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A logic for factive ignorance

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Abstract
In the current debate there are two epistemological approaches to the definition of ignorance: the Standard View and the New View. The former defines ignorance simply as *not knowing*, while the latter defines it as the *absence of true belief*. One of the main differences between these two positions lies in rejecting (Standard View) or in accepting (New View) the factivity of ignorance, i.e., if an agent is ignorant of ϕ, then ϕ is true. In the present article, we first provide a criticism of the Standard View in favour of the New View. Secondly, we propose a formal setting to represent the notion of factive ignorance.

Keywords Factive ignorance · Agnoiology · Epistemic logic · Ignorance representation

1 Introduction

In the contemporary debate, there are two influential epistemological approaches to understanding the notion of ignorance: the *Standard View* and the *New View*.[1] The Standard View states that ignorance coincides with non-knowledge, thus the notion of ignorance is the complement to the notion of knowledge. This definition is often used in moral philosophy and in ethics, for instance by Zimmerman (1988), Driver (1989), Fields (1994) and Le Morvan (2011, 2012, 2013). One of the advantages of conceiving ignorance in this way is that it allows one to use all the theoretical and formal apparatus used to deal with knowledge for analyzing ignorance. The New

[1] We borrow this terminology from Le Morvan and Peels (2016).

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View considers that ignorance corresponds to the *absence of true belief*. This position is defended by Goldman (1986), Goldman and Olsson (2009), Guerrero (2007), van Woudenberg (2009), and Peels (2010, 2011, 2012). The crucial difference between the two positions lies in accepting (Standard View) or in rejecting (New View) two specific situations as genuine cases of ignorance: first, the case where justified true belief does not imply knowledge (this corresponds to the so-called Gettier examples, see Gettier 1963); second, the case of false propositions. Here we focus our attention on the latter and leave the former for further investigation. The remainder of this article is organized as follows. Section 2 develops a criticism of the Standard View based on the case of false propositions. As a consequence, we agree with the New View and reject the idea that ignorance simply corresponds to the complement of knowledge. However, one major shortcoming of the New View in its current form lies in its lack of a precise formal representation. Section 3 provides a formal setting, in which ignorance can be represented in accordance with the New View.

2 Factive ignorance

2.1 Criticism of the Standard View

One of the major disagreements between the Standard View and the New View lies in accepting or rejecting cases of false propositions as cases of ignorance. More precisely, the question is whether it is possible to be ignorant of a false proposition. According to the Standard View, everything that is not knowledge is ignorance. A consequence of this approach is that an agent is ignorant of every false proposition. To clarify the situation, let us consider the following example borrowed from Le Morvan and Peels (2016, pp. 22–23). Consider the following two propositions:

(1) The most popular paid iOS app of 2014 was *Heads Up!*
(2) The most popular paid iOS app of 2014 was NOAA Hi-Def Radar.

The proposition (1) is true whereas (2) is false. Next we consider the state of knowledge of King Herod I (73-4 BCE). Since he lived more than 2000 years ago, he was in no position to have any propositional attitude towards propositions (1) and (2). It is evident that he had no knowledge of these propositions, and thus, according to the Standard View, he was ignorant of the true proposition (1) as well as of the false proposition (2). This example is used by the adherents of the Standard View to show that ignorance of a false proposition is possible. However, the same example may be seen as a weakness of the Standard View. The reason why King Herod I has no knowledge of the truth of (1) and of the falsity of (2) is the same: he does not comprehend either proposition. Indeed, he does not understand what an iOS application is, what the year 2014 is, etc. Thus, according to the Standard View, incomprehension is a case of ignorance. This assumption is not so obvious as it seems and may be challenged. Suppose that an agent does not speak French at all. Thus the agent would not understand the proposition "Deux fois deux égale quatre" ("Two times two equals four," in English), but it is not excluded that the agent is not ignorant of the mathematical expression "$2 \times 2 = 4$." The Standard View should at least clarify the relationship between ignorance and understanding (or incomprehension).

Another weakness of the Standard View can best be shown by the use of simple logical analysis. In epistemic logic, it is standard to formalize statements about an agent's knowledge through the epistemic operator K; thus, for instance, the proposition "an agent knows that ϕ" is represented by $K\phi$. An advantage of the Standard View is that one can adopt the simple and elegant possible world semantics to modeling epistemic logic as a basis for representing ignorance (see Hintikka 1961, 1962, 1963). However, from this perspective, if ignorance is the complement of knowledge, then "an agent is ignorant of ϕ" should be expressed by $\neg K\phi$. Next, assume that knowledge is consistent, i.e., that if an agent knows a proposition, the agent does not know its negation. This can be formally expressed by the axiom D: $K\phi \rightarrow \neg K\neg\phi$. Let p indicate the proposition "Brasilia is the capital of Brazil," and assume that an agent knows the truth of p: Kp. Next, by applying the axiom D, we obtain $\neg K\neg p$, which means that the agent is ignorant of $\neg p$, where $\neg p$ expresses the false proposition "Brasilia is not the capital of Brazil." It seems unnatural to say that the agent is ignorant of this last false proposition just because the agent knows that the capital of Brazil is Brasilia. If the adherents of the Standard View want to avoid such counter-intuitive conclusions about an agent's ignorance, they should at least give up the consistency of knowledge, which is traditionally considered as one of the basic properties of the notion of knowledge.

A further argument against the Standard View can be put forward by adopting the *theory of presuppositions* (see Beaver and Geurts 2011). This theory is based on the idea that an agent often presupposes some kind of information. For instance, by writing "*this* article," we presuppose that the reader speaks English. Moreover, in this context, the word "this" permits to referring to the present article. The use of the word "this" presupposes the existence of that article. Words, the use of which presupposes some supplementary information, are called *presupposition triggers*. Here we are only interested in the verb "to know," which is a factive presuppositional trigger (see Kiparsky and Kiparsky 1970). The central notion of the presupposition theory is the notion of *projection*. As Langendoen and Savin (1971, p. 57), formulate it, "The projection principle for presuppositions, therefore, is as follows: presuppositions of a subordinate clause do not amalgamate either with presuppositions or assertions of higher clauses; rather they stand as presuppositions of the complex sentence in which they occur." For example, the proposition "This article is written in English." presupposes "This article is written." It is possible to construct a complex negative proposition from the initial one: "This article is not written in English." The presupposition of this last proposition remains the same: "This article is written."

It is easy to see that the Standard View is not compatible with the application of the projection principle. In particular, if one defines ignorance as not-knowing, then "a is ignorant of ϕ" should be interpreted as "a does not know that ϕ." The verb "to know" is a factive presuppositional trigger, i.e., if "a knows ϕ," then it is presupposed that ϕ is true. The negation applied to the verb "to know" is a projection of the proposition "a knows that ϕ." Hence, according to the projection principle, it presupposes the truth of ϕ. By bearing in mind that ϕ is an arbitrary proposition, we conclude that one cannot be ignorant of a false proposition whenever ignorance is defined as not-knowing. Recall that according to the Standard View, an agent is ignorant of all false propositions. This makes the application of the projection principle and the Standard View incompatible.

All the arguments we presented point to the idea that it might be interesting to separate situations of simple non-knowledge from situations of genuine ignorance. This would allow us to focus our attention on the notion of ignorance independently from the notion of knowledge. One of the aims of the New View is exactly to introduce such an analysis.

2.2 The New View and its limits

According to the New View, ignorance is the absence of true belief. Moreover, the adherents of the New View consider that one cannot be ignorant of a false proposition. In this way, the New View tries to salvage the *factive* character of the notion of ignorance. Factivity can be expressed as follows:

If an agent is ignorant of ϕ, ϕ is true.

It is clear that the example of being ignorant that Brasilia is not the capital of Brazil because of knowing that Brazil's capital is Brasilia is not problematic for the New View. The proposition about Brasilia not being the capital is not one about which one is ignorant, because it is false. Similarly, once ignorance is conceived of as a factive notion, all the other criticisms we put forward against the Standard View dissolve. Nonetheless, the factivity of ignorance becomes problematic as soon as one tries to capture it formally. If ignorance is identified with the absence of true belief, then it should be the complement of true belief. It is common to formalize the fact that an agent has a true belief of ϕ by using the operator B. More precisely, $B\phi \wedge \phi$ means that the agent believes ϕ and that ϕ is the case. Hence, ignorance should be formalized as $\neg(B\phi \wedge \phi)$. However, Le Morvan and Peels (2016, pp. 24–25) show that this formalization is problematic. Let p be a false proposition. Thus, $\neg p$ is true. Having $\neg p$, we can introduce a disjunction: $\neg Bp \vee \neg p$. According to De Morgan's laws, this last formula is equivalent to $\neg(Bp \wedge p)$. This, in turn, means that the agent is ignorant of p, according to the present formalization of ignorance. In our example, however, p is false. Thus, the falsity of p would be a sufficient condition for stating the ignorance of p. This contradicts the factive character of ignorance. To avoid this counter-example, the adherents of the New View should admit that the absence of true belief is not the complement of true belief. However, this would require some further clarifications of what is the *absence* of a true belief, and how such an understanding of ignorance may be implemented in a formal setting.

Adherents of the New View may propose a different formalization. Ignorance might be represented as the conjunction between a true proposition and the negation of the belief of this proposition, i.e., $\phi \wedge \neg B\phi$. However, such a formalization would not solve the problem of specifying what is the absence of true belief and how it comes about that it is not the complement of true belief, i..e., $\neg(B\phi \wedge \phi)$. Indeed, one may infer $\neg(B\phi \wedge \phi)$ from $\phi \wedge \neg B\phi$, but not vice versa.

Moreover, both the $\neg(\phi \wedge B\phi)$ and $\phi \wedge \neg B\phi$ suggest that ignorance should be represented in terms of doxastic logic. However, in our understanding, both the New

View and the Standard View interpret ignorance as an epistemic notion, and not as a doxastic one. On the one hand, according to the Standard View, ignorance is the complement of knowledge, not of belief. On the other hand, by defining ignorance as an absence of true belief, the New View permits one to distinguish situations of knowing, being ignorant, and both not-knowing and not being-ignorant. Such a distinction is granted by taking ignorance not as complementary to the notion of knowledge, but to the one of true belief, i.e., a specific part of the classical definition of knowledge as justified true belief. Ignorance in this case is an epistemic notion, whose definition is given in doxastic terms, as is the case for the definition of knowledge. However, the nature of ignorance is not the same as the nature of belief, and the nature of knowledge is different from the nature of belief as well.[2] From this perspective, ignorance should be represented independently of the B operator.

Let us consider a candidate for expressing ignorance in epistemic terms: $\phi \wedge \neg K\phi$. Indeed, such a formalization permits one to express the factive character of ignorance, as well as not leading us to a formal doxastic setting. However, such a formalization would not correspond to the New View. Let I be an ignorance operator defined by $\phi \wedge \neg K\phi$. In this case, knowledge can be defined by $\phi \wedge \neg I\phi$. Let us now consider a Gettier-style example, where a true proposition p is not known. Thus we have $\neg Kp$, that is, $\neg(p \wedge \neg Ip)$. By De Morgan's laws and negation elimination we obtain $\neg p \vee Ip$. The agent is ignorant of proposition p since p is true. However, according to the New View, Gettier-style examples are not cases of ignorance, nor are they are cases of knowledge (see for instance Peels 2011). Hence, the formalization $\phi \wedge \neg K\phi$ is not suitable for representing ignorance in accordance with the New View. Moreover, any use of the K operator in defining ignorance brings us back to a mismatch between considering that ignorance is the absence of true belief and its formal representation in terms of knowledge. A possible way out of this impasse is to represent ignorance as a primitive operator. Indeed, the problem of Gettier examples and the definition of knowledge is bypassed in epistemic logic by representing knowledge by the use of a primitive and independent operator K. In the same way, we suggest that ignorance should also be represented by a primitive and independent operator I. This strategy allows one to put aside the problem of specifying what is the absence of true belief, as well as the problem of Gettier-style examples.

On the basis of the foregoing analysis, it is possible to identify three desiderata that must be met by a formal representation of ignorance in accordance with the New View. First, ignorance must be factive. Second, ignorance must be introduced via a primitive operator, i.e., it should not be introduced by a complex formula containing other operators. Third, ignorance must be represented independently of the operators K and B, i.e., it should not be in principle reducible to formulas containing K or B modalities. In the next section we introduce a logic that satisfies all these desiderata.

[2] The difference between the nature of knowledge and belief is acknowledged by many authors, see e.g., Williamson (2000).

3 Syntax and semantics

In accordance with the Standard View, traditional epistemic logic treats ignorance in terms of knowledge with two noteworthy exceptions: the systems **Ig** and **LUT**. The system **Ig**, introduced by van der Hoek and Lomuscio (2004), contains an ignorance operator as the sole primitive modality. To the best of our knowledge, **Ig** is the first logical system explicitly formalizing ignorance through a primitive operator I. However, this system does not contribute to investigations of the notion of ignorance by itself, but rather to the discussion of (non-)contingency logics, where the contingency of a formula ϕ means that both ϕ and $\neg\phi$ are possible (see Humberstone 2013 and references therein). Ignorance in this case is considered as the epistemic counterpart of contingency, and thus not factive. The system **LUT** can be found in Steinsvold (2008), and its Kripke semantics can be found in Gilbert and Venturi (2016). It contains a special consistency operator $\bullet$ which is intended to formalize "is true, but not known." Clearly, the $\bullet$ operator is factive. However, $\bullet\phi$ is defined by $\phi \wedge \neg K\phi$, i.e., ϕ is true and not known. This means that $\bullet$ is not independent of K.

In this section we introduce the system **ELI** (Epistemic Logic of Ignorance) and prove its completeness.[3] This system represents factive ignorance and allows one to conceive ignorance independently of knowledge.

The language $\mathcal{L}$ is defined by the following grammar:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid I\phi$$

Other propositional operators can be defined as usual: $\phi \vee \psi \Leftrightarrow \neg(\neg\phi \wedge \neg\psi)$; $\phi \rightarrow \psi \Leftrightarrow \neg\phi \vee \psi$; and $\phi \leftrightarrow \psi \Leftrightarrow (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$. The formula $I\phi$ has to be read as "the agent is ignorant that ϕ."

The language $\mathcal{L}$ is interpreted by standard Kripke semantics.

Definition 1 (*Frames, models, and satisfaction*) A Kripke Frame $\mathcal{F} = (W, R)$ is a tuple where W is a set of possible worlds, and $R \subseteq W \times W$ is an accessibility relation. A Kripke Model $\mathcal{M} = (\mathcal{F}, v)$ is a tuple where $\mathcal{F}$ is a Kripke frame and $v \colon P \rightarrow 2^W$ is an interpretation for a set of propositional variables P. Given a model $\mathcal{M}$ and a formula ϕ, we say that ϕ is true in $\mathcal{M}$ at world w, written $\mathcal{M}, w \models \phi$ if:

- $\mathcal{M}, w \models p$ if $w \in v(P)$,
- $\mathcal{M}, w \models \neg\phi$ if $M, w \not\models \phi$',
- $\mathcal{M}, w \models \phi \wedge \psi$ if $\mathcal{M}, w \models \phi$ and $\mathcal{M}, w \models \psi$,
- $\mathcal{M}, w \models I\phi$ if for all w' that are not w and such that Rww', $\mathcal{M}, w' \not\models \phi$ and $\mathcal{M}, w \models \phi$.

We say that ϕ is valid on $\mathcal{M}$ and write $\mathcal{M} \models \phi$ if $\mathcal{M}, w \models \phi$ for all w in W. If for all $\mathcal{M}$ based on $\mathcal{F}$ we have $\mathcal{M} \models \phi$, we say that ϕ is valid on $\mathcal{F}$ and write $\mathcal{F} \models \phi$.

The accessibility relation R from a world w to w' can be understood as a weakening of the indistinguishability relation for standard epistemic logic. The standard relation is

[3] To be precise, **ELI** does not correspond to an epistemic logic, simply because the central notion involved is not the one of *epistémě*. Since the theory of ignorance is called agnoiology (from the Greek *ágnoia*) by Ferrier (1856), the logic we are going to introduce would be better characterized as an *agnotic logic*. However, here we stick to the standard terminology.

described by Lewis (1986, p. 27) as follows: "The content of someone's knowledge of the world is given by his class of epistemically accessible worlds. These are the worlds that might, for all he knows, be his world; world W is one of them iff he knows nothing, either explicitly of implicitly, to rule out the hypothesis that W is the world where he lives." We say that a world w' is *weakly indistinguishable* from w if, and only if, from all of that which an agent is not ignorant, the agent cannot rule out the hypothesis that w' is the actual world. The main difference between indistinguishability and weak indistinguishability hinges on the possibility of distinguishing between knowing and not being-ignorant. According to the Standard View, there is no distinction to be made between knowledge and non-ignorance, as the notion of ignorance is the complement of the notion of knowledge. However, from the perspective of the New View, knowledge is included in non-ignorance, but not vice versa. In this case, the agent distinguishes worlds not only on the basis of what the agent knows, but also on the basis of any hypothesis the agent might have.

Following this explanation, an agent is ignorant of ϕ in a world w when first, ϕ is true in w, and, second, for all worlds different from w such that they are weakly indistinguishable for the agent, ϕ does not belong to these worlds, i.e., if ϕ is not part of the agent's knowledge or hypotheses.

Let us introduce the system **ELI**.

Definition 2 (*Proof system* **ELI**) The proof system **ELI** consists of the following axiom schemes and inference rules:

$(TAUT)$	All instances of propositional tautologies
(Ie)	$I\phi \to \phi$
$(I\wedge)$	$(I\phi \wedge I\psi) \to I(\phi \vee \psi)$
(MP)	From ϕ and $\phi \to \psi$ infer ψ
$(I \leftrightarrow)$	From $\phi \leftrightarrow \psi$ infer $I\phi \leftrightarrow I\psi$
(Imp)	From ϕ and $\phi \to \psi$ infer $I\psi \to I\phi$

A derivation of **ELI** is a finite sequence of $\mathcal{L}$-formulas such that each formula is either the instantiation of an axiom or the result of applying an inference rule to previous formulas in the sequence. A formula $\phi \in \mathcal{L}$ is called a theorem, denoted $\vdash \phi$, if it occurs in a derivation of **ELI**.

The axiom scheme (Ie) represents the factive character of the operator I, i.e., if an agent is ignorant of ϕ, then ϕ is true. The axiom scheme $(I\wedge)$ stresses that if an agent is ignorant of ϕ and of ψ, then the agent is ignorant of either ϕ or ψ. This axiom scheme may be also understood in its contrapositive form: $\neg I(\phi \vee \psi) \to (\neg I\phi \vee \neg I\psi)$, i.e., if an agent is not ignorant that ϕ or ψ, then either the agent is not ignorant that ϕ or the agent is not ignorant that ψ. The rule $(I \leftrightarrow)$ is an introduction of the operator I over equivalences, i.e., whenever ϕ is equivalent to ψ, being ignorant of ϕ is equivalent to being ignorant of ψ. The rule (*Imp*) permits introducing the operator I over implication: if ϕ is true and it implies ψ, then being ignorant of ψ implies being ignorant of ϕ. Note that the truth of ϕ is a necessary premiss, because $\phi \to \psi$ can be true in two cases: either ϕ is false, or ψ is true. However, when ϕ is false, the agent cannot be ignorant of it.

In order to prove the completeness of **ELI**, first we show that the system is sound, secondly we prove completeness via the construction of a canonical model.

Lemma 1 (Soundness) *The system* ***ELI*** *is sound with respect to the class of all frames.*

Proof The soundness of **ELI** follows immediately from the validity of the axioms (Ie), $(I\wedge)$ and the inference rules $(I\leftrightarrow)$, (Imp).

For (Ie), assume that for some $(\mathcal{M}, w)$ $\mathcal{M}, w \models I\phi$. By definition of I (see Definition 1), we have $\mathcal{M}, w \models \phi$.

For $(I\wedge)$, assume that (i) $\mathcal{M}, w \models I\phi \wedge I\psi$, but (ii) $\mathcal{M}, w \not\models I(\phi \vee \psi)$. From (i), by Definition 1, we obtain (iii) for all w' that are not w and such that Rww', $\mathcal{M}, w' \not\models \phi$, (iv) $\mathcal{M}, w \models \phi$, (v) for all w'' that are not w and such that Rww'', $\mathcal{M}, w'' \not\models \psi$ and (vi) $\mathcal{M}, w \models \psi$. From (ii) we obtain that either (vii) there exists a w''' that is not w such that Rww''', and that $\mathcal{M}, w''' \models \phi \vee \psi$, or (viii) $\mathcal{M}, w \not\models \phi \vee \psi$. But (vii) contradicts (iii) and (v), and (viii) contradicts (iv) and (vi).

For $(I\leftrightarrow)$, assume that for some $(\mathcal{M}, w)$, (i) $\mathcal{M}, w \models \phi \leftrightarrow \psi$ and (ii) $\mathcal{M}, w \not\models I\phi \leftrightarrow I\psi$. Then, either (iii) $\mathcal{M}, w \not\models I\phi \rightarrow I\psi$, or (iv) $\mathcal{M}, w \not\models I\psi \rightarrow I\phi$. Suppose (iii). Then, (v) $\mathcal{M}, w \models I\phi$ and (vi) $\mathcal{M}, w \not\models I\psi$. Then, by Definition 1, (vii) for all w' that are not w and such that Rww', $\mathcal{M}, w' \not\models \phi$ and (viii) $\mathcal{M}, w \models \phi$; and either (ix) there exists a w'' that is not w such that Rww'', and that $\mathcal{M}, w'' \models \psi$, or (x) $\mathcal{M}, w \not\models \psi$. By (i) and (viii), (x) is contradictory, as well as (vii) and (ix). A contradiction from (iv) can be obtained similarly.

For (*Imp*), assume that for some $(\mathcal{M}, w)$ (i) $\mathcal{M}, w \models \phi$, (ii) $\mathcal{M}, w \models \phi \rightarrow \psi$, (iii) $\mathcal{M}, w \models I\psi$ and (iv) $\mathcal{M}, w \not\models I\phi$. Then, from (iii) and the definition of I, we have (v) for all w' that are not w and such that Rww', $\mathcal{M}, w' \not\models \psi$ and (vi) $\mathcal{M}, w \models \psi$. From (iv) we have either (vii) there exists a w'' that is not w such that Rww'', and that $\mathcal{M}, w'' \models \phi$, or (viii) $\mathcal{M}, w \not\models \phi$. By (ii), it is clear that (vii) contradicts (v), although (viii) contradicts (i). □

By using the rule $(I\leftrightarrow)$ it is possible to show that the inference rule *Substitution of equivalents* (*SUB*) is admissible in **ELI**.

Proposition 1 *The rule*

(*SUB*) *From* $\phi \leftrightarrow \psi$*, infer* $\chi[\phi/p] \leftrightarrow \chi[\psi/p]$

is admissible in ***ELI***.

The proof is by a straightforward induction on the length of the formulas and can be omitted. Note that in this proof it is standard to use the K-axiom scheme,[4] which is not valid in **ELI**. In our system, the inference rule $(I\leftrightarrow)$ fulfils the role of the K-axiom scheme and is thus crucial for proving the admissibility of (SUB).

In order to provide a proof of the Truth-Lemma (Lemma 2), we need to prove a generalization of the axiom scheme $(I\wedge)$ for a number n of conjuncts.

Proposition 2 *For all* $n \geq 1$:

$$\vdash (I\phi_1 \wedge \cdots \wedge I\phi_n) \rightarrow I(\phi_1 \vee \cdots \vee \phi_n)$$

Proof We prove the proposition by induction on n.

[4] In terms of I this would be $I(\phi \rightarrow \psi) \rightarrow (I\phi \rightarrow I\psi)$.

(Basic step) $\vdash (I\phi_1 \wedge I\phi_2) \rightarrow I(\phi_1 \vee \phi_2)$ is obtained by an instantiation into the axiom scheme $(I\wedge)$.
(Inductive step) Assume by induction hypothesis (IH) that the proposition holds for $n = k$. We show that:

$$\vdash (I\phi_1 \wedge \cdots \wedge I\phi_{k+1}) \rightarrow I(\phi_1 \vee \cdots \vee \phi_{k+1})$$

1. $(I\phi_1 \wedge \cdots \wedge I\phi_k) \rightarrow I(\phi_1 \vee \cdots \vee \phi_k)$ (IH)
2. $I\phi_{k+1} \rightarrow I\phi_{k+1}$ (TAUT)
3. $(((I\phi_1 \wedge \cdots \wedge I\phi_k) \rightarrow I(\phi_1 \vee \cdots \vee \phi_k)) \wedge (I\phi_{k+1} \rightarrow I\phi_{k+1}))$
$\rightarrow ((I\phi_1 \wedge \cdots \wedge I\phi_k \wedge I\phi_{k+1}) \rightarrow (I(\phi_1 \vee \cdots \vee \phi_k) \wedge I\phi_{k+1}))$ (TAUT)
4. $((I\phi_1 \wedge \cdots \wedge I\phi_k) \rightarrow I(\phi_1 \vee \cdots \vee \phi_k)) \wedge (I\phi_{k+1} \rightarrow I\phi_{k+1})$ (*TAUT*, 1., 2.)
5. $(I\phi_1 \wedge \cdots \wedge I\phi_k \wedge I\phi_{k+1}) \rightarrow (I(\phi_1 \vee \cdots \vee \phi_k) \wedge I\phi_{k+1})$ $(MP$, 3., 4.)
6. $(I(\phi_1 \vee \cdots \vee \phi_k) \wedge I\phi_{k+1}) \rightarrow I(\phi_1 \vee \cdots \vee \phi_k \vee \phi_{k+1})$ $(I\wedge$, 5.)
7. $((I\phi_1 \wedge \cdots \wedge I\phi_k \wedge I\phi_{k+1}) \rightarrow (I(\phi_1 \vee \cdots \vee \phi_k) \wedge I\phi_{k+1}))$
$\rightarrow (((I(\phi_1 \vee \cdots \vee \phi_k) \wedge I\phi_{k+1}) \rightarrow I(\phi_1 \vee \cdots \vee \phi_k \vee \phi_{k+1}))$
$\rightarrow ((I\phi_1 \wedge \cdots \wedge I\phi_k \wedge I\phi_{k+1}) \rightarrow I(\phi_1 \vee \cdots \vee \phi_k \vee \phi_{k+1})))$ (*TAUT*)
8. $((I(\phi_1 \vee \cdots \vee \phi_k) \wedge I\phi_{k+1}) \rightarrow I(\phi_1 \vee \cdots \vee \phi_k \vee \phi_{k+1}))$
$\rightarrow ((I\phi_1 \wedge \cdots \wedge I\phi_k \wedge I\phi_{k+1}) \rightarrow I(\phi_1 \vee \cdots \vee \phi_k \vee \phi_{k+1}))$ $(MP$, 5., 7.)
9. $(I\phi_1 \wedge \cdots \wedge I\phi_k \wedge I\phi_{k+1}) \rightarrow I(\phi_1 \vee \cdots \vee \phi_k \vee \phi_{k+1})$ $(MP$, 6., 8.) □

Let Γ be a set of $\mathcal{L}$ formulas. We say that Γ is *consistent* if $\Gamma \nvdash \phi \wedge \neg\phi$. Γ is *maximal* if for every $\phi \in \mathcal{L}$, we have $\phi \in \Gamma$ or $\neg\phi \in \Gamma$. Now the canonical model of **ELI** can be constructed.

Definition 3 (*Canonical model*) The canonical model $\mathcal{M}^C$ of **ELI** is the triple (W^C, R^C, v^C), where:

- $W^C = \{w \mid w$ is a maximal consistent set of **ELI**$\}$;
- $R^C ww'$ iff for some ϕ:

  a. $\phi \in w$ and
  b. for all ϕ: if $I\phi \in w$ and w' is not w then $\phi \notin w'$;

- $v^C(p) = \{w \in W^C \mid p \in w\}$.

Every consistent set of **ELI** can be extended to a maximal consistent set of **ELI** (Lindenbaum's Lemma) in a standard way. Let us call the resulting set **ELI**-maximal consistent set.

Lemma 2 (Truth-Lemma) *For all formulas ϕ, and all **ELI**-maximal consistent sets w,*

$$\mathcal{M}^C, w \models \phi \text{ iff } \phi \in w.$$

Proof We prove the lemma by induction on the structure of ϕ. The proof for basic propositional connectives is standard and thus is omitted. We only prove the induction hypothesis for the case of $I\phi$: $\mathcal{M}^C, w \models I\psi$ iff $I\psi \in w$.

($\Rightarrow$) Suppose $I\psi \notin w$. Then $\neg I\psi \in w$ by the consistency of w. We need to show that either there exists a w' that is not w such that $R^C ww'$ and $\psi \in w'$, or $\psi \notin w$. We will prove the first part of the above disjunction under the condition that $\psi \in w$. By observing that every consistent set can be extended to a maximal consistent set (Lindenbaum's Lemma) and the definition of R^C, it is enough to show that the set $\{\psi\} \cup \{\neg\chi \mid I\chi \in w\}$ is consistent.
Suppose that the set is inconsistent. Then there exist $\chi_1, \ldots, \chi_n$ such that $\vdash (\neg\chi_1 \wedge \cdots \wedge \neg\chi_n) \to \neg\psi$, that is, by contraposition and De Morgan's laws, $\vdash \psi \to (\chi_1 \vee \cdots \vee \chi_n)$. By having $\psi \in w$, we obtain by (Imp) $I(\chi_1 \vee \cdots \vee \chi_n) \to I\psi \in w$. By our construction, we have $I\chi_1 \wedge \cdots \wedge I\chi_n \in w$. Thus, by Proposition 2, $I(\chi_1 \vee \cdots \vee \chi_n) \in w$. But this brings us to a contradiction: $I\psi \in w$.
We conclude that either there exists a w' that is not w such that $R^C ww'$ and $\psi \in w'$, or $\psi \notin w$, i.e., $\mathcal{M}^C, w \not\models I\psi$.

($\Leftarrow$) Let $I\psi \in w$. By the definition of R^C (see Definition 3) if $I\psi \in w$, then whenever $R^C ww'$ and w' is not w, we have $\psi \notin w'$. According to the induction hypothesis, the last one means that whenever $R^C ww'$ and w' is not w, $\mathcal{M}^C, w' \not\models \psi$. That is, for all w' that are not w and such that $R^C ww'$, $\mathcal{M}^C, w' \not\models \psi$. This one, taken together with $\psi \in w$ (obtained by axiom scheme (Ie) from $I\psi \in w$), gives $\mathcal{M}^C, w \models I\psi$ (by Definition 1). □

Theorem 1 (Completeness) *The system* **ELI** *is sound and complete with respect to the class of all frames:* $\vdash \phi$ *iff* $\models \phi$.

Proof Soundness was proved in Lemma 1. Completeness follows in a standard way from the Truth-Lemma. Suppose $\nvdash \phi$. Then $\neg\phi$ is consistent, given a maximal consistent set w with $\neg\phi \in w$. By the Truth-Lemma we have $\mathcal{M}^C, w \models \neg\phi$, i.e., $\not\models \phi$.□

It is straightforward to see that **ELI** already satisfies two out of the three desiderata we mentioned at the end of Sect. 2. First, the axiom scheme (Ie) makes explicit the fact that ignorance is factive. Second, I is the only primitive modal operator in the system. We now prove that **ELI** satisfies also the third desideratum: I is independent of K.

Using the method of bisimulations[5] we prove that I cannot be defined in terms of the standard operator K. Let $\mathcal{L}_K$ be a language for standard epistemic logic that contains propositional variables, negation, conjunction and the operator K, and let this language be interpreted by a standard $S5$ model. Consider the models of Fig. 1. From the perspective of $S5$, the models $\mathcal{M}_1$ and $\mathcal{M}_2$ are bisimilar, i.e., they contain the same atomic information and their worlds verify the same formulas. Hence, these models are indistinguishable for K. From this, it immediately follows that I cannot be defined in terms of K, since it distinguishes models $\mathcal{M}_1$ and $\mathcal{M}_2$. For instance, $\mathcal{M}_1, w_1 \models Ip$, but $\mathcal{M}_2, w_2 \not\models Ip$. The same method can be applied to prove that I is independent of B.

5 For more details on this method, see van Ditmarsch et al. (2008, chapter 8).

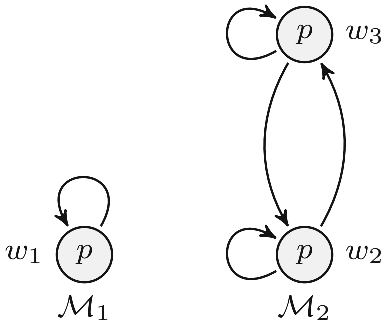

Fig. 1 Models $\mathcal{M}_1$ and $\mathcal{M}_2$

4 Conclusions and future work

In this article we have discussed two possible epistemological approaches to the notion of ignorance: the Standard View and the New View. We provided various criticisms of the Standard View and, in accordance with the New View, we argued that the common root of such criticisms is the lack of an account in which ignorance is factive. To represent the factive character of the notion of ignorance, we introduced the system **ELI** and showed its completeness. We consider **ELI** as a legitimate candidate to represent (at least some) cases of ignorance as conceived by the New View.

To the best of our knowledge, the operator I we have introduced is an original modal operator, not yet considered in the literature. We proved that I is not definable in terms of the operators K and B. From this perspective, the introduction of a multi-modal system in which K and I can be independently defined would contribute both to the logical and epistemological analysis of the relationships between ignorance and knowledge. Such a system can be introduced in a standard way, by fusion of two models for I and K. However, the advantage of independence of I from K is that it allows one to introduce a model with a single accessibility relation both for I and K. Another interesting development of the current work would be the introduction of a multi-agents system for ignorance in which agents can share their ignorance. We leave these tasks for future investigations.

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