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ORIGINAL RESEARCH

Entering the valley of formalism: trends and changes in mathematicians' publication practice—1885 to 2015

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Abstract

Over the last century, there have been considerable variations in the frequency of use and types of diagrams used in mathematical publications. In order to track these changes, we developed a method enabling large-scale quantitative analysis of mathematical publications to investigate the number and types of diagrams published in three leading mathematical journals in the period from 1885 to 2015. The results show that diagrams were relatively common at the beginning of the period under investigation. However, beginning in 1910, they were almost completely unused for about four decades before reappearing in the 1950s. The diagrams from the 1950s, however, were of a different type than those used earlier in the century. We see this change in publication practice as a clear indication that the formalist ideology has influenced mathematicians' choice of representations. Although this could be seen as a minor stylistic aspect of mathematics, we argue that mathematicians' representational practice is deeply connected to their cognitive practice and to the contentual development of the discipline. These changes in publication style therefore indicate more fundamental changes in the period under consideration.

Keywords Mathematical diagrams · Mathematical practice · Mathematical representations · Corpus study

1 Introduction

The recent movement towards a practice-oriented philosophy of mathematics has brought considerable attention to the various representations used in mathematics, especially non-textual representations such as figures, diagrams, and other types of visualizations (e.g. Giaquinto, 2007; Manders, 2008; Kjelsen, 2009; Carter, 2010; Mumma & Panza, 2012; Toffoli & Giardino, 2014; Giardino, 2017; Vold & Schlimm,

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2020). This work spans almost all aspects of the use of visualizations—from the epistemology of diagrammatic reasoning to the use of visualizations in ideation and heuristics—and shows that visualizations play a large and diverse role in mathematics, both historically and in the contemporary period.

This renewed interest in visualizations stands in contrast to much of the twentieth-century philosophy of mathematics, in which logic and foundational issues took center stage, leaving figures and diagrams to minor and purely heuristic roles. This dismissive attitude toward figures and diagrams can be traced back to the late nineteenth century, when it was discovered that earlier mathematicians' reliance on visual intuition and diagrammatic arguments had led to false or unfounded conclusions in several high-profile cases. These cases included the discovery of gaps and missing axioms in the Euclidean framework and the implicit assumption that everywhere continuous functions are differential except in a finite number of points (see Mancosu, 2005). Discoveries such as these led prominent figures like David Hilbert, Moritz Pasch, and Hans Hahn to distance themselves from the use of visualizations and instead recommend a more rigorous axiomatic and formal style.

Although Pasch acknowledged that figures may play a heuristic role in mathematics, he clearly saw their epistemic role as limited at best. In his *Vorlesungen über neuere Geometrie*, where he gives the first rigorous axiomatization of projective geometry, he famously states: "If you are not afraid to spend some time and effort, you can always omit the figure in the proof of any theorem, indeed, a theorem is only really proved if the proof is completely independent of the figure…[T]he theorem can only be justified by reference to a specific previously shown theorem (or definition), and not by reference to the figure" (Pasch & Dehn, 1882, p. 43, our translation). Similarly, in the introduction to a lecture on the foundation of geometry, Hilbert use a well-known false figure proof of the (likewise false) theorem that all triangles are isosceles, to warn against figures "…we will use figures frequently, but never rely on them" (Hilbert, 2004, p. 541, our translation). Along the same line (although slightly more extreme), Bertrand Russell states: "Formerly, it was held by philosophers and mathematicians alike that the proofs in Geometry depended on the figure; nowadays, this is known to be false. In the best books there are no figures at all" (Russell, 1901, p. 93).

For short we will denote the general idea that the epistemology of mathematics should be based in a rigorous formal approach rather than visualizations as the *formalistic ideology*. It is not the aim of this paper to give an in-depth analysis of this ideology or to describe the different specific philosophical programs it was part of (e.g. logicism, various forms of formalism, bourbakism etc.). What is important for us here is simply to point out that a central idea—namely the idea that visualizations are not to be trusted and that rigorous, formal deductions should form the epistemological backbone in mathematical reasoning—emerged and grew popular in the late nineteenth and early twentieth century.

It should be noted, however, that although Hilbert clearly disapproved of the use of some types of figural reasoning, he seems to approve of others. In his 1900 lecture at the International Congress of Mathematicians, he states:

> The arithmetical symbols are written diagrams and the geometrical figures are graphic formulas; and no mathematician could spare these graphic formulas, any

> more than in calculation the insertion and removal of parentheses or the use of other analytical signs.
> The use of geometrical signs as a means of strict proof presupposes the exact knowledge and complete mastery of the axioms which underlie those figures; and in order that these geometrical figures may be incorporated in the general treasure of mathematical signs, there is necessary a rigorous axiomatic investigation of their conceptual content. Just as in adding two numbers, one must place the digits under each other in the right order, so that only the rules of calculation, i.e., the axioms of arithmetic, determine the correct use of the digits, so the use of geometrical signs is determined by the axioms of geometrical concepts and their combinations (Hilbert, 1902, p. 443 English translation).

Here, Hilbert is not dismissive of visual elements per se; the decisive feature of a representation does not seem to be whether it is a visualization or not, but whether it is part of an axiomatic system and supports syntactic manipulation. Along the same lines Hilbert, in his famous lecture *Über das Unendliche*, calls a mathematical proof "a figure which as such must be accessible to our intuition"(Hilbert, 1983, p. 198). He goes on to explain that "a formalized proof, like a numerical symbol, is a concrete and visible object" (Hilbert, 1983, p. 199), thus emphasizing the visual aspects of formal reasoning.

Hilbert is not alone in noticing this connection. Charles S. Peirce's functional definition of iconicity (Stjernfelt, 2007, p. 90) similarly categorizes algebraic or symbolic deductions like those Hilbert has in mind as diagrammatic, and as a more recent example, Marcus Giaquinto points out that "Symbolic thinking typical of algebra, to wit, rule-governed manipulation of symbols, is just as spatial as geometrical thinking" (Giaquinto, 2007, p. 241). The distinction between visual and formal thinking may therefore be less clear from a theoretical point of view than it is from a practical point of view.

We will return to this distinction again in Sect. 4 and 5 below, but for now we will set the more theoretical discussion aside and return to the reality of mathematical practice. Here, there is a very real distinction between algebraic manipulations and visualizations (such as figures and diagrams), and judging from a recent interview study, contemporary working mathematicians feel subjected to a strong value of *formalizability* that restricts the use of diagrams and other visualizations in published work (Johansen & Misfeldt, 2016). So apparently, the dismissive attitude toward visualizations formulated in the late nineteenth century remains influential.

Nevertheless, a cursory look at the papers published in mathematics journals reveals that diagrams and other visualizations have not completely disappeared (as also noted by Giaquinto, 2020). On the contrary, a comparison between recent publications and publications from, say, the 1920s gives the impression that visualizations are *more* prevalent today than before—and that contemporary diagrams are different from those published a hundred years ago. These observations, however, are only first impressions. We do not have any solid empirical knowledge describing the historical variation in either the frequency or the types of visualizations being used.

Although the renewed philosophical interest in visualizations has vastly expanded our understanding of the numerous and varied roles played especially by diagrams in mathematical practice, this work is mostly qualitative in nature, building on case studies and analyses of the function of particular diagram types. In other areas of the philosophy of mathematical practice, more quantitative methods have successfully been applied to investigate various aspects of mathematicians' use of language (e.g. Inglis & Aberdein, 2015), but to the best of our knowledge, no large-scale quantitative studies have been performed investigating the use of diagrams and other visualizations. Consequently, the overall trends and changes in the use of visualizations over time have not been investigated in a systematic and comprehensive way. Although we may have the impression that the use of visualizations has changed over time, we do not have *knowledge*, and we especially do not know the details of how, when, or why the changes occurred. Such questions are of vital importance. As recent work in the philosophy of mathematical practice has shown, visualizations play a central role in mathematical practice. An understanding of the context and an overview of the changes in publication style are indispensable for picking the most relevant examples and cases to analyze more closely. Most importantly, we argue that changes in mathematicians' representational practice may also indicate more profound changes in cognitive style and ideological outlook. Although mathematics has a front and a back, as pointed our by Reuben Hersh (1991), the two realms are not completely disconnected and in some cases improved knowledge about the front may facilitate inferences about the back.

To improve our knowledge of the use of visualizations in mathematical publication practice we conducted a large-scale quantitative investigation. We focused the investigation on the use of one particular type of visualization: diagrams. The goal of the investigation was to give an outline of the trends and changes in the use of diagrams in published mathematical papers throughout the twentieth century. In the following, we will describe our methods and how we operationalized the research goal (Sect. 2), present the results of our investigation (Sect. 3), and give an analysis of these results (Sect. 4). Finally, we will discuss the philosophical implications of our study (Sect. 5).

2 Methods

2.1 Corpus

Although new machine learning tools are in the making, it is currently not possible to automate the kind of investigation envisioned above[1]. Consequently, to operationalize the research goal, we had to compile a corpus of reasonable size and quality. To track influential twentieth-century mathematical works without data breach and avoid tracking mathematical sub-cultures, we decided on the following three inclusion criteria for journals:

1. The journal must have been published continuously from at least the end of the nineteenth century to the present.
2. The journal must have a general scope.
3. The journal must have a high impact.

[1] After—and in part building on—the empirical part of this study machine learning tools for detecting, but not classifying diagrams have been build. For reports on their performance see (Sørensen & Johansen, 2020; Sørensen, 2021).

Based on these criteria and an estimate of impact from (ScienceWatch, 2008), we decided to include in our corpus papers from the following three journals: *Bulletin of the AMS*, *Acta Mathematica*, and *Annals of Mathematics*. Since *Acta Mathematica* is based in Europe and *Bulletin of the AMS* and *Annals of Mathematics* are based in the United States, the three journals will encompass some geographic diversity (although only within the European-American research culture). Furthermore, in terms of style, the three journals range from long and highly prestigious papers (*Annals of Mathematics*) to short research announcements (*Bulletin of the AMS*). By consulting with historians of mathematics at University of Copenhagen, we determined that the only other suitable candidate was *Journal für die Reine und Angewandte Mathematik* (*Crelle's Journal*). However, *Crelle's Journal* is also based in Europe and resembles *Acta Mathematica* in its style, so we elected not to include it.

From the three selected journals, we compiled a corpus consisting of all research papers published in years separated by 5-year intervals beginning in 1885 and ending in 2015 (except for *Bulletin of the AMS*, founded in 1891, from which we included papers from 1895 on). Obituaries, errata, book reviews, and other non-research papers were not included in the study. In total, 2940 papers spanning a total of 55,446 pages were included in the corpus. A full overview of the corpus and all the raw data from the count can be seen in the protocol (available online at https://doi.org/10.17894/ucph.6e18e1c7-7eef-4445-8d9c-3b4d38949079).

2.2 Coding and classification scheme

All papers in the corpus were coded by hand following a code book developed during a pilot study on a small subset of the corpus (see Johansen et al. 2018). This meant that a representation was coded as a mathematical diagram if its two-dimensional structure was a carrier of information (loosely following Larkin & Simon, 1987). However, matrices, tables, and diagrams with non-mathematical content (such as pictures of the author or flowcharts showing a paper's argument) were not coded as mathematical diagrams, even if they fulfilled the criterion of two-dimensionality. In other words, coding considered two-dimensionality as an inclusion criterion and matrices, tables, and non-mathematical content as exclusion criteria. We will further discuss these definitional choices in Sect. 5.

In cases where several typographically unconnected diagrammatic representations were juxtaposed on a page, we treated these as one diagram if there was a clear logical connection between them (e.g., temporal progression, the same diagram at different levels of zoom, or before and after the use of an operator), whereas we treated them as several diagrams if the distinct diagrams contained different information (e.g., showing different variations of a parameter or gradually changing to illustrate steps of inference).

Representations coded as mathematical diagrams were classified as either 'Unknown' or as belonging to one of the following three categories of diagrams (following Johansen et al., 2018):

Resemblance diagram: Diagrams that have a direct likeness (either geometric or topological) to the physical objects that the corresponding mathematical concepts are supposed to model.

Algebraized diagram: Shapes and figures drawn in an algebraized domain.

Abstract diagram: Diagrams that are only meaningful if the mathematical content they represent is understood through a particular conceptual map.

As further explained in Johansen et al. (2018), resemblance diagrams are representations that display some sort of geometric shape or recognizable gestalt that resembles members of the abstraction class of the mathematical objects represented. Prototypical examples of this diagram type are Euclidean diagrams (geometric resemblance) and knot diagrams (topological resemblance). Algebraized diagrams are representations in which a geometric shape or recognizable gestalt is presented in an algebraized context. Prototypical examples are drawings in a Cartesian coordinate system, including graphs of functions, although we use the category quite liberally, incorporating manifolds and diagrams in which the algebraization is only indirectly visible.[2] Finally, abstract diagrams are diagrams that do not resemble the mathematical objects they represent unless these objects are interpreted through a conceptual map or metaphor. Prototypical examples are commutative diagrams, in which sets are conceptualized as locations in space and mathematical maps are conceptualized as paths or fictive motions between such locations (see also Johansen, 2014).

This categorization seeks to capture the different cognitive roles diagrams can play in mathematical practice. From a cognitive point of view there is a clear difference between practices using, say, resemblance diagrams and practices using abstract diagrams, as the latter presupposes the use of conceptual maps and similar cognitive tools (see also Johansen and Misfeldt, 2018, for empirical examples). Another categorization could have been chosen—we do not consider this categorization to be the only right one, and we invite other researchers to try other categorizations of the diagrams in the corpus. We chose this particular categorization because it might give philosophically relevant information about possible changes and variations in the cognitive role played by diagrams in the period under investigation. Furthermore, the selected categories were broad enough to be feasible for use in large-scale hand coding.

The categorization was mainly based on the typographic appearance of the diagrams. For practical reasons, we could not analyze in depth all the diagrams we encountered, but if we were in doubt about the classification of a particular diagram, we consulted the mathematical context within which the diagram appeared and used that to inform our decision.

2.3 Procedure

Beginning with the analysis from Johansen (2014), we developed an initial, tentative version of the previously described classification scheme and tested it in a pilot study of *Annals of Mathematics* in which all papers published every tenth year in the period

[2] In Johansen et al. (2018) we named this category of diagrams "Cartesian diagrams", but we have chosen to change the name to stress that it is a general category and not a particular type of diagrams.

from 1885 to 2005 were coded and the diagrams classified. The pilot test is described and discussed in detail in Johansen et al. (2018). Based on the experience from the pilot study, the code book was adjusted, and the first author of this paper coded the full corpus of 2940 papers. During this first coding, 178 papers were coded as 'in doubt'—for the most part because the coder was in doubt about either the classification or the number of diagrams in the paper. All 178 papers were discussed between the first and the second author, and the codebook was adjusted accordingly. Although most cases of doubt were resolved, the code 'in doubt' was kept for a total of 38 papers. These papers are included in the final data set but clearly constitute a source of uncertainty.[3]

After the adjustment of the code book, the second author revisited all papers coded as containing diagrams and adjusted the classification where necessary such that all codes reflected the final code book. To test the reliability of the coding of papers containing diagrams, all papers coded as containing diagrams were re-coded by a prototype machine-learning agent trained to detect diagrams in mathematical texts, and all discrepancies between the human and the machine coding were inspected (see Sørensen & Johansen, 2020). This investigation led to the inclusion of 36 additional diagrams, corresponding to 0.6% of the total number of diagrams detected in the corpus. To test the reliability of coding for papers not containing diagrams, a random sample consisting of 100 of the papers coded as not containing diagrams in the first round of coding was compiled and recoded. Only one of these papers had to be reclassified as containing one diagram.

3 Results

3.1 Trends and changes in publication practice

In total, we investigated 55,466 pages distributed over 2940 papers. Of these papers, 668 contained at least one diagram, and in total, we coded 6001 representations as mathematical diagrams (see Table 1). Therefore, there were an average of 0.11 diagrams per page in the three journals, and, furthermore, a randomly selected paper had an 23% average probability of containing at least one diagram.

Nonetheless, these diagrams are not equally distributed in time or type. If we begin with the distribution over time, the main results shown in Table 1 can be represented with two different histograms. The first histogram (Fig. 1) represents the development of the average number of diagrams per page, and the second (Fig. 2) represents the development of the percentage of papers containing at least one diagram. Both histograms cover the aggregated data from all three journals in the period from 1885 to 2015.

These two ways of presenting the data reveal connected but slightly different aspects of the trends and changes in the norms governing mathematicians' publication practices. Figure 1 reveals the frequency of diagrams in mathematical publications and thereby tracks the overall prevalence of diagrams in published mathematical work.

[3] The 'in doubt' code is included in the raw data set available at https://doi.org/10.17894/ucph.6e18e1c7-7eef-4445-8d9c-3b4d38949079. The reader can thus see which concrete papers we were in doubt about and how we chose to code them.

Table 1 Number and types of diagrams in the aggregated corpus

Corpus			Statistics on diagrams of all types				Number of diagrams in each type			
Year	Papers	Pages	Number of diagrams	Papers with diagrams	Diagrams per page (average)	% of papers containing diagrams	Algebrized	Resemblance	Abstract	Unknown
1885	33	868	3	3	0.00	9,09	2	1	0	0
1890	28	667	52	4	0.08	14,29	30	21	1	0
1895	53	709	57	12	0.08	22,64	40	17	0	0
1900	55	783	38	11	0.05	20.00	24	14	0	0
1905	60	829	90	16	0.11	26.67	65	25	0	0
1910	52	713	5	5	0.01	9.62	5	0	0	0
1915	51	529	18	7	0.03	13.73	18	0	0	0
1920	74	848	13	5	0.02	6.76	11	2	0	0
1925	96	1342	32	8	0.02	8.33	23	9	0	0
1930	163	2065	35	12	0.02	7.36	22	13	0	0
1935	166	2205	58	6	0.03	3.61	40	12	6	0
1940	168	1888	47	14	0.02	8.33	36	7	1	3
1945	167	1828	46	11	0.03	6.59	34	10	2	0
1950	120	2458	62	16	0.03	13.33	35	8	19	0
1955	85	1982	88	16	0.04	18.82	15	0	73	0
1960	146	2207	140	28	0.06	19.18	13	1	126	0
1965	177	2528	193	34	0.08	19.21	28	9	156	0
1970	248	2993	377	61	0.13	24.60	81	67	229	0
1975	248	2587	281	48	0.11	19.35	108	29	142	2
1980	117	2546	640	49	0.25	41.88	183	133	324	0
1985	101	2135	443	39	0.21	38.61	145	16	282	0

Table 1 continued

Corpus			Statistics on diagrams of all types				Number of diagrams in each type			
Year	Papers	Pages	Number of diagrams	Papers with diagrams	Diagrams per page (average)	% of papers containing diagrams	Algebrized	Resemblance	Abstract	Unknown
1990	104	2484	344	44	0.14	42.31	74	84	185	1
1995	67	2116	293	27	0.14	40.30	72	29	192	0
2000	74	2929	430	34	0.15	45.95	184	58	187	1
2005	91	3957	362	45	0.09	49.45	133	114	114	1
2010	127	5659	719	68	0.13	53.54	319	25	373	2
2015	69	3591	1135	45	0.32	65.22	425	439	271	0
Total	2940	55,446	6001	668	0.11	22.72	2165	1143	2683	10

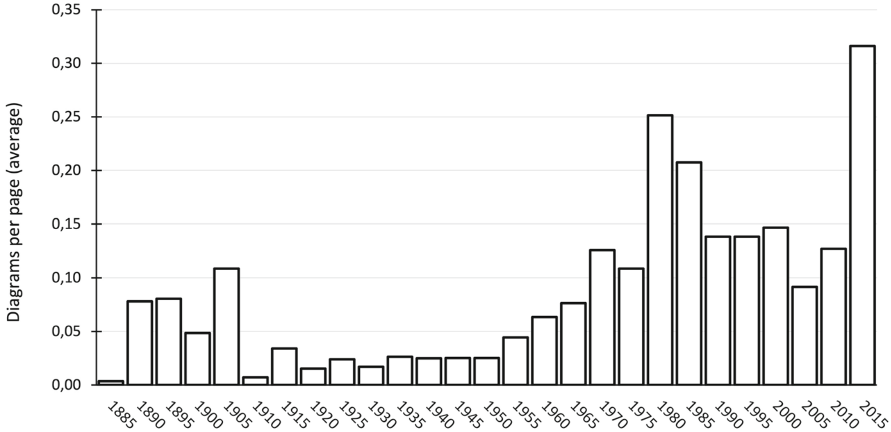

Fig. 1 Average number of diagrams per page in the aggregated corpus

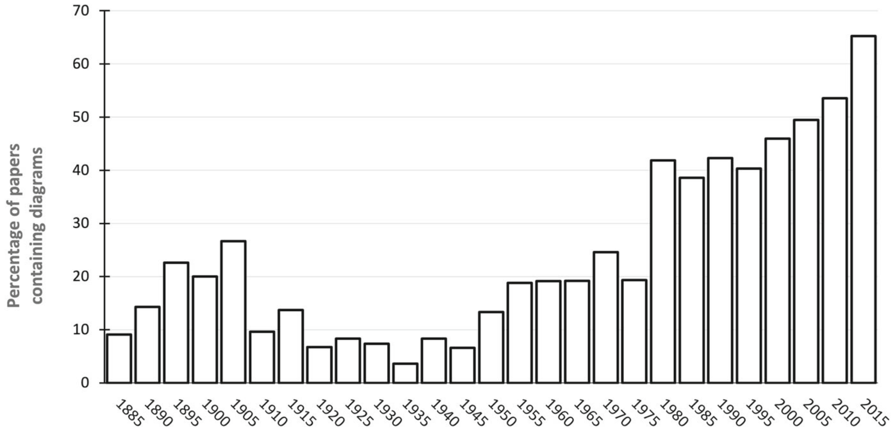

Fig. 2 Percentage of papers in the aggregated corpus containing at least one diagram

This number, however, is vulnerable to outliers in the form of papers containing unusually high numbers of diagrams. To visualize the development of the number of diagrams used in individual papers, we produced a bubble diagram (Fig. 3) displaying for every year in the corpus how many papers included a given number of diagrams. For instance, the bubble diagram shows that more than a third (242) of the 640 diagrams coded in 1980 originated in three papers ((Mandelbaum, 1980) with 109 diagrams, (Jungerman & Ringel, 1980) with 68 diagrams, and (Karoubi, 1980) with 65 diagrams). Similarly, the spike in the frequency of diagrams in 2015 is largely due to a single paper (Buch, 2015) containing almost 400 diagrams.

The second histogram (Fig. 2) does not have this problem since it does not track the number of diagrams. Figure 2 instead reflects the likelihood of authors of mathematical papers to decide to include diagrams in their papers at all. This number may also be subject to confounding factors. In particular, the average length of mathematical papers

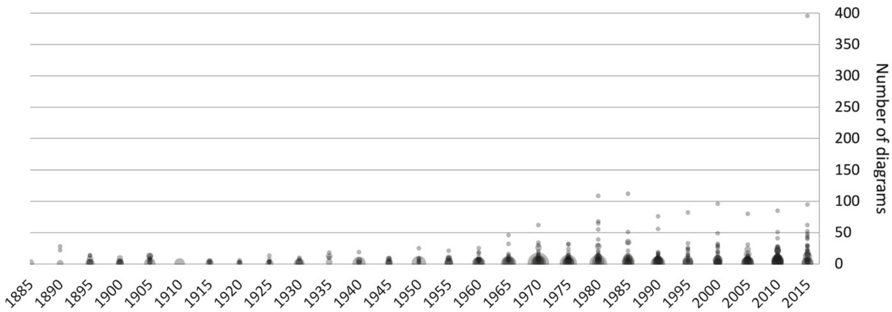

Fig. 3 Bubble diagram showing for every year in the corpus how many papers included a given number of diagrams; the years are indicted on the 1. axis, the number of diagrams on the 2. axis, and the number of papers containing a specific number of diagrams is indicated with the size of the bubble (only papers with one or more diagrams are included)

increased during the period we investigated (see Table 2), which might make authors more inclined to include diagrams, for instance because each individual paper covers more mathematical ground or simply because authors are allowed more space. However, statistical analysis shows that there is only a weak positive correlation between the average length of papers and the percentage of papers containing diagrams (with a correlation coefficient of 0.50). Of course, other factors besides changes in publication norms may have influenced the authors' choices regarding diagrams. We will return to these factors in the analysis and discussion below.

The results thus show that diagrams were relatively common in the period around 1900—with a peak in 1905 when 27% of all the papers in the corpus included at least one diagram. In the period from 1910 to 1950, diagrams almost disappeared from mathematical publications—with 1935 as the global minimum when only 4% of the papers in the corpus included diagrams. Beginning in 1950, diagrams became increasingly common, reaching a peak in 2015, when 65% of the papers in the corpus included at least one diagram. The combination of figures 1 to 3 furthermore shows that the surge in diagrams in the last half of the twentieth century was sparked by a few papers relying heavily on diagrams, after which the use of diagrams gradually became a common feature in research papers; the majority of papers included diagrams but generally in fewer numbers (with 2015 as the exception). Therefore, the impression we noted in the introduction that diagrams are more prevalent in recent publications is correct at least for the corpus we have investigated.

Turning to the changes in diagram type, the main results are shown in Fig. 4, which displays the relative frequency of the three main diagram categories. As is clear from the figure, the mid-century surge in diagrams that we saw in Figs. 1 and 2 is coextensive with a radical shift in the types of diagrams being used: whereas abstract diagrams are rare before 1950 (only 10 abstract diagrams are recorded), they constitute the vast majority of all diagrams in the period from 1955 to 1970, when a more diverse picture with all three categories of diagrams present in roughly equal amounts emerges. These results show that the mid-century return of diagrams is strictly speaking not a return of diagrams as such but rather the introduction of a qualitatively new type of diagram.

Table 2 Number of papers with diagrams compared to the average paper length (in pages) of all papers in the corpus

Year	1885	1890	1895	1900	1905	1910	1915	1920	1925	1930	1935	1940	1945	1950
Papers with diagrams	3	4	12	11	16	5	7	5	8	12	6	14	11	16
Average paper length	26	24	13	14	14	14	10	11	14	13	13	11	11	20
Year	1955	1960	1965	1970	1975	1980	1985	1990	1995	2000	2005	2010	2015	Total
Papers with diagrams	16	28	34	61	48	49	39	44	27	34	45	68	45	668
Average paper length	23	15	14	12	10	22	21	24	32	40	43	45	52	19

Fig. 4 Distribution of diagram types

3.2 Exploring abstract diagrams

The three diagram categories used in the coding are broad, and within each, several sub-categories can be singled out. The category we have dubbed 'abstract diagrams', for instance, covers well-known diagram types such as commutative diagrams, Dynkin diagrams, directed graphs, trees, and numerous diagram types that do not (to the best of our knowledge) have a generally accepted name. To explore the internal development within the abstract diagram category in more detail, we carried out an approximate count of the various types of diagrams falling under that category in the period from 1955 to 2015 (in 30-year intervals; see Table 3). For simplicity, we only distinguish between commutative or similar diagrams (e.g., exact triangles and representationally similar diagrams conceptualizing maps as arrows and sets as locations) and all other types of diagrams without delving into the specific types of other diagrams (examples of the 'other' types can be seen in Fig. 5, 6, 7, 8, 9, and 10).

The table indicates that the mid-century surge in abstract diagrams was largely due to the introduction of a single type of diagram, namely commutative diagrams (and small variations thereof). However, after this introduction, the hegemony of commutative diagrams seems to have been broken, and a larger diversity within the broader category of abstract diagrams developed. It should of course be noted that this result is only exploratory; it is difficult to quantify the diversity in diagram expressions beyond the

Table 3 Distribution of diagram subtypes within the category of 'abstract diagram' in the period 1955–2015

	1955	1985	2015
Total number of abstract diagrams	73	282	271
Commutative (and similar) diagrams	70 (96%)	251(89%)	205 (76%)
Number of subtypes of abstract diagrams	4	10	19

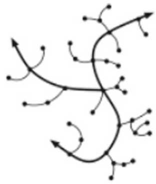

Fig. 5 An abstract diagram (reproduced with permission from (Bishop, 2015, p. 8))

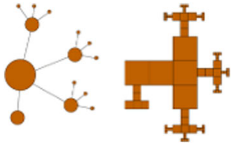

Fig. 6 An abstract diagram (reproduced with permission from (Drasin and Pankka, 2015, p. 224))

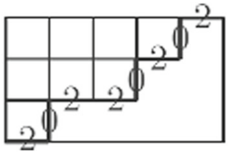

Fig. 7 An abstract diagram (reproduced with permission from (Buch, 2015, p. 181))

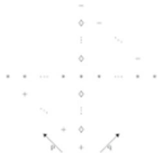

Fig. 8 An abstract diagram (reproduced with permission from (Dolgushev et al., 2015, p. 920))

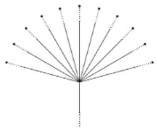

Fig. 9 An abstract diagram (reproduced with permission from (Payne, 2015, p. 230))

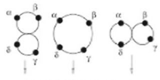

Fig. 10 An abstract diagram (reproduced with permission from (Smith, 2015, p. 443))

overall categories we have used, and a thorough investigation lies beyond the scope of this paper.

4 Analysis

As noted above, there is a characteristic U-shape in the two histograms (Figs. 1 and 2) indicating first a decline in the use of diagrams, then a period where diagrams were relatively uncommon (roughly 1910–1950), followed by a resurge in diagram use from 1950 onwards. The quantitative data available in this study cannot in itself explain the pattern we see in the data. Consequently, to examine some of the possible explanations for the development we will in the following examine three contextual factors connected to the use of diagram.

The increased use of diagrams in contemporary practice could be ascribed to the introduction of new technology that has made it easier for journals and mathematicians to typeset diagrams. Throughout the period under investigation there were technological development in typesetting of mathematical texts, such as the introduction of the 4-line system in the late 1950's and the graudual change to phototypesetting techniques during the 1960's (Knuth, 1979; Wishart, 2003). Although these innovations may go some way in explaining the mid-century increase in the use of diagrams they cannot explain the sudden drop we see around 1910. It is also noteworthy that the contemporary and ubiquitous typesetting system, TeX (and derivates like LaTeX and AMSTeX), which has made it feasible for working mathematicians to typeset certain types of diagrams with relative ease, was only introduced around 1980—that is, well after the initial increase in diagram use during the 1960s. In fact, the frequency of diagrams (understood as diagrams per page) seems to have dropped since the introduction of TeX. If anything, the increasing demand for an easy way to typeset diagrams could serve as a possible explanation for the success of TeX—not the other way around. Of course, this timeline does not mean that TeX is wholly unrelated to the variations in diagram use, a point to which we will return.

Variations in diagram use may also be the result of trends and changes in research agendas—perhaps agendas that depended more on the use of diagrams simply went out of fashion in the first half of the twentieth century, while (possibly new) diagram-dependent fields of research became popular around 1950. This is certainly a possibility. In connection to the data presented above the introduction of commutative diagrams in category theory in 1942 and the subsequent rise in interest for the field is especially worth noting (Krömer, 2007). However, changes in publication norms and research agendas are deeply interconnected: it would be difficult for a research area that depends heavily on diagrammatic representations to come into fashion if the publication norms did not allow the use of diagrams.

This brings us back to the dismissive attitude toward diagrams and other kinds of visualizations expressed by Hilbert, Pasch, Russell, and others in the late nineteenth century and early twentieth century. It is clear that the advent of what we have called "formalist ideology" coincides with the decline in diagram use observed around 1910. Furthermore, since formalist ideology directly attacks the use of visualizations, such as certain types of diagrams, there is a clear causal link between the two phenomena. So although you cannot infer causation from correlation the advent of formalism seems to be a reasonable partial explanation for the decline in diagram use we see on the data.

If we turn to the reappearance of diagrams in the 1950s and 1960s, the story is probably more complicated. As noted above, the mid-century surge in diagram use was mainly due to a new kind of diagrams: what we have called 'abstract diagrams'. Although we should be careful about generalizations since there is much variation within this category, abstract diagrams in general are removed from sensual content, and at least some central types of abstract diagrams allow for rigorous, rule-governed use—or can even be said to be part of a formalism. De Toffoli (2017), for instance, categorizes commutative diagrams as a 'hybrid notation' as the notation shares features with both textual and (traditional) diagrammatic notations and support both algebraic and geometric thinking. We believe similar analysis can be extended to other types of abstract diagrams, such as Dynkin diagrams, Young Tableaus, and several unnamed types, although we will not claim that it can be extended to all abstract diagrams due to the diversity of the category. The point is that a large majority of the abstract diagrams we see in our material support some form of algebraic thinking, a feature which indicates that they come closer to fulfilling the formalist requirements for rigor and emphasis on rule-governed syntactic manipulations of external signs than (most) other types of diagrams.[4] It is especially worth noticing that almost all the abstract diagrams used in the crucial years following 1950 were commutative diagrams (or close derivatives) with a clear formal nature. Abstract diagrams, on the other hand, are still diagrams and allow mathematicians the cognitive economy of visual reasoning, as well as the possibility of exploring and profit from analogies to everyday experiences [by activating the conceptual maps incorporated in the diagram design (e.g. Johansen et al., 2018)]. Thus, the mid-century rise in the use of diagrams was not simply the return of diagrams as we knew them before formalism, and the rise in diagram use does not necessarily imply that formalist ideology has disappear. Rather, in our interpretation the increased use of diagrams was in part the result of the introduction of a new and distinct type of diagram that allowed a compromise between formalist demands for rigor and the cognitive needs of mathematicians. The heavy use of commutative diagrams in the first decades after 1950 furthermore suggests that the development can also in part be attributed to the introduction of a new research agenda—category theory—that depended on exactly this type of diagram.

In the decades around 2000, an explosion of variety occurred both within and between the three main categories of diagrams. Between the categories, resemblance and algebraized diagrams appeared to grow in number (although more data points are needed to reach firm conclusions), and within these categories, our in-depth analysis of abstract diagrams (Table 3) showed a steep rise in the number of distinct types of diagrams falling under this category. As the instruments chosen for this large-scale survey made it difficult to quantify this variety in a meaningful way, we have not estimated the number of different types of diagrams used in the two other categories, but it is our clear impression that a similar growth in diversity occurred in algebraized and resemblance diagrams. Several explanations can be given for this surge in diagram diversity. That diagrams became more normal in the sense that more authors chose to include diagrams in their papers probably contributed to the increased variety of diagrams. The adoption of TeX and similar flexible typesetting systems during this period could also account for the rise in diagram diversity, but it lies beyond the scope of the methods adopted in this paper to investigate these hypotheses further.

Finally, our analysis of the data can be summarized with the following narrative: Diagrams were in use in the publications included in our corpus in the late nineteenth century, but with the advent of formalist ideology at the start of the twentieth century, they more or the less disappeared. In the 1950s diagrams started reappearing and they are a relatively common feature of the more contemporary papers in our corpus, with about two-thirds of the publications containing at least one diagram. The diagrams that reappeared in the 1950s and 1960s were of a qualitatively different kind than

[4] Although there are exceptions. For instance, we consider knot diagrams to be resemblance diagrams due to the clear topological resemblance with the objects they model, but it can be claimed that such diagrams can be used for rigorous, semi-syntactic manipulations (e.g. (Toffoli & Giardino, 2014)).

those that disappeared in the beginning of the century. Most of the new diagrams were abstract commutative diagrams (or closely related types). It can be argued that such diagrams live up to the strict demands for rigor imposed by formalism; diagrams in other words adapted to formalist ideology. Once these new diagrams were accepted in publication practice, they mutated (to borrow language from epidemiology) and spread, and in the beginning of the twentieth century we see an explosion of diagram variety and type, as well as a reemergence of diagrams from the otherwise abandoned categories of algebraized and resemblance diagrams.

Clearly, many details are missing from this rough story and, as indicated above, several in-depth investigations are required to complete this picture, especially concerning the development of typesetting techniques. However, as noted in the introduction, part of the attraction of large-scale quantitative investigations such as this one is precisely to point out relevant cases and questions for further in-depth qualitative investigation. With the questions suggested above, we consider this goal to have been reached.

5 Discussion

Taken at face value, these results only concern an isolated aspect of mathematical practice: namely, how mathematicians express and communicate their results. To assess the philosophical significance of the results, it is necessary to begin with a more general discussion of the roles external representations and notational choice play in mathematics.

It would be tempting to see representations as neutral tools that merely allow us to communicate thoughts and ideas that were worked out independently of the representations. Such a view, however, would critically underestimate the role played by representations, as a number of recent studies in philosophy of mathematical practice have made clear. Not only is our ability to anchor abstract and counterintuitive concepts, such as complex numbers, in external representations crucial for our ability to formulate and work with such concepts (Cruz & Smedt, 2013), but the choice of representation also has consequences for theory choice (Kjelsen, 2009) and conceptual development. For instance, in tool-object conversions, representations originally introduced purely as tools of investigation are turned into objects in their own right (Steensen & Johansen, 2016), and contingent aspects of the typographical design of a representation can inspire new concepts or open new avenues of investigation (Carter, 2010; Steensen & Johansen, pted). Representations, in other words, are not just neutral tools mathematicians use to investigate a pre-given reality but rather co-constitute the objects under investigation. They are not merely used to record results; they also shape in multiple ways the ideas and theories that they express. As Barany and MacKenzie conclude from an observation study of mathematical seminars: “Mathematical ideas are not pregiven as the universal entities they typically appear to be. The most important features of mathematics can be as ephemeral as dust on a blackboard” (Barany & MacKenzie, 2014, p. 124).

Representations also do not only serve as tools of communication. By allowing epistemic actions (Cruz & Smedt, 2013) and by serving as material anchors for conceptual structures, conceptual blends, and metaphors, they play a crucial role in the individual mathematicians' work process. This importance was exemplified in a recent interview study with active research mathematicians (Johansen et al., 2018). The study participants saw mathematical thinking as more or less synonymous with writing (Johansen et al., 2018, p. 6), and they described how the development of mathematical thought is done in close interaction with the representational tools available to them. One of the participants, for instance, said: "I have something in my head, but I need to write it down in order for it to be concrete and correct; that is, sometimes you have a wrong picture in your head…What you have in your head is an attempt to structure information. Or the beginning of it. And then you start writing it down, and it might not be exactly what you had expected. You need to change it before it works, or it might not work. That also happens" (Johansen et al., 2018, p. 8).

Representations thus play a crucial and central role in mathematical cognition and in the theoretical and conceptual development of mathematics, and it is in this light that the results reported in the present study should be understood. In fact, the overall result of the study confirms the importance of representations. Although our data covers a relatively short period, we see frequent shifts in representational practice and (since the 1950s) a dramatic development of new diagrammatic forms. It is difficult to imagine why mathematicians would spend the time and effort it takes to develop and publish new representations if notational choices were inconsequential and representations only played a secondary role in mathematical practice.

With the methods chosen for the investigation, we do not have direct access to the representations mathematicians use in their research practice, only to the representations they choose to publish. The relationship between public and private aspects of mathematical practice is not well understood, but there seems to be a discrepancy between the public and the private, particularly in the sense that mathematicians may refrain from publishing visualizations they have used in their research practice either because they feel forced to or because the visualizations are too idiosyncratic for others to follow. As noted in the introduction, mathematics has a front and a back (Hersh, 1991). On the other hand, the two spheres are not totally disconnected. There is a lot of room for representational innovation—and as noted above innovations are frequently made. Yet, our previous research suggests that mathematicians prefer to use generally accepted representational forms even in their own practice rather than making their own (Johansen & Misfeldt, 2016). This is presumably in part because of the labor involved in creating new notation and in part because the use of completely idiosyncratic representations would make collaboration with other mathematicians as well as the use of previously proved results and proof techniques difficult.[5] The choice of representation is thus in part a social choice, and the representational practice of the individual mathematician is to some extent entangled with publication norms and the general practice of the field.

In the light of the above we suggest that the trends and changes in publication practice demonstrated by our data not only reflect trivial changes in typesetting fashion or similar, but also indicate changes in the underlying cognitive practice of mathematics; mathematics itself changes when the representations mathematicians use are changed.

[5] We are indebted to one of the anonymous reviewers for pointing out the connection between representations and known results and proof tecniques.

Seen in this light, the phase shift in diagram types around 1950 is particularly interesting. It indicates a radical change in cognitive practice: a representational tool with new affordances (such as the improved capacity to support conceptual mapping) suddenly came to play a central role in mathematics. From our purely quantitative study, we cannot discern precisely how this change in tools affected cognitive practice, although it is apparent that something dramatic occurred.

Furthermore, the general development in publication practice is interesting because of the clear connection to formalism. Ideas have consequences, and the data shows that formalist ideology directly affected publication practices during the twentieth century, both in the disappearance of diagrams in the first half of the century and in the dramatic mid-century surge of a new type of diagrams that (to some extent) conformed to the ideals of formalism. As argued above these directly visible consequences indicate other, less visible consequences such as changes to mathematicians' cognitive practices. This episode illustrates how material and cognitive practices centered around the use of diagrams are intertwined with large-scale social negotiations and discussions, in this case converging on the advent of formalism. Ideas certainly matter, and the changes studied here serve as a reminder that careful reflection and discussion over meta-issues such as which representations to allow are not foreign to but an integral part of mathematical practice as well as ongoing discussion about the contentual development of the discipline.

Finally, we may return to the question: What is a diagram? Although this question was not a part of our original research question, we have had to address it in our empirical design for methodological reasons. We previously discussed how the concept of mathematical diagrams has been defined in several ways following different criteria. These range from the very inclusive, such as the C.S. Peirce-inspired functional definition, in which diagrams are defined as representations that make it possible to infer more information than went into their construction, and the more limited, such as Larkin and Simon's definition, in which diagrams are delineated as representations where information is organized in two dimensions.

For this investigation, Larkin and Simon's restrictive definition served as our starting point, as we perceived it to be best aligned with the way the concept is used in mathematical practice. We did, however, encounter several problematic representations. Tables and matrices are clearly two-dimensional, but they are not (in our experience) typically considered to be diagrams in mathematical practice. We also found a number of photos [e.g. Sattinger (1980, p. 780); see also Morgan (1990, p. 297) for a grey zone example], which are also two-dimensional representations but not diagrams. However, these representations raise a good question: where is the line separating diagrams and illustrations? If a photo is not a diagram, what about a computer-generated image of a complex surface (e.g. Hoffman & Meeks, 1990)? At the other end of the spectrum, it is similarly difficult to distinguish clearly between symbols and diagrams. As pointed out by Hilbert and Giaquinto, even pure symbolic derivations may in some sense be considered visualizations, and with the advent of hybrid notations such as commutative diagrams, the border between symbols and diagrams is difficult to discern.

Furthermore, we also encountered several one-dimensional representations with obvious diagrammatic appearances. The distinction between exact sequences and commutative diagrams is minimal, and the fact that one is considered a diagram (because it is two-dimensional) and the other is not (because it is one-dimensional) seems arbitrary. Yet we also encountered examples where all members of a representational type are considered diagrams, even though some are one-dimensional and others two-dimensional (e.g., Dynkin diagrams). In sum, the criteria of two-dimensionality suggested by Larkin and Simon is neither a sufficient nor a necessary condition for a representation to be a diagram in mathematical practice.

This should not be seen as a criticism of Larkin and Simon. Based on our experience, we do not believe it to be possible to formulate necessary and sufficient criteria to describe the concept of a mathematical diagram. In a scientific language it is an important goal to give clear and distinct definitions of basic concepts, but the concept of mathematical diagrams does not belong to a scientific language. It belongs to the everyday language of mathematical practice, and, like other concepts from everyday language, it is context dependent, governed by prototypes, and open ended: the concept develops with the practice of which it is a part, and, as we saw above, the use of representations in mathematical practice is in constant flux and development.

For the same reason, we would not consider definitions that confine diagrams to one of their function as fruitful. For instance, consider (as a hypothetical extreme) a definition focusing solely on diagrams' ability to support rule-governed manipulations. Although such a criterion would explain why, say, photos are not diagrams, it would also make it difficult to investigate the many actual and possible functions diagrams play in mathematical practice. Specifically, if diagrams are defined by their capacity to support syntactic manipulations, the difference between diagrams and symbols disappears, while, for example, some diagrams' capacity to anchor conceptual blending becomes difficult to understand and investigate (as argued in Johansen, 2014). The drastic development we saw in the types of diagrams being published suggests that a more flexible attitude is advisable.

In our investigation, we tried to keep an open mind about the definition of a diagram, and we handled the dilemmas stemming from an open definition with a pragmatic compromise. As indicated in our methods, we accepted two-dimensionality as a necessary, but not a sufficient, criterion for a representation to be a diagram. This means that we do not consider one-dimensional Dynkin diagrams as diagrams (so here we deviate from mathematical practice), but we also do not consider tables and matrices as diagrams (and here we follow the practice and deviate from the abstract criteria). This was of course a pragmatic choice—one that can and should be debated.

Conclusion

In conclusion, mathematics is not a purely mental activity. Mathematicians and mathematical activity depend heavily on and are shaped by external artifacts such as representational systems. We have used a limited section of current representational practice—mathematicians use of diagrams—to explore if and how the practice develops over time. The data shows that the use of diagrams varied heavily in the journals included in our corpus during the period under investigation, a development that we argue was influenced by the philosophical and ideological discussions about what mathematics were during this time. Mathematicians' use of diagrams thus constitutes a vertex where material, cognitive, and social aspects of mathematical practice meet.

Limitations

As described in Sect. 3, tests of the reliability of the coding were performed. Since the main purpose of this investigation is to track the overall trends and changes in publication practice and since the conclusions presented do not rely on detailed statistical analysis, we consider the tests made to be sufficient and the error margin indicated by the tests to be acceptable for the purpose of this paper.

The sampling of the papers included in the corpus constitutes another clear weakness. The sampling relies on the theoretical assumption that the papers published in only three journals give a reliable picture of the publication norms in mathematics. A bigger corpus, a sampling strategy based on random inclusion, or both methods might have made it possible to construct a corpus that reflected the average papers published in the field more accurately. Due to the amount of work such a sampling strategy would require, it was unfortunately not feasible for our investigation. Also, the question remains open of what constitutes the average mathematics paper, as there might be enormous differences between the different subfields of the discipline. In the current investigation, we attempted to avoid this problem altogether by focusing on 'general' journals only and avoiding the inclusion of subfields. Another, and perhaps better, strategy would be to investigate the subfields directly instead of considering mathematics as a monolith. Such a strategy, however, would be labor intense, although it might be feasible with the introduction of new digital tools (as envisioned in Sørensen and Johansen (2020)).

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