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ORIGINAL RESEARCH

Understanding mathematical texts: a hermeneutical approach

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Abstract
The work done so far on the understanding of mathematical (proof) texts focuses mostly on logical and heuristical aspects; a proof text is considered to be understood when the reader is able to justify inferential steps occurring in it, to defend it against objections, to give an account of the "main ideas", to transfer the proof idea to other contexts etc. (see, e.g., Avigad in The philosophy of mathematical practice, Oxford University Press, Oxford, 2008). In contrast, there is a rich philosophical tradition dealing with the concept of understanding and interpreting texts, namely philosophical hermeneutics, represented, e.g., by Schleiermacher, Dilthey, Heidegger or Gadamer. In this tradition, "understanding" generally refers to the integration in a comprehensive (historical, existential, life-worldly, ...) context. In this article, we take some first steps towards exploring the question how the ideas from philosophical hermeneutics presented in Gadamer's "Truth and Method" apply to mathematical texts and what (if anything) can be learned from these for the didactics and presentation of mathematics.

Keywords Hermeneutics · Mathematical proof · Philosophy of mathematics · Understanding · Gadamer

1 Introduction

"Understanding math", and in particular "understanding proofs", is a notoriously difficult task, and the complaints about failing at it are manifold—reason enough, it seems,

Some of the examples and thoughts developed in this paper are contained in Sect. 4.7 of our submitted thesis "Diproche – Entwurf, Umsetzung und Erprobung eines automatischen Systems zur Unterstützung des Beweisenlernens bei StudienanfängerInnen".

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to take up the even more daunting task of “understanding understanding math”. This, of course, has many aspects, such as mastering basic algorithms and being able to justify them, grasping the relation between mathematics and physical reality in the process of modeling, or understanding mathematical proofs. The latter, which is the aspect we will focus on, has been considered in a variety of works from didactics [see, e.g. Conradie and Frith (2000) or Mejia-Ramos et al. (2012, 2017)], linguistics (Carl et al., 2021; Fisseni et al., 2019), cognitive science [see, e.g., Lakoff (2001)] and philosophy [see, e.g., Avigad (2008)]; an implicit model of understanding mathematical (proof) texts is also in the background of software systems that strive for an automated verification of (natural language) proof texts, such as Naproche (Cramer, 2013), SAD (Lyaletski et al., 2004) or their recent crossing, Naproche-SAD (De Lon & Koepke, 2020; Koepke, 2019).

The work done so far on the understanding of mathematical proofs focuses mostly on logical and heuristical aspects; a proof text is considered to be understood when the reader is able to justify inferential steps occurring in it, to defend it against objections, to give an account of the “main ideas”, to transfer the proof idea to other contexts etc. (see, e.g., Avigad (2008) or Mejia-Ramos et al. (2017)). For Lakoff (2001), understanding mathematics in general requires an identification and application of the right structural templates, so-called “frames”, that arise from our everyday experience with the sensual and social world; in this sense, Lakoff’s picture of understanding explicitly invokes a much broader context. One general approach at addressing the understanding of proofs is based on the observation that proofs are usually presented in the form of texts, so that “proof understanding” can be treated as a special case of “text understanding”; this approach, which we will call the “linguistic approach” here, is, e.g., taken in Fisseni et al. (2019), or Carl et al. (2021), by regarding proofs as certain kinds of narratives.[1],[2] In Carl et al. (2021), the role of structural and ontological frames in understanding proof texts is emphasized, while explicitly retaining a “text-immanent” approach.[3] This paper can be seen as complementing (Carl et al., 2021), where hermeneutical approaches, such as the concept of a frame and Schmid’s model of narratives, were applied to mathematical texts, and in particular to proof texts. However, cultural and historical aspects, as well as intertextuality, were explicitly excluded from consideration.[4] In this paper, we want to pursue the question what the tradition of philosophical hermeneutics can tell us about understanding proofs.

Philosophical hermeneutics is a rich philosophical tradition dealing with the concept of understanding and interpreting texts, represented, e.g., by figures such as

[1] There are other approaches, of course. It is certainly a relevant aspect of proofs that they can be represented in other ways than written text, such as using pictures [see, e.g., Nelsen (1997); for an impressive example, also see Löwe and Müller (2008, Fig. 3)] or hand-waving gestures in combination with or even instead of words; and that mathematicians of the caliber of Brouwer could regard mathematics as an “essentially languageless activity of the mind”. Logical arguments may be given as mental manipulations (for example, in geometry) or follow the example of causality reasoning, which will usually be communicated in language, but not necessarily depend substantially on such a representation. Fortunately, the hermeneutical approach to understanding followed in this work is not restricted to understanding texts and has the potential to apply to such contexts as well.

[2] Brouwer (2011, pp. 4–5).

[3] Carl et al. (2021, p. 8).

[4] Carl et al. (2021, p. 10).

Schleiermacher, Dilthey, Heidegger or Gadamer. Very roughly, in this tradition, "understanding" generally refers to the integration into a comprehensive (historical, existential, life-worldly, ...) context. While it is obvious how such aspects are relevant when interpreting religious, philosophical or legal texts or works of art such as poems—which are cases that Schleiermacher and Gadamer focus on, they seem to bear little relevance for mathematics at first sight. Exciting as it may be to learn about the dramatic circumstances under which the first notes on Galois theory were written or the curious habits of Paul Erdős, it hardly contributes to our understanding thereof.

In this situation, our paper can be seen as an experiment: We let the seemingly disparate worlds of philosophical hermeneutics on the one hand and mathematics on the other collide and see what happens. The hoped outcome is that this will reveal, or at least emphasize, aspects of understanding mathematical texts that are usually overlooked in the heuristical/logical paradigm. Thus, in this article, we take some first steps towards exploring the question how the ideas from philosophical hermeneutics apply to mathematical texts and what, if anything, can be learned from these for the didactics and presentation of mathematics.

It will turn out that the conception of understanding from the hermeneutical tradition is indeed rather remote from the operationalizations found in works on understanding mathematical texts in the philosophy or didactics of mathematics. One may wonder whether whatever hermeneutics has to tell us about mathematical texts is relevant for the kind of understanding that mathematicians or math educators are (or should be) interested in; it is conceivable that it is relevant for ways of reading that are required for the history, psychology or sociology of mathematics rather than for mathematics itself.[5]

One aspect of understanding that will play an important role below, but is not particularly emphasized in the accounts of mathematical understanding in works like Avigad (2008) is the ability to "explain, why it is as it is", to "experience as meaningful". From this perspective, we may say that the understanding of a text is the better and fuller the more aspects of it can be explained as being the way they are and experienced as meaningful, the more aspects "can be made sense of". While this occurs, e.g., in Avigad's criteria in the aspect of "the ability to 'motivate' the proof, that is, to explain why certain steps are natural, or to be expected" (Avigad, 2008, p. 328), it will move into the focus much more below. This involves in particular appreciating the complications arising in an argument. A proof text that (unjustly) appears to be unnecessarily involved or to contain superfluous steps is one in which certain aspects

[5] Concerning the history of mathematics, a systematical hermeneutical perspective such as Gadamer's might well shed light on discussions like the one of whether the interpretation of the history of (infinitesimal) calculus should be approached via "geneseology" ["Geneseologie" (Berg, 1990, p. 123)], which is the attempt to interpret historical approaches strictly within the conceptual and methodological contexts of their time or via "resultatism" ["Resultatismus" (Berg, 1990, p. 123)], which applies modern methodology such as Robinson's nonstandard analysis in reconstructing historical texts. This conflict between an absolutization of modern understanding and attempts at reproducing the historical context is a central topic of traditional hermeneutics, and particularly emphasized by Gadamer. See Spalt (1990) for several contributions to this discussions; see in particular Laugwith (1990) for a criticism of "resultatism" (p. 37f) and the impossibility of "reviving" history (p. 37); for an attempted synthesis of these approaches, see Berg (1990).

do not appear as meaningful and should therefore not count as fully understood, even if all steps have been verified.[6]

As some of the figures important in philosophical hermeneutics, we already mentioned Dilthey, Schleiermacher, Heidegger and Gadamer. Clearly, this is much more than we can hope to treat in a single paper. We hence focus on Gadamer, who is the most recent one of these four, because he is informed by and refers critically both to Dilthey and Schleiermacher, and applies Heidegger's much more general view on understanding to texts. The basis for our investigation is his main work "Truth and method" (Gadamer, 2006), where we focus on the second part, the "Grundzüge einer Theorie der hermeneutischen Erfahrung". Still, it would be a daunting task to write a comprehensive Gadamerian hermeneutical theory of proof texts, one we certainly do not feel ready to take up. Instead, we isolate some views and concepts from the hermeneutical tradition that are addressed—although not necessarily embraced—by Gadamer and see how they apply to mathematics; more specifically, we consider the following aspects:

1. The role of the hermeneutical circle in understanding proofs
2. The role of tradition and authority in understanding proofs
3. Understanding proofs in the light of Gadamer's discussion of different concepts of understanding; this involves, in particular, the relevance of the author's intentions, historical context and reconstructing proof texts as answers to certain questions.

Links between traditional hermeneutics and understanding mathematics have been made in various publications on the didactics of in particular elementary mathematics. Brown (1991) emphasizes that mathematical understanding arises from a personal and subjective experience, and thus "necessarily retains a residue from the 'process' through which it has been approached" (Brown, 1991, p. 475), which can be the basis of a hermeneutically informed didactics; this is considerably deepened in Brown (2001), where further approaches of hermeneutics and post-structuralism are applied to the didactics of mathematics. In Stordy (2015), hermeneutics, and in particular the work of Gadamer, is used in order to obtain a fresh view on the teaching situation. Rodin (2006) pointed out that (re-)interpretation is an important motive in mathematics since the 19th century. None of these, however, focuses particularly on proofs or proof texts, as we do.

This article is structured as follows: The task and general approach are already explained above. In Sect. 2, we consider briefly how the hermeneutical tradition explicitly treats mathematical texts and comment on Gadamer's somewhat pessimistic attitude towards the fruitfulness of a hermeneutics of mathematics. The Sects. 3–5, which form the core part of the paper, contain our applications of the three topics (I)–(III) just mentioned to mathematical texts. Each of these Sects. 3–5 closes with a discussion of (further) didactical consequences of our findings. Finally, in Sect. 6, we give a summary and some further possible topics to pursue in this direction.

[6] This point is, for example, made by Gödel in his notes on how to prepare lectures, where he writes: "For understanding mathematical proofs, it is very important to know why it cannot be done in a simpler way!" (Gödel (2020), p. 444)

2 Mathematical texts in the hermeneutical tradition

As Gadamer points out, the tradition of philosophical hermeneutics developed mainly to deal with (i) the exegesis of religious texts, in particular the Bible and the interpretation of (ii) legal texts, (iii) literature and poetry and (iv) historical documents. It is thus not surprising that scientific, and in particular mathematical, texts, appear to have received relatively little attention.

Is it even intended to be applied to this kind of texts? Although this question is of little relevance for our purpose of obtaining impulses from this tradition (even more so as one of Gadamer's points that understanding is not about reconstructing the author's intention), it is still interesting to see what its representatives said about mathematics.

The first place where the understanding of mathematical texts is mentioned in "Truth and Method" is on p. 185 in an exposition of Spinoza's hermeneutics. With respect to the claim that "in interpreting Euclid (...), no one pays any heed to the life, studies, and habits (vita, studium et mores) of that author" (Truth and Method, p. 181), Gadamer mentions that Schleiermacher disagreed in this respect:

> "It is symptomatic of the triumph of historical thought that in his hermeneutics Schleiermacher still considers the possibility of interpreting Euclid subjectively, i.e., considering the genesis of his ideas." (Truth and Method, p. 291)[7]

This passage refers to a distinction that Schleiermacher makes between a "grammatical" and a "subjective" or "psychological" reading. While the former concerns linguistical structure, the latter attempts to get a grasp on the author's personality and inner state,[8] including, e.g., intentions, knowledge, beliefs, motivations and emotions. According to Schleiermacher, both approaches play a role in understanding a text, although in varying proportions.[9] Indeed, Schleiermacher explicitly mentions the possibility of reading Euclid's elements psychologically. Namely, in "Hermeneutik und Kritik" (Schleiermacher, 2012), Schleiermacher alludes to a claim that Euclid's ultimate goal was to show that the regular solids can be enclosed by a sphere and that knowing this intention to be the driving force behind the line of thought presented there would add a new aspect to the interpretation:

> "The same holds true even for mathematics. Considers Euclid's elements, which were long regarded as a textbook on geometry. From these, other views arose, and it has been claimed that it was Euclid's aim to demonstrate that the regular solids can be enclosed in a sphere, and that he does this by starting from the elements, but in a way that he always has this point in view. Naturally, this does not change the objective content, but the subjective content will be understood differently by the one and the other." (my translation) '[10] (Schleiermacher, 2012, p. 881; also see p. 906)

[7] Note that "still" ("immerhin") is used here in the sense of "at least".

[8] See George, Sect. 2.1.

[9] See, e.g., Schleiermacher (1978, p. 5): "Correct interpretation requires a relationship of the grammatical and psychological interpretation, since new concepts can arise out of new emotional experiences."

[10] "Dasselbe gilt sogar in der Mathematik. Denkt man sich die Elemente des Euklidis, die man lange angesehn als Lehrbuch der Geometrie, aus [diesen] sind andre Ansichten zum Vorschein gekommen, und

The idea here is this: There is a historical thesis according to which Euclid's intention in writing the elements was ultimately to give a demonstration that the regular solids possess a circumscribed sphere. While it is certainly possible to read the elements simply as a textbook on geometry, independent of the author's intention, thus "objectively" obtaining the same information from it, this assumption about Euclid's intentions will modify our reading: For example, we will try to relate certain passages to this ultimate goal, we can (try to) regard the choice of topics and the structure and organization of the work in the light of this intention etc., thus becoming able to make sense of more aspects of the text—that is, improving our understanding. A further understanding may then be gained by learning about the role that the platonic solids plaid in the Greek worldview, which explains why working on it would be interesting in the first place.[11]

Gadamer comes back to mathematics and natural science in his treatment of the role of authority and tradition on p. 284f of Gadamer (2006), where he first confirms that his considerations, which are supposed to grasp understanding in a very general sense, apply to these fields as well:

> "Of course none of man's finite historical endeavors can completely erase the traces of this finitude. The history of mathematics or of the natural sciences is also a part of the history of the human spirit and reflects its destinies." Gadamer (2006, p. 284)

This, however, is part of an attempt to carve out the peculiarities of the humanities as opposed to mathematics and the natural sciences; following the quoted passage, Gadamer maintains that "(...) it is only of secondary interest to see how advances in the natural sciences or in mathematics belong to the moment in history at which they took place. This interest does not affect the epistemic value of discoveries in those fields" (Gadamer, 2006, p. 284)

The reason Gadamer gives for this is that "scientific research as such derives the law of its development not from these circumstances [of tradition, my insertion] but from the law of the object it is investigating, which conceals its methodical efforts" (Gadamer, 2006, p. 284). That is, while the objects of the natural sciences are simply "there" for investigation and present themselves as independent from our background, motivation, intentions, interests etc., the humanities play a much more active role for their realm of investigation; namely, in the case of humanities, "the theme and object of research are actually constituted by the motivation of the inquiry" (Gadamer, 2006, p. 285)

In the footnotes to later editions, Gadamer relativizes these remarks in the light of the findings of the history and philosophy of science, in particular the work of Thomas

Footnote 10 continued
man hat gesagt, Euklidis Zweck ist zu demonstriren die Einschließung regelmäßiger Körper in der Kugel, und er thut das indem er von den Elementen beginnend fortschreitet, aber so, daß er diesen Punkt immer im Auge gehabt. Das Objective bleibt dabei natürlich dasselbe, aber das Subjective wird von dem Einen verschieden verstanden als von dem Andern." The last sentence, which is crucial in this context, is omitted in the translation (Bowie, 1998, p. 104); we therefore provided our own translation.

[11] See, e.g. Plato (1960).

Kuhn.[12] Indeed, it is doubtful whether superstrings, electrons of even just electromagnetic fields are just "there" to be seen with no need for a prior conceptualizing work, which, in turn, is motivated by research interests etc.[13] However, and more importantly for our purpose, mathematics appears to have slipped out of focus at this point. For even when one accepts strong versions of mathematical platonism, one can hardly claim that mathematical objects are encountered independently of the constitutive and conceptualizing activity of the working mathematician; this is the reason, after all, why it is possible to have a debate on what mathematical terms refer to or even whether they refer to anything at all![14] Even if some concepts, like those of a natural number or a straight line, may arise out of prescientific intuitions, it happens frequently that mathematical concepts, which become the topic of mathematical investigations, arise out of attempts to conceptualize a new domain of interest. For example, to gain a substantial understanding of abstract algebra, it is quite advisable to consider at least to some degree the motivations behind these concepts.[15] Regarding them as arbitrary stipulations will lead one astray just as much as expecting them to describe an already familiar intuitive concept.

We also point out that the role of historical development for understanding mathematics has been emphasized by various eminent mathematicians. As an example, in Toeplitz (1927), p. 94, Otto Toeplitz describes an approach to teaching the concept of series convergence that starts with the "beautiful, fruitful discoveries in this area that start with Nicolaus Mercator and Newton", goes on to outline the "terrible chaos that Euler and Bernoulli got into through the use of divergent series" and "how they finally did not know what to do, whether the occurring contradictions were due to these or the simultaneous use of the still mysterious imaginary magnitudes"; only then he considers his hearers "ripe for the concept of series convergence"[16].[17]

Moreover, the importance of historical, social, cultural and subjective context for the understanding of mathematics has recently been emphasized both in developments in

[12] See Gadamer (2006, p. 374, footnotes 25 and 27).

[13] For further discussions on the relation between hermeneutics and the natural sciences, see Crease (1997).

[14] The position that they do not is, e.g., defended by Field in Field (2016).

[15] As an example, we mention Graßmann's "dialectical" approach in the first version of his "Ausdehnungslehre" Grassmann (1878) that led to the concept of a vector space.

[16] My translations.

[17] This approach for teaching beginner students is considerably radicalized on p. 99f, where Toeplitz considers the "relation between the genetic and the normative perspective on mathematics" as a "great philosophical question":

"Mathematics is under the spell of the (...) belief in its objective nature. Starting from the presentation of mathematical works and books, it suffers from this spell. This is because the reason, why we all (...) are able to grasp only the smallest portion of the published works is that these works usually hide rather than reveal the motives from which they start. It is not common style to say something subjective about mathematics.' (My translation.) (Toeplitz, 1927, p. 99).

We thus find in a renowned mathematician Toeplitz a proponent of the view that the "subjective" component of mathematics should be taken quite seriously and granted appropriate attention in presentations of mathematical works. His motivation here is clearly not to defend a certain hermeneutical agenda, but rather the facilitating and fostering of mathematical understanding not just for students, but for everyone. For the relevance of Toeplitz' approach in contemprary didactics of mathematics, see, e.g., Schiffer (2019).

the didactics of mathematics (see the next footnote) and the philosophy of mathematics, in particular the tradition now known as the philosophy of mathematical pratice.[18]

From this, it can be seen that Gadamer's hesitance with respect to the applicability of hermeneutics to mathematics is ungrounded; the subjective component plays a role in the understanding of mathematics and mathematical texts, as do aspects of history and tradition.[19]

3 The hermeneutical circle in mathematics

A naive picture of reading and understanding mathematical (proof) texts is the following: A proof consists of successive steps. In reading a proof, these steps are read in their order of appearance. Each step demands a certain mental act of us—such as assuming something, noting (and memorizing) a certain definition or abbreviation, or accepting that a certain statement follows from the work available at this point—and "understanding" consists in carrying out these acts. In particular, in the case of deductive steps, we are challenged to verify that the respective statement is indeed implied by what has been assumed or obtained so far. This is the logic-oriented picture of reading and understanding mathematical texts, which implicitly underlies and is implemented by systems such as Naproche or SAD. In the words of T. George, it is a special case of what we can call a "foundationalist" or "vertical" (or perhaps "architectonic") picture of understanding: The proof text is seen as a kind of building that is erected starting from its foundation. In particular, there is no "somewhat" or "more or less" understanding a text in any interesting sense[20]: one either understands a text—if one succeeds at carrying out all of the mental acts indicated in the text— or one does not (when one fails in at least one step); and if one does not, one can point to at least one specific step where understanding fails.

In contrast to this view, many readers of mathematical texts will be familiar with the experience that one obtains a "successively improving" and "clarified" understanding of a proof by reading it repeatedly, and that, at least in some cases, one can achieve "degrees" of clarity and understanding through repeated reading that were inadmissible at first reading, even a very careful and thorough one. Also, it frequently happens that,

[18] From the many works written in this tradition, we mention the collections (Fereiros & Gray, 2006) for historical and Larvor (2016) for cultural aspects of mathematics.

[19] This has been observed by various authors in the context of the didactics of mathematics, in particular elementary mathematics. See, for example, Tony Brown (1991), where the author points out that mathematical understanding depends on its development, so that "personal interpretation underlies mathematical understanding" (p. 475); this is further elaborated in Brown (2001), where mathematical understanding is further considered from the perspectives of phenomenology, hermeneutics and poststructuralism. Also see Mary Stordy (2015), where the author refers, among other, to Schleiermacher, Dilthey, Husserl, Heidegger and in particular to Gadamer in order to obtain a fresh view on the teaching of elementary mathematics. Concerning social and (micro-)cultural aspects, one example would be the research on "sociomathematical norms" initiated by Yackel and Cobbs (1996).

[20] Proposals such as the number of unverified steps, or the unverified/verified-step ratio can hardly be seen as measures of understanding: There are steps that "completely spoil" an argument, while in other cases, a proof text may occur to us as "essentially right", "revealing" and "explanatory" in spite of being full of mistakes in detail. One example would be Euler's use of infinite series in various arguments which is discussed in the next section.

after reading a proof, one is “lost” or ”perplexed”, feeling that one fails to grasp the argument, but with no specific step to point to; rather, the impression is that one has missed the argument “in general”.[21]

How strongly this feature of reading proof texts is at odds with the logic-oriented one described above becomes apparent when one considers again systems for automatic proof-checking like Naproche. A crucial feature that we demand of such systems is that they should be deterministic; it would be regarded as a fatal flaw if the system would, after repeatedly failed attempts at verification, finally regard a proof text as correct after, say, five iterations of checking!

On the other hand, the role of repetition for understanding mathematics occurs in several places; as an example, we quote Gödel, Time Management (Maxims) 2, p. 376: “It is fruitful to repeatedly reconsider seemingly insignificant and trivial theorems until one understands them perfectly”. It thus seems that the logic-oriented picture of proof reading is missing something. What is going on here? One approach to answer this question comes from the hermeneutical tradition, where the phenomenon of increased understanding through repeated reading is known as an aspect of the “hermeneutical circle”. As George puts it in [George], the hermeneutical tradition critizes the “vertical” picture as insufficient and inadequate: “In hermeneutics, by contrast, the emphasis is on the ‘circularity’ of understanding.” [22]

One of the basic aspects of the hermeneutical circle, (see, e.g., Gadamer, 2006, p. 291, where Gadamer refers back to Schleiermacher) consists in the application of a certain view on the dialectic of part and whole to text understanding. In general, and then also with respect to texts, parts only have meaning—and are thus understandable—as part of a whole to which they belong; on the other hand, the “whole” is only accessible through the parts of which it consists.[23] The goal, then, is to bring the interpretations of the ”parts” into agreement with the interpretation of the ”whole”; and understanding has been obtained once this agreement is achieved:

> “Thus the movement of understanding is constantly from the whole to the part and back to the whole. Our task is to expand the unity of the understood meaning centrifugally. The harmony of all the details with the whole is the criterion of correct understanding. The failure to achieve this harmony means that understanding has failed” (Gadamer, 2006, p. 291)

This interplay between whole and part does indeed form an important part of understanding in a very general sense, applicable not only to texts, but also to pieces of music, technical devices, buildings and perhaps even natural entities like organisms; at the same time, it shows how understanding is obtained by a repeated circular process. In

[21] This is one reason why it is only of limited use for students when they are encouraged to “ask questions”—asking a question about an argument already presupposes a “general” or “overall” understanding; therefore, specific questions are usually to be expected rather from the “strongest” students than from those who experience fundamental difficulties.

[22] George, Sect. 1.3.

[23] The point may be illustrated by the famous duck-or-rabbit-picture of Joseph Jastrow, see, e.g., https://en.wikipedia.org/wiki/Rabbit%E2%80%93duck_illusion: Depending on whether one perceives the “whole” as a rabbit or a duck, one will interpret a certain “part” as either ears or as a beak.

particular, we have here a situation similar to the one described above for mathematics: An "overall" or "general" understanding to be achieved by repeated consideration.[24],[25]

Indeed, it is rather natural to view a proof as a functioning whole. Lemmas, chains of deduction, auxiliary concepts and methods form the parts of this whole. All of these somehow interact in order to achieve a certain purpose. It will be discussed below that the purposes of different proofs can be quite different: is it supposed to, e.g., convince us, by any means necessary, of the truth of a so far undecided statement; or is its purpose to show that certain statements can be obtained within a certain methodological or axiomatic framework? Understanding the proof will involve and require that such a purpose is determined.[26] Once the goal to be achieved is clear, the parts have to be understood in their contributions to that goal; at the same time, an "overall" picture of the argument is obtained.

Consider, for example, the proof of Bertrand's postulate in Hardy and Wrights classical book on number theory (Hardy & Wright, 2009, p. 343f). In this case, the purpose is clear enough: That a prime number will occur between n and $2n$ for every natural number n is a surprising statement, and we want to be convinced that it is true. Taking some auxiliary statements from the previous pages that are used in the proof as part of it, we are then confronted with a number of statements which include writing the prime factorization of factorials (p. 342) and certain binomial coefficients in a certain way (p. 342), estimating infinite sums involving Gauß brackets using logarithms (p. 342), proving that $\binom{2n}{n}$ does not contain prime factors p with $\frac{2n}{3} < p < n$ (p. 343), finding upper bounds for certain sums of binomial coefficients (p. 343, bottom) etc. In the end, a certain involved inequality is shown to be contradictory.

Even when one has verified all parts of this, one may be justified in feeling that one has not understood the proof. As it is said in Carl et al. (2021):

> "(...) when a mathematics student has laboriously checked all details in a complex proof but does not see the big picture of how all these proof steps work together (...) we would normally not ascribe to that student understanding of that proof" (Carl et al., 2021, p. 5)

But now we can work on this understanding: After the first reading, we know that the proof is one by contradiction, and we see that the contradiction is going to be a certain inequality. We can now, in a second reading, focus on how certain parts of the text contribute to this inequality. In this way, the text will receive a rough structuring into steps that immediately enter the final inequality and "secondary" steps that contribute

[24] In a different context, namely that of concept formation, the relevance of the hermeneutical circle for mathematics is pointed out in Löwe (2010, p. 193).

[25] We also mention that the difficulty that the circular account of understanding addresses is already present at the level of sentences; see, e.g., Ganesalingam, 2013, p. 88: "Together, these sentences will show that neither ambiguity in symbolic material nor ambiguity in textual material can be resolved before the other: we are faced with the linguistic equivalent of simultaneous equations."

[26] It may happen that the purpose is not clear from the outset; in that case, understanding requires to read off the purpose from the proof, i.e., to determine the kind of purpose that this particular proof could fulfill; this, as well, will be part of the circular movement of hermeneutics.

to such steps. Going through the text a few more times,[27] we will see that the whole argument works by estimating $\binom{2n}{n}$ in two ways, once by obtaining an upper bound via its prime factorization and assuming that there is no prime between n and $2n$, and once by obtaining a lower bound designed to be as simple as possible in order to finally still yield a contradiction. Once this is achieved, we can even motivate the seemingly arbitrary choice of considering $\binom{2n}{n}$ (p. 342): If there were no primes between n and $2n$, the prime decomposition of $\binom{2n}{n} = \frac{(2n)!}{n! \cdot n!}$ would entirely consist of primes below n, but these are heavily cancelled out by the denominator; it is hence natural (but nevertheless brilliant) to try to show that, under the negation of Bertrand's postulate, this would be too small.

We thus find at least one respect in which the hermeneutical circle plays a role in reading and understanding mathematical proof texts: Regarding the proof text as a functional whole, the understanding of which requires grasping the separate parts in their functioning, i.e., their relation to the whole text, shows how the part-whole-relation is relevant for understanding proof texts. The circular movement described in hermeneutics can thus help us to see the "big picture" mentioned in Carl et al. (2021) and thereby to fulfill some of the criteria of Avigad's operationalized account of proof understanding in Avigad (2008).[28]

This gives at least a partial explanation why repeated reading of proof texts can lead to a degree of understanding that a merely thorough, but linear,[29] reading cannot: At a first (linear) reading, we have no chance to determine the role a certain statement, passage or paragraph plays with respect to the whole text: At best, we may have certain expectations how specific parts will be employed in the overall argumentative structure. It is only after finishing the text at least once (or by occasionally "peaking ahead" in reading) that such expectations can either be confirmed or rejected. We may then revise our assumption on the overall *gestalt*[30] of the proof.[31]

We have thus seen how, based on the traditional hermeneutical analysis of understanding, we can explain the phenomenon that repeated reading of proofs furthers understanding in the sense of Avigad's operationalization. Moreover, this analysis

[27] It is clearly irrelevant for our purposes whether this "going through the text" takes the form of an actual re-reading or merely a reflection on memorized contents.

[28] In particular, the following of Avigad's criteria seem relevant here (all from Avigad (2008), p. 238): (i) "the ability to give a high-level outline, or overview of the proof", (ii) "the ability to 'motivate' the proof, that is, to explain why certain steps are natural, or to be expected", (iii) "the ability to indicate where in the proof certain of the theorem's hypotheses are needed (...)"

[29] A reading involving jumping back and forth is of course well in line with the circular picture of understanding proposed in this section. That expert readers of mathematics actually do this is empirically supported by the eye-tracking study of Inglis et Alcock, see (Inglis & Alcock, 2012).

[30] In a didactical contest, on of the subjects of the study of Moore (2016) mentions the "gestalt point of view" towards grading proof texts.

[31] Some authors of mathematical texts will help us in getting the "big picture" (the "frame", in the terminology of Carl et al. (2021)), for example by using the usual indicators for announcing proof strategies "by induction", "assume for a contradiction", "Case 3:" or even by offering comments on the proof plan and the role of separate parts within the proof (which, unfortunately, is a rather rare convenience).

allows us to regard several of the criteria in Avigad (2008) as special cases of the whole-part-relationship, which is ubiquitous in traditional hermeneutics.[32]

4 The role of tradition and authority in reading and understanding proofs

At first sight, and particularly for logic-minded people, it may seem strange to even ask for the role of tradition and authority in reading proofs. After all, an (attempted) proof text consists of (pretenses of) logical deductions, and these are either sound or not. The proof is (to be) accepted if they are and rejected if they are not; there is little space, it seems, for an influence of the way the proof text arrived at its reader (tradition) or the reputation of the person who wrote it (authority).[33] Of course, there may be different lines of research, in which different methodologies may be regarded as acceptable, such as classical vs. intuitionistic mathematics. But when a work deviates from the received view on acceptable methods, it will usually explicate this (e.g. by stating in the title that it is about "intuitionistic analysis"), thus clarifying that it should be read relative to the respective framework.

Indeed, the same could be said about any other scientific text: At least in principle—ignoring practical complications with, e.g., conducting experiments with a particle accelerator in my office—it should be possible to verify the claims contained in the text without needing to trust its author.[34] Indeed, trust in the author puts us at risk of suspending rational judgement, thus ending up with unsupported (and possibly false) beliefs.

This attitude, which Gadamer describes as the attitude of enlightenment towards authority and tradition (Gadamer, 2006, pp. 273–277), is called into question in the section "Prejudices as conditions of understanding" (Gadamer, 2006, pp. 277–305), in which Gadamer treats tradition and authority as a part of his treatment of the role of prejudices in understanding.

According to the ideal of enlightenment, a "methodologically disciplined use of reason can safeguard us from all error" (Gadamer, 2006, p. 279). This means in partic-

[32] This analysis of mathematical understanding as a circular movement also matches well with the frame view of proof understanding proposed in Carl et al. (2021), where frames were shown to be one way of modeling an "overall" understanding of a proof text: A frame can only be matched to the whole of the text, i.e., after the text has been processed. It is only then that the position of specific parts within this frame can be determined.

[33] The factual influence of authority on the development of mathematics, however, can hardly be doubted. For instance, in his treatment of the history of vector analysis, Crowe writes: "The fact that the idea [of a graphical representation of the complex numbers, my insertion] was neglected until Gauss entered the field should not, I think, be taken as surprising. (...) The men before Gauss were all little known (...). But when Gauss wrote, he wrote with the authority of one who had already acquired fame through impressive work in traditional fields (...). It may be noted now and discussed later that the pattern exhibited in this instance will recur in the later history of vectorial analysis." (Crowe, 1993, pp. 11–12) A more recent example would be the reception—or the long lack thereof—of Royen's proof of the Gaussian correlation inequality, see Wolchover (2017).

[34] This is certainly different, e.g., with reports of personal experiences; in such cases, the authorship may indeed serve as an important criterion of validity.

ular that only such judgements are accepted for which one has seen sufficient evidence and checked that it does indeed support the judgement.[35]

The contrary of the ideal of enlightenment just described are prejudices, i.e., judgements made before sufficient evidence is present or its examination is complete. From this perspective, authority is then one cause of prejudices, as “authority (...) is responsible for one’s not using one’s own reason at all.” (Gadamer, 2006, p. 279).

It is this thoroughly negative view of prejudices in general and authority in particular that Gadamer subjects to a critical investigation. Is it really the case that , in understanding, prejudices and reliance on authority can and should be avoided altogether? Might they not have other, more positive or even essential, functions? If that was the case, i.e., the role of authority should be reconsidered: “If, on the other hand, there are justified prejudices productive of knowledge, then we are back to the problem of authority” (Gadamer, 2006, p. 280)

In his treatment of this question, Gadamer follows the example of his teacher Heidegger by pointing the idealizations underlying epistemological positions and emphasizing precisely those aspects that were traditionally considered as irrelevant. While “in principle”, the critical rationality of idealized cognitive agents may be able to perform “independent” and fully, thoroughly justified judgements in every case, this is not possible for actual, finite human beings. Like Heidegger, Gadamer proposes not to regard this finitude as an inessential disruption of the true, ideal picture of understanding, but as a fundamental formal and structural aspect of understanding:

> “The overcoming of all prejudices, this global demand of the Enlightenment, will itself prove to be a prejudice, and removing it opens the way to an appropriate understanding of the finitude which dominates not only our humanity but also our historical consciousness.” (Gadamer, 2006, p. 277)

This development has quite obvious parallels in mathematics: The “fundamental presupposition of the Enlightenment” (Gadamer, 2006, p. 279) corresponds to the received view that a careful, rational, critical reader of a proof is fully equipped to judge its correctness; mistakes are either due to sloppiness (called “overhastiness” in the tradition of hermeneutics, see, e.g., Gadamer, 2006, p. 279) or to “one’s not using one’s own reason at all” (Gadamer, 2006, p. 279).

It is quite obvious that the way mathematical texts are actually read deviates quite strongly from this ideal. Quite often, proof steps, passages or entire proofs are left out; theorems are accepted and used without reading (and checking) their proofs or even without having the prerequisites for understanding the proofs.[36]

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[35] Gadamer in particular refers to Descartes as a precursor of this viewpoint (Gadamer, 2006, p. 279f); indeed, the *Meditationes* can be regarded as a rather radical expression of it.

[36] Although we believe that most mathematicians and mathematics students will recognize themselves in this description, we have little more than personal and some cases of reported experience to support the first claim. For the second, typical examples would be (uses of) the classification theorem for finite simple groups, the proof of which is likely to transcend the time most algebraists can invest in reading in it. Another example would be the use of Zorn’s lemma to prove the existence of bases for arbitrary vector spaces in beginning linear algebra classes. This omission is occasionally hidden by claiming that Zorn’s lemma is an “axiom” or “equivalent to an axiom”. The former is at least strange in the sense (i) that one will have a hard time in looking for a source in which Zorn’s lemma, rather than the axiom of choice, is stated as axiomatic

In order to get a clearer grasp on how human "finitude" and reliance on tradition and authority enter the picture of understanding proofs, let us look a bit closer at the process of verifying proof steps. If I succeed at verifying a step, there may still be the question whether this primarily points to a property of the given text or of myself (if I was smart enough to supplement the proofs myself, the author could just have written "trivial" everywhere). The more interesting case for our purposes, however, occurs when my attempt at verification fails; this can be due to the following:

1. The proof "itself" has a gap, regarded, not as a particular text, but rather as a "type" of texts that gathers together many presentations of the "same" proof. This may, for example, imply that neither the author nor anyone in the tradition of the proof—its history of presentations, reformulations, rearrangements, reconceptualizations for various audiences and purposes[37]—would have known how to fill it so far.[38]
2. In contrast to (i), it can also be the case that the presentation is insufficient, while the "thinking" itself is sufficient.
3. The problem is on my side: perhaps my background knowledge is insufficient; I may be insufficiently equipped methodologically and be ignorant of the method by which the step works; it may also be the case that I lack the ingenuity to properly apply the tools, even though they are known to me; and finally, I may have all it takes to understand the step, but I did not apply myself sufficiently and should try longer, or harder, or more focused, or more often.

Now, finite human being that I am, I can only allocate a bounded amount of resources to the task of verifying a proof step. The decision whether to accept a proof step, and how to diagnose a failure of verification has to be made within these boundaries.

Thus, to decide between (i)–(iii)—and possibly further options—requires meta-reasoning, which has little to do with the logical correctness of the text—after all, it is exactly this logical correctness that is to be established. It is at least at this point that prejudices, and in particular authority and tradition, enter the picture and play a crucial role in proof understanding. To illustrate this, we consider some example scenarios.

In the first scenario, let us suppose we are reading an account of Euler's proof that the sum of the reciprocal primes diverges in some random internet source. Reading as enthusiasts in number theory, rather than history, our focus is on getting to know a proof that agrees with modern standards or proof, rather than an understanding of the historical text. We may not have much precise knowledge about Euler's biography, but at least we vaguely know that Euler was a great mathematician who lived some

Footnote 36 continued
and (ii) Zorn's lemma, in contrast to the axiom of choice, has little intuitive appeal. Making this claim may well lead to a distorted impression of the nature of axioms. The latter, while true, is of little help: first of all, the same could simply be said about the basis existence theorem without detour via Zorn's lemma (see Blass, 1984). Secondly, only one direction of the equivalence is relevant here, and "follows from the axioms" is hardly a satisfying justification (all the more when, as in the case of an introductory course in algebra "the axioms" are not even given)—if it was, any proof could be replaced by it.

37 For a hermeneutical approach to the question in what sense such reformulations, reconceptualizations and recontextualizations of statements still give the "same" statement, see Rodin (2006).

38 We note that the distinction between a "real" gap and one for which the knowledge how to bridge it has been lost in history is quite subtle, but potentially of crucial importance for our attitude in reading a proof and the effort we are (and should be) willing to invest in understanding and verifying it.

200 years ago; moreover, we know that this proof of said fact is quite famous and has a long history of being taught and presented in textbooks. Yet, we fail to understand our internet source; there is this one step where we just don't see why it would be true. Given both the authority of Euler and the long tradition of his proof, it is highly unlikely that we just discovered a crucial gap in it. Few people will even consider the possibility that the proof is "actually" wrong. This leaves the presentation or ourselves as the reasons for the failed verification. Let us suppose that, being enthusiastic about the beauty of elementary number theory, we have already read quite a bit about this subject; we know, for example, proofs of some "big" theorems like the two-squares theorem or Bertrand's postulate. Thus, we know ourselves to be capable of grasping somewhat intricate arguments in elementary number theory with some confidence; we also have some experience how much effort this usually costs us. In this case, we may consider the particular presentation as the source of our failure; the presentation is either incomplete, or not particularly well explained.

Thus, we go back to (an English translation of)[39] Euler's original work "Variae observationes circa series infinitas"; since our Latin is a bit rusty, we consider the English translation (Euler, 2021) instead.

Unfortunately, this version of the proof is even less comprehensible to us. In the Wikipedia article about Euler's proof, we are told that it contains "a sequence of audacious leaps of logic" and makes use of "questionable means".[40] Reading on, we find that mathematicians in the 18th century, and Euler in particular, had a view on and a way of dealing with infinite series rather different from the modern standards of rigor.[41] Our trust in Euler's text providing a proof according to modern standards is somewhat shaken. But the fact of tradition still remains—his proof has been passed on through many generations of teaching and textbook writing. We conclude that there may well be gaps in the presentation, at least from our modern perspective on analysis, but that they must be repairable; otherwise, why would we still talk of "Euler's" proof? Tradition gives us confidence that, also by modern standards, the proof is "essentially" correct.

Something interesting has happened here: Euler's proof text is judged as "essentially" correct (by modern standards) in light of the fact that the methods used in it could later on be made to fit into the modern conception of limit. The question whether a text represents a correct proof is thus answered depending not merely on the text,

39 We are mindful that we didn't really read the (Latin) original; but given that this is the official translation in the Euler archives, we find it sufficiently implausible that logical correctness was merely "lost in translation".

40 See https://en.wikipedia.org/wiki/Divergence_of_the_sum_of_the_reciprocals_of_the_primes#Euler's_proof (accessed 28.07.2021).

41 See, e.g., Ferraro, 2007, p. 195: Although Euler worked on transforming analysis from an intuitive into a "discursive" discipline, "(...) the foundations of modern (...) analysis are entirely different from the foundations of Euler's analysis" and "the tools Euler used to carry out his program (...) have been rejected or have been deeply modified. It would be premature, however, to see this as a triumph of modern precision over the sloppiness of the past: As Ferraro points out (ibid.), "The decline of Euler's foundations of analysis has often attributed to lack of rigour (...); but this opinion did not grasp the core of the problem." The Eulerian conception of infinite series is discussed, e.g., in Ferraro (1998, 1999). In particular, the efforts for assigning values also to divergent series, such as $1 - 1 + 1 - 1 \pm \dots$ were no "sloppiness", but led to profound pieces of mathematics and left their traces in the modern conception, for example in Cesaro summation (Ferraro, 1999). Also see Burkhardt (1911).

but on later developments, some of which (such as the modern definition of a limit by Weierstrass) took part long after the author's death. We see here an example of what Gadamer calls "history of effect" (Gadamer, 2006, p. 299) in the reception of mathematics.

We thus resort to a modern, but still classical number theory textbook, say, Hardy and Wright's "Introduction to the Theory of Numbers" (Hardy, 2009). Of this book, we know that it meets the modern standards of rigor—it is used in lectures etc. after all—and at the same time, it has been around for quite a while and extensively read and cited. There are several editions, so that apparent mistakes would certainly have been spotted and corrected. Moreover, it has received considerable laud for its presentation.[42] We are thus convinced that neither the presentation nor the argument itself is at fault when we still fail to understand the step that has been bugging us so far. Thus, we devote some extra effort to it, and finally, we are able to resolve our difficulty.[43]

What, then, is the right way to look at tradition and authority when it comes to mathematics? Certainly, the alternative cannot be blind belief in proof texts or results due to the fact that they come from some respectable person or source! While tradition, authority and, more generally, prejudices may help us in guiding our critical and rational examination, both individually and collectively, rationality should clearly remain the ultimate criterion![44]

This is exactly what is achieved in a much more general setting by Gadamer's account of authority: He regards the "authority of persons" as "based not on the subjection and abdication of reason but on an act of acknowledgment and knowledge—the knowledge, namely, that the other is superior to oneself in judgment and insight and that for this reason his judgment takes precedence—i.e., it has priority over one's own. (...) It [the authority, my insertion] rests on acknowledgment and hence on an act of reason itself which, aware of its own limitations, trusts to the better insight of other." (Gadamer, 2006, p. 281)

The right relation of authority and reason is then that authority directs us to something that could be grasped by reason in principle, while our (current) factual limitations (and also different cultural or historical contexts[45]) prevent us from doing so:[46]

[42] See, for example, the review by Bell (1939).

[43] Further examples in this direction abound: Proofs will be read differently when they occur in a student's homework, a paper one is supposed to referee, or a well-known and often-cited research article. But the above should suffice to show that there are quite sensible motives for invoking tradition and authority in reading mathematical proofs, and that indeed such invoking is practically indispensable.

[44] In this respect, we disagree with accounts on social aspects of proofs, as, e.g. described in Kitcher (1985), p. 43: "If I check through a proof in a book, thinking I see how the inferences go, and if the proof is very complex, then, under circumstances in which there is weighty evidence against both book and theorem, it would be unreasonably arrogant and stubborn of me to form the belief." In such a case, one has certainly reason to go through the inferences a few more times, possibly with enhanced critical awareness, and pay particular attention to the "weighty evidence" against the theorem, seeing whether it helps to reveal a gap in the proof. But if this does not yield a particular mistake, then "evidence against" the statement that falls short of a head-on refutation, let alone against the source in general, should not prevent one from becoming convinced of the respective statement.

[45] We are indebted to one of our anonymous referees for proposing this addition. Also cf. the next footnote.

[46] It is certainly true, as one of our referees pointed out, that the standards of rationality in mathematics are themselves subject to historical development; thus, for example, the ideal of proofs as (indications of) formal

> "Thus, acknowledging authority is always connected with the idea that what the authority says is not irrational and arbitrary but can, in principle, be discovered to be true. This is the essence of the authority claimed by the teacher, the superior, the expert." (Gadamer, 2006, p. 281)

This expresses quite well the function that authority can and does legitimately play in understanding mathematical (proof) texts, and in particular in distinguishing which of the factors (i)–(iii) accounts for a failure to understand.

4.1 Didactical consequences

It can hardly be doubted that many students of mathematics rely on authority in learning mathematics, believing statements and arguments to be correct "because the professor/tutor/textbook says so". In very few cases will belief in the contents of a lecture—as it is, e.g., exhibited in the use of statements in working on exercises—be due to the fact that the student has thoroughly and critically examined all of the proofs and decided on this basis whether or not she accepts a statement as true.[47] If this—certainly desirable—state is reached at all, it will rather come out of a reworking of the material after the term has ended. (Indeed, the prospects of success should be expected to improve this way, given our above considerations on the role of the hermeneutical circle in mathematics.)

Based on an idealized picture of understanding mathematics, reliance on authority will typically be discouraged: It is by reason, and reason alone, that conviction in the truth of statements and the correctness of proofs should be gained.

The tension between the need to rely on authority and the discouragement to do so may lead to some rather unwanted consequences: students may become convinced that they are unable to understand mathematics, as what they actually achieve is continually communicated to them as illegitimate and insufficient; seeing that (most) others do not fare any better, the idea that mathematics is inaccessible to all but a few particularly blessed individuals may arise. Seeing that reliance on authority is nevertheless invoked by and occurs on the part of the lecturer as well may yield the impression that mathematics is built on hypocritical double-standards.

In order to avoid the formation or solidification of such detrimental attitudes, it might be a good idea to communicate the role of authority in mathematics openly and explicitly; one should clearly distinguish between the idealized subject of mathematics whose mathematical beliefs are solely based on rational insight in the correctness of logical deductions and actual, resource-bounded (finite) human beings. In particular, understanding and verification of proofs should be presented as something to strive

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Footnote 46 continued
derivations from axioms in classical logic, which developed through such figures as Euclid and Hilbert is quite different from, say, intuitionist mathematics as proposed by Brouwer or experimental approaches to mathematics. In developing and establishing such standards, authority is likely to play an important role. However, it is impossible to give this topic the treatment it deserves in this paper; and so we restrict ourselves to pointing out that, even within an established standard of rationality, authority still has a role to play in understanding.

[47] In fact, determining whether a mathematical argument is sound and supports a certain claim is a challenging talk for many students. See, e.g. Selden and Selden (2003).

for, but also as something that may well take place quite a while after one learns about a statement and its use in further development and proofs.

5 Concepts of proof understanding

In this section, we will focus on Gadamer's account of the "hermeneutical experience", where classical hermeneutical positions, Gadamer's criticism thereof and his own approach are given in the form of a three-step "ascension", where each stage reacts to the shortcomings of the ones preceding it. Gadamer's approach in the section of "'Truth and Method" entitled "Elements of a theory of hermeneutic experience" then, is to decribe hermeneutics as a certain kind of experience. The relevant concept of experience here, however, is not that of empirical experiences that inform us that something is such-and-such, thus forming the background of the natural sciences;[48] this kind of experience would be "closed" or "fulfilled" in a state where everything is known. To this kind of "positive" experiences (that "posit" something), Gadamer opposes "negative" experiences that cannot be processed under the given conditions—such as one's expectations or conceptual system—thus telling us something not only (and not even primarily) about its superficial subject matter, but rather about ourselves, the nature of our knowledge and our limitations. The "fulfillment" of this type of experience is not a completion of knowledge, but rather an "openness for experience".[49]

With respect to the concept of experience, Gadamer proposes a shift of perspective: Namely, a text is not experienced like an external object, which plays a mere passive role in its being observed, but rather like another person, a "Thou", who relates "back" to me whenever I relate to him.[50] This reciprocity of relation must be taken into account in my attempts to understand another person: For trying to perceive the other as a mere object would overlook this essential ontological characteristic of the other, thus severely leading us astray.[51]

Gadamer's goal is then to analyze various ways to relate to another "Thou" and to parallelize these with certain kinds of text understanding.

We will now go through Gadamer's three stages, at the same time giving an example how they can occur in the understanding of a mathematical text.

48 Cf., in particular, Gadamer (2006, pp. 342, 347).

49 Cf. Gadamer (2006, p. 350): "(...) the perfection that we call "being experienced," does not consist in the fact that someone already knows everything and knows better than anyone else. Rather, the experienced person proves to be, on the contrary, someone who is radically undogmatic; who, because of the many experiences he has had and the knowledge he has drawn from them, is particularly well equipped to have new experiences and to learn from them. The dialectic of experience has its proper fulfillment not in definitive knowledge but in the openness to experience that is made possible by experience itself."

50 Gadamer (2006, p. 352).

51 See, for example, Gadamer (2006, p. 352): "A Thou is not an object; it relates itself to us."; "It is clear that the experience of the Thou must be special because the Thou is not an object but is in relationship with us."

5.1 1 + 1 = 2

This claim is frequently cited as an example of a completely uncontroversial statement; one that no one in her right mind will ever deny; one that can be universally agreed upon, independent of cultural, linguistical or religious differences; one that is immediately apparent in intuition and neither in need, nor capable of, a further argument or foundation.

In spite of this, Russell and Whitehead's famous "Principia Mathematica" (Russell & Whitehead, 1935) contains a rather long and involved proof of this statement. It takes several hundred of pages of symbolic manipulations to arrive at a point where a statement equivalent to $1 + 1 = 2$ can finally be derived.[52],[53]

Let us imagine an exceptional smart and persistent student who found this work (possibly with the foreword cut off) and, for some reason, decided to read it up to this point. He quickly grasps the rules by which the derivations proceed; he understands that these rules are truth-preserving; he is ready to explain each of the derivation steps to anyone who has a question about or an objection to one of these; in particular, he will happily and easily fill gaps in the derivations, should they occur. He has also mastered the construction of the kind of derivations found in this work heuristically: He can both identify general strategies behind particular proofs in the PM and construct derivations of this kind of given formulas on his own. He may well satisfy these and all the other criteria listed in Avigad (2008, pp. 11–12). To stress a somewhat overused analogy, he has learned the rules of the PM like one can learn the rules of chess, and become a competent player.[54]

Should he choose to remain at this point, our student would be at the first level described by Gadamer: He regards the text as a phenomenon that can be methodologically studied by extracting manipulation rules and strategies, but without any deeper involvement of his part.[55] In Gadamer's account, the corresponding attitude towards other persons would be a mere observational one with the purpose of predicting (and perhaps controlling) the other's behaviour.

But our student is far from willing to leave it at this point. Indeed, at this point, in spite of all his competences, there is more than one sense in which the text will remain a great mystery to him. It is as if we stood in front of a very strange building: After some investigation, we understand its statics; we may be able to repair it when it is damaged, and we may even be capable of building a copy of it. And yet, it remains alien to us, for, being unsuccessful at grasping its purpose, we cannot regard what we observe as an expression of that purpose. Therefore, our understanding will in a relevant sense be incomplete. What is the point, our student will wonder, in putting all

[52] See (Russell & Whitehead, 1935), volume 2, p. 119, proposition $*110.643$.

[53] It would clearly be inappropriate to reduce the PM up to this point as a proof of $1 + 1 = 2$ and it is certainly not our intention to propose such a reductive interpretation; but the statement can serve as an example of a kind of statement that is derived within the PM with great care and effort and that, if merely assumed as given, would lead to a considerable shortening of the work.

[54] For the comparison of mathematics and chess, see, e.g., Weir (2021).

[55] See, e.g., Gadamer (2006, pp. 352–353): "Someone who understands tradition in this way makes it an object—i.e., he confronts it in a free uninvolved way—and by methodically excluding everything subjective, he discovers what it contains."

of these complicated constructions behind something as obvious as $1 + 1 = 2$? Why all this wordless symbolism? Why the strict adherence to a small set of rules?

This is all part of a foundational program, one wants to answer; an attempt to reduce all mathematical truths, even, and particularly the most simple ones, to mere logic. As such, it should be viewed against the background of founding arithmetic, say, on sensual experience (Mill)[56] or on pure intuition (Kant)[57]. Rather than answering the question whether $1+1 = 2$, this text attempts to answer the question whether $1+1 = 2$ belongs to logic.[58]

Our student feels that he made progress. It is much clearer to him now what drives all of these symbolic manipulations and what holds them together. But even now, something seems to be missing. He has learned that, at some point in history, people wondered whether arithmetic could be reduced to logic; he understands that they regarded logic as "secure" and therefore capable of serving as a "basis" or "foundation" for arithmetic. He has thus obtained a historical understanding of Russell and Whitehead's proof. In this kind of understanding, the proof, and the ideas driving it, appear as a peculiar historical phenomenon. At some point in history, people wrote taxonomies of demons and treatises on witchcraft. At still a different time, they wrote heavy volumes on $1 + 1 = 2$. People are strange, and past people apparently even more so.

This is hardly satisfying. The reduction of the strangeness of the text to the strangeness of an epoque may be a step into the right direction—the direction determined by the subject under consideration—but as long as the reduction leads from strangeness to strangeness, we have not achieved understanding. How did this logicist program come about? What was the reason, our student could ask, why people like Frege, or, for that matter, Russell, devoted so much time to complicated justifications of trivialities? Why was it so important to them whether arithmetic can be reduced to logic?

To a certain extent, an answer can again be given by providing further historical context: Some time ago, mathematics had indeed been in trouble: They were, for instance, struggling with analysis, which rested on imprecise and occasionally controversial concepts of, e.g., function, continuity and limit[59], operated with inconsistent objects like infinitesimals[60] and even produced proofs of statements that were either false or ambigous[61]. This caused an actual and urgent need for conceptual clarification and foundational considerations. But these things could be sorted out without going all the way down to $1 + 1 = 2$. Weierstraß had made great progress towards securing analysis; and whatever difficulties there may have been in analysis apparently never

56 See, e.g., Mill (1974), Book III, §4, p. 609 and §5.

57 See (Kant, 2013), B15.

58 See, e.g. Russell (1919, p. 5): "It is time now to turn to the considerations (...) of Frege, who first succeeded in "logicising" mathematics, i.e. in reducing to logic the arithmetical notions which his predecessors had shown to be sufficient for mathematics."

59 For discussions of the early history of calculus and the corresponding difficulties, controversies and occasional inconsistencies, see, e.g., Lakatos (1976), Bedürftig and Murawski (2019) and Spalt (1981).

60 See, e.g., Colyvan (2008).

61 See, e.g., Rickey (2007).

troubled number theory, let alone elementary arithmetic. Why was that not enough for Russell?

Here, an answer can be given based on Russell's personal motives, which are explained in his autobiographical work "My philosophical development"[62]—that is, by turning to a "subjective" interpretation in the sense of Schleiermacher.

Here, we find the picture of a young man whose "general outlook, in the early years of this century, was profoundly ascetic", who "disliked the real world and sought refuge in a timeless world, without change or decay or the will-o'-the-wisp of progress." and thus "came to think of mathematics, not primarily as a tool for understanding and manipulating the sensible world, but as an abstract edified subsisting in a Platonic heaven and only reaching the world of sense in an impure and degraded form." Russell (2007, pp. 209–210)

The turn to mathematics, then, was apparently an attempt to deal with a fundamental existential insecurity. Mathematics was attractive to Russell for its being "remote from human passions, remote even from the pitiful facts of nature" (Russell, 2007, p. 210, "the study of mathematics") and providing "an ordered cosmos, where pure thought can dwell as in its natural home, and where one, at least, of our nobler impulses can escape from the dreary exile of the actual world." [ibid].

But the certainty that mathematics had to offer, while possibly sufficient for the purpose of "understanding and manipulating the sensible world" (Russell, 2007, p. 209) turned out to be insufficient for securing a basis solid enough for resisting an existential crisis. Mathematical practice might have worked well, but not well enough for someone in the need of and looking for "a splendid certainty" (Russell, 2007, p. 212). Thus, mathematics, as usually taught, was called into question; and here we find the transition that motivates writing up a long and involved proof for $1 + 1 = 2$:

> "This change came about through a wish to refute Mathematical scepticism. A great deal of the argumentation that I had been told to accept was obviously fallacious, and I read whatever books I could find that seemed to offer a firmer foundation for mathematical beliefs. This kind of research led me gradually further and further from applied mathematics into more and more abstract regions, and finally into mathematical logic." (Russell, 2007, p. 209)

This certainly sheds some light on the purpose of the PM. It allows for a psychological interpretation of the Principia Mathematica as a strive for security, or possibly even as a symptom of a disturbed mental condition. And indeed, there have been attempts to interpret logicism in this way; see, e.g., Pambuccian (1992) or Herrmann (1960). Even if our student, rather than adopting the perspective of a personally detached psychoanalyst, develops empathy at this point, appreciating insecurity as a strong and urgent motive, he has now arrived at a purely subjective view, where any claim to respond to an objectively present problem is lost.

If our student views Russell and Whitehead's proof in this way, his understanding still appears to lack an important point. If the text can be seen to flow from a certain program, which in turn comes from a certain motivation, but this motivation itself is

62 Russell (2007).

regarded as "strange" (or even pathological), something remains unexplained. In the worst case, Russell appears as a madman, and the PM as his ramblings.

By regarding the text as an expression of another person's intention while maintaining a distance to it from which the text appears as something to be explained with respect to a historical or psychological context, our student has entered the second of Gadamer's levels of understanding, which, on the intersubjective analogue, corresponds to the attempt to interpret another person's utterances as mere expressions of that person's idiosyncratic character, with no relevance of oneself:

> "One claims to know the other's claim from his point of view and even to understand the other better than the other understands himself. In this way the Thou loses the immediacy with which it makes its claim. It is understood, but this means it is co-opted and pre-empted reflectively from the standpoint of the other person." (Gadamer, 2006, p. 353)

In this way, the other person is "understood" in a way that at the same time conceals or "overlooks" her in an essential sense by failing to take her seriously. Coming back to hermeneutics, regarding a text as a mere "symptom" of a historical period or an emotional state ignores an essential aspect of its content, namely its claim to truth:

> "A person who reflects himself out of the mutuality of such a relation changes this relationship and destroys its moral bond. A person who reflects himself out of a living relationship to tradition destroys the true meaning of this tradition in exactly the same way." (Gadamer, 2006, p. 354)

What is missing here is thus the active participation in the questioning that the PM performs: "I must allow tradition's claim to validity, not in the sense of simply acknowledging the past in its otherness, but in such a way that it has something to say to me." (Gadamer, 2006, p. 355) In order to gain an adequate understanding of the PM, one needs to grasp the specific questionability of arithmetic to which the PM attempts to reply, and to make this one's own question. Indeed, even when we can reproduce the underlying motive of the investigation in a general sense and accept that there could be a meaningful way to doubt $1 + 1 = 2$, we are not fully equipped to make sense of the specific course and methodology of the present investigation. In what sense can we actually hope to "look behind" $1 + 1 = 2$? What can count as "secure" in this sense and why? In order to proceed, we need to determine the specific way of questioning $1 + 1 = 2$ to which the PM attempts to provide an answer.

Once we know this question and then ask it, we enable ourselves to interpret the text as an answer to it: "Thus a person who wants to understand must question what lies behind what is said. He must understand it as an answer to a question." (Gadamer, 2006, p. 363) At this point, the text will make sense. This, however, requires that the question is not merely stated, but actually and seriously posed:

> "Questioning opens up possibilities of meaning, and thus what is meaningful passes into one's own thinking on the subject. Only in an inauthentic sense can we talk about understanding questions that one does not pose oneself (...) " (Gadamer, 2006, p. 368)

When approaching the PM in this way, the text can be adequately appreciated. Reading it will then involve exposing oneself to rather fundamental doubts and thus potentially yield a real insight into the unreliability and uncertainty of human knowledge:

> “The hermeneutical consciousness culminates not in methodological sureness of itself, but in the same readiness for experience that distinguishes the experienced man from the man captivated by dogma.” (Gadamer, 2006, p. 355)

This way of reading a text as an answer to a question corresponds to the final section of the second part of “Truth and Method”, which is entitled “The hermeneutic priority of the question” (Gadamer, 2006, p. 355f).

The basis for understanding a text, then, is to uncover the question to which it is an answer. The mastership of the Principia Mathematica, and the key to understanding it, is exactly this achievement of turning $1 + 1 = 2$ into a question that is, in a certain, very specific sense, open:

> “The significance of questioning consists in revealing the questionability of what is questioned. It has to be brought into this state of indeterminacy, so that there is an equilibrium between pro and contra. The sense of every question is realized in passing through this state of indeterminacy, in which it becomes an open question. Every true question requires this openness (...)” (Gadamer, 2006, p. 357)

By bringing in the dimension of questions, an point about proofs becomes apparent: They are not just proofs of a certain statement X, but they are answers to certain ways of making X questionable. Indeed, several ways of arguing of the “same” statement may answer very different questions: An illustration with the help of building blocks for small children, a derivation in Peano arithmetic, a proof from the set-theoretical definitions of 1, 2, + and “equal cardinality” and the path of the PM are all ways of arguing in favor of, or proving, that $1 + 1 = 2$. But they do not reply to the same question. The first shows how the symbolic expression $1 + 1 = 2$ relates to objects in the physical world. The second shows that the axioms of Peano arithmetic are strong enough to deduce basic arithmetical statements, thus replying to the question “how adequate is PA as a basis for arithmetic?”. The third one shows that the set-theoretical conception of numbers allows us to conceptualize basic arithmetic, showing that this conception captures the expected meaning of natural numbers to a certain degree, thus replying to the question “how does arithmetic work in the set-theoretical framework?” (note that this question is independent of any particular axiomatization of set theory, such as ZFC; it concerns the adequacy of a conceptualization, in this case, the question whether natural numbers can be regarded as sets, not, like the last example, of an axiomatization). The last one, finally, is an attempt at reducing it to “mere logic”; it replies to the question “What is arithmetic” by “arithmetic is (reducible to) logic”.

Indeed, there is a vast variety of questions that can be answered by proofs: A proof of the four colour theorem shows that the theorem is actually true; before, this was an open question about entirely clear and intuitive concepts; it is hardly imaginable that a counterexample would have motivated a revision of the definition of “finite planar graph" or of "coloring", not even in the sense of typical laudatory supplements such as "good", "normal" etc. It would simply have shown that the statement is false. A proof of Jordan curve theorem, on the other hand, shows that the definition of "closed curve" captures the intended semantics and can thus, to a certain degree, be used as a formal replacement of the intuitive concept; a counterexample would certainly have led to a closer analysis, followed by attempts to reconceptualize the domain in such a way that the counterexamples are excluded. The many proofs of certain theorems like the prime number theorem, the fundamental theorem of algebra, Pythagoras' theorem etc. often show a certain kind of framework or methodology is strong enough to obtain some already known statements, possibly in a simpler way than before.[63] Another examples where knowledge of the specific kind of question a proof is supposed to answer is the work of Brouwer on intuitionstic mathematics; this work both as a whole and in specific parts (such as the famous proof of the bar theorem) are hardly comprehensible, although perhaps technically accessible, unless one knows about Brouwer's quarrels with "classical" mathematics and his ideas to reform mathematics, in the light of which statements assume a new meaning and the standards of proof change considerably.

We believe that this differentiation of proofs according to the questions answered by them is didactically highly relevant: If understanding a proof requires understanding the question that it attempts to answer and reading it as a response to this question, but these questions can be quite different, then it is quite plausible that in particular beginner students can profit from support in determining these questions, in the simplest case by making them—and thus the status of the given proofs—explicit. The difference between a proof that there are infinitely many primes and one for the commutativity of addition should be highlighted, along with the changes in standards of what counts as a satisfying argument in each case; and the same support may well help in making sense of proving exercises.

5.2 Understanding mathematical statements example: Gödel's theorem

We close this section by pointing out that historical context, subjective motivation and "revealing the questionability" enters not only the understanding of proofs, but also of statements. As an example, consider the following statement:

> "To every ω-consistent recursive class c of formulae there correspond recursive class-signs r, such that neither v Gen r nor Neg (v Gen r) belongs to Flg(c) (where v is the free variable of r)." Gödel (1992)

Even when one looks up all of the definitions, one can hardly say that one understands this statement when given in isolation. The statement still looks like one arbitrary example out of a large class of messy and complicated statements one could formulate.

[63] See the introduction of Dawson (2015), where these and many further motives for reproving known statements are discussed. A particularly illustrative example treated by Dawson is the proof of the prime number theorem by Dawson (2015, pp. 8, 139f): This proof will probably appear as a pointless complication to—and thus be met with incomprehension by—someone who knows a proof using analytical number theory until she understands that the intention is to give an "elementary" proof for the theorem, what "elementary" means here and why striving for elementarity is interesting (in contrast to imposing some arbitrary handicap, such as never using the letter "e" in the proof).

Here, it is quite obvious that understanding requires knowledge of the historical developments in which the question arose to which the incompleteness theorem provides a most surprising and significant response; and moreover, an understanding can hardly count as satisfying unless it involves at least some of the further developments that were triggered by Gödel's discovery.

5.3 Didactical Consequences

> "A logical presentation of a reasonably advanced part of mathematics (...) bears little relation to the historical development of that subject. (...) On the other hand, a student is handicapped if he has no idea of the forces that figured in the development of his subject." Barwise (1975), p. 1

In understanding proof texts, grasping overarching intentions plays an essential role; consequently, in order to achieve understanding in teaching, one should support students in experiencing as meaningful as many aspects of a proof text as possible, and not merely as "logically sound". This includes raising a desire for proof in the first place; but then further, and more precisely, to point out the particular questionability of the proof goal in the specific sense in which the proof supports the claim. We have seen that putting mathematical proof text in the broader context of its historical background and even personal motivations can considerably enhance our understanding of these texts; they help us to make sense of various aspects of proofs, and they play a role in reconstructing the text as an answer to a question. An obvious consequence for the teaching of proofs is then that proofs should be taught in a way that provides such contexts: That is, mathematical proofs should not be presented as formal objects detached from extra-mathematical contexts, but rather one should strive for teaching mathematics in the style of integrated narratives, in which historical, methodological, existential, personal and heuristical aspects are are combined with a systematical development, logical correctness and formal rigour. This radicalizes the approach of regarding mathematical texts as special kinds of narratives in Carl et al. (2021), where it is stated that mastering such a text "includes the ability to talk about dramaturgical aspects of the narrative and its presentation and to draw intertextual links to other texts" (p. 11). This may appear to be an equally desirable and daunting task: One may well doubt whether it is even possible to combine all of the criteria above in teaching materials that can still be reconciled with the curricula. We will skip at this point the question whether this might not justify a change in the curricula—after all, it may be better to achieve substantial understanding on a reduced amount of content than superficiality for a larger one. Instead, we want to point out two approaches at providing such material that we regard as particularly hopeful signs. The first is Toeplitz' treatment of calculus for beginner students (Toeplitz, 2007). Toeplitz' didactical approach, which he explains in a separate paper, entitled "Das Problem der Universitätsvorlesungen über Infinitesimalrechnung und ihrer Abgrenzung gegenüber der Infinitesimalrechnung an den höheren Schulen" (Toeplitz, 1927) ("The problem of university lectures on infinitesimal calculus and its difference to infinitesimal calculus in secondary schools"), describes the use of narratives for teaching mathematics as follows:

> "Regarding all these basic topics in infinitesimal calculus which we teach today as canonical requisites, (...) the question is never raised 'Why so?' or 'How does one arrive at them?' Yet all these matters must at one time have been goals of an urgent quest, answers to burning questions, at the time, namely, when they were created. If we were to go back to the origins of these ideas, they would lose that dead appearance of cut-and-dried facts and instead take on fresh and vibrant life again." (Toeplitz, 1927, pp. 92–93, translated in Toeplitz, 2007, p. xi)

This passage summarizes various aspects of our hermeneutical considerations very well: We see (i) the aspect of contextualization as a prerequisite of understanding ("why so?", "How does one arrive at them?"), (ii) the relevance of historical context for understanding mathematics in that, specifically, (iii) historical context can help us to uncover the questions to which certain mathematical developments can then be understood as answers. It thus seems that Toeplitz' didactical recommendations agree quite well with the analysis of understanding offered by the hermeneutical tradition; and this analysis then be seen as a theoretical or conceptual underpinning of Toeplitz' approach (which does not appear to be based on any explicit concept of understanding). As Toeplitz emphasizes, such narratives can, but do not necessarily have to, go along with the historical facts; however, the "actual" history should both be seen as an important source in the formation of such narratives.[64] This approach of making concepts, propositions, arguments etc. accessible by referring to their genesis is known as the "genetic" approach, a didactical paradigm that has been discussed in considerable depth in the work of Martin Wagenschein.[65],[66]

Toeplitz' program is implemented in his calculus textbook accordingly entitled "The Calculus—A Genetic Approach" (Toeplitz, 1927), which starts from the first attempts of the ancient Greeks to think about infinity, develops the historical line up to the modern concepts of convergence and limit and, after developing integral and differential calculus in a similar manner, closes with various classical—and natural—applications to physics.

As a second example, we mention Wittenberg's approach to teaching irrational numbers starting from geometry in Wittenberg (1963).

Based on our hermeneutical considerations, we regard these approaches as pointing in the right direction: Historical context and, in particular, key problems—such as Zenon's paradoxes in the work of Toeplitz—are used in order to lead the reader into grasping the questionability of the content under consideration. However, neither Toeplitz nor Wittenberg put a particular emphasis on the way in which proofs answer this questionability, or on proof understanding in general; they are more concerned with motivating core concepts and questions. In part, this may be due to the fact that, in their accounts, the occurring proofs have a rather similar status—they support a statement of a previously unknown truth value—so that the need does not arise. This would

64 In order to prevent the formation of mathematical "urban legends", it may be advisable to point out deviations of didactical narratives from the course of history in teaching.

65 See, e.g., Wagenschein (1965).

66 For a contemporary application of the genetic principle in the didactics of mathematics, see Schiffer (2019), chapter 3, where the history of algebra is analyzed with regard to its didactic use, referring, in particular, both to Wittenberg and the above Toeplitz quote.

be different, for example, in an account of (linear) algebra, where issues of deriving statements from a certain set of axioms come into play. We believe that extending their approach to the presentation of proofs may have a positive influence on proof understanding. We have thus seen how the hermeneutical approach to proof understanding has revealed several aspects of understanding that are naturally addressed by the genetic approach; and thereby, the hermeneutical perspective has provided us with a differentiated account of understanding that explains the (expected) advantages of the genetic approach.

6 Conclusion

By reading Gadamer's "Truth and Method" with a focus on its applicability to mathematical proof texts, we have been directed to three phenomena that are rarely considered in this context: The relevance of the hermeneutical circle (which may explain why re-reading proofs achieves levels of comprehension that close and thorough readings do not), the role of tradition and authority in the understanding and verification of proofs, the significance of historical background and the author's intentions for making sense of proof texts and the interpretation of proofs as answers to questions, which can be very different even for proofs of the apparently same statement. These observations are somewhat orthogonal to the "logical" view of mathematical proofs as formal deductions dressed up in natural language. We hope to have convinced the reader that there is indeed something to be learned from the hermeneutical tradition for mathematical proof texts, and that this has relevant consequences for the didactics of mathematics.

Inevitably, since this is an article and not a book, we had to leave out many other potentially interesting aspects of hermeneutics that could tell us something about mathematical proofs and may be worthwhile to pursue in future work. To mention a few:

1. Surpassing the author's understanding A point frequently made in the hermeneutical literature[67] is that understanding is not a mere reproduction of the author's thoughts, but offers the possibility—which should be pursued—to surpass her understanding. Now, one can certainly say that mathematics has a tradition of re-interpreting proofs with the goal of making their "core ideas" more visible, or revealing the "heart" of an argument: for example, case distinctions may be avoided, or certain crucial points are isolated and explicated as separate theoretical building blocks, as, e.g. in applications of algebra to number theory or geometry, where a "structural core" is revealed by detaching it from the concrete geometrical or number-theoretical context.[68],[69] One can then say that the original proof is now better understood than before. This does not mean that one is approach-

[67] See, e.g., Schleiermacher (1978), p. 9: "The task is this, to understand the discourse just as well and even better than its creator."

[68] The different formulations of Pythagoras' theorem, e.g., in terms of elementary geometry or in terms of vector calculus in Rodin (2006), are an illustrative example of this.

[69] Avigad (2008) explicitly mentions "the ability to cast the proof in different terms, say, eliminating or adding abstract terminology" as a criterion for understanding proofs (p. 328).

ing a fictitious ideal or optimal understanding, but that different interpretations in different theoretical, conceptual and methodological frameworks improve our understanding in some aspect.

2. The significance of "applications". In the section "the hermeneutic problem of application" (Gadamer, 2006, p. 306f), Gadamer discusses the importance of purposes of interpretations in the course of interpreting text, giving, among others, the example of interpreting legal texts for judicial decisions. It is quite obvious that proofs are read for rather different purposes: Out of sheer curiosity, out of doubt in the respective statement, in order to learn from the methods used, as a preparation to convince others, out of historical interest, to present it in a seminar talk or prepare for an examination etc.; it is likely that such different purposes have an effect on the way a proof text is approached (for example, when we merely want to learn the overall method, we may pay less attention to minute details than when we are in actual doubt about the statement in question).
3. In the treatment of Gödel's theorem in the section on concepts of understanding, we already briefly mentioned the aspect of "consciousness of being affected by history" (Gadamer, 2006, p. 301) as an aspect of understanding. Indeed, we regard knowledge of the influence of a certain theorem or proof on the subsequent developments up to the current state of the art as contributing an important aspect of understanding. This potentially touches on a further point made by authors of the hermeneutical tradition such as Schleiermacher, who have pointed out that understanding, in the sense of being able to make sense of every aspect of a text, is an ever ongoing and unfinished, hence an "infinite task". [70]
4. A further, more general point to which the section on concepts of understanding points is the following: Mathematical practice and communication takes place on several levels simultaneously; in particular, there is a layer behind the propositions and proofs in mathematical texts, in which, e.g., foundational, philosophical or ideological points of view, methodological and thematical preferences[71] and possibly even political matters etc. play a role; and these layers constitute an important part of understanding mathematics.[72]

On the didactical side, the obvious challenge is to develop teaching material and course concepts along the lines of the proposal of "mathematical narratives" developed in the last section, to apply them in teaching and to see whether understanding improves.

[70] See, e.g., Schleiermacher (2012), p. 6: "Zwei entgegengesezte Maximen beim Verstehen 1.) Ich verstehe alles bis ich auf einen Widerspruch oder Nonsens stoße 2.) ich verstehe nichts was ich nicht als nothwendig einsehe und construiren kann. Das Verstehn nach der lezten Maxime ist eine unendliche Aufgabe." ("Two opposite maxims for understanding 1.) I understand everything until I hit on a contradiction or nonsense 2.) I understand nothing unless I can see that it is necessary and can construct it. Understanding according to the latter maxim is an infinite task.") (My translation.)

[71] An obvious example would be the debate about the axiom of choice, see, e.g., Cramer (2020), section 5.

[72] For example, proving involves an orientation with respect to what is regarded as requiring proof, what can be assumed as not in need of further verification and what are the valid methods of justification. At this point, many different points of view come into play that may have their roots in a multiplicity of factors such as those just mentioned.

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