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ORIGINAL RESEARCH

How to frame innovation in mathematics

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Abstract

We discuss conceptual change and progress within mathematics, in particular how tools, structural concepts and representations are transferred between fields that appear to be unconnected or remote from each other. The theoretical background is provided by the frame concept, which is used in linguistics, cognitive science and artificial intelligence to model how explicitly given information is combined with expectations deriving from background knowledge. In mathematical proofs, we distinguish two kinds of frames, namely structural frames and ontological frames. The interaction between both kinds of frames can drive mathematical interpretation. We first discuss two examples where structural frames (formulaic notation) drive ontological development (the discovery or exploration of mathematical objects). The development of Boole's Boolean algebra may at first appear as a metaphorical treatment of the (then) new area of logic. In the analysis, we discuss how different (aspects of) certain algebraic frames change in the transfer, how arising difficulties are solved and overall argue that Boole uses the numerical algebra frame as a research template for the discovery of a system for calculations in logic. Following Ifrah, we analyse the discovery of zero as an extension to the number ontology as driven by the development of notation. Both structural and ontological frames are extended and simplified as notation progresses. Finally, we discuss two examples from infinite combinatorics, viz. topological graph theory, and one foundational issue. In both examples, the two simultaneous frames about one object are maintained independently. They motivate different research questions, but may also fruitfully interact: shifting between multiple synchronously maintained perspectives acts as a motor of innovation. The analysis shows how a frame-based approach allows to model how different perspectives

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drive mathematical innovation because they highlight different aspects, questions and heuristics.

Keywords Frames · Mathematical language · Innovation

1 Introduction

This article can be seen as a contribution to the analysis of conceptual change and progress within mathematics. In particular we consider an important kind of innovation in mathematics, namely how tools, structural concepts and representations are transferred between fields that appear to be unconnected or remote from each other.

For modelling this transfer, we use the concept of *frames*. The frame concept (further explained in Sect. 2) is used in linguistics, cognitive science and artificial intelligence to model how explicitly given information is combined with background knowledge. Recently, Kornmesser and Schurz (2020) have applied the frame concept to the philosophy of science. The concept has also been applied in the context of mathematical proofs (Fisseni et al., 2019; Carl et al., 2021), assuming readers interpret proofs by complementing the explicitly given information with two kinds of schematic background knowledge. These can be uniformly modelled through two kinds of frames, namely *structural frames* that specify how proofs using specific proof methods are usually structured, including formulaic notation, and *ontological frames* that specify mathematical structures and objects as well as typical patterns of reference to these.

The interaction between both kinds of frames can drive mathematical interpretation, as we will discuss in this paper. An interesting aspect of the two cases of Boole's algebra (Sect. 3) and (the discovery of) Zero (Sect. 4) is that they can be seen as playful experiments with extending the use of notation and exploring whether the result is helpful. The use of frames for both aspects (notation and ontology) facilitates describing their mutual influence. The frame system we use (Sect. 2) allows to explicitly model hierarchies of (structural and ontological) concepts and the interaction between frames. This is in our view essential in treating the phenomena at hand.

The development of (Boole's) Boole an algebra may at first appear as a metaphorical treatment of the new area of logic. In their distinction between grounding and linking metaphors Lakoff and Nûñez (2000) characterize Boole's metaphorical transfer of concepts as the latter. However, Boole's approach exceeds the usual consequences of applying a concept c to a new domain D metaphorically: Usually, aspects of a c that do not fit D are 'covered' or 'suppressed' (Bühler, 1934, p. 349, 392) so that only an active zone of the concepts remains (Langacker, 2008, e.g., p. 114).[1] We will argue that Boole thus uses the numerical algebra frame as a research template for the discovery of a system for calculations in logic.

The discovery of zero as an extension to the number ontology is, if we follow Ifrah (2000), basically driven by the development of notation. Both structural and ontological frames are extended and simplified as notation progresses.

[1] See Fisseni (2023+) for an attempt of formally modelling metaphors as 'partial perspectives'.

We will furthermore discuss cases where two frames about one object are maintained independently (Sect. 5). We will focus on one example from infinite combinatorics, namely topological graph theory and one foundational issue. In those cases no new properties are introduced. The two original perspectives continue to be valid perspectives but they may differ in non-core elements and make different questions feasible. Even more so, some questions which are interesting from one perspective are not really meaningful from the other. We argue that shifting between multiple synchronously maintained perspectives can be a motor of innovation.

2 Frames

The frame concept originates from linguistics, cognitive science and artificial intelligence. It models how explicitly given information is combined with (expectations deriving from) background knowledge. In the context of mathematical proofs, readers of proofs have expectations due to their mathematical training. These allow them to interpret a mathematical text and to complement it with additional relevant information.

The classic report by Minsky (1974) introduces frames as “a data-structure for representing a stereotyped situation, like being in a certain kind of living room, or going to a child’s birthday party” (p. 1), organised in a hierarchical *frame-system*. Frames contain slots, which can be (sub-)frames again. They can have default values, and constraints can apply to them; *slots* are *filled* by concrete values. The concept of *feature structures* (see, e.g., Carpenter, 1992) can be seen as a formalization of frames; it uses an an inheritance hierarchy of types, where each type carries constraints on the applicability of features (= slots) and their possible values.

FrameNet[2] is an important approach within linguistics. Its hierarchical structure of (verb) semantics contains frames for each verb and (groups of verbs like *Commercial Transaction*) with features for semantic roles ultimately filled by participants. The distinction between core and non-core roles models that within a frame, roles have different degrees of salience, and some may even be optional. In the frame *Commerce_buy*, *Buyer* and *Goods* are core roles; non-core roles include *Seller*, *Money* and *Means* (e.g., cash vs. check). Frame elements which are explicitly realised are labelled in the following examples.

(1) a. [$_{Buyer}$ John] bought [$_{Goods}$ a beautiful medieval book] [$_{Time}$ yesterday].
b. [$_{Seller}$ Peter] sold [$_{Goods}$ a beautiful medieval book] to [$_{Buyer}$ John] for [$_{Money}$ twenty Euros].

The role assignments are displayed in the feature-value matrix in (2); its TIME slot also contains a subframe. We indicate core roles by an exclamation mark, and the semantics of the *expression* is `expression`. *point-in-time*, *person*, *money*, *purpose* are type labels; these types constrain how the slots can be filled. One can see that the SELLER slot need not be filled in the buying frame. Like with most verbs, the TIME slot is non-core in these frames. Ellipsis dots indicate that some slots have not been

[2] See, e.g., Ruppenhofer et al. (2006) and the project’s website at https://framenet.icsi.berkeley.edu.

given explicitly. It is customary to provide partial descriptions, so that we will omit the ellipsis dots from now on.

(2) a.

$$\begin{bmatrix} \textit{buy} & \\ \textsc{Buyer!} & [\![\textit{John}]\!] \\ \textsc{Goods!} & [\![\textit{a beautiful medieval book}]\!] \\ \textsc{Time} & [\![\textit{yesterday}]\!] \\ \textsc{Seller} & \textit{person} \\ \textsc{Money} & \textit{money} \\ \textsc{Purpose} & \textit{purpose} \\ \ldots & \end{bmatrix} = \begin{bmatrix} \textit{buy} & \\ \textsc{Buyer!} & \textit{j} \\ \textsc{Goods!} & \textit{b} \\ \textsc{Time} & \begin{bmatrix} \textit{point-in-time} & \\ \textsc{year} & \textit{2018} \\ \textsc{month} & \textit{02} \\ \textsc{day} & \textit{28} \\ \textsc{hour} & \{1, \ldots, 24\} \\ \textsc{minute} & \{0, \ldots, 60\} \\ \ldots & \end{bmatrix} \\ \textsc{Seller} & \textit{person} \\ \textsc{Money} & \textit{money} \\ \textsc{Purpose} & \textit{purpose} \\ \ldots & \end{bmatrix}$$

b.

$$\begin{bmatrix} \textit{sell} & \\ \textsc{Seller!} & [\![\textit{Peter}]\!] \\ \textsc{Buyer!} & [\![\textit{John}]\!] \\ \textsc{Goods!} & [\![\textit{a beautiful medieval book}]\!] \\ \textsc{Time} & \textit{point-in-time} \\ \textsc{Money} & [\![\textit{twenty Euros}]\!] \\ \textsc{Purpose} & \textit{purpose} \\ \ldots & \end{bmatrix} = \begin{bmatrix} \textit{sell} & \\ \textsc{Seller!} & \textit{p} \\ \textsc{Buyer!} & \textit{j} \\ \textsc{Goods!} & \textit{b} \\ \textsc{Time} & \textit{point-in-time} \\ \textsc{Money} & \textit{20€} \\ \textsc{Purpose} & \textit{purpose} \\ \ldots & \end{bmatrix}$$

In the following sections we will limit our considerations to core slots. Therefore, we will not use exclamation marks any more to differentiate between core and non-core slots.

More recently, the concept of frame has been developed further in linguistic and philosophic projects, most notably the SFB 991: *Die Struktur von Repräsentationen in Sprache, Kognition und Wissenschaft* in Düsseldorf (see e.g., Gamerschlag et al., 2014, 2015). The research elaborated the connection of the concept of *frames* to the semantic category of *functional concepts*, and also the history of scientific language, while also connecting to discourse analysis (see, e.g., Ziem, 2008, 2014). Regarding the formal representation of frames, Petersen (2015) develops a model using feature structures closely related to Carpenter's (1992) and highlighting the connection between frames and *functional concepts* (see Löbner, 2015).

In the realm of mathematics the frame concept was first used in the context of didactics by Davis (1984). From a constructivist perspective Davis models the possibly erroneous and over-generalized individual knowledge of learners by using frames. In this approach frames are a tool for the modelling of individual conditions for success or difficulties in learning processes. Many of the frames causing errors are overgeneralizations of mathematical knowledge acquired up to a certain stage, as e.g., a frame of binary operations equipped with features of addition which is transferred to multiplication and leads to invalid calculations like $4 \times 4 = 8$ (Davis, 1984, 111ff).

In others cases Davis argues that explanations for certain mathematical phenomena given by students depend on very basic pre-mathematical frames.

Modelling inductive proofs, Fisseni et al. (2019) see frames as guiding the processing of proofs based on usual mathematical practice. Carl et al. (2021) relate the frames approach to concepts of understanding and especially Avigad's ability-based account of understanding mathematical proofs.

The approach to applying the frame concept to mathematical proofs taken in this paper builds on the two aforementioned papers. It hypothesises, as stated in the introduction, that (at least) two kinds of frames play a crucial role. *Structural frames* on the one hand schematically model the structure of proofs and definitions, as exemplified by Engel's (1999) *Problem-Solving Strategies*. *Ontological frames* on the other model domain knowledge: mathematical structures and how these structures and their elements are usually referred to.

The mathematical subfield and the type of text determine expectations regarding presence and explicitness of proof elements, as attested to by the differences between, e.g., textbooks articles in a mathematical journal. Frames, formalizing these expectations, bridge the gap between the text and more formal representations. Handbook articles on specific areas often present proof techniques; these can be understood as building blocks for frames that can also be used in innovative ways.

In frame systems, frames can interact: slots of one frame can be filled from neighbouring, sub- or superordinate frames. For instance, the form of an induction depends on the underlying inductive type. The general structure in Fig. 1 gives the typical parts of the induction proof such as the BASE- CASE, the INDUCTION- STEP and the INDUCTION- VARIABLE. As the form of an induction on some type is also dependent on the structure of said type, the mentioned slots are complemented by those for the INDUCTION- SIGNATURE and the INDUCTION- DOMAIN. The non-recursive BASE-CONSTRUCTORS of the latter provide the cases in the BASE- CASE of the induction, and it RECURSIVE- CONSTRUCTORS derive the 'successors' of the base values, thus informing the INDUCTION- STEP.

This explicit frame structure (defined in the appendix of Fisseni et al., 2019) allows multiple BASE- CONSTRUCTORS and STEP- CONSTRUCTORS. Types with one base constructor and one recursive constructor allow for a simpler, more specialised frame.

Structurally, a typical induction on natural numbers will contain one BASE- CASE and one INDUCTION- STEP. An induction on complex formulas contains CASE-PROOFS for each kind of atomic formulas in the PROOF of the BASE- CASE, and one element in the CASE- PROOFS for each connective of the formal language in the PROOF of the INDUCTION- STEP. Ontologically, the form of hypotheses of the BASE- CASE and INDUCTION- STEP are constrained by the BASE- CONSTRUCTORS and RECURSIVE- CONSTRUCTORS: The former by default concerns the value of one of the BASE- CONSTRUCTORS and the latter applications of the RECURSIVE- CONSTRUCTORS.

In the view taken in Fisseni et al. (2019) and Carl et al. (2021), frames—both ontological and structural ones—have a conceptual and a form dimension. The latter consists in text-structuring elements, notational conventions and linguistic triggers. An example of a linguistic trigger are certain plural constructions like "L_1 and L_2 are parallel", which presuppose a symmetric relation and thus trigger the ontological frame of symmetric relations (see, e.g., Cramer & Schroder, 2012).

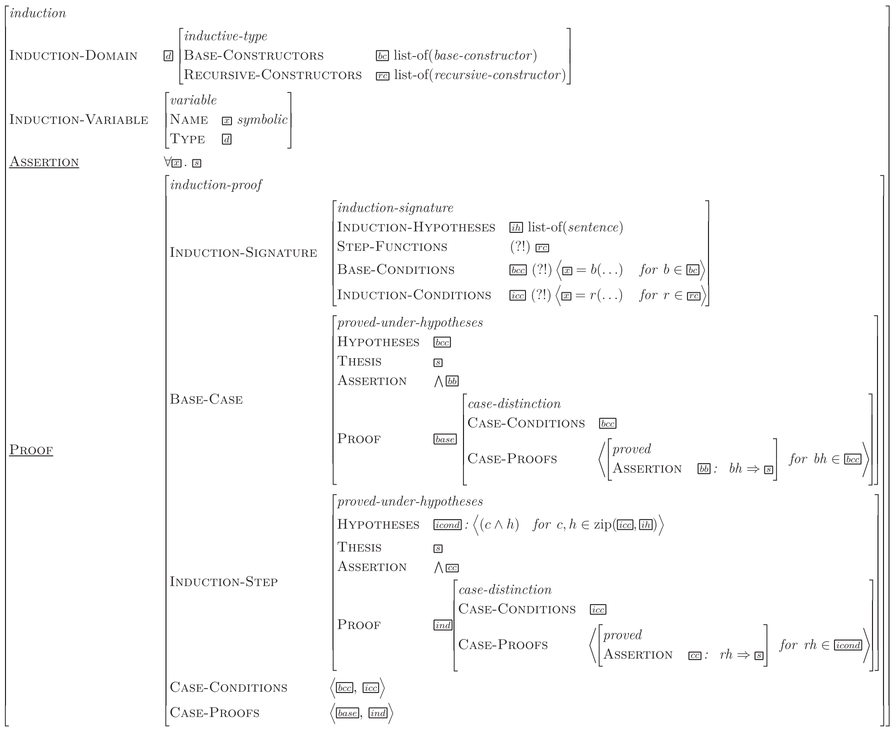


Fig. 1 Induction frame. (?!) marks default fillers of a slot

In this bilateral sign-like conception frames resemble linguistic constructions in the sense of construction grammar as advocated by Goldberg (2006) and others: Like constructions, frames are organized in type hierarchies of more general and more specialized frames. Similar to the acquisition of linguistic constructions in constructivist language acquisition approaches (cf. Tomasello, 2005), more general frames are usually acquired by generalization from more specific frames. Despite these parallel conceptualisations, frames differ from usual concepts of linguistic constructions as the restrictions on the form side are less strict. Most frames can be evoked by a huge range of formal realisations and hints, such as certain argumentation patterns. The form-concept relation is less transparent and more abstract compared with constructions usually discussed in linguistics.

For the cases studied in Sects. 3 and 4 below it is essential to assume that frames do not only represent conceptual structures in complete abstraction from their symbolic representation, but also reflect conventional linguistic and notational realisations. Certain mathematical symbols and notations can hint to certain frames and, conversely, frames make expectable certain linguistic and notational realisations.

3 Boole's algebra of logic

George Boole (1847, 1854) employs notation from algebra to represent logical formulae. This choice of notation reflects algebraic properties of logical connectives.

In (Boole, 1847) symbols as x, y, z are interpreted as functions which select from a given class those entities which are contained in classes X, Y, Z respectively. The given class is the "Universe", if these symbols occur alone. Otherwise, if they are concatenated, as in xy,[3] the left argument of the concatenation (here x) selects the subclass of those entities selected by the right argument (here y) which are contained in x. Boole (1854) gives the symbols a different semantics which interprets them as classes directly. It is obvious that the operation denoted by concatenation is commutative under both interpretations

$$xy = yx \tag{2.1}$$

(the second law in Boole, 1847, p. 15) and associative

$$(xy)z = x(yz) \tag{2.2}$$

Although he does not mention this latter property explicitly, his treatment of exponentiated symbols shows that he assumes associativity.

While semantically an interpretation of symbol concatenation as a logical conjunction or class intersection is intended clearly, the notation reminds of and is referred to as multiplication. The parallelism with multiplication is driven forward by denoting the "Universe" with 1 and the empty class with 0, such that

$$x1 = x \tag{2.3}$$

$$x0 = 0 \tag{2.4}$$

hold. With "+" as the union operator for disjoint classes or as logical disjunction (with presupposed incompatibility of the arguments) a further commutative and associative operator is introduced and we get distributivity for concatenation and "+", i.e.,

$$x(u + v) = xu + xv \tag{2.5}$$

as Boole states in his first law in (Boole, 1847, p. 15). By adding reverse operations like "−" and fraction lines, algebraic transformations of equations become available.

The parallelism with numerical algebra is limited of course. Terms are idempotent with respect to the concatenation operator, i.e., $xx = x$ or more general

$$x^n = x \tag{2.6}$$

holds and is valid for all x. There are further costs of his approach: The application of algebraic transformations may lead to uninterpretable symbols. Boole expresses the

[3] Boole (1854) uses the symbol "×" for this operation, too.

exclusive OR of x and y as the "sum" of $x(1-y)$ and $y(1-x)$ being equal with 1 which yields by factoring out

$$x - 2xy + y = 1 \tag{2.7}$$

He does not give any semantics for 2 or the expression $2xy$. (2.7) can be regarded as an intermediate step in a formal calculation without an independent semantics. By instantiating y with 1 he gets

$$x - 2x + 1 = 1 \tag{2.8}$$

hence

$$-x = 0 \tag{2.9}$$

or

$$x = 0 \tag{2.10}$$

And using the idempotence of concatenation, (2.7) may be rendered as

$$x^2 - 2xy + y^2 = 1 \tag{2.11}$$

making the second binomial formula applicable, which yields

$$(x - y)^2 = 1 \tag{2.12}$$

with the two solutions

$$(x - y) = \pm 1\,. \tag{2.13}$$

Boole (1854) provides $x - y$ with a meaning for the case that y is a subclass of x. But neither -1 in isolation nor $x - y$ in the general case get an interpretation. Boole raises the question, "whether it is necessary to restrict the application of these symbolical laws and processes by the same conditions of interpretability under which the knowledge of them was obtained" (Boole, 1854, p. 48) and immediately answers this question negatively, arguing that cancelling such restrictions is presupposed by symbolic reasoning, as uninterpretable symbols are needed in intermediate steps of certain calculations. He compares uninterpretable symbols in his logic with $\sqrt{-1}$ in algebra for which he claims the same intermediate character, evidently unaware of proposals for the interpretation of complex numbers at the time (cf. Burris & Javier, 2021).

Boole describes his enterprise in two ways, as a new system on its own built in analogy to classical algebra as pointed out in the following passage:

> There is not only a close analogy between the operations of the mind in general reasoning and its operations in the particular science of Algebra, but there is to a considerable extent an exact agreement in the laws by which the two classes of operations are conducted. Of course the laws must in both cases be determined independently; any formal agreement between them can only be established *a posteriori* by actual comparison. (Boole, 1854, p. 6)

The view of a special algebra of logic is supported by a characterization of this algebra as a numerical system limited to 1 and 0:

> Let us conceive, then, of an Algebra in which the symbols x, y, z, &c. admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, [sic!] of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. (Boole, 1854, 37f)

A seemingly divergent characterization is given in chapter V:

> *We may in fact lay aside the logical interpretation of the symbols in the given equation; convert them into quantitative symbols, susceptible only of the values 0 and 1; perform upon them as such all the requisite processes of solution; and finally restore to them their logical interpretation.* [emphasis in original] (Boole, 1854, p. 70)

This quote seems to support a reading that logical problems are translated into numerical ones, which then are solved by numerical algebraic transformations and retranslated into logical statements. But contrary to this interpretation, Boole is fully aware that his algebra of logic is determined by similar but partly diverging laws. Especially the quasi-empirical approach to the new algebra in the above quote from (Boole, 1854, p. 4) shows that Boole regards his algebra of logic as an independent structure on its own. This is consistent with his view that "Symbolic Algebra" is not tied to one interpretation, as he points out:

> Every system of interpretation which does not affect the truth of the relations supposed, is equally admissible, and it is thus that the same process may, under one scheme of interpretation, represent the solution of a question on the properties of numbers, under another, that of a geometrical problem, and under a third, that of a problem of dynamics or optics. (Boole, 1847, p. 3)

And in (Boole, 1854, p. 11) he stresses the similarity, but also the deviation "in a single point" between the "laws of Logic" and "the laws of Number".

The translation view in the quote from (Boole, 1854, p. 70) may be read as emphasizing the ancestry of the algebra of logic and its tool character which is part of the justification of his approach.

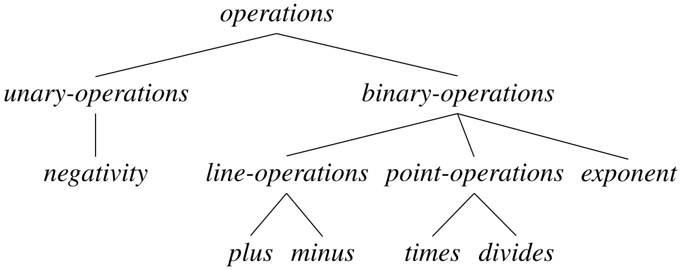
With reference to the concept of frames, his approach can be reconstructed as starting from a frame of numerical algebra which contains a set, namely the real numbers, and at least the binary operations $+$, $-$, $\times$, $\div$, the exponential operation, a unary "$-$" operator for the additive inverse and the binary relation "$=$" as slots. We conceive the semantics of the operations as constrained by the usual of field axioms and definitions of the operations on the basis of "$+$" and "$\times$". The frame is, however, not only the abstract mathematical structure of a field, but enriched with knowledge about mathematical notation of the numbers, of variables, of the operations, etc., and the grammatical behaviour of expressions built with these notational elements, e.g., precedence rules for the operators and notational conventions, for instance the omission of the operator for multiplication. Also natural language terminology should be part of the frame description of the slots and slot parts.

numerical-algebra	
BASICSET	*real-numbers*
UNARY- OPERATIONS	⟨*negativity*⟩
BINARY- OPERATIONS	⟨*plus, minus, times, fraction, exponentiation, …*⟩
RELATIONS	⟨*equality*⟩
CONSTRAINTS	⟨*field-axioms*⟩
DEFINITIONS	list-of(*definition-of-derived-operation*)
TRANSFORMATIONS	⟨*binomial-formulas, multiplying-summands, …*⟩

Here, the slot values and the elements of the sequences in the slots, respectively, are conceived as frames themselves representing semantical as well as notational features of the slot fillers. The frame *real-numbers* reflects notational conventions for writing real numbers, e.g., as decimals, besides [oder: in addition to] the mathematical properties of $\mathbb{R}$. The operation *times* may be thought as the frame

times		
ARGUMENTS	ARGUMENT1	[*number*, NOTATION $\boxed{a1}$]
	ARGUMENT2	[*number*, NOTATION $\boxed{a2}$]
NOTATION	$\boxed{a1} \times \boxed{a2}$	
NL- EXPRESSIONS	⟨*nl-times, nl-multiply, ...*⟩	
PRECEDENCE- OVER	*line-operations*	
TRANSFORMATIONS	⟨*associativity, commutativity, …*⟩	

where the slot *nl-expressions* contains various natural language verbalizations of the times operation, which again form linguistic frames of their own, and the set of operations is organized in a type hierarchy like the following:



The slot *transformations* refers to frequent reformulations of terms and equations.

Boole's algebra of logic is largely identical with the numerical algebra, it only differs in certain restrictions on the set of field axioms and adds a general idempotence axiom in the sense of (2.6).

$$\begin{bmatrix} \textit{algebra-of-logic} & \\ \text{BASICSET} & \textit{sets} \\ \text{UNARY- OPERATIONS} & \left\langle \textit{negativity} \right\rangle \\ \text{BINARY- OPERATIONS} & \left\langle \textit{plus, minus, times, fraction, exponentiation, } \ldots \right\rangle \\ \text{RELATIONS} & \left\langle \textit{equality} \right\rangle \\ \text{CONSTRAINTS} & \left\langle \textit{restricted-field-axioms, idempotence-times} \right\rangle \\ \text{DEFINITIONS} & \text{list-of}(\textit{definition-of-derived-operation}) \\ \text{TRANSFORMATIONS} & \left\langle \textit{binomial-formulas, multiplying-summands, } \ldots \right\rangle \end{bmatrix}$$

Boole (1854) compares laws of his algebra of logic with those of the "ordinary" algebra. Evidently driven by numerical-algebraic laws he develops a number of transformations for equations on the basis of semantic considerations. For each such law it is assessed if it is "in accordance with the laws of ordinary algebra" (Boole, 1854, p. 34) or if "the analogy of the present system with that of algebra, as commonly stated, appears to stop" (Boole, 1854, p. 36) as in the case of the numerically valid, but logically invalid equivalence transformation from $zx = zy$ to $x = y$.

Boole's approach is—as pointed out by Lakoff and Nûñez (2000)—a metaphorical transfer of a frame to a new domain. It is a *linking metaphor* as it allows "to conceptualize one mathematical domain in terms of another mathematical domain“ (Lakoff & Nûñez, 2000, p. 150). He transfers comprehensive parts of the numerical algebraic frame to the logical domain, preserving much of the inferential apparatus. Therefore, it is a *conceptual metaphor*.

More precisely, Lakoff and Nûñez distinguish two stages of metaphors. They call the transfer from arithmetic to the target domain of classes Boole's first-stage metaphor. But, as pointed out, the system had to be revised in order to work for logic. Lakoff and Nûñez describe this process in the following way:

> Since Boole was an algebraist, he turned to algebra as a source domain for his metaphor. In algebra, he was not limited to exactly the arithmetic properties of numbers. In algebra he could find abstract symbols that could obey the laws he observed—those laws that were true of numbers and others that were not. (Lakoff & Nûñez, 2000, p. 127)

In this view the final metaphor has a new algebra as a source domain, the Boolean Algebra with the adapted properties. It is arguable whether it is compelling to describe the relation between the Boolean Algebra and the logic of classes as a metaphor. At least, the link between the Boolean Algebra and the logic of classes is not a unidirectional transfer. The Boolean Algebra is designed to meet the requirements of the target domain. According to the quote from Boole (1847, p. 3) cited above he regards an algebra as an abstract—in the sense of not referring to a specific domain—

structure (represented by a system of symbols with combinational rules and relations among them) which could be applied to different domains and is therefore not limited to numbers—a view shared by others like Galois in the 19th century. This view is certainly a precursor of structuralist views in the philosophy of mathematics, as e.g. Hilbert (1899) or Shapiro (1997), but due to its quasi-empirical approach the grounding of Boole's approach in the target domain is more obvious. So a structuralist view like that by Sneed (1971) seems closer to Boole's description of his approach. In this view theories are considered a pairs of structures and application domains. This allows to see a Boolean Algebra and numerical algebra as two specializations of a more general concept of algebra with two different application domains.

But, however the relation between numerical algebra and Boole's algebra might be conceptualized, it goes far beyond the usual transfer of certain features of the source frame to the target domain in maximizing the contents of the transfer by a quasi-empirical procedure of selecting axioms and admissible transformations. We would argue that frames in the way we conceive them are suitable tool to describe Boole's discovery procedure closely and explicitly. The organization of frames in type hierarchies helps to represent the abstraction of the concept of algebra from its numerical origins to a general structural concept with numerical algebra and the algebra of logics as branches. The notational and linguistic as well as other structural features are included in the frame system and provide the structure for the common part between the two algebras. The two domains of interpretation of these structures can thus be conceptualized as interactions with the ontological domains of numbers as well as truth values and classes, respectively. The two domains of interpretation could be seen as interactions with ontological domains. Deviations between the frames could be localized quite precisely in their rich structuring, and therefore the quasi-empirical research process could be described on their basis. Frames thus proved to be a versatile tool to model the metaphor-driven transfer processes as well as quasi-empirical specialization processes when adapting the algebra to the new domain.

With his innovative framing of logic, Boole initiated the view of logic as a lattice, still used in modern logic, when Boolean algebras (quite different from Boole's in some ways) or Boolean-valued models are studied.

4 Zero

This section sketches how the integration (and development) of the concept of zero can be cast in a frame system. It follows Ifrah's (2000) analysis: The crucial point of this analysis is that it presents the concept of zero as a fortuitous result of the representation of the numbers in a positional system. In a nutshell, the development of a positional system forces signalling of empty positions, and this signal for empty places can then later be reinterpreted as a number, viz. zero. This extension of number space allows to extend arithmetics and algebra.

A more detailed reconstruction is as follows. This is a reconstruction of the (minimal) structure of mathematical knowledge that is necessary to develop a certain number (representation) system. We do not claim that it is historically correct, and we do not claim to model the psychological reality. We will provide a sketch of the (structural) frames for the notation of numbers and especially the digit systems, and we will explain what consequences experimenting with notation has for the mathematical ontology, especially the extension of the numbers that can be referred to and the extension of the natural numbers by zero (and into negativity).

According to Ifrah (2000), zero developed in the context of positional number representation from a symbol that signalled that a certain power of the base was missing. While in non-positional systems like Roman numerals the digits indicate their value completely and hence *MI* is unambiguously 1001, in positional systems the position indicates the power of the base with which to multiply the digit, e.g., modern $1001 = 1 \times 10^3 + 1 \times 10^0$ with no (or zero) 10^2 and 10^1. Just writing 11 or 1 1 for one thousand and one is at best ambiguous (see also Schlimm & Skosnik, 2011, Sects. 2 and 3).

But such ambiguous representations appear on Babylonian tablets (cf. Ifrah, 2000, pp. 150–154). While in many practical situations, disambiguation is possible (cf. Schlimm & Skosnik, 2011, Sects. 2 and 3, for a discussion), the defect was recognised and to disambiguate, a placeholder was introduced, which was first used only between 'real' digits (Schlimm and Skosnik 2011, Sect. 2.1, speak of "intermediate zeros"), but later also at the end of number representations (Schlimm and Skosnik 2011, Sect. 2.1, speak of "final zeros") and, as Ifrah puts it, as a "mathematical" (p. 428) or "arithmetic operator" (p. 342) for multiplying by the base (in the Babylonian case, sixty). But it was not a number of its own, i.e., could not be argument or result of a calculation, as in $6 + 0 = 6$ or $6 - 4 - 2 = 0$.

An anonymous reviewer points out that Ifrah's story may be simplified and outdated in the sense that it can be read as suggesting that number representations were always used to effectuate calculations, while they actually may only have served as devices for recording numbers, whereas calculations were done with abaci and similar devices where positional ambiguity did not arise (cf. for an overview on the Babylonian situation Schlimm & Skosnik, 2011). We feel that this does not change the overall picture, as Schlimm and Skosnik (2011) discuss the deficiencies of the positional systems in a very similar manner to Ifrah, even if they emphasise the ability of experienced readers to compensate for the ambiguities in routine situations. Even though, looking 'backwards' from our use of numerals, we seem to see 'progress' towards zero and modern Western notation in history, our concept of the number representations discussed here is teleological (cf. also the discussion by Chrisomalis, 2010, pp. 5–7) only in the sense that we try to trace one plausible route to where we are today, leaving aside some of the complexity that can be observed and classified (for a more recent version, see Chrisomalis, 2010, whose distinction of cumulative, ciphered and multiplicative systems we mostly leave aside for the sake of brevity).

In any case, precursors of a zero concept as discussed above depended on a number representation that was positional. One consequence of using such a number representation was that numbers could now be arbitrarily large (or small, as fractions) (cf. Ifrah, 2000, p. 355).

According to Ifrah, an important step was understanding that zero can also be seen as a quantity. This is an extension of the number system, but also of the concept of number in the sense that also the non-countable can be represented by numbers.[4]

The last step of making zero a number, following Ifrah (2000, chapters 23 and 24), was taken by the Indian number system and was helped by the fact that in this system, the digits were in themselves no longer compositional but what Chrisomalis (2010, p. 10 f.) calls *ciphered* (i.e., 1 to 9 were not derived from each other, unlike, e.g., in Mayan or Roman digits) and thus, we could say, fully symbols in Peirce's sense. In consequence, it became possible to calculate with zero as we do today (only division giving some problems in the beginning), and to extend the number system to negative numbers (Ifrah, 2000, p. 439; for division by 0, see p. 479).

In the following, we develop some of the frames that come into play; they all have a (minimal) representation side and an abstract structure. For brevity's sake, we only mention, but do not model, constraints on the shape of single digits and on sequences of digits.

4.1 Ordinal scale

An ordinal scale is a (potentially closed) ordered type, e.g., with fingers or body parts (Ifrah, 2000, chapter 1) in a specific order as values, for instance for a scale of ten:

ordinal-finger-scale := ⟨left thumb, left index, . . . , right pinky⟩.

The initial representation side can be some kind of visual marking of the items of the scale. This can later be transferred to more abstract reference, such as writing. Ifrah (2000, pp. 17 ff.) speaks of the ordinal aspect.

Evidently, it is not useful to speak of a zeroth position in such a system.[5]

4.2 Quantity

Independently of this, we also need the concept of quantity. Ifrah (2000, pp. 17 ff.) speaks of the cardinal aspect. Cardinality is measured without numbers when one indiscriminately uses counter stones rather than, e.g., body parts (or numbers) in a certain order.

To model a cardinality in a type system almost forces us to use a type identical to (natural) numbers. That is, the VALUE field of a number representation frame will

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[4] Regarding the Greeks, who were extremely progressive geometers, e.g., Logan (1979, p. 18) claims that "Certain theoretical ideas such as infinity, infinitesimals, atoms, and the vacuum were also rejected by the mainstream of Greek thought because of considerations of logic." and of treating 'nothing' as something: "Once the interval of 'play' (as between wheel and axle) was removed by logical rigour and connectedness, there was no more possibility of zero." (Logan, 1979, p. 21).

[5] When we sometimes do speak of the zeroth element, or even the minus first element, this is generally a sloppy application of the scale of index numbers to the scale of order: We can probably all agree that in many programming languages, the first element has an index of zero. But would anybody really insist that in others, the zeroth element has an index of one? In a different vain, Rojo-Garibaldi et al. (2021, p. 4) mention the discussion regarding a potential zeroth day of the Mayan Long calendar. See Blume (2011, 66f) for a discussion for a discussion (and rejection) of the Maya's 'seating of the month' day being a zeroth day, and Mayan zeros in general.

always contain the number corresponding to the representation under discussion. The VALUE feature of a digit will contain the value the digit has if occurring on its own. Somewhat paradoxically, but conveniently for modern readers, the values will be given in modern-day Western-style Arabic digits.

$$\begin{bmatrix} \textit{quantity} & \\ \text{VALUE} & \mathbb{N} \end{bmatrix}$$

At this stage, a zero quantity is an oxymoron, as it is is precisely the case when quantity cannot be measured.

In Ifrah's (2000) examples, cardinality and ordinality may be fused in a way that one knows for instance how to count a certain amount of reparations on one's body, without being able to calculate or give a number. This can be modelled as follows. Note that by using numbers in the representation, this model suffers from the same paradox as the *quantity* type above. We present below an explicit form which, as in the case of modelling induction (see Sect. 2), allows to plug any scale into the type; the more psychologically realistic representation is the second one, which has the scale 'compiled in'.

The following defines the type; $\boxed{s}$ is a type constraint, while $\boxed{f}$ refers to the concrete value. For a concrete application, see below:

$$\begin{bmatrix} \textit{ordinal-cardinality} & \\ \text{SCALE} & \boxed{s}\ \textbf{\textit{ordinal-scale}} \\ \text{FROM} & \boxed{f}\ \text{start}(\boxed{s}) \\ \text{TO} & \boxed{t} \in \boxed{s} \\ \text{RANGE} & \left\langle \boxed{f}, \ldots, \boxed{t} \right\rangle \end{bmatrix}$$

In other words: We assume that any scale comes with a conventional starting point, and we can then take the start of the list of ordinal values up to the endpoint. We can define the restriction to the finger counting scale from above as follows:

$$\begin{bmatrix} \textit{ordinal-finger-cardinality} & \\ \text{SCALE} & \boxed{s}\ \textbf{\textit{ordinal-finger-scale}} \\ \text{FROM} & \boxed{f}\ \textit{left-thumb} \\ \text{TO} & \boxed{t} \in \textit{ordinal-finger-scale} \\ \text{RANGE} & \boxed{r}\left\langle \boxed{f}, \ldots, \boxed{t} \right\rangle \\ \text{CARDINALITY} & \text{length}(r) \end{bmatrix}$$

When we this integrate this specific *ordinal-finger-scale* frame into the general *ordinal-cardinality* frame, the only variable feature is the endpoint. By this interaction between the frames, in a feature description of the cardinality of seven, it suffices to specify the endpoint:

$$\begin{bmatrix} \textit{ordinal-finger-cardinality} & \\ \text{TO} & \textit{left-index} \end{bmatrix}$$

4.3 Numbers with bases

The next step we take is to use a base for counting (Ifrah, 2000, cf. chapter 2). Essentially, quantity is divided in groups of items of equal size, which can again be grouped.

Besides the main base, there can be an auxiliary base (or sub-base, see Chrisomalis, 2010), for instance Sumerians had a number system based on 60, but first grouped tens.[6] Similarly, Romans and Mayas used groups of five in their number systems[7] based on 10 and 20, respectively, in the sense that Roman *V*, *L* and *D* represent five "units" of the first, second and third order of magnitude of base 10 (i.e., units, tens and hundreds).

Ifrah presents the use of different tokens for such cardinality portions as paving the way for developing a writing system for numbers. As tokens (and digits) are different, the non-presence of a certain portion is not indicated, remember Roman *MI*. The lack of cardinality (or, as we might say: cardinality zero) is expressed verbally, and hence not part of the numbers.

Whether written numbers or tokens, we can model the system as follows, focussing on additive systems with one base:[8]

$$\begin{bmatrix} \textit{number-with-base} & \\ \text{BASE} & \boxed{b}\ b \in \mathbb{N} \\ \text{DIGITS} & \boxed{d}\ \ \text{ne-list-of}(\textit{digit}) \\ \text{VALUE} & \sum_{1<i<\text{length}(\boxed{d})} \boxed{d}_i \end{bmatrix}$$

All of this presupposes a digit type. We assume that it has a definition of the digit symbol and its value.

$$\begin{bmatrix} \textit{roman-digit} & \\ \text{SYMBOL} & \textit{“}\boldsymbol{I}\textit{”} \\ \text{VALUE} & \textit{1} \end{bmatrix}
\begin{bmatrix} \textit{roman-digit} & \\ \text{SYMBOL} & \textit{“}\boldsymbol{D}\textit{”} \\ \text{VALUE} & \textit{500} \end{bmatrix}$$

A symbol can be any of the Roman digit symbols (we skip the formal definition); the value of a digit is a cardinality.

$$\begin{bmatrix} \textit{roman-digit} & \\ \text{SYMBOL} & \textit{roman-digit-symbol} \\ \text{VALUE} & \textit{cardinality} \end{bmatrix}$$

6 We leave aside another ambiguity in such number representations with auxiliary bases and compositional digits if they are written in free form, namely where one digit starts and another ends. This ambiguity can be solved by introducing "separation zeros" (see Schlimm & Skosnik, 2011, Sect. 2.2).

7 Note that the Sumerian and Mayan number systems were positional (see Sect. 4.4).

8 Subtractive rules as in Roman *IX* or *XL* can be easily added; similarly so for auxiliary bases. We also disregard constraints on digit sequences (no more than four *I*s, only one *V* etc.).

Note that the summation cannot deliver 0, as 0 cannot be represented by a digit (and the list of digits must be non-empty).

4.4 Positional numeral systems

In a positional system (Ifrah, 2000, from p. 168 onwards), the symbols are the same for the counting of higher and lower powers, but the position in a series of symbols determines the value. Empty columns in tables or a space in running text may indicate 'missing' powers. The value represented by a digit now depends on the position in the number representation and the base.[9]

For instance, denoting bases by subscripts, decimal (i.e., base 10, as usual today) $1234_{10} = 1 \times 10^3 + 2 \times 10^2 + 3 \times 10^1 + 4 \times 10^0$. Changing the base to 16, as often used in computer science to talk about bytes, hexadecimal $1234_{16} = 1 \times 16_{10}^3 + 2 \times 16_{10}^2 + 3 \times 16_{10}^1 + 4 \times 16_{10}^0$, which corresponds to decimal 4660_{10}. In the vigesimal system used, e.g. by the Mayas, we find that analogously $1234_{20} = 8864_{10}$, while in the sexagesimal system used, e.g., by the Sumerians, Babylonians and often by Arab astronomers, $1234_{60} = 223384_{10}$. Modelling a positional number system in our frame system is straightforward. A single digit is an instance of a digit type; again, the value is a cardinality; an example is the digit *1*.[10]

$$\begin{bmatrix} \textit{digit} & \\ \text{SYMBOL} & \textit{digit-symbol} \\ \text{VALUE} & \textit{cardinality} \end{bmatrix} \qquad \begin{bmatrix} \textit{digit} & \\ \text{SYMBOL} & \text{“}\mathbf{\textit{1}}\text{”} \\ \text{VALUE} & \textit{1} \end{bmatrix}$$

The following frame describes the relation between a sequence of digits (with disambiguated positions) and its meaning.[11] The digit frame interacts with the number frame in the way that the digit values are multiplied with the power of the base number indicated by their position, as explained above.

$$\begin{bmatrix} \textit{written-number} & \\ \text{BASE} & \boxed{b}\ b \in \mathbb{N} \\ \text{DIGITS} & \boxed{d}\ \text{ne-list-of}(\textit{digit}) \\ \text{VALUE} & \sum_{1 \le i < \text{length}(\boxed{d})} \boxed{d}_i \cdot \boxed{b}^{i-1} \end{bmatrix}$$

We can now add an explicit placeholder, the precursor of zero, which does not have a value; for the rest of the digit nothing changes. We do not spell out here the constraint that it can only be used between other digits, and potentially only in a second step of development at the end of digit sequences, and, evidently, never alone.

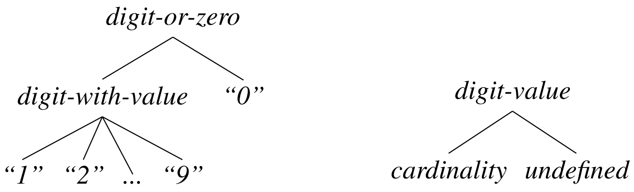
[9] As noted above, there may be ambiguities with respect to the positions in the written representation. These do not change the frame, on the contrary, they emphasise the point that the reader of a number must determine the position of a digit before building the correct number.

[10] We leave aside the aspect that for base b, one needs $b - 1$ (without zero) or b (with zero) digits, as this is merely a practical issue.

[11] Though we use b^0 in our sketch, it is of course not necessary to possess the concept of zero, power or particularly zeroth power.

$$\begin{bmatrix} \textit{written-number-with-placeholder} & \\ \text{BASE} & \boxed{b}\; b \in \mathbb{N} \\ \text{DIGITS} & \boxed{d}\ \text{ne-list-of}(\textit{digit-or-zero}) \\ \text{VALUE} & \sum_{1 \le i < \text{length}(\boxed{d})}^{\textit{where}\ \text{value}(\boxed{d}_i)\ \textit{defined}} \boxed{d}_i \cdot \boxed{b}^{i-1} \end{bmatrix}$$

$$\begin{bmatrix} \textit{zero-placeholder-digit} & \\ \text{SYMBOL} & \textit{``}\mathbf{\textit{0}}\textit{''} \\ \text{VALUE} & \textit{undefined} \end{bmatrix} \quad \begin{bmatrix} \textit{digit} & \\ \text{SYMBOL} & \textit{digit-or-zero} \\ \text{VALUE} & \textit{digit-value} \end{bmatrix}$$



In the introduction to this section we mentioned the use as an arithmetic operator. For conciseness' sake, we do not model this frame more explicitly.

4.5 Positional numeral system with zero

Once we have established a placeholder, it would make the number representation more uniform if we associated a value with this placeholder. However, this step demands that we change the underlying type of digit values $\mathbb{N}$ do $\mathbb{N}_0$. A benefit of this change is that we can simplify the frame for written numbers as follows[12]

$$\begin{bmatrix} \textit{written-number-with-zero} & \\ \text{BASE} & \boxed{b}\; i \in \mathbb{N}_0 \\ \text{DIGITS} & \boxed{d}\ \text{ne-list-of}(\textit{digit-including-zero}) \\ \text{VALUE} & \sum_{1 \le i < \text{length}(\boxed{d})} \boxed{d}_i \cdot \boxed{b}^{i-1} \end{bmatrix}$$

The digit zero now has a defined value, following the minor adaption of the cardinality type:

$$\begin{bmatrix} \textit{quantity-with-zero} & \\ \text{VALUE} & \mathbb{N}_0 \end{bmatrix}$$

We can now again use the same basic concept of a digit as before the introduction of a placeholder, but must plug in the new concept of cardinality:

$$\begin{bmatrix} \textit{digit-including-zero} & \\ \text{SYMBOL} & \textit{digit-symbol} \\ \text{VALUE} & \textit{cardinality-with-zero} \end{bmatrix} \quad \begin{bmatrix} \textit{zero-digit-with-value} & \\ \text{SYMBOL} & \textit{``}\mathbf{\textit{0}}\textit{''} \\ \text{VALUE} & \textit{0} \end{bmatrix}$$

12 Rojo-Garibaldi et al. (2021, Sects. 4–6) discuss alternative positional systems with a digit value from 1 to the base number instead of 0 to one below the base number, and present it as a gateway to discovering zero the Mayas may have followed.

In this simplified number representation system, a single zero is not only permitted, but has value 0.[13] So in the next step, we can try to plug this new value into our existing arithmetic procedures and see how this plays out. It seems that in India, such exploration led to more consistent arithmetics and the discovery of negative numbers (Ifrah, 2000, p. 439; for division by 0, see p. 479).[14]

But what about ordinal scales? Our ordinal scale will probably still normally start at 1, but we can imagine that it could also start at 0, as it does in many programming languages. This leads to a mismatch between indices and 'naive' numbering,[15] and may foster the investigation of scales. The definition remains basically the same, but it is based on different cardinality types.[16]

ordinal-number-scale := $\langle 1, 2, 3, \ldots \rangle$
ordinal-number-zero-based-scale := $\langle 0, 1, 2, \ldots \rangle$

In sum, the use of notation for numbers led to discovery of new mathematical territory, and the extension of structural frames for notation drove the development of new aspects of ontological frames for mathematical objects.

5 Two perspectives on one object

In this section we will analyze two case studies to show how it is possible to have two original perspectives on a mathematical object (this should not be read with a strong ontological commitment). Many high profile prizes are actually awarded for the connection of seemingly far apart areas of mathematics, or the transfer of methods between those. To give a few examples, The Abel Prize for Hillel Furstenberg and Gregory Margulis was awarded "for pioneering the use of methods from probability and dynamics in group theory, number theory and combinatorics".[17] The Fields Medal of Akshay Venkatesh was awarded "for his synthesis of analytic number theory, homogeneous dynamics, topology, and representation theory, which has resolved long-standing problems in areas such as the equidistribution of arithmetic objects." (Mori, 2018, p. 15) Sometimes a whole program is devoted to a the connection of different areas like the *Langlands Program*, which aims to connect number theory and geometry and where partial success yielded many prizes, among those are the Fields Medal for Lafforgue in 2002 or Ngô in 2010.

But how is this even possible? Why should one object pop up in different domains? Why do we actually strive for such developments? We argue that the frame approach can offer an explanation for all this. Let us start with an example where two different

13 Harking back to the remark on the use of number representation not for calculation, but for storing numbers (cf. the beginning of Sect. 4), the fact that such systems were used in calculations (and therefore later replaced Roman numerals in Europe) may be connected to this unified way of treating the other digits and zero.

14 For hypotheses regarding the Maya, see Rojo-Garibaldi et al. (2021).

15 See also Footnote 5.

16 In a way, this is now the reverse from the finger counting we discussed above.

17 See the press release of the Norwegian Academy of Science and Letters, https://abelprize.no/sites/default/files/2021-04/2020_citation_english_Abel_Margulis_Furstenberg.pdf.

Fig. 2 The adjacency matrix of $K_{2,2}$

$$\mathrm{a} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

perspectives on one question can be helpful to ask new kinds of questions. To do this we will introduce the *topological viewpoint* to infinite graph theory. We then talk about the question why we actually would strive for such different framings of one subject area and finally talk about disagreements of such different perspectives in the context of foundations. The key takeaway is that there are blendings in these cases as well.

We also study a motor of innovation, i.e., the development of notions, that has received less attention in the philosophy of mathematical practice than it deserves, in particular we show how different perspectives lead to tensions that can mathematically meaningfully be resolved by positive theorems (equivalence of two notions), or can inspire research programs to map out their actual differences.

We take it as evidence for the frame approach in general that it can at least capture meaningful positions in many debates of the philosophy of mathematical practice.

5.1 The topological viewpoint and progress by switching perspective

Graphs are *prima facie* discrete objects consisting of a set V of vertices and a set $E \subseteq V^2$ of edges connecting two vertices. We call two vertices connected by an edge *adjacent*. Normally we demand that edges connect two distinct vertices and in undirected graphs that the adjacency relation is symmetric. This basic object could be represented in different ways. We could code the set of edges with an adjacency matrix. To do this we enumerate the vertices of the graph. Each row/column of the matrix represents one vertex, so the first row and first column stands for the first vertex, the second row/column for the second vertex etc. Then we put a 1 as an entry in the matrix if the vertices corresponding to the column and row are adjacent. Otherwise we put a 0. Those matrices would be symmetrical and have zeros on the diagonal. This holds since we talk about undirected graphs, where edges connect two *distinct* vertices. So, for instance the graph containing two subsets of vertices each of size n, where all edges are present that have end vertices in different subsets, but no other edges, would look like Fig. 2 for $n = 2$. Observe how the entry $a_{1,2}$ is a 0 as those two vertices belong to the same subset but $a_{1,3}$ is a 1 as the corresponding vertices belong to different subsets.

This looks as a graph as depicted in Fig. 3:

This reformulation does of course not introduce any new properties to the original concept of a graph, but it makes new questions and concepts very salient. A first-year math student could come to think about the eigenvalues of such an adjacency matrix. This is indeed a fruitful field to study (cf. Biggs, 1993; Cvetković et al., 1995).

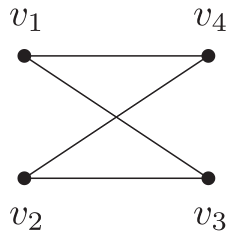


Fig. 3 The $K_{2,2}$

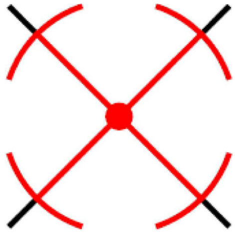


Fig. 4 An open neighbourhood around a vertex

Fig. 5 An open neighbourhood around a point within an edge

Another shift of perspective is given by the 'topological viewpoint' on infinite graphs as introduced by Diestel and Kuhn (2003, 2004) (see also Diestel 2010, 2011). We do not need to discuss the exact details here. It suffices to understand that we interpret graphs as a topological space in a particular way and to note that this space is compact for finite graphs but is not compact for infinite graphs. For those interested in the details, we offer the construction here for the case of locally finite graphs, i.e., graphs where each vertex is only adjacent to finitely many vertices. There might be slight differences in the case of general graphs. A topology can be defined on G, we interpret every vertex as a point and an edge as a copy of the unit interval $[0, 1]$. Slightly more formally: We construct $|G|$ as a 1-dimensional CW complex. The CW stands for *closure-finite weak*. CW complexes are defined inductively, but we only need to concern ourselves with one step. A 0-dimensional CW complex is just a set of discrete points with the discrete topology, i.e., every subset of those points forms an open set. A 1-dimensional CW complex is constructed by taking the disjoint union of a 0-dimensional CW complex and adding copies of the unit interval. Each of those copies is added such that we can identify its endpoints with two vertices. The topology of the CW complex is the topology of the quotient space defined by these gluing maps. So, what are the resulting open sets exactly? We will describe a base of the space, i.e., a set of open sets, such that all open sets are (possibly infinite) unions of sets in the base. For each vertex an ε-part of every edge that is adjacent to it, see Fig. 4. And for each point within an edge, just an ε-neighborhood around it, not containing the end points of that edge, see Fig. 5. It happens that for every finite graph the topological space $|G|$ is actually a compact space.

But what does such a space look like for an infinite graph? We can note that the corresponding space of an infinite graph is not always compact. Take for instance a

Fig. 6 The simplest locally finite, infinite graph: A ray

ray, i.e., a one-way infinite path, see Fig. 6. Every infinite sequence of vertices does not converge, but runs towards infinity.

Here something like a reflex kicks in: A topologist knows about compactification, so the topological perspective makes it very salient to think about compactifications of such a space, and indeed one particular compactification—the Freudenthal compactification, i.e., the maximal pointwise compactification of the space—is often used (in particular in the locally-finite case) to work with a better-behaved topological space. A way to think about this is to add successive points at infinity. So in the case of the ray, we established that it contains a sequence without a limit point within the ray. So we add an infinite far point at the *end of the ray*. This can be done in different ways. The simplest one is the pointwise compactification (or Alexandroff extension) here we really add one point 'infinitely far away'. The mentioned Freudenthal compactification allows to differentiate between topologically different ways to run into infinity. Say you have a double ray (or a copy of the real numbers). You could run through the positive numbers or through the negative numbers away. The Alexandroff compactification would say both go to the same added compactification point, while the Freudenthal compactification would distinguish these directions to infinity. This is of course handwavy, but it suffices to understand our general point. Those compactifications are also both pointwise, in principle the points at infinity could have a topologically more complex structure, but this is was so far not really studied in the context of graphs.

In other words, we embedded graphs into the known context of topological spaces. As discussed above, frames are organized in hierarchies. We depicted a part of the hierarchy of mathematical object in Fig. 7.

This makes it salient to explain a factor, that occurs in practice (which can now be explained within the logic of frames).

The preference for compactification can be modelled as a COMPACTIFICATION slot within the *topological space* frame, as it corresponds to an *expectation* we have. Two different framings of one situation can differ with respect to optionality of slots, like the seller in a buying frame and a selling frame, respectively. In our case the topological graph frame would have a compactification slot, which a discrete conception of graph would not have, or only as a very optional one. But let us focus on a different point here: There is a notion of compactification, one defined by Halin (1964) in purely combinatorial terms. An end is an equivalence class of rays, where two rays are equivalent if they cannot be separated by finitely many vertices. It happens that both notions fully agree for locally finite graphs, but they may differ for graphs with vertices with infinitely many neighbours. But what does this tell us about mathematical practice? We can see that this motivates productive research in mathematics, like when we read

> For graphs with vertices of infinite degree, however, the two notions of an end differ, and it is the purpose of this note to clarify their relationship. This has become relevant in the context of our papers [...], where we found that some

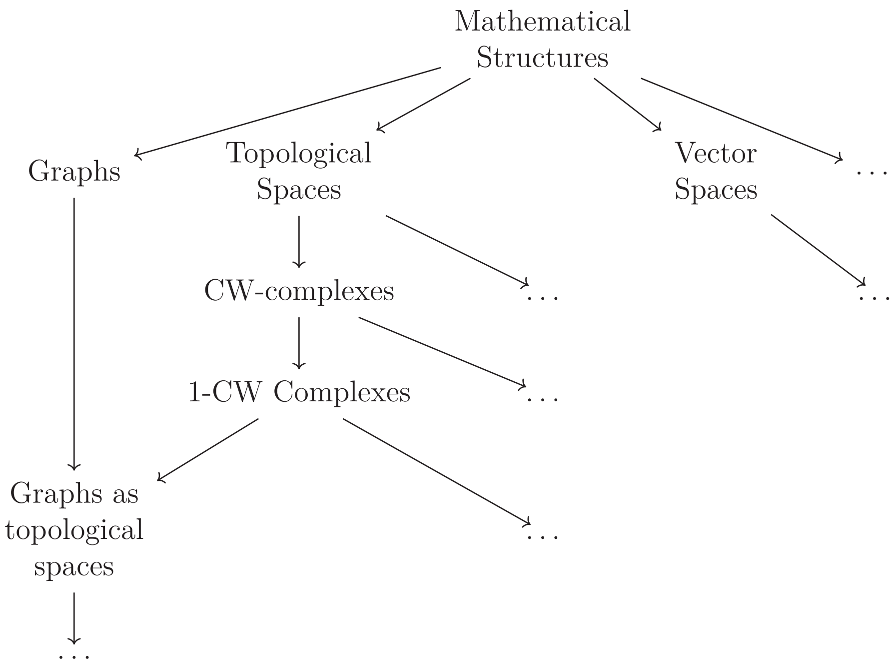


Fig. 7 A part of the frame hierarchy of mathematical objects, containing both topological spaces and graphs

> ends of arbitrary infinite graphs behaved better than others. It now turns out that these are precisely their topological ends. We shall bridge the gap between the topological and the graph-theoretical notion of an end in two ways. We prove that, when a graph G is viewed as a 1-complex, its topological ends correspond naturally to those of its graph-theoretical ends ω that are not dominated, i.e., for which there is no vertex sending an infinite fan to a ray in ω. (Diestel & Kühn, 2003, p. 198)

We should note that this interplay that we observe in practice can again be directly explained by the inheritance hierarchy. The different kind of ends constitute one slot each for a topological infinite graph that was in one case inherited from the combinatorial side of its hereditary and the other from the topological one.

So these different perspectives need to agree on some phenomena, in the sense that they actually lead to similar questions, methods and solutions. But they also motivate conflicting notions as seen in the case of ends here. We should stress that there is no direct contradiction between the two notions of *end*. There is a dissent concerning the normative question "which notion should we study" or concerning the slightly fuzzy question which formally defined notion captures better its informal counterpart. The quote above also gave one idea why this is so productive, as the clarification of the relation of partly agreeing concepts is an interesting field of studies.

Another productive element are new questions we can ask. Stein for instance stresses an aspect of empowerment of the topological approach, when she writes:

> Until now, extremal graph theory usually meant finite extremal graph theory. New notions, as the *end degrees* (...), *circles* and *arcs*, and the topological viewpoint (...), make it possible to create the infinite counterpart of the theory. (Stein, 2011)

So, this new language just allows us to create new areas, like infinite cycles which are defined as circles, i.e., homeomorphic images of S^1. Their partial agreement also help mathematicians to lift theorems from one context to the other, often with slight modifications, like in the case of circles, where there is a plentitude of results transferred from finite graphs to locally finite (but infinite) graphs (cf. Chan, 2015; Hamann et al., 2016; Heuer, 2015, 2016, 2018, 2022; Lehner, 2014; Pitz, 2018; Heuer & Sarikaya, 2023, 2023+).

This shows that new perspectives can be a motor of innovation.[18] The sociological factors are hereby also decisive. A framing of one object in terms of a remote theory opens it to a whole new community of mathematicians and allows for the import of tools and knowledge from different mathematical subfields. This mirrors sources of innovation in other fields. Different communities bring in both hard knowledge and tools. As shown the shift to the topological viewpoint imports knowledge about compactifications. The individuals also bring in new cognitive backgrounds, making other lemmas and theorems salient for usage. The same story can be told for softer tools and heuristics. Again, the frame approaches offer the possibility to model these sociological aspects of individual differences of the frame systems, but for now this remains future work.

In this section we saw the fruitful tension between two perspectives on graphs. In the next section we focus on a tension between two perspectives that seems stronger, because it relates to ontological questions. While we could ask the ontological questions in the domain discussed in this section, i.e. whether a graph is a topological space or ‘just’ a graph, a more prominent example is the approach to natural numbers in mathematical foundations. These can be approached set-theoretically or viewed as objects *sui generis*. As we will see, conflicting paths of such perspectives are (nowadays) often deemed to be non-mathematical, but, for instance, merely metaphysical in nature, as with the example we will discuss in the following subsection.

5.2 Foundations

There are more formal endeavours in the sense, that we formalize more parts of mathematics within automate or interactive theorem provers. These approaches have apparently historical predecessors, like Frege or Russel and Whitehead. We will read the act of formalization as a change in perspective towards a (in our case first order) model or representation of a mathematical theory so to speak. By formalization we mean the shift from an informal (or quasi-formal) mathematical field into a formal theory in the sense of logic. To see an axiomatized theory as a model is no new position, e.g., David and Hersh refer to

[18] Similarly, the cognitive attitude towards technologies was discussed as a main innovator in practice. Our images of technology is a factor that decides which modification we would even consider. Where we see potential of improvement and might help us to imagine new applications. Cf. Virginia (2004); Orlikowski and Gash (1994); note that they talk explicitly about *technological frames*.

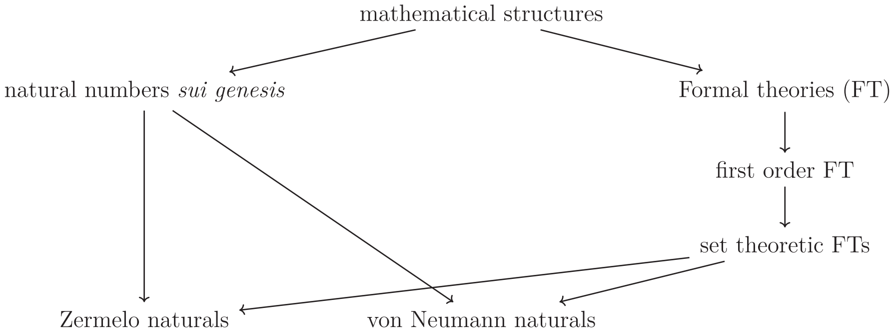


Fig. 8 Hierarchy containing different natural numbers

> the error of identifying mathematics itself (what real mathematicians really do in real life) with its model or representation in metamathematics, or, if you prefer, first-order logic. (Davis et al., 1998, p. 354)

Or in slightly more inner-mathematical vocabulary we hear

> What is the first order Arithmetic? [... It] is not a collection of rules for operating with numbers. No! It's something entirely different. It's a mathematical object, which belongs to a class of mathematical objects, which are called formal theories. (Voevodsky, 2010)

But this change in perspective often causes problems, VOEVODSKY hints at the problem: Asked what the theory of the natural numbers is, how can the logician answer: "a formal theory", and seemingly disagree with the number theorist? Even more, the framing of the natural numbers within set theory and first order logic comes with many decisions, both the Zermelo and the von Neumann ordinals are a model of the PEANO axioms, but in one $2 \in 12$ holds, while in the other it does not.[19] So there are two frames. The first frame would model a formal theory of natural numbers, which could actually be exemplified by different formal theories, like the von Neumann ordinals. It would contain slots like which logic we are using (fillers could be: first order logic or second order logic) and what the successor relation is. The other frame would conceive of the natural numbers as basic objects *sui generis*. In this frame, we would have a zero element, successor function, addition, multiplication and so on. A point where all such perspectives have to disagree is the ontological question. Again we deal with a whole hierarchy of frames, partially depicted in Fig. 8.

While this problem is philosophically challenging, it seems not to be problematic for the actual practice of working mathematicians. But why? Well, since it is a stupid question to ask when we talk about the natural numbers. It only becomes meaningful as a question about details of our logical model. In other words it is a category mistake to put in the membership relation $\in$ between natural numbers. It is a little like a *gestalt* figure. We cannot have the conflicting intuitions at the same point in time, but are alternating between standpoints none of which alone is contradictory.

19 Cf. Benacerraf (1973) for an early problematization of this point.

We find similar considerations within the philosophy of sciences in general. Fleck (1981) argued famously that our cognitive background, social and historical factors play an important role in the generation of knowledge. He coined the terms *thought collective* (*Denkkollektiv*) and similarly Kuhn (2012) (Fleck's influence is noted in the preface) had the idea of paradigms as a kind of perspective on the real world. It is important to note, that while Fleck has very social constructivist tendencies, he was by no means a relativist:

> *Such a stylized solution, and there is always only one, is called truth.* Truth is not "relative" and certainly not "subjective" in the popular sense of the word. It is always, or almost always, completely determined within a thought style. One can never say that the same thought is true for A and false for B. If A and B belong to the same thought collective, the thought will be either true or false for both. But if they belong to different thought collectives, it will just *not* be *the same* thought! It must either be unclear to, or be understood differently by, one of them. (Fleck, 1981, p. 100)

The situation in mathematics is dissimilar in the respect that the mathematician usually shifts perspectives frequently, often within one proof. So are we talking about inconsistencies in mathematics? We argue that a mathematician is not using conflicting ideas but is only using one perspective at a time without revoking conflicting claims from both perspectives. The prime example here are again ontological claims. The case of the whole mathematical community is even more likely to hold such conflicting ideas: For every equivalence result one can expect that some people are more familiar with one side of the equivalence and other people with the other side. To some extend conflicting ideas are actually complemented out of mathematics as mere metamathematics. This includes in particular ontological questions or questions of priority of areas. There might be quite some difference between the notion of number in ancient Greek mathematics and our current notion. But we would still claim, that every mathematical theorem of ancient times still holds. The severity of such conflicts is strongly debated among philosophers and historians of mathematics. While Crowe holds that "there are no revolutions in mathematics" (Crowe, 1975), Pourciau holds that it is (historically speaking) possible that two perspectives are totally incommensurable. Most participants of the Crowe-Dauben debate accept a kind of distinction between object and meta level where the actual objects of mathematics are unchangeable, but our beliefs about them or things like standards of rigour are subject to change (cf. Dunmore, 1992, p. 211). The frame approach cannot settle this dispute but is sufficiently expressible to model different accounts. The question what we would do, if we encountered a strict inner mathematical contradiction between two such perspective can be found in Lakatos' work.

> For instance, we may some day face a situation where some machine churns out a formal proof in a formal set theory of a formula whose intended meaning is that there exists a non-Goldbachian even number. At the same time a number theorist might prove informally that all even numbers are Goldbachian. If his proof can be formalised with our system of set-theory, then our theory will be inconsistent. But if it cannot be thus formalised, the formal set theory will not have been shown

> to be inconsistent, but only to be a false theory of arithmetic (while still being possibly a true theory of some mathematical structure that is not isomorphic with arithmetic). Then we may call the informally proved GOLDBACH theorem a heuristic falsifier, or more specifically, an *arithmetical falsifier* of our formal set theory. (Lakatos, 1976, p. 214)

While Lakatos was in particular anti-formalist in his work, we would conjecture that the priority would be given to the more established view in general. In the case of number theory and logic, this is clearly number theory. So today we would probably like to keep central fields like algebraic geometry intact and would be willing to sacrifice, say, large cardinals.

6 Conclusion

In this paper, we have shown how the exploration of transferring and extending frames can drive innovation in mathematics.

We presented three different cases: In Sect. 3, we discussed an example where viewing logic and logical values as a 'traditional' algebra and exploring the consequences lead to innovation. The development of zero, in the account we reconstructed it in Sect. 4, showed the interplay of the development of writing numbers and the number system. Finally, we discussed in Sect. 5 how the two simultaneously maintained perspectives allow to drive research in different research communities and mutually enrich each other.

Boole's transfer of structures from algebra might be the most obvious case of a linking metaphor in the sense of Lakoff and Nûñez (2000) among the examples in this paper. Moreover, it might be fruitful to consider the cases of Sect. 5 as metaphorically linked, too, with the reservations expressed in the introduction. We, however, do not regard them as prototypical cases of metaphorical transfer. Also the development of number systems could be seen as having its metaphorical grounding in basic counting operations. But we will not claim that every transfer of frames from one domain to another has to be analyzed as metaphorical. Other relations might give more suitable descriptions as the application of a more abstract concept to a more concrete case. On the other hand not every metaphor is to be considered as a frame transfer in the sense advocated here. Frames in our sense have a quite explicit conceptual structure, which the source of a metaphor might lack. The source domain of a metaphor might be an area of preconceptual perceptual or bodily experience without the conceptual structure we regard as defining for frames. Explanations of mathematical conceptual transfer based on metaphorical transfer therefore are partially overlapping and partially complementary to frame-based explanations.

We used frames to address these examples, and have shown how important aspects can be formalized, and discussed the interaction of structural and ontological frames, of formulations and formulaic elements on the one hand, and mathematical objects and categories on the other. In this, we exemplified the role of frame-based structural as well as ontological knowledge, and how different perspectives make different aspects, questions and heuristics salient. The fact that modelling of hierarchies of ontological and structural frames is uniform made it more obvious to what extent they interact;

this is an important asset of frame modelling. It remains to implement a completely formal model.

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