



The relationship between strong belief and assumption

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Abstract

We define two maps, one map from the set of conditional probability systems (CPS's) onto the set of lexicographic probability systems (LPS's), and another map from the set of LPS's with full support onto the set of CPS's. We use these maps to establish a relationship between strong belief (defined on CPS's) and assumption (defined on LPS's). This establishes a relationship at the abstract level between these two widely used notions of belief in an extended probability-theoretic setting.

Keywords Conditional probability systems · Lexicographic probability systems · Strong belief · Assumption

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1 Introduction

In the analysis of games, *rationality and common belief of rationality* is a benchmark model of strategic reasoning. Under that analysis, a player, say Ann, holds a probabilistic belief about the strategies and beliefs of the other player, say Bob.¹ Rationality requires that Ann maximizes her subjective expected utility, given her belief about how Bob plays the game. Belief of rationality requires that Ann assign probability one to Bob's rationality. And so on.

It is well established that, for certain game-theoretic analyses, this benchmark model does not suffice. Two prominent examples involve sequential games and admissibility. We begin by reviewing why those analyses require going beyond ordinary probabilities and the solutions offered by the literature.²

Sequential Games. In simultaneous-move games, Ann makes a choice without observing any action chosen by Bob. By contrast, in sequential games, Ann may gain information—specifically, she may observe actions taken by Bob—before making (some of) her own choices. (That is, she does not commit to a strategy upfront.) If Ann observes Bob choose an action to which she assigned *ex ante* positive probability, she can simply update her belief about Bob before making her own choice. However, if she is surprised by Bob's action—that is, if she observes Bob choose an action to which she assigned *ex ante* zero probability—she will need to form a new belief. These surprise events may arise naturally: If Ann believes *ex ante* that Bob is “strategic,” she may be forced to assign zero probability to his choosing certain actions. At the same time, Bob's own strategic considerations may depend on what he believes Ann will believe conditional upon observing a surprise event.

Battigalli and Siniscalchi (2002) provide a model of strategic reasoning that is well suited to capturing these endogenous surprise events. They endow players with conditional probability systems (Myerson, 1986a, 1986b; Rényi, 1955), instead of ordinary probabilistic beliefs. A *conditional probability system* (CPS) is a sequence of hypotheses—one for each information set—that satisfies the rules of conditional probability when possible. Instead of requiring that Ann simply believes that Bob is rational at the start of the game, Battigalli and Siniscalchi ask Ann to believe Bob is rational, so long as this is consistent with the information she has learned. In this case, Ann is said to *strongly believe* that Bob is rational. Strong belief of rationality captures a form of forward-induction reasoning.

Admissibility. The admissibility criterion says that a rational player should not choose a weakly dominated strategy. Formally, this is equivalent to the requirement that a rational Ann maximizes her subjective expected utility, given a full-support belief about the strategy Bob plays. From this, rationality appears to require that Ann considers all possibilities. At the same time, belief of rationality requires that Ann rules out the possibility that Bob chooses a weakly dominated strategy. So, incorporating an admissibility criterion into the analysis of games creates a tension: On the one hand,

¹ We frame the discussion as a two-player game, but this is immaterial.

² The reader should consult Brandenburger (2007) and Dekel and Siniscalchi (2015) for more complete discussions.

Ann's belief should allow all possibilities; on the other hand, it should rule out certain possibilities (Samuelson, 1992).

Brandenburger et al. (2008) provide a model of strategic reasoning that resolves this tension.³ They endow players with lexicographic probability systems (Blume et al., 1991), instead of ordinary probabilistic beliefs. A *lexicographic probability system* (LPS) is a sequence of hypotheses, i.e., some $(\mu_0, \dots, \mu_{n-1})$: Ann begins with a primary hypothesis μ_0 and only abandons the hypothesis if it does not determine strict preference; in that case, she turns to her secondary hypothesis μ_1 , etc. Informally, she deems her primary hypothesis μ_0 infinitely more likely than her secondary hypothesis μ_1 , which is, in turn, deemed infinitely more likely than μ_2 , etc. Ann considers all possibilities if the LPS has full support. But, she can also rule out the possibility that Bob is irrational, if she views his rationality as infinitely more likely than his irrationality. In this latter case, Ann is said to *assume* that Bob is rational.

CPS's and LPS's both involve a sequence of hypotheses. Likewise, strong belief and assumption both capture the idea that Ann holds on to a hypothesis so long as she can. This raises the question: What is the relationship between strong belief and assumption? This paper aims to address just that question.

We address the question at the level of abstract probability theory. In particular, we define two maps—one from CPS's to LPS's and the other from LPS's to CPS's—that enable us to go from strong belief to assumption, and vice versa. Section 3 shows that (under appropriate hypotheses) an event will be assumed under an LPS σ if and only if it is strongly believed under some pre-image of σ . Section 4 shows that (under appropriate hypotheses) an event will be strongly believed under a CPS p if and only if it is assumed under some pre-image of p .

To the best of our knowledge, this is the first paper to explore the relationship between strong belief and assumption. However, there are other papers that explore the relationship between extended probabilities, importantly, Hammond (1994) and Halpern (2010). Hammond focuses on finite spaces and considers three notions of extended probability: complete CPS's (where there is conditioning on all events); full-support LPS's (where every state gets positive probability under some measure); and logarithmic likelihood ratio functions (due to McLennan (1989a, 1989b)). Halpern considers Popper spaces (a particular type of CPS) and structured LPS's (a generalization of LPS's). Section 5 elaborates on the connection to these papers.

2 Main definitions

This section gives the main definitions used throughout the paper. Let Ω be a Polish space, and let \mathcal{A} be the Borel σ -algebra on Ω . Elements of \mathcal{A} are called **events**. A set of **conditioning events** is a nonempty collection of sets in \mathcal{A} that does not include the empty set. Write \mathcal{B} for a set of conditioning events. Throughout, we keep the measurable space (Ω, \mathcal{A}) fixed; we will vary the set of conditioning events.

³ Several other papers build on this work and modify how the tension is resolved. See, e.g., Lee (2016), Catonini and De Vito (2014), and Dekel et al. (2016).

In the game-theoretic application, the space Ω typically incorporates the set of strategies and the (relevant) beliefs of the other players. In a finite game, this can be an uncountable set, since the set of all beliefs of the other players is uncountable. The set of conditioning events \mathcal{B} is determined by the information sets. In a finite game, the set \mathcal{B} is finite and each conditioning event is clopen.

Definition 2.1 A **conditional probability system (CPS)** on $(\Omega, \mathcal{A}, \mathcal{B})$ is a map $p : \mathcal{A} \times \mathcal{B} \rightarrow [0, 1]$ such that:

- (i) for all $B \in \mathcal{B}$, $p(B | B) = 1$;
- (ii) for all $B \in \mathcal{B}$, $p(\cdot | B)$ is a probability measure on (Ω, \mathcal{A}) ;
- (iii) for all $A \in \mathcal{A}$ and $B, C \in \mathcal{B}$, if $A \subseteq B \subseteq C$ then $p(A | C) = p(A | B)p(B | C)$.

Let $\mathcal{C}_{\mathcal{B}}$ be the set of all CPS's on $(\Omega, \mathcal{A}, \mathcal{B})$.

Conditions (i)–(ii) require that a CPS maps each conditioning event $B \in \mathcal{B}$ to a probability measure on B . Condition (iii) requires that the CPS satisfies the rules of conditional probability, when possible. Because the underlying measure space (Ω, \mathcal{A}) is fixed throughout, we refer at times to a CPS on $(\Omega, \mathcal{A}, \mathcal{B})$ as simply a **CPS on \mathcal{B}** .

Now we can say what it means for a CPS to “reason” that an event is true. We take this to mean strong belief:

Definition 2.2 (Battigalli & Siniscalchi, 2002) Fix a CPS $p \in \mathcal{C}_{\mathcal{B}}$. An event $E \in \mathcal{A}$ is **strongly believed** under p if, for all $B \in \mathcal{B}$, $E \cap B \neq \emptyset$ implies $p(E | B) = 1$.

So, a CPS strongly believes an event E if it assigns probability 1 to the event, whenever it is consistent with the conditioning event.

Next is the definition of an LPS. This will be defined on the measure space (Ω, \mathcal{A}) without making reference to any conditioning events.

Definition 2.3 A **lexicographic probability system (LPS)** on (Ω, \mathcal{A}) is a finite sequence $\sigma = (v_0, \dots, v_{n-1})$ of probability measures on (Ω, \mathcal{A}) such that:

- (i) for $i = 0, \dots, n - 1$, there are Borel sets U_i in Ω with $v_i(U_i) = 1$ and $v_i(U_j) = 0$ for $j \neq i$ (that is: σ is **mutually singular**).

Let \mathcal{L} be the set of all LPS's on (Ω, \mathcal{A}) .

An LPS is a sequences of measures (v_0, \dots, v_{n-1}) for some n . (We refer to n as the **length** of the LPS.) Condition (i) requires that the measures are mutually singular, in the sense that each measure v_i is associated with a hypothesis U_i that is probability 1 under v_i and probability 0 under each v_j for $j \neq i$.⁴ With this in mind, we think informally of the measure v_i as the hypothesis itself. An LPS is often interpreted as a sequence of hypotheses, where the i th hypothesis v_i is infinitely more likely ($i + 1$)th hypothesis v_{i+1} .

Now we can say what it means for an LPS to “reason” that an event is true. We take this to mean assumption.

⁴ The literature often refers to mutually singular LPS's as lexicographic conditional probability systems. To emphasize that there is no conditioning, we simply use the term LPS.

Definition 2.4 (Brandenburger et al., 2008) Fix an LPS $\sigma = (v_0, \dots, v_{n-1})$. An event $E \in \mathcal{A}$ is **assumed** under σ if there exists some j with $n > j \geq 0$ so that:

- (i) $v_i(E) = 1$ for all $i \leq j$;
- (ii) $v_i(E) = 0$ for all $i > j$;
- (iii) for each open set U with $E \cap U \neq \emptyset$, there is some i with $\mu_i(E \cap U) > 0$.

In this case, say that E is **assumed at level j** under σ .

Conditions (i)–(ii) require that there is some hypothesis v_j so that v_i assigns probability 1 to the event if $i \leq j$ and v_i assigns probability 0 to the event otherwise. Condition (iii) requires that all of E is considered possible under some hypothesis. Brandenburger et al. (2008) shows that Condition (iii) can be replaced by the requirement that E is contained in the union of the supports of v_0, \dots, v_j .⁵

Lemma 2.1 Fix an LPS $\sigma = (v_0, \dots, v_{n-1})$. An event E is **assumed** at level j under σ if and only if σ satisfies conditions (i)–(ii) of Definition 2.4 and:

$$(iii') E \subseteq \bigcup_{i=0}^j \text{Supp } v_i.$$

3 From CPS's to LPS's

In this section, we introduce a natural function f that maps a class of CPS's surjectively onto the LPS's. For full-support CPS's, the mapping will translate strong belief to assumption. That is, we will show that, if a CPS p has full support and strongly believes E , then $f(p)$ has full support and assumes E . We will use the mapping to provide a characterization of assumption.

Importantly, the mapping will only be defined when the set of conditioning events \mathcal{B} is a **finite algebra**, i.e., when $\mathcal{B} \cup \{\emptyset\}$ is a finite Boolean subalgebra of \mathcal{A} . Thus, the characterization will link assumption to the requirement that there are enough conditioning events. Specifically, fix a full-support LPS σ . The characterization will say: An event E is assumed under σ if and only if there is a finite algebra \mathcal{B} and a full-support CPS $p \in f^{-1}(\sigma)$ on \mathcal{B} such that E is strongly believed under p .

With this in mind, call a CPS p on \mathcal{B} **finitary** if \mathcal{B} is a finite algebra. Observe that, for a given finite algebra \mathcal{B} , all CPS's on \mathcal{B} are finitary. But, when Ω has at least two elements, there are finitary CPS's that do not belong to $\mathcal{C}_{\mathcal{B}}$. To see this, fix finite algebras \mathcal{B} and \mathcal{D} with $\mathcal{B} \neq \mathcal{D}$. Then, $\mathcal{C}_{\mathcal{B}}$ and $\mathcal{C}_{\mathcal{D}}$ are disjoint, because their members have different domains. Members of $\mathcal{C}_{\mathcal{D}}$ are finitary CPS's that are not contained in $\mathcal{C}_{\mathcal{B}}$.

The requirement that the conditioning events \mathcal{B} form a finite algebra is natural from the perspective of establishing a map from CPS's to LPS's. This said, the conditioning events in a game tree do not, in general, constitute a finite algebra. Accordingly, in that case, there is no immediate natural way to go from CPS's to LPS's.

⁵ Recall, the support of v_i is the smallest closed set that gets probability 1.

3.1 The natural map

Fix a finite algebra of conditioning events \mathcal{B} . In this case, \mathcal{B} is generated by a finite partition. We refer to the cells of the partition as **atoms** of \mathcal{B} .

Fix a CPS p on a finite algebra \mathcal{B} . We will construct an LPS $f_{\mathcal{B}}(p)$. To do so, set $V_{-1} = \emptyset$. Inductively define sets $W_i, V_i \in \mathcal{B}$ and a probability measure μ_i as follows: Let $W_0 = \Omega \setminus V_{-1}$ and $\mu_0 = p(\cdot \mid W_0)$. Take V_0 to be the union over all atoms $B \in \mathcal{B}$ with $\mu_0(B) > 0$. Since \mathcal{B} is an algebra, $W_0, V_0 \in \mathcal{B}$. Assume we have defined sets $W_i, V_i \in \mathcal{B}$ and a probability measure μ_i . If $\Omega = \bigcup_{k \leq i} V_k$, set $f_{\mathcal{B}}(p) = (\mu_0, \dots, \mu_i)$ and put $n = i + 1$. Otherwise, set

$$W_{i+1} = \Omega \setminus \bigcup_{k \leq i} V_k = W_i \setminus V_i,$$

and, note, since \mathcal{B} is an algebra, $W_{i+1} \in \mathcal{B}$. Set $\mu_{i+1} = p(\cdot \mid W_{i+1})$. Let V_{i+1} be the union over atoms $B \in \mathcal{B}$ with $\mu_{i+1}(B) > 0$ and note that $V_{i+1} \in \mathcal{B}$ since \mathcal{B} is an algebra. Also, observe that the construction terminates at some finite n , since \mathcal{B} is finite. In particular, the number $n - 1$ will be less than or equal to the number of atoms of \mathcal{B} .

We will first show that $f_{\mathcal{B}}$ defines a mapping from CPS's on a finite algebra \mathcal{B} to LPS's.

Lemma 3.1 *Fix a CPS $p \in \mathcal{C}_{\mathcal{B}}$, where \mathcal{B} is a finite algebra. Then $f_{\mathcal{B}}(p)$ is an LPS on (Ω, \mathcal{A}) .*

To prove Lemma 3.1, it will be useful to have two lemmas.

Lemma 3.2 *For each $i < n$, $p(\cdot \mid V_i) = p(\cdot \mid W_i) = \mu_i$.*

Proof Fix an event $A \in \mathcal{A}$. It suffices to show that $p(A \mid V_i) = p(A \mid W_i)$. Note, $V_i \subseteq W_i$ with $p(V_i \mid W_i) = 1$. Thus,

$$p(A \mid W_i) = p(A \cap V_i \mid W_i) = p(A \cap V_i \mid V_i)p(V_i \mid W_i) = p(A \mid V_i),$$

where the second equality comes from condition (iii) of a CPS. □

Lemma 3.3 *The set $\{V_0, \dots, V_{n-1}\}$ is a partition of Ω with*

- (i) $\mu_i(V_i) = 1$ for all $i = 0, \dots, n - 1$; and
- (ii) $\mu_i(V_j) = 0$ for all $j \neq i$.

Proof By construction $\Omega = \bigcup_{i < n} V_i$. Moreover, for each i , $V_i \subseteq W_i = \Omega \setminus \bigcup_{k < i} V_k$. Thus, V_i and V_j are disjoint whenever $i \neq j$.

By Lemma 3.2, $\mu_i(V_i) = 1$ for each i . For $j < i$,

$$W_i = \Omega \setminus \bigcup_{k \leq i} V_k \subseteq \Omega \setminus V_j,$$

so that $\mu_i(V_j) = p(V_j | W_i) = 0$. For $j > i$,

$$V_j \subseteq W_j = \Omega \setminus \bigcup_{k < j} V_k \subseteq \Omega \setminus V_i.$$

Since $\mu_i(V_i) = 1$, $\mu_i(\Omega \setminus V_i) = \mu_i(V_j) = 0$. □

Now notice that Lemma 3.1 is a corollary of Lemma 3.3. We next show that this mapping is injective, but need not be surjective.

Lemma 3.4 *Let \mathcal{B} be a finite algebra.*

- (i) *The mapping $f_{\mathcal{B}} : \mathcal{C}_{\mathcal{B}} \rightarrow \mathcal{L}$ is injective.*
- (ii) *If Ω is infinite, then $f_{\mathcal{B}} : \mathcal{C}_{\mathcal{B}} \rightarrow \mathcal{L}$ is not surjective.*

Proof To show $f_{\mathcal{B}} : \mathcal{C}_{\mathcal{B}} \rightarrow \mathcal{L}$ is injective, fix two distinct CPS's p, p' on \mathcal{B} . Then there exists some $B \in \mathcal{B}$ and $A \in \mathcal{A}$ with $A \subseteq B$ so that $p(A | B) \neq p'(A | B)$. Write $f_{\mathcal{B}}(p) = (\mu_0, \dots, \mu_{n-1})$ and $f_{\mathcal{B}}(p') = (\mu'_0, \dots, \mu'_{m-1})$. We will show that there exists some i , so that $\mu_i \neq \mu'_i$.

Observe that $W_0 = W'_0 = \Omega$. Thus, $A \subseteq B \subseteq W_0 = W'_0$. Suppose it is not the case that $p(B | W_0) = p'(B | W'_0) = 0$. Then either $p(A | W_0) \neq p'(A | W'_0)$ or $p(B | W_0) \neq p'(B | W'_0)$. In either case, $\mu_0 \neq \mu'_0$ and we are done. So suppose that $p(B | W_0) = p'(B | W'_0) = 0$. Note, for each atom $D \subseteq B$, $p(D | W_0) = p'(D | W'_0) = 0$. So, $A \subseteq B \subseteq W_1 = W'_1$. Repeating the preceding argument, either $p(B | W_1) = p'(B | W'_1) = 0$ or $p(\cdot | W_1) \neq p'(\cdot | W'_1)$. In the latter case, $\mu_1 \neq \mu'_1$ and we are done. In the former case, we can repeat the argument above. Since the algebra \mathcal{B} is finite, eventually we will hit some i with $p(\cdot | W_i) \neq p'(\cdot | W'_i)$.

Next, note, when Ω is infinite, for each $k \geq 1$, there is an LPS of length strictly greater than k . However, $f_{\mathcal{B}}(\mathcal{C}_{\mathcal{B}})$ only includes LPS's of length less than or equal to the number of atoms in \mathcal{B} . □

One takeaway of Lemma 3.4 is that, when the state space is infinite, looking at the CPS's defined on a given finite algebra does not suffice to capture all of the LPS's. With this in mind, we introduce a mapping $f : \mathcal{C}_{\text{FIN}} \rightarrow \mathcal{L}$ so that, for each finitary CPS p on \mathcal{B} , $f(p) = f_{\mathcal{B}}(p)$.

Proposition 3.1 *The mapping $f : \mathcal{C}_{\text{FIN}} \rightarrow \mathcal{L}$ is surjective.*

Proof Fix $\sigma = (v_0, \dots, v_{n-1}) \in \mathcal{L}$. We begin by constructing a finite algebra \mathcal{B} . To do so, note that, there are Borel sets U_i such that $v_i(U_i) = 1$ and $v_j(U_i) = 0$ for $i \neq j$. Let $X_0 = U_0$; for $0 < i < n - 1$, let $X_i = U_i \setminus \bigcup_{k < i} X_k$; and let $X_{n-1} = \Omega \setminus \bigcup_{k < n-1} X_k$. The set $\{X_0, \dots, X_{n-1}\}$ is a partition of Ω . Moreover, for each i , X_i is Borel, $v_i(X_i) = 1$, and, for each $j \neq i$, $v_j(X_i) = 0$. Let \mathcal{B} be the unique (finite) algebra with the atoms X_0, \dots, X_{n-1} .

Define a mapping $p : \mathcal{A} \times \mathcal{B} \rightarrow [0, 1]$ as follows: For each $B \in \mathcal{B}$, there is a least $i < n$ such that $X_i \subseteq B$. Fix that minimal i and let $p(\cdot | B) = v_i(\cdot)$. It suffices to show that $p \in \mathcal{C}_{\text{FIN}}$ and $f(p) = f_{\mathcal{B}}(p) = \sigma$.

First, we show that p is a CPS on \mathcal{B} . Note, conditions (i)–(ii) are immediate. To prove (iii), fix $A \in \mathcal{A}$ and $B, C \in \mathcal{B}$ with $A \subseteq B \subseteq C$. Take the least i and j such that $X_i \subseteq B$

and $X_j \subseteq C$. Since $B \subseteq C, i \geq j$. If $i = j$, then $p(B | C) = v_j(B) = v_i(B) = 1$, so

$$p(A | C) = v_j(A) = v_i(A) \cdot 1 = p(A | B)p(B | C),$$

and (iii) holds. If $i > j$, we have $p(B | C) = v_j(B) = 0$, so

$$p(A | C) = v_j(A) = 0 = p(A | B)p(B | C),$$

and (iii) holds in this case as well. Thus $p \in \mathcal{C}_{\mathcal{B}}$ and, since \mathcal{B} is a finite algebra, $p \in \mathcal{C}_{\text{FIN}}$.

Finally, starting with p , we have by induction that $V_i = X_i$ for each $i < n$. By Lemma 3.2, $\mu_i = p(\cdot | W_i) = p(\cdot | V_i) = p(\cdot | X_i) = v_i$, and thus $f_{\mathcal{B}}(p) = \sigma$. \square

While the mapping $f : \mathcal{C}_{\text{FIN}} \rightarrow \mathcal{L}$ is surjective, it is not injective. The following example illustrates this point.

Example 3.1 To show that $f : \mathcal{C}_{\text{FIN}} \rightarrow \mathcal{L}$ is not injective, let $\Omega = \{a, b\}$, $\mathcal{A} = 2^\Omega$, and $\mu(\{a\}) = \mu(\{b\}) = 1/2$. Consider two sets of conditioning events $\mathcal{B} = \{\Omega, \{a\}, \{b\}\}$, and let $\mathcal{D} = \{\Omega\}$. Let p be the CPS on $(\Omega, \mathcal{A}, \mathcal{B})$ obtained by conditioning μ on each member of \mathcal{B} and let q be the CPS on $(\Omega, \mathcal{A}, \mathcal{D})$ obtained by conditioning μ on each member of \mathcal{D} . Then $f(p) = f(q) = (\mu)$, but $p \neq q$.

3.2 Full support

Say a CPS p has **full support** if, for each atom B of \mathcal{B} , we have $B \subseteq \text{Supp } p(\cdot | B)$. An LPS $\sigma = (v_0, \dots, v_{n-1})$ has **full support** if $\Omega = \bigcup_{i < n} \text{Supp } v_i$. This section establishes a relationship between full-support CPS's and full-support LPS's.

Proposition 3.2 *The map $f : \mathcal{C}_{\text{FIN}} \rightarrow \mathcal{L}$ takes the set of finitary full-support CPS's surjectively onto the set of full-support LPS's. That is, an LPS σ has full support if and only if there exists a finite algebra \mathcal{B} and a full-support CPS p on \mathcal{B} with $\sigma = f(p)$.*

Proposition 3.2 has two parts. First, if p is a finitary full-support CPS, then $f(p)$ is a full-support LPS. Second, if σ is a full-support LPS, there exists a finitary full-support CPS so that $f(p) = \sigma$. Note, the proposition allows that a full-support CPS and a non-full-support CPS may be mapped to the same full-support LPS. This is illustrated by the following example.

Example 3.2 Let Ω be the real interval $[0, 2]$, and let \mathcal{A} be the algebra of Borel subsets of Ω . Pick a point $a \in (1, 2)$ and let $Y = [0, 1] \cup \{a\}$ and $\mathcal{B} = \{[0, 2], Y, [0, 2] \setminus Y\}$. Let p be a CPS on \mathcal{B} , where $p(\cdot | \Omega)$ is the uniform Borel measure conditioned on $[0, 1]$ and $p(\cdot | [0, 2] \setminus Y)$ is the uniform Borel measure conditioned on $[1, 2]$. (Because $p(Y|\Omega) > 0$ and $Y \subseteq \Omega$, $p(\cdot|Y)$ is the uniform Borel measure conditioned on $[0, 1]$.) Then $f(p) = (\mu_0, \mu_1)$ is such that $\mu_0 = p(\cdot | \Omega)$ and $\mu_1 = p(\cdot | [0, 2] \setminus Y)$. The LPS $f(p)$ has full support, but p does not have full support because $a \in Y \setminus \text{Supp } p(\cdot | Y)$.

Note, there exists another finite algebra $B' = \{[0, 2], [0, 1], (1, 2]\}$ and a full-support CPS p' on $(\Omega, \mathcal{A}, B')$ so that $f(p') = (\mu_0, \mu_1)$. Let $p'(\cdot | \Omega)$ be the uniform Borel measure conditioned on $[0, 1]$ and let $p'(\cdot | (1, 2])$ be the uniform Borel measure conditioned on $(1, 2]$. Then, p' has full support and $f(p') = f(p) = (\mu_0, \mu_1)$.

To prove Proposition 3.2, we will make use of the following lemmas.

Lemma 3.5 Fix a CPS $p \in \mathcal{C}_B$ and events $B, C \in \mathcal{B}$ with $B \subseteq C$ and $p(B | C) > 0$. Then $\text{Supp } p(\cdot | B) \subseteq \text{Supp } p(\cdot | C)$.

Proof Let $A = \text{Supp } p(\cdot | C)$. We have $p(A | C) = 1$ and $p(B | C) > 0$, so $p(A \cap B | C) = p(B | C)$. Since p is a CPS,

$$p(A \cap B | C) = p(A \cap B | B)p(B | C).$$

Therefore $p(A | B) \geq p(A \cap B | B) = 1$. Since A is also closed, it must contain $\text{Supp } p(\cdot | B)$. □

Lemma 3.6 Fix a finitary full-support CPS $p \in \mathcal{C}_B$ and the associated LPS $f(p) = (\mu_0, \dots, \mu_{n-1})$. Then $V_i \subseteq \text{Supp } \mu_i$ for each $i < n$.

Proof Each of the sets V_i can be written as the union of the atoms $B \in \mathcal{B}$ where $p(B | W_i) > 0$, or, equivalently, using Lemma 3.2, where $p(B | V_i) > 0$. So Lemma 3.5 implies $\text{Supp } p(\cdot | B) \subseteq \text{Supp } p(\cdot | V_i)$. Since p has full support, $B \subseteq \text{Supp } p(\cdot | B) \subseteq \text{Supp } p(\cdot | V_i)$. Finally, by Lemma 3.2, $\text{Supp } p(\cdot | V_i) = \text{Supp } \mu_i$. □

Proof of Proposition 3.2 Fix a finitary full-support CPS p and let the associated LPS be $f(p) = (\mu_0, \dots, \mu_{n-1})$. By Lemmas 3.3 and 3.6,

$$\Omega = \bigcup_{i \leq n-1} V_i \subseteq \bigcup_{i \leq n-1} \text{Supp } \mu_i,$$

as required.

Now let $\sigma = (v_0, \dots, v_{n-1})$ be a full-support LPS. In the proof of Proposition 3.1, the sets X_i may be taken so that $X_i \subseteq \text{Supp } v_i$ for each $i < n$. This can be done by first replacing each of the original sets X_i by its intersection with $\text{Supp } v_i$. Then put each point ω of $\Omega \setminus \bigcup_{i < n} X_i$ into X_j where j is the first integer such that $\omega \in \text{Supp } v_j$. The proof of Proposition 3.1 will now give us a full-support CPS p with $f(p) = \sigma$. □

3.3 From strong belief to assumption

For a finitary full-support CPS, strong belief goes over to assumption, under the mapping f .

Proposition 3.3 Fix a finitary full-support CPS p . If E is strongly believed under p then E is assumed under $f(p)$.

Proof Suppose E is strongly believed under p . Then $\mu_0(E) = p(E \mid \Omega) = 1$. Also note that, if $E \cap W_i = \emptyset$, then $E \cap W_{i+1} = \emptyset$. So, there is $j \geq 0$ with: (i) $\mu_i(E) = 1$ for all $i \leq j$, and (ii) $E \cap W_i = \emptyset$ and $\mu_i(E) = 0$ for all $i > j$. Now note that $E \cap W_{j+1} = \emptyset$ implies $E \subseteq \bigcup_{i \leq j} V_i$. By Lemma 3.6, $E \subseteq \bigcup_{i \leq j} \text{Supp } \mu_i$ as required. \square

The assumption that p has full support is important for Proposition 3.3. This is illustrated in the following example.

Example 3.3 If a finitary CPS p does not have full support, then an event E can be strongly believed under p even though it is not assumed under $f(p)$. To see this, let $\Omega = \{a, b, c\}$ and $\mathcal{B} = \{\Omega, \{a, b\}, \{c\}\}$. Let $p(a \mid \Omega) = p(a \mid \{a, b\}) = 1$ and $p(c \mid \{c\}) = 1$. Then $f(p) = (\mu_0, \mu_1)$ where $\mu_0(a) = 1$ and $\mu_1(c) = 1$. Take $E = \{a, b\}$ and note that p strongly believes E but $f(p)$ does not assume E . (Condition (iii) is violated.)

Even if p has full support, $f(p)$ may assume an event even though p does not strongly believe the event. The following example illustrates this point.

Example 3.4 Take $\Omega = [0, 2]$ and let \mathcal{B} be generated by the partition $\{\{0\}, (0, 1), [1, 2], \{2\}\}$. Also, choose a CPS p so that $p(\cdot \mid \Omega) = p(\cdot \mid [1, 2])$ is uniform on $[1, 2]$, $p(\cdot \mid (0, 1))$ is uniform on $(0, 1)$, and $p(0 \mid \{0, 2\}) = p(2 \mid \{0, 2\}) = \frac{1}{2}$. Note, p has full support. Moreover, $f(p) = (\mu_0, \mu_1)$ is such that μ_0 is uniform on $[1, 2]$ and μ_1 is uniform on $(0, 1)$. Note, $\text{Supp } \mu_0 = [1, 2]$ and $\text{Supp } \mu_1 = [0, 1]$, so that $[1, 2]$ is assumed under $f(p)$. But $[1, 2] \cap \{0, 2\} \neq \emptyset$ and $p([1, 2] \mid \{0, 2\}) = \frac{1}{2}$. So, E is not strongly believed under p .

That said, we do have the following converse:

Proposition 3.4 Let $\sigma = (v_0, \dots, v_{n-1})$ be an LPS with full support. A set E is assumed under σ if and only if there exists a finitary full-support CPS $p \in f^{-1}(\sigma)$ so that E is strongly believed under p .

Proposition 3.4 says that, for each full-support LPS σ that assumes E , we can find a finite algebra \mathcal{B} and a full-support CPS p on \mathcal{B} with $f_{\mathcal{B}}(p) = \sigma$, so that p strongly believes E if and only if σ assumes E . Note, this does not contradict Example 3.4: In that example, a given full-support CPS q on a finite algebra \mathcal{D} generated an LPS $f_{\mathcal{D}}(q) = \sigma$ that assumed E , even though q did not strongly believe E . Proposition 3.4 says that, in that case, we can always find some distinct finite algebra \mathcal{B} and a full-support CPS p on \mathcal{B} , so that $f_{\mathcal{B}}(p) = \sigma$ and p strongly believes E . To prove Proposition 3.4, it will be useful to have the following lemma.

Lemma 3.7 Suppose $\sigma = (v_0, \dots, v_{n-1})$ is an LPS with full support and the set E is assumed by σ at level j . Then there is a partition of Ω into Borel sets U_0, \dots, U_{n-1} such that $E \subseteq U_0 \cup \dots \cup U_j$ and, for each $i < n$, $v_i(U_i) = 1$ and $U_i \subseteq \text{Supp } v_i$.

Proof We argue by induction on j . Suppose E is assumed by σ at level 0. By the proof of Proposition 3.2, there is a partition X_0, \dots, X_{n-1} of Ω such that for each $i < n$,

$v_i(X_i) = 1$ and $X_i \subseteq \text{Supp } v_i$. Take $U_0 = E \cup X_0$, and $U_i = X_i \setminus E$ for $0 < i < n$. Since E is assumed at level 0, certainly $v_0(U_0) = 1$ and $U_0 \subseteq \text{Supp } v_0$. For $i \geq 1$, note that, using the fact that E is assumed, $v_i(X_i \cap E) = 0$, so that $v_i(U_i) = 1$. Moreover, $X_i \setminus E \subseteq X_i \subseteq \text{Supp } v_i$, as required.

Now suppose E is assumed by σ at level $j \geq 1$. Choose a set F which is assumed by σ at level $j - 1$. Let $G = (E \setminus F) \cap \text{Supp } v_j$. Then $v_j(F) = 0$ so that $v_j(G) = 1$. Certainly, $G \subseteq \text{Supp } v_j$. Also note that, for $k \neq j$, $v_k(G) = 0$. For $k < j$, this follows from the fact that $v_k(F) = 0$ so that $v_k(E \setminus F) = 0$. When $k > j$, this follows from the fact that E is assumed at level j .

Using these facts, the set $E \setminus G$ is assumed by σ at level $j - 1$. To see this, note that $j - 1 \geq 0$. For all $k \leq j - 1$, $v_k(E) = 1$ and $v_k(G) = 0$, since $j \neq k$, so that $v_k(E \setminus G) = 1$. Moreover, $v_j(G) = 1$, so that $v_j(E \setminus G) = 0$. For all $k > j$, $v_k(E) = 0$, so that $v_k(E \setminus G) = 0$. Finally, by assumption, $E \setminus G \subseteq \bigcup_{k \leq j} \text{Supp } v_k$. By construction, $(E \setminus G) \cap \text{Supp } v_j = \emptyset$. So, $E \setminus G \subseteq \bigcup_{k \leq j-1} \text{Supp } v_k$.

The inductive hypothesis for $E \setminus G$ gives us a partition of Ω into Borel sets Y_0, \dots, Y_{n-1} such that $E \setminus G \subseteq Y_0 \cup \dots \cup Y_{j-1}$, and $v_i(Y_i) = 1$ and $Y_i \subseteq \text{Supp } v_i$ for each $i < n$. The required sets are $U_i = Y_i$ for $i < j$, $U_j = Y_j \cup G$, and $U_i = Y_i \setminus G$ for $i > j$. □

Proof of Proposition 3.4 The “if” direction follows from Proposition 3.3. We focus on the “only if” direction.

Suppose that E is assumed under σ at level j . Let U_0, \dots, U_{n-1} be the partition given by Lemma 3.7. Let \mathcal{B} be the finite algebra of sets with atoms U_0, \dots, U_{n-1} . The proof of Proposition 3.2 gives us a CPS $p \in \mathcal{C}_{\mathcal{B}}$ with full support such that $f(p) = \sigma$ and, for each i , $v_i = p(\cdot \mid W_i) = p(\cdot \mid U_i)$. (The last equality follows from Lemma 3.2.) The CPS p satisfies the following properties: For each $i \leq j$, $p(E \mid U_i) = p(E \mid W_i) = v_i(E) = 1$. For each $i > j$, $E \cap U_i = \emptyset$.

Fix a conditioning event $B \in \mathcal{B}$ and note that B can be taken to be a union of atoms. Suppose $E \cap B \neq \emptyset$. Then there exists some $U_i \subseteq B$ with $E \cap U_i \neq \emptyset$. It follows that $i \leq j$ and $p(E \mid U_i) = 1$. Choose U_i so that there does not exist some $U_k \subseteq B$ with $E \cap U_k \neq \emptyset$ and $k < i$. Then $B \subseteq W_i = \Omega \setminus \bigcup_{k < i} U_k$. Using condition (iii) of a CPS,

$$p(E \cap B \mid W_i) = p(E \cap B \mid B)p(B \mid W_i).$$

We have that $p(U_i \mid W_i) = v_i(U_i) = 1$ and so $p(B \mid W_i) = 1$. Also,

$$p(E \cap B \mid W_i) = p(E \cap B \mid U_i) = p(E \mid U_i) = 1,$$

where the first inequality follows from Lemma 3.2 and the second follows from the fact that $U_i \subseteq B$. But then,

$$1 = p(E \cap B \mid B) \times 1,$$

or $p(E \mid B) = 1$. □

4 From LPS's to CPS's

Section 3 showed how we can map a given finitary CPS to an LPS, via a mapping f . The inverse image of f gives a correspondence from LPS's to finitary CPS's. Importantly, for a given LPS σ , there may be multiple finitary CPS's that map to σ . However, if two different finitary CPS's map to same LPS, they must be associated with different conditioning events. This reflects the fact that LPS's do not come with a fixed set of conditioning events.

This section is concerned with a converse. For a given LPS σ , we look for conditioning events \mathcal{B} so that there exists a CPS $p \in \mathcal{C}_{\mathcal{B}}$ associated with σ . The key will turn out to be whether or not the conditioning events are **covered** by the given LPS.

Definition 4.1 An LPS $\sigma = (v_0, \dots, v_{n-1})$ **covers** \mathcal{B} if, for each $B \in \mathcal{B}$, there is an $i < n$ with $v_i(B) > 0$.

Write $\mathcal{L}_{\mathcal{B}}$ for the set of all LPS's that cover \mathcal{B} .

To better understand this definition, notice that, if Ω is finite or countable, then a full-support LPS covers any set of conditioning events \mathcal{B} . However, if Ω is uncountable, this need not hold. The following example illustrates this point.

Example 4.1 Let $\Omega = [0, 1]$ and let \mathcal{A} be the Borel σ -algebra. Let $\mathcal{B} = \{\Omega, [0, 1), \{1\}\}$. Consider the LPS $\sigma = (\mu_0)$, where μ_0 is the uniform distribution on Ω . Note, σ has full support even though σ does not cover \mathcal{B} .

Important to Example 4.1 is that the conditioning events are not open. Call \mathcal{B} **open** if each $B \in \mathcal{B}$ is open.

Lemma 4.1 *Let \mathcal{B} be open. Then every full-support LPS covers \mathcal{B} .*

Proof Suppose not, i.e., there exists some full-support LPS $\sigma = (\mu_0, \dots, \mu_{n-1})$ and some open set $B \in \mathcal{B}$ so that, for each i , $\mu_i(B) = 0$. Since σ has full support, there exists i so that $B \cap \text{Supp } \mu_i \neq \emptyset$. Using the fact that B is open, $X = \text{Supp } \mu_i \setminus B$ is a closed set with $X \subsetneq \text{Supp } \mu_i$ and $\mu_i(X) = 1$. This contradicts $\text{Supp } \mu_i$ being the support of μ_i . □

4.1 The natural map

We begin by a defining a function $g_{\mathcal{B}}$ from the LPS's that cover \mathcal{B} to the set of CPS's on \mathcal{B} . To do so, fix an LPS $\sigma = (v_0, \dots, v_{n-1}) \in \mathcal{L}_{\mathcal{B}}$. Define $g_{\mathcal{B}}(\sigma)$ to be the mapping $p : \mathcal{A} \times \mathcal{B} \rightarrow [0, 1]$ that satisfies the following: For each $B \in \mathcal{B}$, $p(\cdot \mid B) = v_k(\cdot \mid B)$, where k is the least integer such that $v_k(B) > 0$. To show that $g_{\mathcal{B}}$ maps $\mathcal{L}_{\mathcal{B}}$ to $\mathcal{C}_{\mathcal{B}}$, it suffices to show that $g_{\mathcal{B}}(\sigma)$ is a CPS.

Lemma 4.2 *For each LPS $\sigma = (v_0, \dots, v_{n-1}) \in \mathcal{L}_{\mathcal{B}}$, $g_{\mathcal{B}}(\sigma)$ is a CPS on $\mathcal{C}_{\mathcal{B}}$.*

Proof Write $g_{\mathcal{B}}(\sigma) = p$. It is immediate that p satisfies conditions (i)–(ii) of a CPS. To show condition (iii), fix $A \in \mathcal{A}$ and $B, C \in \mathcal{B}$ with $A \subseteq B \subseteq C$. Let j be the minimum i with $v_i(B) > 0$ and let k be the minimum i with $v_i(C) > 0$. (This is well

defined since σ covers \mathcal{B} .) Note $k \leq j$ since $B \subseteq C$. First suppose that $v_k(B) > 0$. Then using the fact that $A \subseteq B \subseteq C$,

$$p(A | C) = \frac{v_k(A)}{v_k(C)} = p(A | B)p(B | C).$$

Next suppose that $v_k(B) = 0$. Then $v_k(A) = 0$ so that $p(A | C) = 0 = p(A | B)p(B | C)$. □

The mapping $g_{\mathcal{B}}$ goes from the set of LPS's that cover \mathcal{B} to the set of CPS's on \mathcal{B} . It is defined even if \mathcal{B} is not a finite algebra. However, as we will now see, when \mathcal{B} is a finite algebra, the mapping $g_{\mathcal{B}}$ can be seen as an "extended inverse" of the mapping $f_{\mathcal{B}}$.

To see this, we begin with the observation that $f_{\mathcal{B}}$ maps CPS's on a finite algebra \mathcal{B} to LPS's that cover \mathcal{B} .

Lemma 4.3 *Fix a finite algebra of conditioning events \mathcal{B} . The function $f_{\mathcal{B}}$ maps $\mathcal{C}_{\mathcal{B}}$ into $\mathcal{L}_{\mathcal{B}}$.*

Proof Fix $p \in \mathcal{C}_{\mathcal{B}}$ and let $f_{\mathcal{B}}(p) = (\mu_0, \dots, \mu_{n-1})$. By Lemma 3.4, it suffices to show that $f_{\mathcal{B}}(p) = (\mu_0, \dots, \mu_{n-1})$ covers \mathcal{B} .

Since the sets V_0, \dots, V_{n-1} belong to \mathcal{B} and form a partition of Ω (Lemma 3.3), each atom B of \mathcal{B} is contained in exactly one of the sets V_i . Fix one atom B and assume $B \subseteq V_i$. By the definition of $f_{\mathcal{B}}(p)$, $\mu_j(B) > 0$ if and only if $j = i$. Thus $f_{\mathcal{B}}(p) \in \mathcal{L}_{\mathcal{B}}$. □

While $f_{\mathcal{B}}$ maps CPS's on \mathcal{B} into LPS's that cover \mathcal{B} , it does not map onto all LPS's that cover \mathcal{B} . The following example illustrates this point.

Example 4.2 Let $\Omega = \{a, b, c, d\}$ and \mathcal{B} be the subalgebra with atoms $\{a, b\}$ and $\{c, d\}$. Let $\sigma = (\mu_0, \mu_1)$ be the full-support LPS with $\mu_0(a) = \mu_0(c) = 1/2$ and $\mu_1(b) = \mu_1(d) = 1/2$. Note, σ covers \mathcal{B} . Yet, there is no $p \in \mathcal{C}_{\mathcal{B}}$ such that $f_{\mathcal{B}}(p) = \sigma$. If there were such a p , then $p(\cdot | \Omega) = \mu_0$. But, because μ_0 assigns positive probability to both atoms, $f_{\mathcal{B}}(p) = (\mu_0)$, a strict subsequence of σ .

This suggests a sense in which $g_{\mathcal{B}}$ can be seen as an inverse of $f_{\mathcal{B}}$.

Proposition 4.1 *Fix a finite algebra of conditioning events \mathcal{B} .*

- (i) For each $p \in \mathcal{C}_{\mathcal{B}}$, $g_{\mathcal{B}}(f_{\mathcal{B}}(p)) = p$.
- (ii) For each $\sigma \in \mathcal{L}_{\mathcal{B}}$, $f_{\mathcal{B}}(g_{\mathcal{B}}(\sigma))$ is a subsequence of σ .
- (iii) For each $\sigma \in f_{\mathcal{B}}(\mathcal{C}_{\mathcal{B}})$, $f_{\mathcal{B}}(g_{\mathcal{B}}(\sigma)) = \sigma$.

For a given CPS p on a finite algebra \mathcal{B} , $f_{\mathcal{B}}$ maps p to an LPS σ that covers \mathcal{B} . Part (i) says that $g_{\mathcal{B}}$ maps σ back to p . Conversely, for a given LPS σ , $g_{\mathcal{B}}$ maps σ to a CPS p on \mathcal{B} . Part (ii) says that $f_{\mathcal{B}}$ maps p back to a subsequence of σ . Part (iii) adds that this subsequence is only a strict subsequence of σ , if σ is not in the image of $f_{\mathcal{B}}$.

Proof of Proposition 4.1 Begin with part (i). Fix some CPS p on \mathcal{B} and write $f_{\mathcal{B}}(p) = (\mu_0, \dots, \mu_{n-1})$. Note that, for each atom A of \mathcal{B} , there is an i with $\mu_i(A) > 0$. So, for any event B in \mathcal{B} , there is a minimum i with $\mu_i(B) > 0$. For this i , we have $\mu_i(B) = p(B \mid W_i)$. Then, for any event E ,

$$g_{\mathcal{B}}(f_{\mathcal{B}}(p))(E \cap B \mid W_i) = g_{\mathcal{B}}(f_{\mathcal{B}}(p))(E \cap B \mid B)g_{\mathcal{B}}(f_{\mathcal{B}}(p))(B \mid W_i),$$

or

$$g_{\mathcal{B}}(f_{\mathcal{B}}(p))(E \mid B) = \frac{\mu_i(E \cap B)}{\mu_i(B)}.$$

But note that

$$p(E \mid B) = \frac{\mu_i(E \cap B)}{\mu_i(B)}.$$

Now we turn to part (ii). Let $\sigma = (v_0, \dots, v_{n-1})$. From the definitions of $f_{\mathcal{B}}$ and $g_{\mathcal{B}}$, $f_{\mathcal{B}}(g_{\mathcal{B}}(\sigma))$ is the subsequence of σ consisting of those v_i such that, for some atom B of \mathcal{B} , i is the least integer with $v_i(B) > 0$.

Finally, we turn to part (iii). Fix some $\sigma \in f_{\mathcal{B}}(\mathcal{C}_{\mathcal{B}})$. Then there exists some CPS $p \in \mathcal{C}_{\mathcal{B}}$ so that $f_{\mathcal{B}}(p) = \sigma$. By part (i), $g_{\mathcal{B}}(\sigma) = g_{\mathcal{B}}(f_{\mathcal{B}}(p)) = p$. So, $f_{\mathcal{B}}(g_{\mathcal{B}}(\sigma)) = f_{\mathcal{B}}(p) = \sigma$. \square

4.2 Full support

This section revisits the relationship between full-support CPS's and full-support LPS's, when the set of conditioning events is both a finite algebra and consists of open sets. In that case, each conditioning event is clopen.

Recall, for a given full-support LPS σ , there exists a finite algebra \mathcal{B} and some full-support CPS $p \in \mathcal{C}_{\mathcal{B}}$ so that $f(p) = f_{\mathcal{B}}(p) = \sigma$. (See Proposition 3.2.) However, it is not the case that for each finite algebra \mathcal{B} and each $p \in \mathcal{C}_{\mathcal{B}}$ with $p \in f^{-1}(\sigma)$, p is a full-support CPS on \mathcal{B} . (See Example 3.2.) The next lemma establishes that the conclusion does hold, when sets in the finite algebra \mathcal{B} are open.

Lemma 4.4 *Let \mathcal{B} be an open finite algebra. If $p \in \mathcal{C}_{\mathcal{B}}$ and $f_{\mathcal{B}}(p)$ has full support, then p has full support.*

Proof Fix $f_{\mathcal{B}}(p) = (\mu_0, \dots, \mu_{n-1})$, where $(\mu_0, \dots, \mu_{n-1})$ has full support. Let B be an atom of \mathcal{B} , and let U be a nonempty open subset of B . We must show that $p(U \mid B) > 0$.

By Lemma 3.3, there is a unique $i < n$ such that $B \subseteq V_i$. Since V_i is clopen and $\mu_j(V_i) = 0$ when $j \neq i$, $\text{Supp } \mu_j \cap V_i = \emptyset$ when $j \neq i$. Therefore $V_i \subseteq \text{Supp } \mu_i$. By definition, $\mu_i = p(\cdot \mid V_i)$, so $V_i \subseteq \text{Supp } p(\cdot \mid V_i)$. It follows that $p(U \mid V_i) > 0$. Since $U \subseteq B \subseteq V_i$, we have $p(U \mid V_i) = p(U \mid B)p(B \mid V_i)$, and hence $p(U \mid B) > 0$ as required. \square

We now consider the reverse implication. We ask if for a given CPS on \mathcal{B} (whether or not it is full support), there exists a full-support LPS that maps onto the CPS. The answer is yes, if \mathcal{B} is an open finite algebra.

Lemma 4.5 *Let \mathcal{B} be an open finite algebra. Then $g_{\mathcal{B}}$ maps the set of full-support LPS's that cover \mathcal{B} surjectively onto $\mathcal{C}_{\mathcal{B}}$.*

Proof Take $p \in \mathcal{C}_{\mathcal{B}}$. If $f_{\mathcal{B}}(p) = \sigma$ has full support, then we are done since $g_{\mathcal{B}}(\sigma) = p$. (See Proposition 4.1(i).) So, suppose that $\sigma = (\mu_0, \dots, \mu_{n-1})$ with $S = \Omega \setminus (\bigcup_{i < n} \text{Supp } \mu_i) \neq \emptyset$. Note, S is an open set. Since Ω is Polish, $\Omega \setminus S$ has a countable dense subset U . We may choose a probability measure μ_n such that $\mu_n(U) = 1$ and $\mu_n(\{a\}) > 0$ for each $a \in U$. Then $\tau = (\mu_0, \dots, \mu_n)$ is an LPS with full support, and $g_{\mathcal{B}}(\tau) = g_{\mathcal{B}}(f_{\mathcal{B}}(p)) = p$. \square

4.3 From assumption to strong belief

We first show that, assumption goes over to strong belief under the mapping $g_{\mathcal{B}}$, provided the set of conditioning events is open.

Proposition 4.2 *Fix some open \mathcal{B} . If $\sigma \in \mathcal{L}_{\mathcal{B}}$ and E is assumed under σ , then E is strongly believed under $g_{\mathcal{B}}(\sigma)$.*

Proof Suppose E is assumed under $\sigma = (v_0, \dots, v_{n-1})$ at level j . Let $B \in \mathcal{B}$ with $E \cap B \neq \emptyset$. By condition (i) of assumption, for each $i \leq j$, $v_i(B) = v_i(B \cap E)$. So, using conditions (ii)–(iii) of assumption, there is some $i \leq j$ with $v_i(B) > 0$. Let k be the minimum i with $v_i(B) > 0$. Then $g_{\mathcal{B}}(\sigma)(\cdot | B) = v_k(\cdot | B)$. We also have

$$v_k(E | B) = \frac{v_k(E \cap B)}{v_k(B)} = 1,$$

so that $g_{\mathcal{B}}(\sigma)(E | B) = v_k(E | B) = 1$ as required. \square

As a corollary of Proposition 4.2 we get a partial converse to Proposition 3.3.

Corollary 4.1 *Fix some \mathcal{B} that is an open finite algebra. If $p \in \mathcal{C}_{\mathcal{B}}$ and $f_{\mathcal{B}}(p)$ assumes E , then p strongly believes E .*

Corollary 4.2 *Fix an open finite algebra \mathcal{B} and some $p \in \mathcal{C}_{\mathcal{B}}$ that has full support. Then E is assumed under $f(p)$ if and only if E is strongly believed under p .*

Corollary 4.1 is an immediate implication of Proposition 4.1(i) and Proposition 4.2. Corollary 4.2 is an immediate implication of Proposition 3.3 and Corollary 4.1. To better understand Corollary 4.1, it will be useful to compare it with Example 3.4. There, we began with a finite algebra \mathcal{B} and a CPS p on \mathcal{B} . We saw that $f_{\mathcal{B}}(p)$ assumed an event even though p did not strongly believe the same event. Importantly, the conditioning events there were not open.

Corollary 4.2 might suggest a converse to Proposition 4.2. However, this proposition cannot be improved to say: An event E is assumed under σ if and only if it is strongly believed under $g_{\mathcal{B}}(\sigma)$. The next example illustrates this point.

Example 4.3 Take $\Omega = (0, 2)$ and $\sigma = (\nu_0, \nu_1, \nu_2)$ where ν_0 is uniform on $(0, 1)$, $\nu_1(1) = 1$, and ν_2 is uniform on $(1, 2)$. Let $E = (0, 1) \cup (1, 2)$. Then E is not assumed under σ . But, for any open \mathcal{B} , $g_{\mathcal{B}}(\sigma)$ strongly believes E . To see this, fix an open \mathcal{B} and some $B \in \mathcal{B}$ with $B \cap [(0, 1) \cup (1, 2)] \neq \emptyset$. Write $U_1 = B \cap (0, 1)$ and $U_2 = B \cap (1, 2)$. Suppose $U_1 \neq \emptyset$. Then $\nu_0(B) \geq \nu_0(U_1) > 0$, so

$$g_{\mathcal{B}}(\sigma)(E | B) = \frac{\nu_0([(0, 1) \cup (1, 2)] \cap B)}{\nu_0(B)} = 1,$$

as required. If $U_1 = \emptyset$ then $U_2 \neq \emptyset$ and again $\nu_2(B) \geq \nu_2(U_2) > 0$, so

$$g_{\mathcal{B}}(\sigma)(E | B) = \frac{\nu_2([(0, 1) \cup (1, 2)] \cap B)}{\nu_2(B)} = 1,$$

as required.

The example shows that we may have a full-support LPS so that, for each open \mathcal{B} , $g_{\mathcal{B}}(\sigma)$ strongly believes E even though σ does not assume E . That said, if \mathcal{B} is also a finite algebra, we will be able to find some LPS $\tau \in g_{\mathcal{B}}^{-1}(\{g_{\mathcal{B}}(\sigma)\})$, so that τ does assume E . We next show this result.

Proposition 4.3 *Suppose \mathcal{B} is an open finite algebra. Fix $p \in \mathcal{C}_{\mathcal{B}}$. Then E is strongly believed under p if and only if E is assumed under some full-support LPS σ such that $g_{\mathcal{B}}(\sigma) = p$.*

Proof Fix \mathcal{B} that is an open finite algebra. First suppose that E is assumed under a full-support LPS σ . That LPS covers \mathcal{B} . (See Lemma 4.1.) By Proposition 4.2, E is strongly believed under $g_{\mathcal{B}}(\sigma)$.

Now suppose that E is strongly believed under p . Let

$$f_{\mathcal{B}}(p) = (\mu_0, \dots, \mu_{n-1}).$$

Then, there exists some $j < n$ so that (i) $E \subseteq \bigcup_{i=0}^j V_i$, and (ii) $\mu_i(E \cap V_i) = 1$ for each $i \leq j$. We will construct an LPS τ which is a sequence of the form

$$\sigma = (\mu_0, \nu_0, \dots, \mu_j, \nu_j, \mu_{j+1}, \dots, \mu_{n-1}, \kappa),$$

where some of the ν_i and κ may be missing. The idea is that ν_i will capture the part of $E \cap V_i$ which is outside the support of μ_i , and κ will capture the part of Ω outside the supports of the remainder of τ . For $i \leq j$, if $E \cap V_i \subseteq \text{Supp } \mu_i$, then ν_i is missing. Otherwise, take a countable dense subset U_i of $E \cap V_i \setminus \text{Supp } \mu_i$, and let ν_i be a measure such that $\nu_i(U_i) = 1$ and ν_i gives positive measure to each point of U_i . Now let

$$Y = \bigcup_{i=0}^{n-1} \text{Supp } \mu_i \cup \bigcup_{i=0}^j \text{Supp } \nu_i.$$

If $Y = \Omega$, κ is missing. Otherwise take a countable dense subset Z of $\Omega \setminus Y$, and let κ be a measure such that $\kappa(Z) = 1$ and κ gives positive measure to each point of Z . Since each U_i is outside the support of μ_i , σ is a sequence of mutually singular measures, and hence is an LPS. We have $Z \subseteq \text{Supp } \kappa$, so σ has full support. Also, $E \subseteq \bigcup_{i=0}^j (\text{Supp } \mu_i \cup \text{Supp } \nu_i)$. Therefore E is assumed under σ . Finally, $g_{\mathcal{B}}(\sigma) = p$, because for each i and each atom $B \subseteq V_i$, μ_i is the first measure in the sequence σ which gives positive measure to B . \square

5 Literature

To establish a relationship between strong belief and assumption, the paper began by establishing relationships between CPS's and LPS's. This is not the first paper to draw connections between forms of extended probabilities. We now review two prominent approaches. (Neither paper addresses the relationship between strong belief and assumption.)

1. Hammond (1994) This work focuses on a state space Ω that is finite and a set of conditioning events \mathcal{B} that is **complete**, in the sense of including every nonempty subset. The choice of a complete set of conditioning events follows Myerson (1986a, 1986b). In the finite setting, a complete \mathcal{B} is an example of an open finite algebra. Hammond exhibits an equivalence between three spaces: the space of full-support LPS's, the space of complete CPS's, and the space of logarithmic likelihood ratio functions (McLennan, 1989a, 1989b).⁶ Indeed, he shows that the composition of these three maps is the identity, so that, given any member of one of these three spaces, there is a unique corresponding member of the the other two spaces.

2. Halpern (2010) This work focuses on a measurable state space, without imposing Polishness. Halpern considers a particular type of CPS, which he calls a Popper space, and a generalization of an LPS, that he calls a structured LPS (SLPS). A **Popper space** is a CPS that adds two requirements: (i) if $B \in \mathcal{B}$, $B' \in \mathcal{A}$, and $B \subseteq B'$, then $B' \in \mathcal{B}$; and (ii) if $B \in \mathcal{B}$ and $p(C|B) > 0$ then $B \cap C \in \mathcal{B}$.⁷ A **structured LPS** is a (possibly infinite) sequence of probability measures (ν_0, ν_1, \dots) so that there are sets U_i in \mathcal{A} with $\nu_i(U_i) = 1$ and $\nu_i(U_j) = 0$ for $j > i$. (So a structured LPS need not be an LPS.)

Halpern establishes a bijection between SLPS's and Popper spaces. This bijection is natural in preserving conditional probabilities: Given a conditioning event $B \in \mathcal{B}$, the associated Popper measure $p(\cdot|B)$ is equal to the conditional probability $\nu_{i_0}(\cdot|U)$, where ν_{i_0} is the first measure in the SLPS (ν_0, ν_1, \dots) that gives U strictly positive probability. Unlike us, Halpern treats finitely as well as countably additive probability measures.

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⁶ For a finite set Ω , the logarithmic likelihood ratio function is defined by $\psi(\omega, \omega') = \ln[p(\omega)/p(\omega')]$, where ψ is allowed to take values in the extended reals $(+\infty$ or $-\infty)$.

⁷ Note, despite the terminology, a Popper space is not a space but a function.

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