably, decisiveness is typically used to represent the power under a rule. Given a pair (x, y) of distinct alternatives, a coalition $A \subseteq N$ is decisive over (x, y) if

$$(x, y) \in \bigcap_{i \in A} P(R_i) \Rightarrow (x, y) \in P(f(\mathbf{R}))$$

for all $\mathbf{R} \in \mathcal{R}^N$. That is, A is said to be decisive over (x, y) if x is socially better than y, as long as the former is preferred over the latter by all coalition members. Let us denote $\mathcal{D}(x, y)$ be the family of decisive coalitions over (x, y). Notably, different rules (may) have different decisive coalitions.

In general, decisive coalitions might be different across pairs of alternatives. However, as shown by Sen (1969, 1970), the field expansion lemma holds under transitivity. According to this lemma, if some group is decisive over some pairs of alternatives, it is decisive over all pairs.[10] This lemma simplifies the decisive structure because all pairs of alternatives have the same family of decisive coalitions. In other words, alternative dependency is not possible, and a type of neutrality over alternatives holds in some sense. More precisely, the following holds:

$$[A \in \mathcal{D}(x, y) \text{ for some } (x, y) \text{ with } x \neq y] \Rightarrow [A \in \mathcal{D}(x, y) \text{ for all } (x, y) \text{ with } x \neq y].$$

Therefore, it is sufficient to consider "global decisiveness," which is defined as follows: a coalition $A \subseteq N$ is (globally) decisive if it is decisive for all pairs of distinct alternatives. Arrow's proof uses this fact as its key step. The remaining step is to show that for any decisive set that include at least two individuals, there is a proper subset that is decisive.[11]

According to Sen (1969, 1970), the field expansion lemma holds even if transitivity is weakened to quasi-transitivity. Without the lemma, the structure of collective choice rules can be complicated; for instance, see Brown (1975), Banks (1995), Gibbard (2014a, b), and Schwartz (2007, 2018). Notably, the field-expansion lemma does not hold for Suzumura consistency. Largely due to this violation of the lemma, Suzumura consistency has not been intensively examined in the theory of collective choice, despite its importance. Decisiveness is not a very helpful concept for Suzumura-consistent social choice; thus, it is difficult to examine Suzumura consistency with

[10] Notably, a crucial step in the general aggregation problem of individual judgments is also to show a variation of this lemma; see Dietrich and List (2007a, b).

[11] These observations are closely related to the concepts of filters and ultrafilters, as shown by Kirman and Sondermann (1972), Hansson (1976), and Cato (2012, 2013). A collection $\mathcal{F}$ of subsets of N is said to be a *filter* if the following properties hold: (i) $N \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$; (ii) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$; (iii) if $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$. Moreover, a collection $\mathcal{F}$ of subsets of N is called an *ultrafilter* if it is a filter that satisfies the following property: (v) $A \in \mathcal{F}$ or $N \setminus A \in \mathcal{F}$. See Willard (1970) for detailed explanations of filters and ultrafilters.

weak Pareto and independence of irrelevant alternatives. Due to this problem, existing works by Bossert and Suzumura (2008, 2010) impose anonymity and neutrality in addition to Arrow's axioms. These two axioms simplify the decision-making structure. Specifically, Bossert and Suzumura (2008) completely identify a class of rules that satisfy anonymity (impartiality), neutrality, and a stronger version of the Pareto principle. However, if neutrality is imposed as an axiom, alternative dependency, which is the fundamental characteristic considered in this study, disappears.

3.2 Alternative dependency

As mentioned earlier, Suzumura consistency is equivalent to transitivity in the presence of completeness. This means that completeness is also dropped as a property for social relations. When social relations are not assumed to be complete, the concept of a decisive coalition for a pair of alternatives is not enough to characterize the power structure behind the process of collective decision-making. For each pair (x, y), one needs to have a *pair* of disjoint sets to represent the set of voters who put x above y and the set of voters who put y above x. We examine what it is for a collective choice rule to be represented by a family of sets of pairs of sets indexed by pairs of alternatives. Our characterizations of rules are expressed in terms of pairs of coalitions that determine the ranking of the alternatives in X for any given profile. For every $(x, y) \in X \times X$, let $\mathcal{G}(x, y)$ be a family of pairs of disjoint subsets of N. An element of this family $\mathcal{G}(x, y)$ is a pair of (G, J) of disjoint coalitions, where G is intended to represent the set of individuals who put x above y, while J is intended to represent the set of voters who put y above x. Given a collection $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$, a collective choice rule f *is represented by* $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$ if and only if

$$(x, y) \in f(\mathbf{R}) \Leftrightarrow$$
$$\left[\exists (G, J) \in \mathcal{G}(x, y) \text{ such that } G = \{i \in N | (x, y) \in P(R_i)\} \text{ and } J = \{i \in N | (y, x) \in P(R_i)\}\right]$$

for all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{R}^N$.

We investigate what an indexed family of sets of pairs of sets would have to be like to represent a Suzumara-consistent collective choice rule that satisfies weak Pareto and independence of irrelevant alternatives. As an auxiliary step, define $\mathcal{P}(\mathcal{G}(x, y))$ by letting

$$(G, J) \in \mathcal{P}(\mathcal{G}(x, y)) \Leftrightarrow \Big[(G, J) \in \mathcal{G}(x, y) \text{ and } (J, G) \notin \mathcal{G}(y, x)\Big].$$

Although operator $\mathcal{P}$ is not exactly the one generating the asymmetric part of each binary relation, it plays a similar role. An *alternative-dependent coherent collection* on N is a collection $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$ such that

(C1) $(N, \emptyset) \in \mathcal{G}(x, y)$ for all pairs (x, y) of distinct alternatives;
(C2) $(\emptyset, N) \notin \mathcal{G}(x, y)$ for all pairs (x, y) of alternatives;
(C3) for all $K \geq 2$, for all distinct $x_1, \ldots, x_K \in X$, and for all pairs $(G_1, J_1), \ldots, (G_K, J_K)$ of disjoint coalitions, if $(G_k, J_k) \in \mathcal{G}(x_k, x_{k+1})$ for

all $k \in \{1, \ldots, K-1\}$, and $(G_K, J_K) \in \mathcal{P}(\mathcal{G}(x_K, x_1))$, then

$$G_k \nsubseteq \bigcup_{j \neq k} J_j \text{ for some } k \in \{1, \ldots, K\},$$

or

$$J_k \nsubseteq \bigcup_{j \neq k} G_j \text{ for some } k \in \{1, \ldots, K\}.$$

This notion has several distinguishable features. First, the structure is dependent on a pair of alternatives. That is, $\mathcal{G}(x, y)$ can be different from $\mathcal{G}(z, w)$ if (x, y) is not necessarily the same as (z, w). This feature suggests that the field expansion lemma is violated. Next, this notion utilizes the asymmetric part (of a kind) of $\mathcal{G}(x, y)$. As explained in the following section, an element in the asymmetric part of $\mathcal{G}(x, y)$ can make x to be socially strictly preferred to y if the pair (G, J) in the asymmetric part coincides with the pair of the set of those who prefer x to y and the set of those who prefer y to x. This is particularly important for the case of Suzumura consistency because its nature is sensitive to the difference between strict preferences and indifferences.

The following is an example that illustrates the structure of an alternative-dependent coherent collection.

Example 1 For simplicity, we assume here that $X = \{x, y, z\}$. Consider the following collection $\langle \mathcal{G}^*(x, y) \rangle_{(x,y) \in X \times X}$:

$$\mathcal{G}^*(x, y) = \{(N, \emptyset), (\{1, 2\}, \{3\})\}; \mathcal{G}^*(y, x) = \{(N, \emptyset), (\{1, 2\}, \{3\})\};$$
$$\mathcal{G}^*(y, z) = \{(N, \emptyset), (\{2, 3\}, \{1\})\}; \mathcal{G}^*(z, y) = \{(N, \emptyset), (\{2, 3\}, \{1\})\};$$
$$\mathcal{G}^*(z, x) = \{(N, \emptyset)\}; \mathcal{G}^*(x, z) = \{(N, \emptyset)\}.$$

It is easy to check that this is an alternative-dependent coherent collection. Now, note the following:

$$(\{1, 2\}, \{3\}) \in \mathcal{G}^*(x, y), (\{2, 3\}, \{1\}) \in \mathcal{G}^*(y, z), \text{ and } (N, \emptyset) \in \mathcal{P}(\mathcal{G}^*(z, x)).$$

This instance is compatible with (C3) because $N \nsubseteq \{3\} \cup \{1\}$. The other instances can be checked similarly; thus, (C3) is satisfied.

In the next section, we show that an alternative-dependent decisive structure is established for the potential rationality of Suzumura consistency. However, this is not the first study to examine alternative dependency. A pioneering work by Ferejohn and Fishburn (1979) examines transitivity, quasi-transitivity, and acyclicity of social preferences. Bossert and Suzumura (2012) also propose the concept of a product filter that incorporates alternative dependency in a specific way. A substantial difference between their research and this study is that our representation can apply to *all* rules that satisfy weak Pareto and independence of irrelevant alternatives, but the existing works by Bossert and Suzumura (2012) and Ferejohn and Fishburn (1979) can only apply to a *restricted* classes of rules.[12]

4 Main representation theorem

This section provides a main result, which offers a representation of a class of all Suzumura-consistent collective choice rules that satisfy weak Pareto and independence of irrelevant alternatives. The ensuing result states that the connection to an alternative-dependent coherent collection is necessary and sufficient for an Arrovian collective choice rule to be Suzumura-consistent/potentially rational.

Theorem 1 *A Suzumura-consistent collective choice rule f satisfies weak Pareto and independence of irrelevant alternatives if and only if there exists an alternative-dependent coherent collection $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$ on N such that f is represented by $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$.*

Proof 'Only if.' Assume that f is a Suzumura-consistent collective choice rule satisfying weak Pareto and independence of irrelevant alternatives. Let $\mathcal{O}$ be the set of all pairs of disjoint coalitions. For each $(x, y) \in X \times X$, define

$$\begin{aligned}\mathcal{G}_f(x, y) :=&\{(G, J) \in \mathcal{O} | \exists \mathbf{R} \in \mathcal{R}^N \text{ such that} \\ &G = \{i \in N | (x, y) \in P(R_i)\}, J = \{i \in N | (y, x) \in P(R_i)\}, \text{ and } (x, y) \in f(\mathbf{R})\}.\end{aligned}$$

By independence of irrelevant alternatives, $\langle \mathcal{G}_f(x, y)\rangle_{(x,y)\in X\times X}$ is such that f is represented by $\langle \mathcal{G}_f(x, y)\rangle_{(x,y)\in X\times X}$.

It suffices to show that $\langle \mathcal{G}_f(x, y)\rangle_{(x,y)\in X\times X}$ satisfies (C1)–(C3). (C1) and (C2) immediately follow from weak Pareto. Hence, we need to show only (C3). On the contrary, suppose that there exist $K \geq 2, x_1, \ldots, x_K \in X$, and $(G_1, J_1), \ldots, (G_K, J_K) \in \mathcal{O}$ such that $(G_k, J_k) \in \mathcal{G}_f(x_k, x_{k+1})$ for all $k \in \{1, \ldots, K-1\}$, and $(G_K, J_K) \in \mathcal{P}(\mathcal{G}_f(x_K, x_1))$, and both

$$G_k \subseteq \bigcup_{j\neq k} J_j \text{ for all } k \in \{1, \ldots, K\}, \tag{1}$$

and

$$J_k \subseteq \bigcup_{j\neq k} G_j \text{ for all } k \in \{1, \ldots, K\}. \tag{2}$$

[12] This is confirmed by noting the point that Bossert and Suzumura, (2012) divided their results into two theorems (one for necessity and the other for sufficiency). Their concept of product filter is regarded as a special case of the concept set out in this paper. Also, in their theorems, Ferejohn and Fishburn, (1979) explicitly stated a restriction on the class of collective choice rules.
We offer a representation theorem in a more general class. Moreover, this study does not impose finiteness on a set of individuals; any population structure, including the infinite case, is encompassed in our findings.

Let $(\hat{R}_i)_{i\in N}$ be such that $\hat{R}_i \subseteq \bigcup_{k=\{1,\ldots,K\}}\{(x_k, x_{k+1}), (x_{k+1}, x_k)\}$ for all $i \in N$ and the following holds: for all $k \in \{1, \ldots, K-1\}$,

$$G_k = \{i \in N | (x_k, x_{k+1}) \in P(\hat{R}_i)\} \text{ and } J_k = \{i \in N | (x_{k+1}, x_k) \in P(\hat{R}_i)\}$$

and

$$G_K = \{i \in N | (x_K, x_1) \in P(\hat{R}_i)\} \text{ and } J_K = \{i \in N | (x_1, x_K) \in P(\hat{R}_i)\}.$$

We will show that there exists a profile $\mathbf{R} \in \mathcal{R}^N$ such that each individual preference R_i is an ordering extension of $\hat{R}_i$, i.e., $R_i \in \mathcal{R}$, $\hat{R}_i \subseteq R_i$ and $P(\hat{R}_i) \subseteq P(R_i)$ for all $i \in N$.

For convenience, x_{K+1} stands for x_1. By Lemma 1, there exists a profile of ordering extensions of $(\hat{R}_i)_{i\in N}$ if and only if every binary relation $\hat{R}_i$ is Suzumura-consistent. By way of contradiction, suppose that for some $i \in N$, $\hat{R}_i$ violates Suzumura consistency. That is, there exist $i^* \in N$ and $k \in \{1, \ldots, K\}$ such that (i)

$$(x_\ell, x_{\ell+1}) \in \hat{R}_{i^*} \text{ for all } \ell \in \{1, \ldots, K\} \text{ and } (x_{k+1}, x_k) \notin \hat{R}_{i^*},$$

or (ii)

$$(x_{\ell+1}, x_\ell) \in \hat{R}_{i^*} \text{ for all } \ell \in \{1, \ldots, K\} \text{ and } (x_k, x_{k+1}) \notin \hat{R}_{i^*}.$$

Consider the case (i). Since $(x_k, x_{k+1}) \in \hat{R}_{i^*}$ and $(x_{k+1}, x_k) \notin \hat{R}_{i^*}$, we have $i^* \in G_k$. By (1), $i^* \in \bigcup_{j\neq k} J_j$, and thus we have $(x_{m+1}, x_m) \in P(\hat{R}_{i^*})$ for some $m \in \{1, \ldots, K\}$. This implies $(x_m, x_{m+1}) \notin \hat{R}_{i^*}$ for some $m \in \{1, \ldots, K\}$. This contradicts the assumption of (i).

Consider the case (ii). Since $(x_{k+1}, x_k) \in \hat{R}_{i^*}$ and $(x_k, x_{k+1}) \notin \hat{R}_{i^*}$, we have $i^* \in J_k$. By (2), $i^* \in \bigcup_{j\neq k} G_j$, and thus we have $(x_m, x_{m+1}) \in P(\hat{R}_{i^*})$ for some $m \in \{1, \ldots, K\}$. This implies $(x_{m+1}, x_m) \notin \hat{R}_{i^*}$ for some $m \in \{1, \ldots, K\}$. This contradicts the assumption of (ii).

Let $\mathbf{R} \in \mathcal{R}^N$ be such that each individual preference R_i is an ordering extension of $\hat{R}_i$. By construction of $\mathbf{R}$ and by definition of $\mathcal{G}_f(x, y)$, we have

$$(x_k, x_{k+1}) \in f(\mathbf{R}) \text{ for all } k \in \{1, \ldots, K-1\}$$

and

$$(x_K, x_1) \in f(\mathbf{R}) \text{ and } (x_1, x_K) \notin f(\mathbf{R}).$$

This contradicts Suzumura consistency.

'If.' Let $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$ be an alternative-dependent coherent collection on N. Suppose that f is represented by $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$. From (C1) and (C2), the collective choice rule f satisfies weak Pareto. Furthermore, by the definition of f, the axiom of independence of irrelevant alternatives is satisfied. Thus, it suffices to show that the collective choice rule f is Suzumura-consistent. By way of contradiction, suppose that there exist $K \geq 3$, $x_1, \ldots, x_K \in X$, and $\mathbf{R} \in \mathcal{R}^N$ such that

$$(x_k, x_{k+1}) \in f(\mathbf{R}) \text{ for all } k \in \{1, \ldots, K-1\},$$

and

$$(x_K, x_1) \in f(\mathbf{R}) \text{ and } (x_1, x_K) \notin f(\mathbf{R}).$$

Since f is represented by $\langle \mathcal{G}(x, y) \rangle_{(x,y) \in X \times X}$, we have $(G_k, J_k) \in \mathcal{G}(x_k, x_{k+1})$ for all $k \in \{1, \ldots, K-1\}$, $(G_K, J_K) \in \mathcal{G}(x_K, x_1)$, and $(J_K, G_K) \notin \mathcal{G}(x_K, x_1)$, where

$$G_k = \{i \in N | (x_k, x_{k+1}) \in P(R_i)\} \text{ and } J_k = \{i \in N | (x_{k+1}, x_k) \in P(R_i)\}$$
$$\text{for all } k \in \{1, \ldots, K-1\},$$

and

$$G_K = \{i \in N | (x_K, x_1) \in P(R_i)\} \text{ and } J_K = \{i \in N | (x_1, x_K) \in P(R_i)\}.$$

(C3) implies that (i)

$$G_k \nsubseteq \bigcup_{j \neq k} J_j \text{ for some } k \in \{1, \ldots, K\},$$

or (ii)

$$J_k \nsubseteq \bigcup_{j \neq k} G_j \text{ for some } k \in \{1, \ldots, K\}.$$

We distinguish the two cases (i) and (ii). In the rest of this proof, x_{K+1} stands for x_1. Consider the case (i). There exists an individual $i^* \in N$ such that $i^* \in G_k$ and $i^* \notin \bigcup_{k \neq \ell} J_\ell$. Since $i^* \notin \bigcup_{k \neq \ell} J_\ell$, we have $i^* \notin J_\ell$ for all $\ell \neq k$. The completeness of individual preference implies that

$$(x_{k+1}, x_{k+2}) \in R_{i^*}, (x_{k+2}, x_{k+3}) \in R_{i^*}, \ldots, (x_{K-1}, x_K) \in R_{i^*}, (x_K, x_1)$$
$$\in R_{i^*} \ldots, (x_{k-1}, x_k) \in R_{i^*}.$$

However, because $i^* \in G_k$, we have

$$(x_k, x_{k+1}) \in P(R_{i^*}).$$

Obviously, R_{i^*} is not Suzumura-consistent, and thus there is no ordering extension of it by Lemma 1. This contradicts the requirement that each individual preference is an ordering.

Consider the case (ii). There exists an individual $i^* \in N$ such that $i^* \in J_k$ and $i^* \notin \bigcup_{k \neq \ell} G_\ell$. Since $i^* \notin \bigcup_{k \neq \ell} G_\ell$, we have $i^* \notin G_\ell$ for all $\ell \neq k$. The completeness of individual preference implies that

$$(x_k, x_{k-1}) \in R_{i^*}, (x_{k-1}, x_{k-2}) \in R_{i^*}, \ldots, (x_2, x_1) \in R_{i^*}, (x_1, x_K) \\ \in R_{i^*} \ldots, (x_{k+2}, x_{k+1}) \in R_{i^*}.$$

However, because $i^* \in J_k$, we have

$$(x_{k+1}, x_k) \in P(R_{i^*}).$$

Obviously, R_{i^*} is not Suzumura-consistent, and thus there is no ordering extension of it by Lemma 1. This contradicts the requirement that each individual preference is an ordering. ■

As shown in Arrow's impossibility theorem, it is not possible to have a plausible aggregation method if social preferences are required to be transitive (actually rational in our terminology). Theorem 1 implies that if social preferences are required to be potentially extended to transitive and complete relations (potential rationality or Suzumura consistency), there can be a non-dictatorial collective decision rule. Moreover, the collective decision process can be achieved in an alternative-dependent manner.

To understand how an alternative-dependent coherent collection works in decision processes and how it can avoid Arrow's impossibility theorem, let us consider f^* represented by $\langle \mathcal{G}^*(x, y) \rangle_{(x,y) \in X \times X}$ in Example 1. Theorem 1 guarantees that f^* satisfies both weak Pareto and independent of irrelevant alternatives and is Suzumura-consistent. However, there is no dictator for f^*. To observe this, note that $(\{i\}, N \setminus \{i\})$ must be included in $\mathcal{G}^*(x, y)$ for all x, y whenever i has dictatorial power. It is easy to see that there are many alternative-dependent coherent collections that do not have any dictator.

In order to illustrate how f^* generates Suzumura-consistent social preferences, we consider a "Condorcet" profile, which yields a cyclic social preference for the simple majority rule; this preference profile is considered problematic for social choice. Assume that $X = \{x, y, z\}$ and $N = \{1, 2, 3\}$. Let $\mathbf{R}$ be such that

$$P(R_1) = \{(z, x), (x, y), (z, y)\};$$
$$P(R_2) = \{(x, y), (y, z), (x, z)\};$$
$$P(R_3) = \{(y, z), (z, x), (y, x)\}.$$

We have the following social ranking:

$$(x, y) \in P(f^*(\mathbf{R})) \text{ and } (y, z) \in P(f^*(\mathbf{R})).$$

However, x and z are socially incomparable under f^*, that is,

$$(x, z) \notin f^*(\mathbf{R}) \text{ and } (z, x) \notin f^*(\mathbf{R}).$$

This social ranking is indeed Suzumura-consistent. Therefore, there must be an ordering extension (or transitive extension). In this case, it is uniquely determined. Since x is socially better than y and y is socially better than z, x must be socially better than z under any ordering/transitive extension.

Let us explain the general implications of Theorem 1. First, it should be emphasized that any generated social preference can be extended to a complete and transitive preference (ordering), as illustrated for f^*, because of the potential rationality of Suzumura consistency. Thus, it is possible to obtain social preferences with full rationality using an extension procedure. Extended ordering (more precisely, the process of generating orderings through a non-dictatorial Suzumura-consistent rule) is obviously non-dictatorial. No one can impose their preference on social preferences. Moreover, extended orderings always reflect the unanimous agreement of people (thus, being compatible with weak Pareto) because they are extensions. This may be considered puzzling because there is a gap between potential rationality (Suzumura consistency) and actual rationality (transitivity). The key lies in independence of irrelevant alternatives. To determine whether a certain pair (x, y) is included in an extension, a third alternative can be crucial, and thus, the condition of independence of irrelevant alternatives is violated in the procedure of generating full rational preferences through potentially rational ones. To demonstrate, consider the case where x is better than y, and y is indifferent to z (but x and z are not comparable). Notably, x is better than z for any ordering extension of this. For this consideration, the rankings over other pairs (in this case, (x, y) and (y, z)) are necessary. This implies that one will use information other than that on the two alternatives (x and z) in question.

This observation shows a significant trade-off between rationality and the independence axiom. It means that one can impose independence of irrelevant alternatives under potential rationality, but not under actual rationality. However, if independence of irrelevant alternatives is abandoned, then actual rationality is achieved; thus, there is no need to consider potential rationality. For example, social preferences generated by the Borda rule and scoring rules are actually rational; it is known that none of these rules satisfy independence of irrelevant alternatives. We stress that there are many collective decision/choice problems for which independence of irrelevant alternatives is plausible. There are many cases whose alternatives are unknown; however, the Borda rule and scoring rules require ranking such alternatives. Because independence of irrelevant alternatives allows us to make decisions based on each pair of alternatives, unavailable alternatives do not need to be known to determine the rankings of available alternatives. The theorem demonstrates that potential rationality is a reasonable candidate for rationality in such cases.

5 Impartial collective decision-making and potential rationality

This section examines impartial collective decision-making with potential rationality. Notably, impartiality requires decision-making to be invariant for a certain class of permutations. The following axiom is full anonymity, which requires that any permutation of individual names does not change social rankings. Non-dictatorship obviously follows from this axiom.

Full Anonymity: For all bijections $\rho : N \to N$ and for all $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^N$,

$$R_i = R'_{\rho(i)} \ \forall i \in N \Rightarrow f(\mathbf{R}) = f(\mathbf{R}').$$

We note that the implication of anonymity is dependent on the population structure. Consider the set of shuffled individuals; it corresponds to $\{i \in N \mid \rho(i) \neq i\}$. If $\{i \in N \mid \rho(i) \neq i\}$ is finite, a permutation ρ is said to be finite. If the set of individuals is finite, anonymity simply applies to the set of all finite permutations. In contrast, if the set of individuals is infinite, anonymity can include infinite permutations; that is, it can be the case where $\{i \in N \mid \rho(i) \neq i\}$ is infinite. As shown by Lauwers (1997), infinite permutations result in a serious impossibility result. Thus, studies in infinite aggregation often employ a version of anonymity where only finite permutations are allowed to apply; see, for example, Wilkinson (2021).[13]

Finite Anonymity: For all bijections $\rho : N \to N$ such that $\{i \in N \mid \rho(i) \neq i\}$ is finite and for all $\mathbf{R}, \mathbf{R}' \in \mathcal{R}^N$,

$$R_i = R'_{\rho(i)} \ \forall i \in N \Rightarrow f(\mathbf{R}) = f(\mathbf{R}').$$

Now, we examine how the anonymity axioms restrict the structure of alternative-dependent coherent collections. Notably, cardinality is crucial under the anonymity axioms. For each finite set G, its cardinality, $\#G$, represents the number of elements in G. For any infinite set G, let $\#G = \infty$. Therefore, any infinite set includes a proper subset that has the same cardinality, and moreover, any set can be partitioned into two or more subsets such that each subset has the same cardinality as the original one. An alternative-dependent coherent collection $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$ on N is said to be *symmetric* if it satisfies the following:

(C4) for all $(x, y) \in X \times X$, and for all pairs (G, J), (G', J') of disjoint coalitions, if $\#G = \#G'$, $\#J = \#J'$, and $\#N \setminus (G \cup J) = \#N \setminus (G' \cup J')$, then

$$(G, J) \in \mathcal{G}(x, y) \Leftrightarrow (G', J') \in \mathcal{G}(x, y).$$

We now state a representation result with full anonymity.

Theorem 2 *A Suzumura-consistent collective choice rule f satisfies weak Pareto, independence of irrelevant alternatives, and full anonymity if and only if there exists a symmetric alternative-dependent coherent collection $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$ on N such that f is represented by $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$.*

Proof 'Only if.' Theorem 1 implies that there exists an alternative-dependent coherent collection $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$ on N such that f is represented by $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$. That is, it holds that

$(x, y) \in f(\mathbf{R}) \Leftrightarrow$

[13] See also Vallentyne and Kagan (1997) and Cato (2017, 2021).

$$\left[\exists (G, J) \in \mathcal{G}(x, y) \text{ such that } G = \{i \in N | (x, y) \in P(R_i)\} \text{ and } J = \{i \in N | (y, x) \in P(R_i)\}\right]$$

for all $x, y \in X$ and for all $\mathbf{R} \in \mathcal{R}^N$.

It suffices to show that (C4) holds. As in the proof of Theorem 1, $\mathcal{O}$ denotes the set of all pairs of disjoint coalitions. Let $x, y \in X$ and $(G, J) \in \mathcal{O}$ be such that $(G, J) \in \mathcal{G}(x, y)$. Take any $(G', J') \in \mathcal{O}$ such that $\#G' = \#G$, $\#J' = \#J$, and $\#N \setminus (G \cup J) = \#N \setminus (G' \cup J')$. Let $\mathbf{R}' \in \mathcal{R}^N$ be such that $R_i = R'_{\rho(i)}$ for some bijection ρ, and

$$G' = \{i \in N | (x, y) \in P(R'_i)\} \text{ and } J' = \{i \in N | (y, x) \in P(R'_i)\}.$$

Independence of irrelevant alternatives and full anonymity imply that $(x, y) \in f(\mathbf{R}')$, and thus $(G', J') \in \mathcal{G}(x, y)$. Under independence of irrelevant alternatives and full anonymity, if $(G, J) \in \mathcal{G}(x, y)$ for some $(G, J) \in \mathcal{O}$, then $(G', J') \in \mathcal{G}(x, y)$ for all $(G', J') \in \mathcal{O}$ with $\#G = \#G'$ and $\#J = \#J'$. Thus, (C4) holds.

'If.' Suppose that there exists a symmetric alternative-dependent coherent collection $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$ on N such that f is represented by $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$. By Theorem 1, f is Suzumura-consistent and satisfies weak Pareto and independence of irrelevant alternatives. It suffices to show that it also satisfies full anonymity. Take any $\mathbf{R} \in \mathcal{R}^N$ and a bijection $\rho : N \to N$. Let $(a, b) \in f(\mathbf{R})$ and $\mathbf{R}' \in \mathcal{R}^N$ such that $R'_i = R_{\rho(i)}$. It holds that

$$\#G = \#J, \;\; \#G' = \#J', \;\; \#N \setminus (G \cup J) = \#N \setminus (G' \cup J')$$

where

$$\begin{aligned} G &= \{i \in N | (a, b) \in P(R_i)\}; \\ J &= \{i \in N | (b, a) \in P(R_i)\}; \\ G' &= \{i \in N | (a, b) \in P(R'_i)\}; \\ J' &= \{i \in N | (b, a) \in P(R'_i)\}. \end{aligned}$$

Since $(a, b) \in f(\mathbf{R})$, it follows that $(G, J) \in \mathcal{G}(a, b)$ because f is represented by $\langle \mathcal{G}(x, y)\rangle_{(x,y)\in X\times X}$. From (C4), it follows that $(G', J') \in \mathcal{G}(a, b)$. Thus, $(a, b) \in f(\mathbf{R}')$. Full anonymity is satisfied. ■

We subsequently move on to a representation of finitely anonymous collective choice rules. Let us consider the following variant of (C4).

(C4') for all $(x, y) \in X \times X$, and for all pairs $(G, J), (G', J')$ of disjoint coalitions, if $\#(G \setminus G') = \#(G' \setminus G) < \infty$, $\#(J \setminus J') = \#(J' \setminus J) < \infty$, and $\#((G \cup J) \setminus (G' \cup J')) = \#((G' \cup J') \setminus (G \cup J)) < \infty$, then

$$(G, J) \in \mathcal{G}(x, y) \Leftrightarrow (G', J') \in \mathcal{G}(x, y).$$

First, we note that (C4') is logically implied by (C4). This is because the antecedent of (C4) is implied by that of (C4'). That is, if $\#(G \setminus G') = \#(G' \setminus G)$, $\#(J \setminus J') = \#(J' \setminus J)$, and $\#((G \cup J) \setminus (G' \cup J')) = \#((G' \cup J') \setminus (G \cup J))$, then $\#G = \#G'$, $\#J = \#J'$, and $\#N \setminus (G \cup J) = \#N \setminus (G' \cup J')$. However, the converse is not true; (C4') does not imply (C4).

Example 2 Assume that N is countably infinite. Define

$$G = \{1, 3, 5, 7, \ldots, \} \text{ and } J = \{2, 4, 6, 8, \ldots, \}.$$

Let $\langle \mathcal{G}(x, y) \rangle_{(x,y) \in X \times X}$ be such that

$$(G, J) \in \mathcal{G}(x, y) \text{ and } (J, G) \in \mathcal{G}(x, y)$$

for all $x, y \in X \times X$. Let

$$G' = J \text{ and } J' = G.$$

We note that $\#G = \#J = \infty$ and $\#N \setminus (G \cup J) = 0$. Moreover, $\#G' = \#J' = \infty$ and $\#N \setminus (G' \cup J') = 0$. Moreover, $G \setminus G'$, $G' \setminus G$, $J \setminus J'$, and $J' \setminus J$ are infinite. Thus, $(G, J) \in \mathcal{G}(x, y) \Leftrightarrow (G', J') \in \mathcal{G}(x, y)$ holds under (C4), but it does not have to hold under (C4'). In sum, (C4') does not imply (C4).

Since N is allowed to be infinite, a difference between (C4) and (C4') is significant. An alternative-dependent coherent collection $\langle \mathcal{G}(x, y) \rangle_{(x,y) \in X \times X}$ on N is said to be *finitely symmetric* if (C4') is satisfied. We state the result for a representation of a class of finitely anonymous collective choice rules.

Theorem 3 *A Suzumura-consistent collective choice rule f satisfies weak Pareto, independence of irrelevant alternatives, and finite anonymity if and only if there exists a finitely symmetric alternative-dependent coherent collection $\langle \mathcal{G}(x, y) \rangle_{(x,y) \in X \times X}$ on N such that f is represented by $\langle \mathcal{G}(x, y) \rangle_{(x,y) \in X \times X}$.*

Notably, the difference between (C4) and (C4') disappears when N is finite. This corresponds to the fact that finite anonymity is equivalent to full anonymity in the finite case. Moreover, another condition, which is simpler than (C4) and (C4'), can be used for representing anonymous collective choice rules. Let us consider the following:

(C4") for all $(x, y) \in X \times X$, and for all pairs (G, J), (G', J') of disjoint coalitions, if $\#G = \#G'$ and $\#J = \#J'$, then

$$(G, J) \in \mathcal{G}(x, y) \Leftrightarrow (G', J') \in \mathcal{G}(x, y).$$

First, we note that we note that if N is finite, then (C4") logically implies (C4) because

$$[\#G = \#G' \text{ and } \#J = \#J'] \Rightarrow \#N \setminus (G \cup J) = \#N \setminus (G' \cup J').$$

It is easy to verify that (C4") logically implies (C4) (and (C4")). Thus, it is the strongest condition among the three. To show that (C4) does not imply (C4"), we present the following example.

Example 3 Assume that N is infinite. Let us consider

$$G = \{1, 3, 5, 7, \ldots, \} \text{ and } J = \{2, 4, 6, 8, \ldots, \}.$$

Thus, $\#G = \#J = \infty$ and $\#N \setminus (G \cup J) = 0$. Now, let us consider

$$G' = \{3, 5, 7, 9, \ldots, \} \text{ and } J' = \{4, 6, 8, 10 \ldots, \}.$$

Thus, $\#G' = \#J' = \infty$ and $\#N \setminus (G' \cup J') = 2$. Thus, (C4) does not imply (C4”).

In the finite case, all (C4), (C4’), and (C4”) are equivalent.

6 Concluding remarks

In this study, we argue that the distinction between actual and potential rationality, introduced here, is generally important for various fields of philosophy and ethics, including axiology and political philosophy. Some impossible results may be avoided by shifting from actual to potential rationality. In this study, we considered Arrow’s impossibility theorem, which is at the core of social choice theory. The resultant theorem is helpful not only for political philosophy and axiology, but also for the philosophy of science; see Okasha (2011). The findings establish a representation of a class of potential rational collective choice rules that satisfy weak Pareto and the independence of irrelevant alternatives. To achieve this, the concept of an alternative-dependent coherent collection is introduced. The theorem presented here implies that numerous possibilities arise under potential rationality.

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Declarations

Conflicts of Interest I have no conflicts of interest to disclose. This work does not involve any data, and thus, there is no participant.

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