Synthese (2018) 195:1509–1527
https://doi.org/10.1007/s11229-016-1278-7

Husserl on completeness, definitely

Mirja Hartimo[1]

Received: 4 March 2016 / Accepted: 15 November 2016 / Published online: 28 November 2016

Abstract The paper discusses Husserl's notion of definiteness as presented in his Göttingen Mathematical Society Double Lecture of 1901 as a defense of two, in many cases incompatible, ideals, namely full characterizability of the domain, i.e., categoricity, and its syntactic completeness. These two ideals are manifest already in Husserl's discussion of pure logic in the *Prolegomena*: The full characterizability is related to Husserl's attempt to capture the interconnection of things, whereas syntactic completeness relates to the interconnection of truths. In the *Prolegomena* Husserl argues that an ideally complete theory gives an independent norm for objectivity for logic and experiential sciences, hence the notion is central to his argument against psychologism. In the Double Lecture the former is captured by non-extendibility, that is, categoricity of the domain, from which, so Husserl assumes, syntactic completeness is thought to follow. In the so-called 'mathematical manifolds' the expressions of the theory are equations that are reducible to equations between elements of the theory. With such an equational reduction structure individual elements of the domain are given criteria of identity and hence they are fully determined.

Keywords Husserl · Completeness · Definiteness · Formalization · Psychologism

In 1901 Klein and Hilbert invited Husserl to present two lectures, now called a 'Double Lecture', in the Göttingen Mathematical Society. The topic of the lectures was definiteness of axiomatic systems. The exact meaning of Husserl's 'definiteness' has been a topic for much controversy (Hill 1995; Majer 1997; Da Silva 2000; Hartimo 2007;

✉ Mirja Hartimo
mirjahartimo@gmail.com

1 School of Economics and Business, Norwegian University of Life Sciences, Aas, Norway

Centrone 2010; Okada 2013; Hartimo and Okada 2016, etc.). Most recently, Jairo da Silva (2016) has argued that Husserl's definiteness should be understood as syntactic completeness, i.e., that the theory can prove for every sentence in the language of the theory either it or its negation. According to him, the notion best captures Husserl's philosophical intentions. In this paper, I will take up da Silva's challenge and argue that syntactic completeness alone cannot account for Husserl's philosophical intentions. An analysis of Husserl's writings shows that Husserl also intended to capture a full characterization of the *domain* of the theory in question. He wanted the axiomatic theories to be categorical, i.e., to define the intended structure uniquely, so to say 'up to isomorphism', so that any two of its realizations are isomorphic to each other. This is a semantic property, which to Husserl entails, and sometimes is conflated with, syntactic completeness. Husserl's definite theories thus aim to embrace two ideals: full description of a structure *and* syntactic completeness.[1] Such aims are typical for the kind of mathematics of pure structures that Husserl was advocating. It was only some years later that the needed concepts were in place for mathematicians to distinguish the two ideals and consider their mutual compatibility or incompatibility. Furthermore, many definite manifolds are what Husserl calls 'mathematical manifolds,' discussed in Hartimo and Okada (2016). The elements of the 'mathematical manifolds' are further determined by an equational reduction structure. In it the complex expressions, Husserl's example is '18 + 48,' are mechanically reducible to irreducible elements of the domain. Thus in the case of mathematical manifolds, the equational reduction gives criteria of identity to the elements of the domain. Accordingly, the present paper amounts to a synthesis of Hartimo (2007) and Hartimo and Okada (2016). Whereas the former argues for definiteness as categoricity, the latter explains the equational reduction structure of the 'mathematical manifolds'. The present paper shows how Husserl combines these two strategies in his aspiration to fully determine abstract objects.

To show how all this conforms to Husserl's philosophical views and for example how it relates to his argument against pscyhologism in logic I will first discuss Husserl's view of the idea of logic and the role of the concept of definiteness in Husserl's philosophy especially around the turn of the century.

1 The philosophical context of the concept of definiteness

Husserl discussed the notion of the definite manifolds already at the beginning of the 1890's (*Ideen I*, §72, in Husserl 1950, p. 168, n1, English translation, p. 164, n17), and possibly already before the publication of his *Philosophie der Arithmetik* (1891) (for the early development of the notion, see Hartimo and Okada 2016). The notion remains central to him ever since: It is discussed in the *Ideen I* (*Ideen I*, §72), where it is presented as an ideal norm for scientific rationality. He writes, for example, that "the closer an experiential science comes to the 'rational' level, the level of 'exact,' of

[1] Hintikka (1998) has put these ideals in terms of two functions of logic for mathematics, namely a descriptive and a deductive function. Whereas the former aims at giving an analysis to mathematical concepts, the latter function is to study the relations of logical consequence (pp. 1–21).

nomological science - ….- the greater will become the scope and power of its cognitive-practical performance” (*Ideen* I, §9). In the *Formal and Transcendental Logic* (1929) Husserl explains that the concept of the definite manifold “has continually guided mathematics from within (FTL, §31).[2] In the *Crisis* with the definite manifolds “the formal-logical idea of a ‘world-in-general’ is constructed” (*Crisis*, §9f). In the *Crisis* the notion represents the culmination of what Husserl calls ‘mathematization of the world’.

Originally, Husserl was interested in definiteness to find justification for the usage of imaginary numbers in calculations; but already in the *Prolegomena* the notion (made explicit in the Double Lectures few years later) underlined Husserl’s idea of the ideal essence of a theory. This change reflects Husserl’s own development from a mathematician to a philosopher: In the Double Lecture Husserl explains that definiteness is important for mathematicians because they are concerned about the reliability of their methods, whereas philosophers are interested in the general essence of the deductive theories (Schuhmann and Schuhmann 2001, p. 92). Husserl’s early interest in the notion of definiteness is clearly of the former type whereas his later interests are philosophical in nature. The early development of the notion has been discussed in (Hartimo and Okada 2016). Here I want to focus on its role in the *Prolegomena to the Logical Investigations* in order to understand its role in Husserl’s philosophy around the time of the Double Lecture (1901). I will briefly touch upon the later development of the notion towards the end of the paper.

The term ‘definiteness’ is not explicitly mentioned in the *Prolegomena*, but it underlies Husserl’s argument against psychologism. In order to avoid the pitfalls of psychologism, Husserl claims, we have to have an idea of pure logic. Accordingly Husserl promises in the introduction to the *Prolegomena* that

> [t]he final outcome of these discussions is a clearly circumscribed idea of the disputed discipline’s essential content, through which a clear position in regard to the previous mentioned controversies will have been gained. (Prolegomena, §3.)[3]

Husserl’s aim is to arrive at a view of the essential content of the idea of logic, or as he puts it later in the same work, of the ideal of theory as such (§66b). The ideal Husserl has in mind is the axiomatic, or in his terms, ‘nomological’ science, “which deals with the ideal essence of science as such” (§66b). While not all sciences are nomological, Husserl holds that the nomological sciences are basic and from their “theoretical stock the concrete sciences must derive all that theoretical element by which they are made sciences” (Prolegomena, §64). The nomological science rescues

---

[2] In what follows I will use the following abbreviations: FTL for Husserl (1974); *Briefwechsel* I for Husserl (1994); *Crisis* for Husserl (1976); Hua XX/1 for Husserl (2002); and *Ideen I* for Husserl (1950); and *Prolegomena* for Husserl (1975). I will use existing published translations, however so that I will translate the term ‘Mannigfaltigkeit’ systematically as a ‘manifold‘. Other possible changes will be noted separately.

[3] „Als letzter Erfolg dieser Überlegungen resultiert eine klar umrissene Idee von dem wesentlichen Gehalt der strittigen Disziplin, womit von selbst eine klare Position zu den aufgeworfenen Streitfragen gegeben ist“ (§3).

us from psychologism about logic by giving an independent norm of objectivity for logic and experiential sciences in general.

Husserl's argument against psychologism in the *Prolegomena* is thus at the same time a defense of an axiomatic approach. Distinctive to Husserl's approach is that he derives the notion from mathematics. According to him, "the mathematical form of treatment is … the only scientific one, the only one that offers us systematic closure and completeness, and a survey of all possible questions together with the possible forms of their answers" (*Prolegomena,* §71). In the Double Lecture, given two years after the *Prolegomena* was sent to be printed, Husserl aims to give a concrete formulation to what is meant by deductive or nomological system (FTL, §31). Husserl's interest in the essence of the axiomatic systems ties remarkably with the then state of mathematics: Hilbert was at the time focused on roughly the same topic, too. Indeed, in a meeting on November 5, 1901, two weeks before Husserl's first lecture in the Göttingen Mathematical Society, Hilbert lectured on the axiom of continuity and the Archimedean axiom in geometry and arithmetic (Gutzmer 1902, p. 72).

2 The two-sided essence of the theory in the *Prolegomena*

In the *Prolegomena* Husserl explains the role of the axiomatization to be to ground knowledge by unifying the otherwise separate facts. The sciences owe their unity to "a certain objective or ideal interconnection which gives these acts a unitary objective relevance, and, in such unitary relevance, an ideal validity" (Prolegomena, §62). This unity, according to Husserl, refers to both interconnection of the things to which our thought-experiences are directed to and the interconnection of truths, in which this unity of things comes "to count objectively as being what it is" (Prolegomena, §62). In other words, the theoretical unity refers to the unity of things and to truths about them, thus combining an *ontological* unity with an *apophantic* unity, i.e., unity related to judgments about the objects. The two are correlated for "[n]othing can be without being thus or thus determined, and that it is, and that it is thus and thus determined, is the self-subsistent truth which is the necessary correlate of the self-subsistent being" (ibid.).

The two-sidedness carries over to Husserl's conception of the idea of logic. Note that 'logic' to Husserl is not, e.g., propositional or predicate logic, as the term is understood today, but it is something much more encompassing as it covers all a priori formal truths. Hence, it includes mathematics. Logic has three tasks each of which has the apophantic as well as the ontological aspect: First, the nomological theories are built out of the concepts whose origin has been phenomenologically clarified. The concepts ought to be "unambiguous, sharply distinct verbal meanings" (§67). These concepts can be combined according to certain fixed grammatical laws. Correlated to the *apophantic* notions such as concepts of 'concept', 'sentence', 'truth', there are concepts of objects, e.g. 'object', 'state of affairs,' 'unity,' 'plurality,' 'number,' 'relation,' 'connection,' etc.

The second task of the pure logic is to search for the laws and theories to which the concepts give rise. On the apophantic side there are theories of inferences and on the side of the objects there are number theory, set theory, etc. All these theories are nomological, i.e., axiomatic. They are ideally complete, by which Husserl means that they aim to capture the unique, pure structure[4] common to several particular theories:

> The ideal completeness of the categorial theories and laws in question, rather yields the all-comprehensive fund from which each particular valid theory derives the ideal grounds of essential being appropriate to its form. These are the laws to which it conforms, and through which, as a theory validated by its form, it can be ultimately justified (Prolegomena, §68).[5]

The ideal completeness of the 'categorial' theories and laws is thus the source for justification for particular theories that share the same form. The particular theories are interpreted theories, and the ideal completeness of the 'categorial' theories refers to their categoricity. The quote shows how the 'ideal completeness' is the culmination of Husserl's more general argument against psychologism: the ideal completeness is needed to overcome psychologism about logic and achieve 'ultimate justification' for particular theories. It provides us with an independent pure structure that gives the 'mold' to which the individual theories should conform. The completeness at this point thus suggests that there is an isomorphism between the individuals and relations of each domain so that they are ideally of the same form. Furthermore, the theory should be all-comprehensive, which also suggests syntactic completeness.

Husserl's formulation of the relationship between the theories and their objective correlates becomes clearer when he introduces the term 'manifold':

> The *objective correlate* of the concept of a possible theory, definite only in respect of form, is *the concept of a possible field of knowledge over which a theory of this form will preside.* Such a field is, however, known in mathematical circles as a *manifold*. It is accordingly a field which is uniquely and solely determined by falling under a theory of such a form, whose objects are such as to permit of *certain* associations which fall under certain basic laws of this or that *determinate* form (here the only determining feature). The objects remain quite indefinite as regards their matter, to indicate which the mathematician prefers to speak of them as 'thought-objects'. They are not determined directly as individual or specific singulars, nor indirectly by way of their material species or genera, but solely by the form of the connections attributed to them. These laws then, as they

4 To clarify the notion of pure structure I will follow Steward Shapiro (1997) and use the term 'system' for "a collection of objects with certain relations" and the word 'pure structure' for an abstract form of a system (Shapiro 1997, pp. 73–74). Pure structure can then be formalized as

$$[\mathrm{S}] = [\mathrm{S}_0] \leftrightarrow \mathrm{S} \cong \mathrm{S}_0,$$

as done by Øystein Linnebo (in a presentation given in Munich, October 2016). Whereas the system can be any collection of objects with any relations on them, in Husserl the systems are created by axiomatic theories. The theories can thus be said to be satisfied by the systems. Hence, in Husserl's case, the systems are models and pure structures can be said to be domains of categorical theories in the model theoretical sense.

5 "Vielmehr bilden jene Theorien in ihrer idealen Vollendung den allumfassenden Fond, aus dem jede bestimmte (sc. Wirkliche, gültige) Theorie die gehörigen idealen Gründe ihrer Wesenhaftigkeit schöpft: es sind die Gesetze, denen gemäß sie verläuft, und aus denen sie als gültige Theorie, ihrer ‚Form' nach, vom letzten Grund aus gerechtfertigt werden kann" (§68, in accordance to A edition).

> determine a *field* and its *form*, likewise determine the theory to be constructed, or more correctly, the theory's form. In the theory of manifolds, e.g. '+' is not the sign for numerical addition, but for any connection for which laws of the form $a + b = b + a$ etc., hold. The manifold is determined by the fact that its thought-objects permit of these 'operations' (and of others whose compatibility with these can be shown *a priori*) (Prolegomena, §70).[6]

On the basis of this passage it should be clear that Husserl views a 'manifold' as a purely formal, unique domain of an axiomatic theory, i.e., a 'pure structure' as specified in the footnote 6. Specializations of manifolds are domains of actual theories, i.e., systems. Husserl's manifold is hence a pure structure, that is, the abstract form of a system.

The study of nomological theories, according to Husserl, "points beyond itself to a completing science, which deals *a priori* with the *essential sorts (forms) of theories and the relevant laws of relation*. " (Prolegomena, §69). This is a science of theory in general in which possible theories and their relationships to each other are investigated generally. The third task of logic is accordingly mathematics of pure structures, in which the 'forms' of theories, i.e., are related to each other. Husserl envisions that "[t]here will be a definite, ordered procedure which will enable us to construct the possible forms of theories, to survey their law-governed connections, and to pass from one to another by varying their basic determining factors etc." (Prolegomena, §69 translation modified).[7] This is the "highest goal for a theoretical science of theory in general" (Prolegomena, §69),[8] which has also major methodological significance. Husserl's theory of theory is a study of different kinds of pure structures that can be related to each other in various ways:

> The *most general Idea of a Theory of Manifolds* is to be a science which definitely works out the form of the essential types of possible theories or fields of theory, and investigates their rule-governed relations with one another. All actual theories are then specializations or singularizations of corresponding forms of

6 The passage reads in its entirety as follows: "Das *gegenständliche Korrelat* des Begriffes der möglichen, nur der Form nach bestimmten Theorie ist der Begriff eines *möglichen*, *durch eine Theorie solcher Form zu beherrschenden Erkenntnisgebietes überhaupt*. Ein solches Gebiet nennt aber der Mathematiker (in seinem Kreise) eine *Mannigfaltigkeit*. Es ist also ein Gebiet, welches einzig und allein dadurch bestimmt ist, daß es einer Theorie solcher Form untersteht, bzw. daß für seine Objekte gewisse Verknüpfungen möglich sind, die unter gewissen Grundgesetzen der und der *bestimmten* Form (hier das einzig Bestimmende) stehen. Ihrer Materie nach bleiben die Objekte völlig unbestimmt—der Mathematiker spricht, dies anzudeuten, mit Vorliebe von ‚Denkobjekten'. Sie sind eben weder direkt als individuelle oder spezifische Einzelheiten, noch indirekt durch ihre inneren Arten oder Gattungen bestimmt, sondern ausschließlich durch die *Form* ihnen zugeschriebener Verknüpfungen. Diese selbst sind also inhaltlich ebensowenig bestimmt, wie ihre Objekte; bestimmt ist nur ihre Form, nämlich durch die Form für sie als gültig angenommener Elementargesetze. Und diese bestimmen dann, wie das *Gebiet*, so die aufzubauende *Theorie* oder richtiger gesprochen, die *Theorienform*. In der Mannigfaltigkeitslehre ist z. B. + nicht das Zeichen der Zahlenaddition, sondern einer Verknüpfung überhaupt, für welche Gesetze der Form a + b = b + a usw. gelten. Die Mannigfaltigkeit ist dadurch bestimmt, daß ihre Denkobjekte diese (und andere, damit als *a priori* verträglich nachzuweisenden) ‚Operationen' ermöglichen" (§70, according to A edition).

7 "Es wird eine bestimmte Ordnung des Verfahrens geben, wonach wir die möglichen Formen zu konstruieren, ihre gesetzlichen Zusammenhänge zu überschauen, also auch die einen durch Variation bestimmender Grundfaktoren in die anderen überzuführen vermögen usw."(§69).

8 "Dies ist ein letztes und höchstes Ziel einer theoretischen Wissenschaft von der Theorie überhaupt."(§69)

> theory, just as all theoretically worked-over fields of knowledge are *individual* manifolds. If the formal theory in question is actually worked out in the theory of manifolds, then all deductive theoretical work in constructing all actual theories of the same form has been done (Prolegomena, §70).[9]

Thus, if we have the envisioned theory of theories, we only need to investigate which pure structure satisfies the given individual theory because the rest of the deductive work has already been carried out. Husserl thinks that the general idea of the theory of manifolds is partially already realized in the work of Grassmann, Hamilton, Lie, Cantor, or Riemann. His own formulation suggests mathematics of pure structures such as natural numbers or three-dimensional Euclidean manifold.[10]

The mathematical reality for Husserl consists of pure structures. We have a grasp of the mathematical reality through ideation of the basic concepts, we then construct a theory around them. Through 'formalization', i.e., by looking at the theories up to isomorphism, their common, purely formal and unique structure can be grasped. All this will be further elaborated in Husserl's Double Lecture to which I will now turn.

3 Husserls double lecture

Two years after having sent the *Prolegomena* to the printer (Schuhmann 1977, pp. 58–59) Husserl delivered the notorious Double Lecture in the Göttingen Mathematical Society in 1901. The topic of the lectures was definiteness, that is, what Husserl later referred to as the concretization of the Euclidean ideal. The first lecture took place on November 26, 1901. It was entitled „Der Durchgang durch das Unmögliche und die Vollständigkeit eines Axiomensystems". The second lecture, with the title, "Vor allem waren die Begriffe des 'definiten' und des 'absolut definiten' Systems auseinandergesetzt" was held on December 10, 1901 (Gutzmer 1902, p. 72, 147). The exact order and composition of especially the second lecture is unclear. I will here draw from the passages that are shared by all existing editions of the Double Lecture

9 "*Die allgemeinste Idee einer Mannigfaltigkeitslehre* ist es, eine Wissenschaft *zu* sein, welche die wesentlichen Typen möglicher Theorien [(bzw. Gebiete) added to the B edition] bestimmt ausgestaltet und ihre gestzmäßigen Beziehungen zueinander erforscht. Alle wirklichen Theorien sind dann Spezialisierungen bzw. Singularisierungen ihnen entsprechender Theorienformen, so wie alle theoretisch bearbeiteten Erkenntnisgebiete *einzelne* Mannigfaltigkeiten sind. Ist in der Mannigfaltigkeitselhre die betreffende formale Theorie wirklich durchgeführt, so ist damit alle deduktive theoretische Arbeit für den Aufbau aller wirklichen Theorien derselben Form erledigt" (§70).

10 This goes well with his self-declared Platonism about mathematics. The categorical theories describe the well-determined reality. In a 1905 letter to Brentano Husserl claimed that already the *Prolegomena* had been influenced by Lotze's interpretation of Plato (Briefwechsel I, 39). In his attempt to rewrite the introduction to the 1913 edition of the Logische Untersuchungen Husserl elaborates on this as follows: "The fully conscious and radical turn and the related 'Platonism' I owe to the study of Lotze's Logik. As little as Lotze himself could overcome contradictions and psychologism, as much his genial interpretation of Platonic ideas helped me and my further studies. Lotze's discussion of truths in themselves suggested to me the thought to place all mathematics and a good part of traditional logic into the realm of ideality" (Hua XX/I, 297).

[that is, Schuhmann and Schuhmann (2001), and one by Lothar Eley, translated into English by Dallas Willard (Husserl 2003)].

The exact interpretation of what Husserl means by definiteness has created much discussion. One reason behind the confusion is that the rigorous concepts of logic, and that of consequence and deduction, were not in place at the time. Hence Husserl's discussion does not directly translate to the present day terminology, and when he for example speaks about decidability it should be taken as an *informal* decidability that does not distinguish between model theoretical consequence relation and proof theoretical deducibility relation. At times the used logic could be captured in first order predicate calculus, at other times the second order is needed.

It has also been claimed that Husserl is not that clear himself: As Centrone puts it, Husserl oscillates between a semantical and syntactical characterizations of definiteness (2010, p. 168). Centrone gives as an example of a more syntactical characterization the following:

> An axiom-system that delimits a domain is said to be 'definite' if every proposition intelligible on the basis of the axiom-system, understood as a proposition of the domain, … either … follows from the axioms or contradicts them (Husserl 2003, p. 438).[11]

This characterization refers to syntactic completeness. The following definition is a good example of a semantic formulation:[12]

> An irreducible axiom system is definite which delimits (or grounds as existing) a formal domain of objects in such a way that for this domain—that is, if one preserves the identity of the axiom system, and if one presupposes that no new objects are defined and thereby assumed as existing—no independent axiom can be added which is constructed purely from the concepts already defined (of course, also, none can be withdrawn, since otherwise the axiom system would not be irreducible). (Husserl 2003, 434)[13]

The idea of maximal determination of the domain referred to here is, I will argue, a semantic notion. It aims at characterizing the domain of the theory exhaustively and unambiguously. I will here propose that this 'oscillation' between syntactic and semantic definitions of definiteness is largely intentional on Husserl's part. It reflects the two sidedness of the idea of pure logic discussed in the *Prolegomena*. Whereas the

---

[11] „Ein Axiomensystem, das ein Gebiet umgrenzt, heiße definit, wenn jeder aufgrund des Axiomensystems verständliche Satz, als Satz für das Gebiet aufgefaßt, entweder … er folgt aus den Axiomen oder er widerspricht ihnen" (Schuhmann and Schuhmann 2001, p. 111).

[12] *Pace* Centrone (2010), who takes this definition as a syntactic one.

[13] „Definit ist ein irreduktibles Axiomensystem, welches ein formales Objektgebiet so umgrenzt (als existierend begründet), daß für dieses Gebiet, d.h. unter Festhaltung der Identität des Axiomensystems und unter Voraussetzung, daß keine neuen Objekte definiert werden und hierdurch als existierend angenommen werden, kein independentes Axiom hinzugefügt werden kann, welches sich rein aus den schon definierten Begriffen aufbaut… Ich kann aber auch sagen: Definit ist ein Axiomensystem, welches ein Objektgebiet formal so definiert, daß jede für dieses Objektgebiet sinnvolle Frage durch das Axiomensystem seine Antwort fände oder daß jeder durch die Axiome sinnvolle Satz, wenn wir ihn ausschließlich auf die durch die Axiome als existierend begründeten Objekte beschränken, entweder aus den Axiomen folgt oder ihnen widerschpricht" (Schuhmann and Schuhmann 2001, p. 108).

syntactic notion above refers to the unity of the interconnection of truths, the semantic notion refers to the unity of the interconnection of things. The first one captures the idea that the theory decides all the sentences of the theory as true or false. The second refers to the uniqueness, or pureness, of the manifold and its exhaustive determination. In other words, Husserl's ideal is to have both deductive power as well as expressive power at once. Furthermore, as I will argue below, the expressive power is prior to deductive power of the theory, for one needs the axioms with which to capture the domain of the theory before one can examine what follows from them.

In a recent paper, Jairo da Silva argues that Husserl's notion of definiteness has to be syntactical completeness for epistemological reasons. According to him, for Husserl, axiomatic theories are founded on conceptual intuition. The conceptual intuition is confined to the axioms of the theory, and from the axioms theorems follow by logical deduction. According to da Silva, the axiomatization is adequate, that is, serves the purpose it was designed for, only if all the remaining truths follow from the axioms (da Silva 2016, esp. pp. 1935–1936). This is true, but as I will argue below, it captures Husserl's philosophical aspirations only partially. Husserl also intended to capture a full characterization of the domain of the theory in question. Hence, syntactic completeness alone does not account for all of Husserl's philosophical intentions.

In what follows I will first briefly explain Husserl's line of thought in the first lecture. It will again show Husserl's conception of mathematics as mathematics of unique structures. After this I will move on to discuss the second lecture and discuss the definiteness of different kinds of manifolds.

4 Husserl's lecture of November 26, 1901: „Der Durchgang durch das Unmögliche und die Vollständigkeit eines Axiomensystems"

Husserl begins his first lecture by describing mathematicians' aspiration for formal theory, free from all specific domains of knowledge (Erkenntnisgebieten). According to him

> Mathematics in the highest and most inclusive sense is the science of theoretical systems in general, in abstraction from that which is theorized in the given theories of the various sciences. If for some given theory, for some given deductive system, we abstract from its matter, from the particular species of objects whose theoretical mastery it has in view, and if we substitute for the materially determinate representations of objects the merely formal ones—thus the representation of objects in general—which are mastered through such a theory, through a theory of this form, then we have carried out a generalization that grasps the given theory as a mere singular case of a class of theories, or rather of a form of theories, which we grasp in a unified way and in virtue of which we then can say that all these particular scientific domains have, in form, the same theory. (Husserl 2003, p. 410.)[14]

[14] "Mathematik im höchsten und umfassendsten Sinn ist die Wissenschaft von den theoretischen Systemen überhaupt und in Abstraktion von dem, was in den gegebenen Theorien der erschiedenen Wissenschaften theoretisiert wird. Vollziehen wir bei irgendeiner gegebenen Theorie, bei irgendeinem gegebenen

Husserl thus repeats what he stated already in the *Prolegomena*. Mathematicians' aim is to extract pure forms of theories from the 'material,' interpreted theories. The domain of an individual theory is regarded as an individual case of the pure structure defined by all individual theories of the same form. Husserl then immediately states how a theory, when "systematische durchgearbeitete", systematically worked out, encompasses the two ideals:

> A systematically elaborated theory in this sense is defined by a totality of formal axioms, i.e., by a limited number of purely formal basic propositions, mutually consistent and independent of one another. Systematic deduction supplies in a purely logical manner, i.e., purely according to the principle of contradiction, the dependent propositions, and therewith the entire totality of propositions that belong to the theory defined. But the object domain is defined through the axioms in the sense that it is delimited as a certain sphere of objects in general, irrespective of whether real or Ideal, for which basic propositions of such and such forms hold true. An object domain thus defined we call a determinate, but formally defined, manifold. (Husserl 2003, p. 410.)[15]

A systematically worked out theory is a syntactically complete theory. But, furthermore, the axioms characterize a domain of objects uniquely. The determinate, formally defined manifold is a pure structure.

Neil Tennant has introduced a term 'monomathematics' for this kind of view of mathematics. It seeks both expressive power to describe structures exactly and deductive power to prove whatever follows logically from one's description. After Gödel's work it was clear that the combination of these two ideals cannot be achieved in the interesting cases. Tennant argues that the noncompossibility of the two ideals could have been stated and proved by the end of the World War I. At that time the needed concepts were in place, namely categoricity of a theory, and completeness of a proof system. (Tennant 2000, pp. 257–258). Hintikka on his part thinks that the noncompossibility of the ideals should lead to seeking new deductive methods (1998, p. 99).

Husserl is a perfect example of a 'monomathematician.' Indeed, what he calls 'formalization,' means a move from a domain of individual, interpreted concrete theory

Footnote 14 continued
deduktiven System [Abstraktion] von seiender Materie, von den besonderen Gattungen von Objekten, auf deren theoretische Beherrschung sie es abgesehen hat, und substituieren wir den materiell bestimmten Objektvorstellungen die bloß formalen, also die Vorstellung von Objekten überhaupt, die durch solch eine Theorie, durch eine Theorie dieser Form beherrscht wird so haben wir eine Verallgemeinerung vollzogen, welche die gegebene Theorie als einen bloßen Einzelfall einer Theorienklasse auffaßt oder vielmehr einer Theorienform, die wir einheitlich auffassen und um deren willen wir dann sagen können, alle diese einzelnen wissenschaftlichen Gebiete hätten der Form nach dieselbe Theorie" (Schuhmann and Schuhmann 2001, p. 91).

15 "Eine systematisch durchgearbeitete Theorie in diesem Sinn ist definiert durch einen Inbegriff von formalen Axiomen, d.h. durch eine begrenzte Anzahl rein formaler, miteinander konsistenter und voneinander independenter Grundsätze; die systematisch Deduktion liefert rein logisch, d.i. rein nach dem Prinzip vom Widerspruch, die abhängigen Sätze und damit den Gesamtinbegriff von Sätzen, die zu der definierten Theorie gehören. Das Objektgebiet aber ist durch die Axiome in dem Sinn definiert, daß es umgrenzt ist als irgendeine Sphäre von Objekten überhaupt, gleichgültig ob realen oder idealen, für welche Grundsätze solcher und solcher Formen gelten. Ein so definiertes Objektgebiet nennen wir eine bestimmte, aber formal definierte Mannigfaltigkeit" (Schuhmann and Schuhmann 2001, p. 91).

to the pure structure it shares with other domains of theories isomorphic with it. Husserl takes this to settle the old question whether the numbers should be viewed as cardinals or ordinals or as something else, since both are mere concrete examples of the structure of natural numbers (Schuhmann and Schuhmann 2001, p. 91). In the theory of theories one can examine the relationships of different kinds of theory forms [Theorienformen], i.e., pure structures a.k.a. manifolds, e.g., natural numbers, integers, etc. with one another.

In Husserl's example of geometry, the concrete Euclidean geometry is first formalized into the form of the theory, which defines the three-dimensional Euclidean manifold. This can further be regarded as an individual, albeit formal, case of all manifolds of different curvatures. The outcome is what Husserl calls 'formal mathematics,' which thus means mathematics of pure structures. (Schuhmann and Schuhmann 2001, pp. 91–92.) Note that 'formal mathematics' in this sense is not 'formalistic' mathematics in the sense that mathematics is understood as mechanical operations on the signs in accordance to the given rules. Husserl's formal mathematics is formal because it deals with pure structures that are 'formal' as opposed to 'material', not because of being merely syntactic.

But, according to Husserl, there are unsolved difficulties in the application of formal mathematics in the concrete fields (realen Mathematik, bzw. in den besonderen Erkenntnisgebieten). Husserl has in mind the problem of the imaginaries elements (including also negative and irrational numbers). This gives the rise to the problem that is the topic of Husserl's talk:

> Suppose a domain of objects given in which, through the peculiar nature of the objects, forms of combination and relationship are determined that are expressed in a certain axiom system A. On the basis of this system, and thus on the basis of the particular nature of the objects, certain forms of combination have no signification for reality, i.e., they are absurd forms of combination. With what justification can the absurd be assimilated into calculation—with what justification, therefore, can the absurd be utilized in deductive thinking—as if it were meaningful? How is it to be explained that one can operate with the absurd according to rules, and that, if the absurd is then eliminated from the propositions, the propositions obtained are correct? (Husserl 2003, p. 412)[16]

A concrete set of axioms determines the field of objects, which does not, however, include imaginary elements. The problem is what justifies the usage of such imaginary elements in calculations. Husserl considers five alternative answers to the problem. After refuting the first four ones Husserl proposes the fifth one. The end of the lecture notes is very fragmentary and no clear statement of Husserl's solution can be found

[16] "Es sei ein Gebiet von Objekten gegeben, in welchem durch die besondere Natur der Objekte Verknüpfungs- und Beziehungsformen bestimmt sind, die sich in einem gewissen Axiomensystem A aussprechen. Aufgrund dieses Systems, also aufgrund der besonderen Natur der Objekte, haben gewisse Verknüpfungsformen keine reale Bedeutung, d.h. es sind widersinnige Verknüpfungsformen. Mit welchem Recht darf das Widersinnige rechnerisch verwertet, mit welchem Rechte kann also das Widersinnige im deduktiven Denken verwendet werden, als ob es Einstimmiges wäre? Wie ist es zu erklären, daß sich mit dem Widersinnigen nach Regeln operieren läßt und daß, wenn das Widersinnige aus den Sätzen herausfällt, die gewonnenen Sätze richtig sin¿' (Schuhmann and Schuhmann 2001, p. 93)

in it. Husserl's strategy seems to be to formalize the given theory, then extend it so that its domain is fully definite, which then is supposed to guarantee the calculations with imaginary elements. To this effect, Husserl discussed the concept of the domain and of the axiom system and how the axioms define the manifold. Husserl considers constructing the domains by adding to it the unique results of the operations of the domain. He discusses more and less strict cases of such construction. The strictest case is such where the domain is defined constructively by an individual operation from a finite number of objects. (Schuhmann and Schuhmann 2001, p. 99). All in all, the first lecture does not manage to give a full nor clear account of definiteness. Let us hence turn to the second lecture.

5 Husserl's lecture on December 10, 1901: "Vor allem waren die Begriffe des 'definiten' und des 'absolut definiten' Systems auseinandergesetzt"

Both Schuhmann and Schuhmann (2001) edition as well as the earlier Lothar Eley 1970 edition (Husserl 1950, translated in Husserl 2003) hold that Husserl started the second lecture by discussing the question of definiteness separately for different types of manifolds:

1. A definite manifold through an inessential closure axiom will be ruled out (Schuhmann and Schuhmann 2001, p. 99).[17]
2. Can a purely algebraic manifold, which defines no individual of the domain whatever—can such a manifold have the character of a definite manifolds? (Husserl 2003, p. 422; Schuhmann and Schuhmann 2001, p. 100).
3. Operationsystems, which do not exclude individuals in the generation of the domain; these systems are divided further into two: (1) not every generally defined and existing operational result belongs in the sphere of the operationally producible and distinguished individuals. (2) mathematical systems, where everything that exists is operatively uniquely determined. (Schuhmann and Schuhmann 2001, p. 100).

The first case refers to postulating completeness with a closure axiom, such as Hilbert's axiom of completeness. Husserl was critical of the idea and held that completeness should rather be an 'inner' property of the theory (see e.g., Centrone 2010, pp. 170–171).

To sort out the rest I will start from the consideration of the mathematical systems that have a rather clear definition and that have been discussed in (Hartimo and Okada 2016, pp. 962–965). The mathematical systems or constructible systems, according to Husserl, are defined recursively so that their complex expressions are reducible to expressions of equalities among the elements of the domain. Husserl maintained this idea already in the *Philosophy of Arithmetic* (1891) where he formulated it as a "general postulate of arithmetic: the symbolic formations that are different from the systematic numbers must, wherever they turn up, be reduced to the systematic

[17] The translation deviates from that of Willard (2003, p. 422). The original is as follows: „Definite Mannigfaltigkeit durch das unwesentliche Schließungsaxiom wird ausgeschlossen" (Schuhmann and Schuhmann 2001, p. 99).

numbers equivalent to them, as their normative forms" (Husserl 2003, p. 277, 1970, p. 262). In other words, the expressions of the theory are typically equations that are mechanically reducible to equations among natural numbers. Mitsuhiro Okada (2013), and, few years later, Hartimo and Okada (2016) have shown mathematical systems to be 'constructor-based rewrite systems', in modern terms of rewriting theory. Centrone construes the same idea as "arithmetizability of a manifold" (2010, pp. 191–192). Husserl's idea is that certain manifolds generate what he calls an 'operation system,' which means that it can be interpreted arithmetically, with an equational reduction.

Husserl's discussion of purely algebraic manifolds presumably refers to Cantor's theory of transfinite cardinals. Husserl was well familiar with Cantor's work. Husserl had for example discussed the so-called Schröder–Cantor–Bernstein theorem with Cantor in 1898 (Schuhmann 1977, p. 52). Husserl's discussion is not entirely clear here, and instead of the definiteness of the manifold he talks about the definiteness of addition and multiplication. His conclusion is that the known laws of addition and multiplication are definite on the condition of "des genannten Zusatzaxioms", without any specification of what these 'Zusatzaxioms' are. The 'Zusatzaxioms' could refer to axioms such as Hilbert's axiom of completeness, which simply posits the non-extendibility of the axiom system.[18] It thus seems that in the case of purely algebraic manifolds, Husserl thinks that, after all, we need something like Hilbert's closure axioms. In any case, without a clear understanding of the Zusatzaxiome not much more about the definiteness of the purely algebraic manifolds can be said.

I will thus move to Husserl's definition of the relative and absolute definiteness of the axiom systems, which according to the minutes taken from the lecture were the very topic of the lecture.

The definitions of the relative and absolute definite axiom systems are given in the context of discussing Hilbert's approach. Husserl, critical of Hilbert's axiom of completeness, writes that completeness should not be an axiom, but a theorem for definite axioms systems and manifolds. He then proceeds to define:

> An axiom system is relatively definite if, for its domain of existence it indeed admits of no additional axioms, but it does admit that for a broader domain the same, and then of course also new, axioms are valid. New axioms, since the old axioms alone in fact determine only the old domain. Relatively definite is the sphere of the whole and the fractional numbers, of the rational numbers, likewise of the discrete sequence of ordered pairs of numbers (complex numbers). I call a manifold absolutely definite if there is no other manifold which has the same axioms (all together) as it has. Continuous number sequence, continuous sequence of ordered pairs of numbers (Husserl 2003, p. 426). [19]

[18] One considered possibility is a mistake in the transcription of Husserl's shorthand and that he actually means 'Zuordnungsaxiome', which Cantor discusses in the *Beiträge*. However, Husserl's notation is rather clear on this point as Thomas Vongehr of Husserl Archive Leuven kindly showed to me: Husserl writes 'Zusatzaxiome,' not 'Zuordnungsaxiome.'

[19] "Relativ definit ist ein Axiomensystem, wenn es zwar für sein Existentialgebiet keine Axiome mehr zuläßt, aber es zuläßt, daß für ein weiteres Gebiet dieselben und dann natürlich auch neue Axiome gelten. Neue Axiome, denn die bloß alten Axiome bestimmen ja nur das alte Gebiet. Relativ definit ist die Sphäre der ganzen, der gebrochenen Zahlen, der rationalen Zahlen, ebenso der diskreten Doppelreihenzahlen

The relative definiteness is thus a property of a theory that can be extended, but not without changing the domain. An example of the relatively definite domain is that of natural numbers that can be defined by Dedekind–Peano axioms.[20] The theory contains the maximum amount of independent and with each other consistent axioms to define uniquely the structure of natural numbers. If one extends it with an independent axiom, consistent with the existing ones, one starts to ‘determine’ another domain, say that of rationals. In that sense, the existing axioms capture the structure of natural numbers. Accordingly Husserl writes as follows:

> The natural numbers are what they are only through the definitions. Since the definitions univocally determine the numbers (in virtue of the axioms), then a number or group of numbers can indeed have infinitely many properties, but none which is not grounded in the definitions and axioms and determined through them. It would be a contradiction against the determinateness of the natural numbers if one wished to reckon among their properties those which are not covered by the definitions. In fact, it is the peculiar property of the natural numbers that they are ‘determined’ in this sense. Not only are they in general univocally determinate objects of the domain, but rather they are determinate in such a way that they can undergo no other determination, i.e., that for them fixed as they are by the axioms, no additional property can be newly adjoined axiomatically. But that must be proven. (Husserl 2003, p. 443) [21]

Husserl thus clearly thinks that the axioms define natural numbers univocally and exhaustively.

Absolute definiteness then relates to a theory that cannot be extended consistently. Husserl writes that it is the same as completeness in Hilbert’s sense (Schuhmann and Schuhmann 2001, p. 103; Husserl 2003, p. 427). [22]

Footnote 19 continued
(komplexen Zahlen). Absolut definit nenne ich eine Mannigfaltigkeit, wenn es keine andere Mannigfaltigkeit gibt, welche dieselben Axiome hat wie sie (alle zusammen). Kontinuierliche Zahlenreihe, kontinuierliche Doppelzahlenreihe“ (Schuhmann and Schuhmann 2001, p. 102).

[20] Husserl speaks generally of axiom systems. The example of Dedekind–Peano Axioms is mine. It should be noted that Husserl did not refer to Peano. However, Husserl was aware of Dedekind’s “Was sind und was soll die Zahlen” (1888).

[21] „Die natürlichen Zahlen sind, was sie sind, nur durch die Definitionen. Da die Definitionen die Zahlen eindeutig bestimmen (vermöge der Axiome), so kann eine Zahl oder Zahlengruppe zwar unendlich viele Eigenschaften haben, aber keine, die nicht in den Definitionen und Axiomen gründet und durch sie bestimmt ist. Es wäre ein Widerspruch gegen die Bestimmtheit der natürlichen Zahlen, wenn man Eigenschaften zählen wollte, die nicht durch die Definitionen beschlossen sind: Ja, das ist die eigentümliche Eigenschaft der natürlichen Zahlen, daß sie in diesem Sinn “bestimmt” sind. Nicht nur sind es überhaupt eindeutig bestimmte Objekte des Gebietes, sondern so bestimmte, daß sie keine andere Bestimmung erfahren können, d.h. daß für sie, die durch die Axiome so festgelegt sind, keine Eigenschaft mehr axiomatisch neu hinzugefügt werden kann. Das aber muß bewiesen werden.“ (Schuhmann and Schuhmann 2001, p. 115)

[22] Centrone (2010) defends an interpretation of relative definiteness as syntactic completeness and absolute definiteness as categoricity (2010, pp. 149–213). The present approach is in agreement with her account of absolute definiteness, but holds that also the former, relative definiteness should be understood as categoricity. Centrone’s motivation for her interpretation seems to follow from Centrone’s understanding of the impossibility of adding new axioms as a kind of maximality, which she thinks corresponds to syntactic completeness: “As to the impossibility, on pain of inconsistency, of adding new axioms while preserving

With Centrone (2010, p. 185) I believe that Husserl at least at times distinguishes the mathematical manifolds from other definite manifolds. Mathematical manifolds are the ones that can be given an arithmetical interpretation. As mentioned above, Hartimo and Okada (2016) show how in them the axiomatic structure gives a rise to an ‘operation system’ with which every expression of the theory is reducible to an equality between elements of the domain. This determines the objects of the domain even more determinately, down to ‘*infimae species*’ as Husserl sometimes says (e.g. Husserl 2003, p. 446; Schuhmann and Schuhmann 2001, p. 117). The equational reduction thus gives the criteria of identity for conceptualizing the natural numbers, which are then fully determined. Thus while Husserl thinks there are unique Platonistically existing structures, such as the one of natural numbers, he also wants to be able to individuate the elements of the domain if possible. This completes determination of the elements of the domain.

Husserl thus seems to think that the elements of pure structures that are in the end mere pure positions, need to be enriched and further determined if possible. For this reason, for example, he talks about the richer content given to [inhaltsreicher fixiert] the axiom system. He does this by means of the existence axioms which establish the existence the result of applying any combination to the determinate elements of the domain (Husserl 2003, p. 420; Schuhmann and Schuhmann 2001, p. 98). This is presumably also what he means when he at one point claims that he wants to “operate not only with general, indeterminate concepts of objects, but rather I also introduce individually designating concepts of objects—as it were proper names for objects (or species of objects)” (Husserl 2003, p. 445).

Definiteness, for Husserl, thus embraces in the end three ideals of completeness: “pureness” and non-extensibility as captured by categoricity, syntactic completeness, and often also a kind of computational completeness to aspire for richer determination

Footnote 22 continued
the independence of the system, this is a property which exactly corresponds—as is easily seen—to the property nowadays known as maximality of (sometimes) saturatedness of a formal system: informally speaking, a formal system T is maximal when it proves all that can be proved, on pain of inconsistency; that is, formally, when for each closed formula α of the language of the theory it holds that if α is not derivable from T then the system $T + \alpha$ is inconsistent” (2010, p. 169). This is precisely where I disagree with Centrone’s approach. She ignores Husserl’s attempt at a full and unambiguous characterization, which aims at capturing the domain purely and uniquely. True, Husserl thought that from such full determinability syntactic completeness follows and sometimes he even equates the two. But still, his intention to characterize the domain maximally cannot be reduced away when interpreting the passage. Indeed, for this reason maximal expressibility is prior to syntactic completeness: it is understandable that one infers the latter from the former, but from syntactic completeness one cannot derive Husserl’s goal to fix the “Existentialgebiet” unambiguously. In Husserl’s formal mathematics the non-extendibility of the axiom system shows that they are maximally determined and hence unique. Centrone analyzes the relative definiteness as syntactic completeness but so that the domains of the syntactically complete theory are structurally very similar (2010, p. 178). It is little unclear to what she refers to, but it seems that she means second-order equivalence (2010, p. 194, 199). This is an interesting analysis of Husserl’s view of definiteness, but to me it seems that if Husserl’s philosophical intentions are taken seriously, it is more straightforward to think that he aimed at categoricity from which he thought syntactic completeness follows. Only after one has analyzed a domain with certain axioms, can one examine what follows from the axioms. Whereas Centrone thinks that the essence of mathematics to Husserl is primarily about proving theorems, I find Husserl’s primary intentions to be in capturing the pure structures. That Centrone ignores Husserl’s concern with the domain has been pointed out before by Mark van Atten in his review of her book (2012, p. 375).

of the ‘existential domain’.[23] As we saw above, the first two ideals are already sought for in Husserl’s *Prolegomena* discussion of the nomological theories, and form a cornerstone of his argument against psychologism. Hence, his conception of definiteness is a continuation of his *philosophical* intentions as expressed in the *Prolegomena*. The computational notion of completeness Husserl entertained from early on (cf. Hartimo and Okada 2016), and it reflects Husserl’s idea of how computations can be of help for genuine thought (cf. Husserl 1975, §§54–56). Husserl’s ideals of completeness make perfect systematic sense as well: syntactic completeness, categoricity, and the criteria of identity for the abstract objects—any mathematician would ascribe to such ideals, the problem shown by the later development of logic is that we typically cannot have them all.

6 Later development

Let us briefly consider Husserl’s later views about definiteness. He discusses definite manifolds in print for the first time in the *Ideen I*. In it he gives several definitions for a ‘definite manifold’, such as:

> Such a manifold is characterized by the fact that a *finite number of concepts and propositions* derivable in a given case from the essence of the province in question*, in the manner characteristic of purely analytic necessity completely and unambiguously determines to totality of all the possible formations belonging to the province* so that, *of essential necessity, nothing in the province is left open* (§72).[24]

This concept is equivalent with the following claims:

> Any proposition which can be constructed out of the distinctive axiomatic concepts, regardless of its logical form, is either a pure formal-logical consequence of the axioms or else a pure formal-logical anti-consequence—that is to say, a proposition formally contradicting the axioms, so that its contradictory opposite would be a formal-logical consequence of the axioms. *In the case of a mathematically definite manifold the concepts ‘true’ and ‘formal-logical consequence of the axioms’ are equivalent*; and so are the concepts ‘false’ and ‘formal-logical anti-consequence of the axioms (§72). [25]

[23] These three senses of completeness are mentioned in connection of Husserl in the notes of Charles Parsons‘ seminar on structuralism given at Harvard in 2003. I have had a privilege to look at these incredibly rich and interesting notes, but unfortunately only after having first submitted this paper.

[24] ”Sie ist dadurch charakterisiert, daß eine *endliche Anzahl*, gegebenenfalls aus dem Wesen des jeweiligen Gebietes zu schöpfender *Begriffe und Sätze die Gesamtheit aller möglichen Gestaltungen des Gebietes in der Weise rein analytischer Notwendigkeit vollständig und eindeutig bestimmt*, so daß also in ihm *prinzipiell nichts mehr offen* bleibt” (§72).

[25] ”Jeder aus den ausgezeichneten axiomatischen Begriffen, nach welchen logischen Formen immer zu bildende Satz ist entweder eine pure formallogische Folge der Axiome, oder eine ebensolche Widerfolge, d.h. den Axiomen formal widersprechend; so daß dann das kontradiktorische Gegenteil eine formallogische Folge der Axiome wäre. *In einer mathematisch-definiten Mannigfaltigkeit sind die Begriffe ‚wahr‘ und ‚formalogische Folge der Axiome‘ äquivalent*, und ebenso die Begriffe ‚falsch‘ und ‚formallogische Widerfolge der Axiome“‘ (§72).

The *Ideen I* version thus still captures the two ideals: the first quote expresses Husserl's aim to capture an unambiguously determined totality of whatever belongs to the domain of the theory. The latter quote holds this to be equivalent with its syntactic completeness.

In the *Formal and Transcendental Logic* Husserl cites extensively from the *Prolegomena* and summarizes his *Prolegomena* view of the mathematics. He first discusses 'formalization', in which

> determinate *object-province* made up of spatial data becomes *the form of a province*; it becomes, as the mathematician says, a *manifold.* It is not just any manifold whatever (that would be the same as any set whatever); not it is the *form*, 'any infinite set whatever'. On the contrary, it is a set whose peculiarity consists only in the circumstance that it is thought of with empty-formal universality, as 'a', province determined by the complete set of Euclidean postulate-forms—that is to say, determined in a deductive discipline having a *form* derived from Euclidean space-geometry by formalization (§29).[26]

Formalization yields a form that is "equiform" with the original theory. Husserl's formulation suggests that he still thinks of mathematics to be about pure structures. The following quote is even clearer in this regard:

> Naturally all the materially concrete manifolds subject to axiom-systems that, on being formalized, turn out to be equiform are manifolds that have the same deductive science-form in common; in their relationship to it, these manifolds themselves are equiform (§31).[27]

Formalization thus is still a move to consider theories up to isomorphism. By means of it a formal domain, a pure structure, becomes created. Also in *Formal and Transcendental Logic* definite theories are syntactically complete:

> any proposition… that can be constructed, in accordance with the grammar of pure logic, out of the concepts (concept-forms) occurring in that system, is either 'true'—that is to say: an analytic (purely deducible) consequence of the axioms—or 'false'—that is to say: an analytic contradiction—*tertium non datur* (§31). [28]

[26] "Aus dem bestimmten *Gegenstandsgebiet* räumlicher Gegebenheiten wird die *Form eines Gebietes*, oder wie der Mathematiker sagt, eine *Mannigfaltigkeit*. Es ist nicht schlechthin eine Mannigfaltigkeit überhaupt, was so viel wäre wie eine Menge überhaupt, auch nicht die Form ‚unendliche Menge überhaupt', sondern es ist eine Menge, die nur ihre Besonderheit darin hat, daß sie in leer-formaler Allgemeinheit gedacht ist als ‚ein' Gebiet, das bestimmt sei durch den vollständigen Inbegriff *Euklidischer* Postulatformen, also in einer deduktiven Disziplin von der aus der *Euklidischen* Raumgeometrie durch jene Formalisierung hergeleiteten *Form*" (§29).

[27] "Natürlich haben alle sachhaltig konkret vorzulegenden Mannigfaltigkeiten, deren Axiomensysteme sich bei der Formalisierung als äquiform herausstellen, dieselbe deduktive Wissenschaftsform gemein, sie sind in Beziehung auf sie selbst äquiform" (§31).

[28] "jeder aus den in diesem auftretenden Begriffen (Begriffsformen natürlich) rein-logisch-grammatisch zu konstruierende Satz (Satzform) entweder ‚wahr', nämlich eine analytische (rein deduktive) Konsequenz der Axiome, oder ‚falsch' ist, nämlich ein analytischer Wiedrspruch: *tertium non datur*" (§31).

Even though Husserl still seems to emphasize the uniqueness of the formal domain, Husserl is not entirely consistent about that. For example, he explicitly wants to refrain from saying that there is only one 'universe of experience' (FTL, §89b), which could be a reference to alternative 'domains' of the theory. Nevertheless, Husserl refers to definite manifolds also in his posthumously published *Crisis*, where it "gives a special sort of totality in all deductive determinations to the formal substrate-objects contained in them. With this sort of totality, one can say, the formal-logical idea of a 'world-in-general' is constructed" (§9f).

Acknowledgements I greatly acknowledge the support of Centre for Advanced Study in Oslo, Norway that funded and hosted our research project "Disclosing the Fabric of Reality—The Possibility of Metaphysics in the Age of Science" during the academic year of 2105/16, when the present article was written. Special thanks are due to the other members of the project, especially Frode Kjosavik and Øystein Linnebo. I also want to thank the support of the project "Conceptual Engineering" (250654) funded by The Research Council of Norway.

References

Cantor, G. (1895). Beiträge zur Begründung der transfiniten Mengenlehre (1). *Mathematische Annalen*, *46*, 481–512. doi:10.1007/bf02124929.

Centrone, S. (2010). *Logic and philosophy of mathematics in the early Husserl Synthese library* (Vol. 345). Dordrecht: Springer.

Da Silva, J. J. (2000). Husserl's tow notions of completeness, Husserl and Hilbert on completeness and imaginary elements in mathematics. *Synthese*, *125*, 417–438.

da Silva, J. J. (2016). Husserl and Hilbert on completeness, still. *Synthese*, *193*, 1925–1947.

Ewald, W. (1996). *From Kant to Hilbert: A source book in the foundations of mathematics* (Vol. II). Oxford: Clarendon Press.

Gutzmer, A. (1902). *Jahresbericht der Deutschen Mathematiker-Vereinigung*. Leipzig: Druck und Verlag von B. G. Teubner.

Hartimo, M. (2007). Towards completeness: Husserl on theories of manifolds 1890–1901. *Synthese*, *156*, 281–310.

Hartimo, M., & Okada, M. (2016). Syntactic reduction in Husserl's early phenomenology of arithmetic. *Synthese*, *193*(3), 937–969.

Hill, C. O. (1995). Husserl and Hilbert on completeness. In Hintikka (Ed.), *From Dedekind to Gödel, synthese library* (Vol. 251). Dordrecht: Kluwer.

Hintikka, J. (1998). *The principles of mathematics revisited*. Cambridge, UK: Cambridge University Press. [1996].

Husserl, E. (1950). *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie. Erstes Buch. Allgemeine Einführung in die reine Phänomenologie.* Husserliana Band III. Herausgegeben von Walter Biemel. Haag: Martinus Nijhoff. English translation: *Ideas pertaining to a pure phenomenology and to a phenomenological philosophy. First book. General Introduction to a Pure Phenomenology.* The Hague, Boston, Lancaster: Martinus Nijhoff (1983).

Husserl, E. (1970). Philosophie der Arithmetik. Mit ergänzenden Texten (1890–1901). Edited by Lothar Eley. Den Haag: Martinus Nijhoff.

Husserl, E. (1974). *Formale and transzendentale Logik. Versuch einer Kritik der logischen Vernunft. Husserliana Band 17.* Edited by Paul Janssen. Martinus Nijhoff, The Hague, Netherlands 1974. English translation: *Formal and Transcendental Logic*, transl. by Dorion Cairns. Martinus Nijhoff, The Hague. 1969

Husserl, E. (1975). *Logische Untersuchungen. Erster Band. Prolegomena zur Reinen Logik.* Husserliana Band XVIII. Herausgegeben von Elmar Holenstein. Den Haag. Martinus Nijhoff. English translation: *Logical Investigations. Prolegomena to pure logic*, transl. by J. N. Findlay. London and New York: Routledge [1970], 2001, 1-161.

Husserl, E. (1976). *Die Krisis der Europäischen Wissenschaften und die Transzendentale Phänomenologie. Eine Einleitung in die phänomenologische Philosophie.* Herausgegeben von Walter Biemel. Husserliana Band VI. Den Haag: Martinus Nijhoff. English translation: *The Crisis of European Sciences and Transcendental Phenomenology. An Introduction to Phenomenological Philosophy.* Translated by David Carr. Evanston: Northwestern University Press, 1970.

Husserl, E. (1994). *Briefwechsel. Band I, Die Brentanoschule*. Dordrecht, Boston, London: Kluwer: Herausgegeben von Karl Schuhmann.

Husserl, E. (2002). *Logische Untersuchungen. Ergänzungsband. Erster Teil. Entwürfe zur Umarbeitung der VI. Untersuchung ud zur Vorree für die Neuauflage der Logischen Untersuchungen (Sommer 1913)*. Herausgegeben von Ullrich Melle. Dordrecht, Boston, London: Kluwer Academic Publishers.

Husserl, E. (2003). *Philosophy of arithmetic*. Transl. by Dallas Willard. Dordrecht: Kluwer.

Majer, U. (1997). Husserl and Hilbert on completeness: A neglected chapter in early twentieth century foundations of mathematics. *Synthese*, *110*, 37–56.

Okada, M. (2013). Husserl and Hilbert on Completeness and Husserl's Term Rewrite-based Theory of Multiplicity. *Invited paper, 24th International Conference on Rewriting Techniques and Applications (RTA'13)* (pp. 4–19). Edited by Femke van Raamsdonk. The Netherlands: Eindhoven.

Schuhmann, E., & Schuhmann, K. (2001). Husserls Manuskripte zu seinem Göttinger Doppelvortrag von 1901. *Husserl Studies*, *17*, 82–123.

Schuhmann, K. (1977). *Husserl-Chronik. Denk- und Lebensweg Edmund Husserls*. Den Haag: Martinus Nijhoff.

Shapiro, S. (1997). *Philosophy of mathematics: Structure and ontology*. Oxford: Oxford University Press.

Tennant, Neil. (2000). Deductive versus expressive power: A pre-Gödelian predicament. *The Journal of Philosophy*, *97*(5), 257–277.

van Atten, M. (2012). Stefania Centrone, logic and the philosophy of mathematics in the Early Husserl. *The New Yearbook for Phenomenology and Phenomenological Philosophy*, *XII*, 370–376.