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Husserl and Hilbert on completeness, still

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Abstract In the first year of the twentieth century, in Gottingen, Husserl delivered two talks dealing with a problem that proved central in his philosophical development, that of imaginary elements in mathematics. In order to solve this problem Husserl introduced a logical notion, called "definiteness", and variants of it, that are somehow related, he claimed, to Hilbert's notions of completeness. Many different interpretations of what precisely Husserl meant by this notion, and its relations with Hilbert's ones, have been proposed, but no consensus has been reached. In this paper I approach this question afresh and thoroughly, taking into consideration not only the relevant texts and context, as others have also done before, but, more importantly, Husserl's philosophy, his intuition-based epistemology in particular. Based on a system of clearly defined concepts that I here present, I reinforce an interpretation—definiteness as a form of syntactic completeness—that has, I believe, some advantages vis-à-vis alternative interpretations. It is in conformity with the available texts; it makes clear that Husserl's notion of definiteness is indeed close to Hilbert's notions of completeness; it solves the important problem of imaginaries for which it was created; and last, but not least, it fits naturally into Husserl's system of concepts and ideas.

Keywords Husserl · Hilbert · Definiteness · Completeness · Imaginary elements in mathematics

Few things are more frustrating than trying to make things fit where they don't. This is the risk one takes when attempting to interpret concepts and ideas of yesterday within conceptual systems of today. Anyone facing this challenge must carefully consider the advantages and pitfalls of the task.

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The problem that will concern me here is the concept of logical completeness (or definiteness) of both axiomatic systems and manifolds as introduced by Edmund Husserl, its correct interpretation vis-à-vis our modern conceptions of completeness and its relation to other notions of completeness of the time (end of the nineteenth century, beginning of the twentieth), in particular those of Hilbert. This problem has been approached by many authors before, among them Hill (1995), Majer (1997), Silva (2000), Hartimo (2007), Centrone (2010), and Okada (2013), and my reason to revisit it are multiple. First, the issue is far from being settled[1]; second, a clarification of this problem can illuminate some aspects of the development of modern logic; third, it helps to understand Husserl's and Hilbert's differing approaches to axiomatics and the influence one may have exerted on the other; fourth, the problem has intimate connections with a central question in Husserl's epistemology, the justification of symbolic knowledge. We simply cannot afford to be unclear about so important a question and my goal here is to investigate it more thoroughly than the authors just mentioned have done.

Contrary to some of them, who emphasize textual and contextual analyses, I prefer to concentrate on *conceptual* analyses. Of course, text and context are important and will be taken into consideration, but sometimes they can mislead rather than lead. A good deal of attention must be given first to the problems Husserl was facing and what they involved conceptually. From my point of view, two stand out, the problem of imaginary elements in mathematics and Husserl *epistemological* dissatisfaction with the way Hilbert secured completeness of axiomatic systems. Husserl's notions of definiteness (completeness) were born in the attempt to solve these problems and must be approached from this perspective.

From his early days as a philosopher, Husserl was deeply concerned with axiomatics and its role in mathematics. It all began with his efforts to provide logical-epistemological justification for arithmetic, that of natural numbers, but also that of more general numerical concepts, from 1887 (the date of his *Habilitationsschrift*) onwards. After realizing that general concepts of number could not be derived from the concept of natural number, that is, that the bottom-up "genetic" approach was no longer the adequate one, Husserl turned to the top-down strategy of axiomatization.[2] A particularly puzzling question was, for him, the *successful* use of numerical concepts

[1] There is no consensus on what Husserl meant by definiteness or what exactly was the relation of his and Hilbert's notions of completeness related to his axioms of completeness, a relation that Husserl believed should be obvious to all.

[2] Initially, Husserl had the idea of basing more general concepts of number on the notion of cardinal number, favoring, that is, a genetic, bottom-up approach. This was the project for the second volume of *Philosophy of Arithmetic* (Husserl 2003), which never saw the light of day for, supposedly, Husserl realized that this could not be done. He, then, turned to the axiomatic, top-down approach [see, for instance, Husserl (1970a, p. 378) , where general arithmetic is characterized as a formal science]. Husserl axiomatic approach to set theory of 1891 (Husserl 1970a, pp. 385–407) and geometry of 1893 (Husserl 1983, pp. 285–293), where he shows how the fundamental concepts and truths of axiomatic geometry, such as congruence, are grounded on intuition) show how much involved with the axiomatic method the early Husserl was [see Centrone (2010) for a careful historical account of the philosophical development of the early Husserl and Ortiz Hill, in Hill and Silva (2013, pp. 93–114), for the role Husserl allowed to the axiomatic method in his approach to arithmetic]. For the purposes of this paper, however, this discussion is irrelevant; Husserl's notion of definiteness, which is a property of axiomatic systems, is without doubt part of his treatment of the axiomatic method, particularly in its complex relations with intuition.

of greater generality in the theoretical investigation of numerical concepts of lower degrees of generality (for instance, complex numbers in the theory of real numbers). He called this the problem of "imaginaries"; the terminology is justified, for "imaginary" entities not only do not exist, but *could not* exist from the perspective of the domains into where they are introduced.

In his efforts to justify logically and epistemologically the use of imaginaries, Husserl introduced two notions of completeness (or definiteness, as he called them), relative and absolute, each splitting in two related notions, the *apophantic* one, relative to axiomatic systems, and the *ontological* one, relative to their "domains". The discussion and interpretation of these notions will occupy most of this essay. The notion of a definite system of axiom and the correlate notion of a definite manifold (*definit Mannigfaltigkeit*) served also, from Husserl's perspective, as alternatives to the "inauthentic" axioms of completeness of Hilbert's axiomatization of arithmetic and geometry (the term "inauthentic" is Husserl's own).[3]

Clearly, Husserl's privileged interlocutor on these issues was Hilbert, particularly after he moved to Gottingen in 1901 and became close to the great mathematician. Hilbert had just then provided axiomatic foundations for the arithmetic of real numbers (1900) and Euclidean geometry (1899), which, given Husserl's interest in axiomatics, were bound to attract his attention. The issues of extendibility and non-extendibility of theories and their "domains", and "completeness", in some sense or other, were salient in both Hilbert's and Husserl's approaches to axiomatics at the time (around 1900).

Hilbert is to this day considered a champion of the axiomatic method, a reputation that his contributions to it—the above mentioned axiomatizations, the invention of metamathematics, which turned metamathematical questions such as the consistency of axiomatic systems and the independence of the axioms of a system with respect to each other into real mathematical problems, and the project of axiomatization of physics that he launched with the sixth problem of his famous list—undoubtedly justify. Husserl's contributions to axiomatics, on the other hand, which may have influenced Hilbert, and which were certainly influenced by him, were almost completely forgotten for a long time, resurfacing only recently. They are, however, still absent from history of logic books and articles.[4] To interpret Husserl's views on matters of axiomatics, and confront them with Hilbert's against the background of modern

[3] "Dieser [i.e. Hilbert's] Begriff der Vollständigkeit soll als *unechte* [*my emphasis*] Vollständigkeit bezeichnet werden." (Husserl 1970a, p. 442). I translate "unecht" as "inauthentic", but it can also be translated as "false", "artificial", "counterfeit", in short, inadequate.

[4] Awodey and Reck (2002) and Corcoran (1980), for example, who follow with care the appearance and maturing of certain logical notions, categoricity and completeness in particular, in early efforts of axiomatization from Dedekind onwards, completely ignore Husserl and his work on axiomatics. This is to some extent understandable, since Husserl did not produce a mathematically relevant pioneering axiomatization that made its way into the mainstream of mathematical logic. But *philosophically* it is an error to ignore him, for Husserl was probably the philosopher better equipped to provide the *correct* interpretation of the axiomatic method advanced by Hilbert and himself, much better than the "game" interpretation that gained prominence [see, for instance, Hill and Silva (2013), in particular chapters 3 and 5, and Silva (2012, pp. 115–136)].

axiomatics is then an effort that I believe justified, either from the perspective of the development of modern formal logic and some of its central concepts or that of the development of Husserl's thought.

Husserl's work on axiomatics does indeed occupy a central position in his philosophy. Husserl's struggle with the problem of imaginaries is, I believe, essential to understand his change of philosophical orientation around the middle nineties of the XIX century, which brought the problem of objectless representations and the role of symbolic thinking, among other logical questions, to the center of his philosophical concerns.[5] These issues are recurrent in his philosophy, from his opera magna of 1900–1901, the *Logical Investigations*, to his last book, the *Crisis of European Science and Transcendental Phenomenology* (*Hua VI* 1954, containing previously published and unpublished texts from 1935 to 1936). So, a clear understanding of the problems Husserl was facing at the beginning of his philosophical career and how he dealt with them, including his contributions to axiomatics, is central to the correct assessment of Husserl's philosophical development and the genesis of some of his main philosophical ideas (particularly that of intentionality, a concept inherited from Brentano which Brentano himself had borrowed from medieval thinkers to characterize the mental in opposition to the physical, and the use Husserl made of it, after having conveniently "depsychologized" the concept, in his theory of knowledge in general and science in particular).

So, I believe, historical and exegetical interests in logic and philosophy justify that we take the risk of interpreting Husserl's (and Hilbert's) contributions to axiomatics from the perspective of modern logic.

Let us begin with explicit definitions and fixing the terminology. Let L be a language in which mathematical statements can be written. L need not be a formal language, either in Husserl's sense of a language devoid of material meaning (but preserving the formal meaning implicit in the rules for the "blind" manipulations of its symbols) or in the modern sense of a non-interpreted symbolic language in which the notions of symbol, term and formula are decidable. L can simply be the traditional language of mathematics. Let us suppose that a notion of deduction is available, not necessarily in the modern formal sense of a "mechanical" output of formulas of a formal language by the action of explicit rules of inference on a basis of logical and non-logical axioms, but in a larger sense, closer to that familiar to mathematicians, that is, mathematical reasoning based on "obvious" truths and largely implicit rules of derivation (including maybe higher-order and infinitary rules such as ω-rule). I will denote the fact that a statement φ of L is derivable from a set Γ of statements from L by $\Gamma \vdash \varphi$.

5 "Above all it was its [*i.e. arithmetic's, my note*] purely symbolic procedural techniques, in which the genuine, original insightful sense seemed to be interrupted and made absurd under the label of the translation through the 'imaginary', that directed my thoughts to the significance and to the purely linguistic aspects of the thinking – and knowing – processes and from that point on forced me to general 'investigations' which concerned universal clarification of the sense, the proper delimitation, and the unique accomplishment of formal logic" (Husserl's draft introduction to *Logical Investigations apud* Moran 2005, 90). There is, then, a direct link between the problem of "imaginaries" and the *Logical Investigations* and further attempts of clarification and delimitation of formal logic; for example, *Formal and Transcendental Logic*.

Let us also suppose that notions of *interpretation* of L and *satisfaction* of sentences of L in some interpretation are available. An interpretation is a *specific* mathematical context where sentences of L acquire a sense and a truth-value, not necessarily a model in the Tarskian sense (for this reason I use the term "interpretation" not "model"). I will take the notion of a sentence φ of L being true in some interpretation as primitive; "true" means "according to the facts". I will denote the fact that a sentence φ of L is true in an interpretation A by $A \models \varphi$. Sentences of L can be devoid of interpretation and differently reinterpreted, more or less in the sense that Hilbert reinterprets geometrical assertions arithmetically. If a set Γ of sentences is given, by $\Gamma \models \varphi$ I mean that φ is true in all interpretations where *all* the sentences of Γ are also true.

By *axiomatic system* I mean a set of axioms, the basic unproved truths of a theory. For Husserl, axiomatic systems must be *finite*, but I will not presuppose this much here (or even that the notion of axiom is decidable).

Definition 1 $\vdash$ is *complete with respect to* $\models$ if for any set Γ of sentences and any sentence φ of the language: $\Gamma \models \varphi \Rightarrow \Gamma \vdash \varphi$. I will take the converse for granted; that is, that deduction preserves truth. The reason is that no deduction "technology" is acceptable in mathematics or science that does not preserve truth.

Definition 2 An axiomatic system T is *logically complete (l-complete)* if, for any sentence φ, $T \models \varphi$ implies $T \vdash \varphi$. In words, what is *true* in all interpretations of a theory must follow *deductively* from the axioms of this theory.

Definition 3 An axiomatic system T is *categorical* if all its interpretations are isomorphic in the usual *algebraic* sense. Although the notion of categoricity is not clear until, as some argue, Huntington's works of 1902 and 1903, it is implicit in Dedekind, the Hilbert of 1899–1900 and others.[6] Boole, Husserl, and even Leibniz were aware that mathematical theories can have different interpretations (Husserl explicitly notes that Schroder's calculus can be differently interpreted as a calculus of classes and a calculus of judgments). Some of them realized (I am not sure that Husserl did) that different *interpretations* can, nonetheless, be "essentially the same". By "essentially the same" *they* almost certainly had in mind what *we* call isomorphic, although this is never explicitly stated. Husserl sometimes refers to a mathematical theory that has been devoid of interpretation and formalized (thus becoming a theory-form or the form of a theory) as "equiform" with the original theory. But by this he only means that both, the theory and its form have the same *logical* form.[7]

Definition 4 An axiomatic system (or *theory*) T is *semantically complete (s-complete)* if for any sentence φ, either $T \models \varphi$ or $T \models \neg \varphi$ (either the sentence or its negation, but not both if the system is consistent, is a *semantic* consequence of the axioms of the system; that is, either the sentence is true in all interpretations of T or it is false in all of them).

6 For details and references see Awodey and Reck (2002).

7 Mahnke (1923) uses terms that were translated by "logically isomorphic" and "formally equivalent" to refer to the relation between contentual theories and their formalizations, derived for metamathematical purposes (he mentions consistency proofs).

Theorem 1 *The following assertions are equivalent (proofs are trivial)*:

1. T *is* s-*complete.*
2. *All interpretations of* T *are logically equivalent, that is, exactly the same set of sentences of L is satisfied in any interpretation of* T. *In a sense, s-complete theories characterize their interpretations uniquely as far as the expressive powers of the language are concerned.*
3. *Given any sentence* φ*, either* $T \cup \{\varphi\}$ *(for short:* T, φ*) or* $T, \neg\varphi$ *does not have an interpretation.*
4. *For no sentence* φ*, both* T, φ *and* $T, \neg\varphi$ *have interpretations.*

Theorem 2 *If* T *is categorical, then* T *is* s-*complete.*

Proof If T is categorical, then (2) of Theorem 1.

Note 1: The converse is not true; for example, the first-order theory of real closed fields is s-complete but not categorical.[8] In fact, no first-order theory with *infinite* models is categorical [by the cardinality theorem–see Shoenfield (1967, p. 88)]. So, axiomatic systems can *completely* characterize interpretations up to *logical* equivalence without characterizing them categorically, i.e. up to *algebraic* equivalence (isomorphism). □

Definition 5 T is *syntactically or deductively complete* (*d-complete*) (relative to $\vdash$) if, for any sentence φ of the language, either $T \vdash \varphi$ or $T \vdash \neg\varphi$. Since, as supposed, $T \vdash \varphi$ implies $T \models \varphi$ (deductions preserve truth), then d-completeness implies s-completeness.

Definition 6 T is d-*complete relative to an interpretation A of T* (d-*A-complete*) if all sentences that are true in A are provable in T.

Theorem 3 *The following are equivalent*:[9]

1. *For some interpretation* A *of* T *(also written* $A \models T$*),* T *is* d-A-*complete.*
2. T *is* d-*complete.*
3. *For all sentences* φ*: either* $T \vdash \varphi$ *or* T, φ *is inconsistent (a theory is inconsistent if it proves both a sentence and its negation).*
4. *There is no sentence* φ *such that* T, φ *and* $T, \neg\varphi$ *are both consistent.*

Proof (1) $\Rightarrow$ (2): Let φ be any sentence of the language. Either $A \models \neg\varphi$ or $A \models \varphi$. In the first case $T \vdash \neg\varphi$, in the second $T \vdash \varphi$. (2) $\Rightarrow$ (1): Let A be an interpretation of T and φ a sentence such that $A \models \varphi$ (φ is true in A). Suppose *ad absurdum* that T does not prove φ; then it must prove $\neg\varphi$ and so $\neg\varphi$ must be true in A. Contradiction, since φ is true in A. Therefore, $T \vdash \varphi$.

2) $\Rightarrow$ 3): If T does not prove φ, it must prove $\neg\varphi$, by definition of d-completeness. So, T, φ is inconsistent.

8 Vide Shoenfield pp. 87–88.

9 In order to show that 4) implies 2) I assume from the start that a theorem of deduction is valid for the calculus; i.e. if $T, \varphi \vdash \psi$, then $T \vdash \varphi \rightarrow \psi$, for any sentences φ and ψ.

3) ⇒ 4): Suppose T, φ and T, ¬ φ are both consistent for some φ. The consistency of T, ¬ φ implies that T cannot prove φ; therefore T, φ is consistent *and* T does not prove φ, denying 3).

4) ⇒ 2): If T is not complete, there is a sentence φ such that neither φ nor ¬ φ are theorems of T. I claim both T, φ and T, ¬ φ are consistent. Indeed, suppose that T, ¬ φ is *inconsistent*; then T, ¬ φ ⊢ F, for some absurd sentence F (F $= \psi \wedge\neg \psi$, for some ψ). By the deduction theorem (see Note 9), T ⊢ ¬ φ → F, which is logically equivalent to φ, a contradiction (because T does *not* prove φ). Analogously for T, φ.

Note 2: Neither Peano arithmetic (first and second order) nor the theory of complete ordered fields (first and second order) are d-complete. But Tarski's axiomatization of the arithmetic of real numbers is both s- and d-complete.

Note 3: If T is d-complete, then the set of all deductive consequences of T (the apophantic domain of T) is *maximal* in the sense that for any sentence φ, either φ or ¬ φ (but not both if T is consistent) belongs to it. So, the apophantic domain of a d-complete theory is *non-extendable* in the sense that no sentence that does not belong to it can be consistently added to it. □

Theorem 4 *Suppose* T *is* s-*complete. Given two interpretations* A *and* B *such that the universe of* A *is contained in the universe of* B *and there is a formula* $\psi(x)$ *in one free variable such that* B $\models \exists x((\psi(x) \wedge \forall y(\psi(y) \rightarrow \mathrm{y} = \mathrm{x}))$ *(i.e. there is only one element in* B *satisfying* ψ*), then the element in* B *satisfying* ψ *must belong to* A.

Proof Let *b* the only element in B satisfying ψ. By (2) of Theorem 1 there must be a *unique* element in A satisfying ψ, call it *a*. Since A is contained in B, *a* is in B. Since *b* is the only element in B satisfying ψ, $a = b$ and so *b* belongs to A. In general, if an existential assertion is satisfied in some interpretation of T (maybe not uniquely), and T is s-complete, it is also satisfied in any other interpretation (by possibly different elements). So, if an element satisfying a certain condition expressible in the language of an s-complete T does not exist in some interpretation of T, no such element exists in any other interpretation. This means that interpretations of s-complete theories are *non-extendable* to interpretations of the *same* theory by the *adjunction* of *new definable* elements. The same is a fortiori true of d-complete theories, since d-completeness implies s-completeness. □

Definition 7 Let S be a set of sentences of the language, T is *d-complete relative to S* (*d-S-complete*) is for any φ in S, either T ⊢ φ or T ⊢ ¬ φ.

Note 4: If T is d-S-complete, where S contains all existential assertions of the language, then no interpretation of T is extendable to other interpretations by the adjunction of elements definable by formulas of the language. The reason is that no element *a*, definable by a condition φ, can exist in one but not in another interpretation of T (an interpretation can only be *extended* by the adjunction of a *new* element to its domain).

Definition 8 Let's introduce a unary predicate symbol R in L. For any sentence φ, let φ_{R} (φ restricted to R) denote the sentence φ with $\forall x(\ldots x\ldots)$ replaced by $\forall x(\mathrm{R}(x) \rightarrow \ldots x\ldots)$ and $\exists x(\ldots x\ldots)$ replaced by $\exists x(\mathrm{R}(x) \wedge \ldots x\ldots)$. Let A be an interpretation of T and D a subset of the universe of A, we say that φ_{R} *refers* to D in A if $\mathrm{R}^{\mathrm{A}} = \mathrm{D}$. In particular, φ_{R} refers to the universe of an interpretation A if $\mathrm{R}^{\mathrm{A}} = \mathrm{A}$.

Note 5: Obviously, if R refers to A, then φ_R and φ are either both true or both false in A (notation: $\varphi_R \equiv_A \varphi$).

Note 6: Suppose $t(x)$ and t' are terms such that $\exists x(t(x) = t')$ is *false* in an interpretation A of T. So, T cannot prove $\exists x(t(\mathrm{x}) = t')$ and, a fortiori, $\exists x(\mathrm{R}(\mathrm{x}) \wedge t(\mathrm{x}) = t')$. Suppose that T is d-S-complete, where S is the set of all the sentences of the language restricted to R. T must then prove $\forall x(\mathrm{R}(x) \rightarrow t(x) \neq t')$; so, in *no* interpretation M of T there is an element a in R^{M} such that $t(a) = t'$. This, however, does *not* rule out the existence of interpretations B whose universe extends that of A, where R^{B} = the universe of A and in which there is b in B-A so that $t(b) = t'$. Suppose there is such an interpretation. Then, T cannot prove $\forall \mathrm{x}(t(\mathrm{x}) \neq t') = \neg\exists x(t(\mathrm{x}) = t')$. Hence, $\exists x(t(\mathrm{x}) = t')$ is deductively *undecidable* in T (despite the fact that its restriction $\exists x(\mathrm{R}(\mathrm{x}) \wedge t(\mathrm{x}) = t')$ is decidable). It is then possible an axiomatic system T to exist that is d-S-complete with respect to the set S of sentences restricted to R, but not d-complete.

Example Let T be the set of all sentences φ_D in the language $\{\mathrm{D}, c, +, \mathrm{x}\}$, D a unary predicate symbol and c a constant, true in $\mathbf{Z} = \langle \mathrm{Z}; \mathrm{Z}, -1, +, \mathrm{x}\rangle$, where $\mathrm{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, $\mathrm{D}^{\mathrm{Z}} = \mathrm{Z}$ and $c^{\mathrm{Z}} = -1$. Note that T is actually d-S-complete, because given any sentence of this language, either this sentence or its negation (but not both) is true in Z, and so belongs to T. Putting $\mathrm{t}(x) = x^2$ and $t' = c$, $\exists x(t(\mathrm{x}) = t')$ is *false* in **Z**. Consider now Z[i], the ring of numbers $a+b$i, where a, b are integers and i is the imaginary unit. Now, let $\mathbf{Z}' = \langle \mathrm{Z}[i]; \mathrm{Z}^*, -1 + 0i, +, \mathrm{x}\}$, where $\mathrm{Z}^* = \{a + 0i : a \text{ is in } \mathrm{Z}\}$; $\mathbf{Z}'$ is also an interpretation of T, but $\exists x(t(\mathrm{x}) = t')$ is *true* in Z[i] (take $x = 0 + 1i$). So, T decides any sentence that *refers* to Z, but it is not d-complete, for the sentence $\exists x(t(\mathrm{x}) = t')$ is undecidable in T.

Definition 9 A theory T is *d-R-complete* if it is d-S-complete where $\mathrm{S} = \{\varphi_R$: R a fixed unary predicate of the language$\}$.

Note 7: Since a theory can be deductively complete with respect to all the sentences restricted to some specific predicate and not be deductively complete, it is interesting, considering Theorem 3, which asserts the equivalence of d-A-completeness with d-completeness, to compare the notion of d-R-completeness with that of d-A-completeness. The latter guarantees T-deducibility of all sentences that are true in A; the former, of all sentences whose quantifiers range only over the domain of a specific unary predicate. If φ is true in A, φ_R is also true in A - it is enough to interpret R as A -, but φ_R true in A (for *some* other interpretation of R) does *not* imply that φ is also true in A. So, even if T can deductively decide all the sentences φ_R (d-R-completeness) and so *prove* all the sentences φ_R that are *true* in A, it may not be capable of proving all sentences φ that are true in A (d-A-completeness). On the other hand, if T is d-A-complete it must, in particular, *prove* all the φ_R's that are true in A; hence, it must *decide* all the φ_R's (d-R-completeness), for either φ_R or $\neg\,\varphi_R$ is true in A. Then, in this sense, d-R-completeness is a weaker notion than d-A-completeness. However, if the axiom $\forall x\mathrm{R}(x)$ is added to a d-R-complete theory the resulting theory is d-complete (and so d-A-complete), since the enlarged theory proves that any sentence φ is equivalent to its restriction φ_R.

Definition 10 An interpretation A of a theory T is *non-extendable* if there is no interpretation of T of which A (or, in general, some interpretation isomorphic to A) is a sub-interpretation.

Note 8: This is a very strong condition, which no first-order theory with infinite models, not even s-complete or d-complete theories, satisfies. Indeed, let A be an infinite model of a first-order theory T, consider the diagram Γ of A, that is, the set of all quantifier-free sentences of the language of the theory, extended with constants $\underline{a}$ for all the elements a of A, that are true in A. Add to the extended language a new constant c and to Γ all the sentences $c \neq \underline{a}$. Any *finite* subset of this extended theory has a model, and so, by the theorem of compactness, the entire theory has a model B'. The reduction B of B' to the original language (without the extra constants) is a model of T in which A can be isomorphically embedded; moreover, B extends A, since the interpretation of c does not belong to the copy of A in B.

If T is a categorical theory, all interpretations of T are trivially non-extendable. However, non-extendibility of interpretations does *not* imply the categoricity of the theory. For example, consider a language with only a binary relation symbol R and the theory T with the axioms (1) there are *exactly* two things and (2) R is reflexive. T obviously has non-isomorphic models (for example, R can be symmetric in one and asymmetric in another), but each of them is non-extendable. So, although categoricity guarantees non-extendibility, the converse is not true.

Let's now define an interpretation to be *non-extendable by definitions* if no interpretation can be extended by the adjunction of definable elements. More precisely, an interpretation A of T is *non-extendable by definitions* if (1) there is an interpretation B of T such that A (or, more generally, an isomorphic copy of it) is contained in B, (2) there is an element b in B-A and a formula $\psi(x)$ in one free variable such that $\psi(x)$ is *not* satisfiable in A but $\psi(b)$ is true in B. The idea is that an interpretation is extendable by definitions when it can be extended into another interpretation of the *same* theory by the adjunction of elements satisfying certain properties that are expressible in the language but not satisfied in A. For example, Z, the ring of integers, can be extended by definition to the ring Z$[i]$ by the adjunction of the element i satisfying $\psi(x) \equiv (x^2 = -1)$ in Z$[i]$; $\psi(x)$, however, is not satisfied in Z.

Now, as already observed, if T is either s- or d-complete, no interpretation of T is extendable by definitions.

I will offer below, within the conceptual context sketched above, an interpretation of the many notions of completeness that Hilbert and Husserl have put forward. But first we must understand the purpose for which they have introduced them. Although the conceptions are essentially the same, they were differently motivated. Hilbert was primarily concerned with providing adequate axiomatizations of fundamental mathematical concepts (number and space, for example), and his notions of completeness were devised as means to this end; Husserl, rather, was driven by philosophical concerns related to the logical-epistemological justification of the axiomatic method and the problem of imaginaries, and his notions of completeness were devised with this goal in mind. Both, however, probably believed that the philosophical approach of one was a necessary complement to the foundational perspective of the other.

From Husserl's point of view, Hilbert's axiomatics was not immune to philosophical criticism. Although Hilbert allowed desinterpretation and reinterpretations of axiomatic systems for the sake of logical investigations—the relative consistency and logical independence of axioms, for example, he did not raise the important problems relative to the ontological correlates and epistemological relevance of *purely formal* axiomatic systems that figured so prominently in Husserl's logical-philosophical agenda.[10]

Husserl did not believe that all axioms of Hilbert's axiomatizations were adequately justified either. Despite the latter's claim that the choice of axioms of his *Grundlagen der Geometrie* (1899), for example, was determined by "a logical analysis of our perception of space" (Hilbert 1971), Husserl had difficulties in accepting that the axiom of completeness could indeed be so justified. For Husserl, as I read him (Husserl 1970a, pp. 441–442), this axiom was a purely ad hoc accretion to an otherwise intuitively justified system. The completeness of the system must, Husserl thought, be attained by other means. But before seeing how, we must get a better understanding of what both understood by "completeness".

For Hilbert, axiomatic systems must provide complete descriptions of their domains; geometry (by which I think Hilbert meant *physical* geometry), for example, is for him a description of *perceptual* space in terms of *relations* such as "lies on", "between" and "congruent" as applied to *objects* such as "points", "lines" and "planes"; this description, moreover, must be *complete*. By such a requirement Hilbert clearly implies some notion of deductive completeness. For instance, in the following quote:[11]

> The necessary task then arises of showing the consistency and the completeness of these axioms, i.e. it must be proved that the application of the given axioms can never lead to contradictions, and, further, that *the system of axioms is adequate to prove all geometrical propositions* [*my italics*]. We shall call this procedure of investigation the axiomatic method.

Awodey and Reck (2002) offers two possible readings of Hilbert's completeness requirement; first, the axioms should be sufficient to derive all *known* theorems of geometry; second, they should be sufficient to derive *all* theorems of geometry. If by "theorems of geometry" Hilbert meant, as is reasonable to suppose, assertions true of our representation of space, then the first reading of "complete" coincides with d-S-completeness where S is the set of "known" theorems, and the second with d-A-completeness, where A is our representation of space (in view of Theorem 3, this coincides with d-completeness).

But, to complicate the matter, the notion of completeness associated with the axiom of line completeness seems to point in a different direction. This axiom requires that it should "not be possible to extend the system of points on a line" in such a way that order and congruence relations are preserved and all the other axioms still satisfied.

[10] For Husserl, purely formal (i.e. non-interpreted) axiomatic systems are theory-forms, that is, forms of theories, whose objective correlates are formal manifolds determined as to form but indeterminate as to (material) content. Hence, for him, formal theories belong to formal ontology, the province of formal logic concerned with the formal properties of objects, conceived exclusively as such and merely as possibilities.

[11] From Hilbert's *On the Concept of Number*, in Ewald (1996, pp. 1092–93).

Clearly, *this* notion of completeness is related to *non-extendibility*, in the following way: the only acceptable interpretation of the notion of "line" is one in which "lines" are maximal sets of "points". Let us be more precise. Let A be a given interpretation of the axioms of geometry *excluding* line-completeness. The question is whether A can be taken as an interpretation of the whole system, *including* line-completeness. The answer is that it can, *provided* that there is *no* interpretation B extending A in which B-lines, as sets of B-points, properly include A-lines as sets of A-points.

This was not the first version of the axiom of completeness. An earlier version required that there should be no two interpretations, one extending the other, of the remaining axioms. By imposing non-extendibility Hilbert's axiom of completeness grants uniqueness of domain, or, to use a concept neither Hilbert nor Husserl possessed, categoricity. But although *implying* categoricity, the axiom is not *aimed at* categoricity; in either version, the axiom is included to secure non-extendibility of the domain, uniqueness is a consequence.

However, as observed before, neither d-(or s)-completeness guarantees non-extendibility of the domain of the theory, nor does non-extendibility guarantee d-completeness. If Hilbert wanted, as indicated in the quote above, d-completeness, why did his axioms require non-extendibility? The answer, I believe, must be sought along the following lines. Hilbert obviously realized that d-completeness could not be granted directly by axiomatic stipulation; it is a condition of adequateness of axiomatic systems, not itself an axiom; axioms refer to domains of objects, not domains of axioms. It is to be expected, however, that an exhaustive or, better, *complete* intuitive survey of the concept that governs the relevant domain should be sufficient for a *complete* set of axioms to be obtained (for how else could it be obtained?). Hilbert probably believed that a system *strong enough to single out essentially one interpretation, the intended one, would necessarily be complete (d-complete)*. If I am correct, Hilbert implicitly presupposed that non-extendibility (categoricity) implied d-completeness.[12]

Husserl agreed that *adequate* axiomatizations must be d-complete. This can actually be *shown*: for him, a priori theories are conceptual theories founded on conceptual intuition.[13] Now, axiomatization can only be considered *adequate*. i.e. to serve the purpose for which axiomatizations are devised, if conceptual intuition is confined to the axiomatic basis, theorems following by logical deduction from the axioms (this divi-

[12] This is certainly true if logical consistency implies the existence of interpretations (for example, in first-order logic). For if a system is categorical but not d-complete, one can obtain two consistent extensions of it, a particular sentence belonging to one and its negation to the other. If they both have interpretations, one has a contradiction; the same sentence being true in one interpretation but false in another, both interpretations, however, being isomorphic to each other.

[13] See Husserl (1970a, p. 382), where after claiming that arithmetic is, by general consensus, an a priori science, he says that "darin liegt, dass sie nicht mit singulären Tatsachen beginnt, um sich durch Induktion zu wahrscheinlichen Allgemeinheiten zu erheben, sondern alsbald mit gewissen, und zwar apodiktisch gewissen und unmittelbar evidenten Allgemeinheiten, die sie durch blosse Vergegenwärtigung gewisser ‚Grundbegriffe' und die auf dem Wege mittelbarer Evidenz und Gewissheit alle weiteren Sätze der Wissenschaft liefern". In short, a priori theories do not begin with induction but conceptual intuition, from where they proceed by the way of logic alone. See also Husserl (1984, Sect. 13) and Husserl (1994, p. 37).

sion of labor *characterizes* modern axiomatics).[14] So, for Husserl, conceptual intuition *must* be capable of delivering enough conceptual truths from which *all* the remaining truths *should follow* by logical deduction. In other words, adequate axiomatizations must be d-complete; d-completeness stands as the ideal to which axiomatization must strive. Despite formalist readings, for Hilbert too axioms are *conceptual* truths obtained by inquiring *intuitively given* concepts (that of space in the case of geometry); desinterpretation serving metalogical purposes only.[15] For both Husserl and Hilbert, axiomatic systems describe their domains by describing the *concepts* under which they fall. Axiomatization proceeds by assembling self-evident axioms in the hope of eventually obtaining a basis of just enough truths (not too many, not too few) from which *all* relevant conceptual truths can be obtained by logical means (intuition leaving, then, the scene). Hence, axiomatizations that are not d-complete fall short of an *ideal*, for Husserl, Hilbert, and, for that matter, ourselves.

Consider, for example, Dedekind's informal axiomatization of the arithmetic of natural numbers. The most salient structural property of natural numbers (to be arranged in a *chain* in Dedekind's terminology), despite its intuitive foundation, does not single out the chain of natural numbers; Dedekind had then to add the requirement that the intended chain be the "smallest" one (which requires second-order logic to be fully expressed).[16] Dedekind's second-order arithmetic is categorical, but in virtue of Gödel's first incompleteness theorem it is not d-complete. Obviously, neither Husserl nor Hilbert had, around 1900, the information provided by Gödel's theorem, and would have no reason to believe that d-completeness could not be universally attainable or that non-extendibility (categoricity) would not be sufficient for attaining it (although, for Husserl, illicitly so). But even if categoricity logically implied d-completeness, completeness could not be obtained, or so Husserl believed, as one can infer from his reaction to Hilbert's completeness axiom, the way Dedekind did, by simply fixing one intended interpretation axiomatically. The goal is *d-completeness*, but by means of *conceptual analyses*.

Hilbert also had to add "selection" axioms (also second-order) to his axiomatizations of geometry and the arithmetic of the real numbers. He, however, contrary to Dedekind, wanted to ensure that the intended domain was the "largest", with a maximum of elements. So, his "selection" axiom was designed to pick among the many interpretations that which was non-extendible.

Husserl was very critical of such ad hoc axioms, which works, so to speak, from the outside.[17] For him, as already observed, axioms had to be obtained by a conceptual analysis of the concept presiding over the domain the axiomatic system is designed

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[14] See e.g. Husserl (1984, Sect. 8), where Husserl says that "after mathematicians [...] have formalized their work , they proceed pure mechanically [...]" (ibid, 26); see also ibid, 32. As for the secondary literature, see, for example, Ortiz Hill's "Husserl on Axiomatization and Arithmetic", in Hill and Silva (2013, pp. 93–114) .

[15] Likewise Husserl, Hilbert gave conceptual intuition a fundamental role in mathematical axiomatics. In *metamathematics*, however, one is allowed to take axiomatic systems as intuition-free "games" with symbols.

[16] See his *Was sind und was sollen die Zahlen*, of 1888.

[17] See Husserl (1970a, pp. 441–442).

to master *theoretically*. The *only* way an axiomatic system can be erected entirely *a priori* is by submitting the concept presiding over its domain to conceptual analysis in conceptual intuition. Only by reflecting on a concept given in full clarity can a system of fundamental conceptual truths be derived from which all the truths of the domain over which this concept presides can be deduced by strictly logical means. Axioms, for Husserl, are not based on induction, nor can they be, as for Dedekind and Hilbert, means of getting what one wants by brute force. According to Husserl, despite starting on the right track, by submitting "our perception of space" to "logical analysis", Hilbert eventually abandoned that approach by adding an axiom of completeness to the system. Husserl believed that *epistemologically adequate* axiomatizations could not resort to selection axioms, *even at the price of admitting non-intended interpretations*. In short, for Husserl, as for Hilbert, d-completeness was still the goal, *even if, and here is where Husserl's logical approach diverges from Hilbert's, uniqueness of domain cannot be granted*.

Since Husserl insisted on *intuitive foundations* for axiomatics, but did not believe that selection axioms qualified as such, he had to face a problem that Hilbert did not: how to guarantee that the theory designed to describe a given domain (circumscribed by its ruling concept), *and that has failed to singularize it* (by allowing interpretations that *extend* the intended one), can nonetheless accomplish the task of theoretically mastering *the intended domain* in a complete manner? In what follows I shall present what I believe to be Husserl's treatment of this problem, translated into the conceptual apparatus I introduced earlier.

Let T be our theory and A its *intended* interpretation. *Ideally*, from Husserl's perspective, the concept presiding over A (that of number, for example, or space) must have been "dissected" to the point of providing T with enough conceptual truths, its axioms, so that it can be the undisputed "master" of its domain.[18] Such a theory is, in *some* sense, *complete*. But there are, as we know, many senses of completeness. One, however, imposes itself, given Husserl's understanding of the *purpose* for which theories are *designed*. For him, a theory is *first and foremost a provider of truths*, of a given concept or, derivatively, its extension (its domain). The primary task of an axiomatic system is to organize a domain of truths (an apophantic domain) so that *all* truths would follow by logical necessity from a basis of axiomatic (intuitive) truths, not necessarily, and certainly not primarily, to capture the intended interpretation descriptively.[19] So, ideally, T should be d-A-complete, capable, that is, of deciding as to its "truth" or "falsity" any relevant assertion concerning its domain A (whatever T proves must be true in A and any assertion true in A must be provable in T). So, the condition of *completeness* for a theory translates naturally, given Husserl's own understanding of axiomatic theories, into d-A-completeness or, equivalently, d-completeness.

[18] By "domain" I here mean the intended interpretation or, more specifically, its universe, the collection of objects falling under the concept the theory is designed to scrutinize. Husserl often uses this term (*Gebiet*) in a slightly different sense, as the collection of objects the theory explicitly or implicitly *requires* to exist. We must keep this ambiguity in mind. The concept of relative definiteness applies to either conception of domain.

[19] Since we know from the start to which domain the theory refers, to capture it descriptively (categorically) can only be a means to an end, namely, to describe it exhaustively (completely).

Now, Husserl might have realized, given what he knew of the efforts of Dedekind and Hilbert to axiomatize three of the most basic mathematical theories, the arithmetic of natural numbers, the arithmetic of real numbers (analysis) and geometry, whose basic concepts are, respectively, natural number, real number, and space, that this goal is not easily attainable. Dedekind and Hilbert could only fulfill the ideal by adding ad hoc axioms which guaranteed d-completeness by the way of *categoricity*.[20] But Husserl *explicitly* rejected such an approach; "selection axioms" for the sake of completeness were, for him, out of the question. He then had to face the possibility that a theory, no matter how diligently and carefully designed, had *unintended* interpretations, in particular interpretations extending the intended one.

Husserl always believed that d-completeness is an *ideal* to be pursued by diligent conceptual analyses.[21] But while still maintaining this ideal he had to find a solution for the problem posed by d-incomplete theories, particularly if they admitted *unintended* interpretations, since he could not accept "selection axioms". The problem he was facing can, then, be put thus: can theories with non-intended interpretations (extending, in particular, the intended one), and so d-incomplete, be nonetheless d-complete at least *relative to their intended interpretations*?

My interpretation of Husserl's treatment of this question takes as starting point the fact that, for him, *a priori* axiomatic systems, to the extent they have *intended* interpretations, are *conceptual* systems, that is, systems of conceptual truths.[22] A concept, like any *intentional object* in the phenomenological perspective, is *conceived* (or conceptualized) with a certain sense, its *intentional meaning*; it counts as a characterization of the concept (or, if put into words, an *explicit* definition of it). But the intentional meaning is only a sort of identity card of its associated intentional object. In order to "bring out" the truth the object may contain, we must go beyond its intentional meaning, bring the object *itself* as meant to *intuition*, if the theory of the object is to have intuitive foundations, and examine it from up close. There is, then, a *meaning* associated with a concept and *conceptual truths* spelling out this meaning. As we shall see, (superficial) meaning, as well as (deeper) conceptual truths, play a role in my reading of Husserl's treatment of the phenomenon of d-incompleteness, to which I now turn.

Let us add to the language of T an extra unary relation symbol R whose task is to restrict the domain of quantification to the domain of the theory. In order to guarantee that the scope of the quantifiers is confined to the domain of the theory, extra axioms (in the extended language including the new symbol R) are added to T, which, in some sense, "express" the *intentional meaning* attached to the concept presiding over the

[20] Dedekind, of course, did not come up with a d-complete theory, but he did not know that and we can conjecture that he believed he had attained this ideal.

[21] Gödel, a thinker influenced by Husserl, also believed so. For him, the independence of certain assertions concerning set theory (given its d-incompleteness) could be overcome by a better intuition of its ruling concept, that of set (the incompleteness of the theory, however, as he himself had shown, can never be completely overcome). Gödel took very seriously Husserl's notion of conceptual intuition and its role in axiomatics.

[22] A priori non-interpreted axiomatic systems, those that do not have intended interpretations, are an altogether different matter. For Husserl, as already mentioned, they belong to formal ontology, that part of logic that cares about the *form* with which object *in general* can present themselves to us.

domain. Let us call this extended theory T,R. These extra axioms are added to T with a *single* purpose, to guarantee that the interpretation of R in A is the universe of A and that in any interpretation B *extending* A the interpretation of R in B is still the universe of A. Even if T admits non-intended interpretations *extending* A, they can only be expanded to interpretations of T,R if R is interpreted as the universe of A. In short, in all interpretations of T,R extending A, the interpretation of R contains the same collection of objects, the "intended" ones. I claim that according to Husserl a theory T is *definite relative to its domain* A (the intended interpretation, its universe or, still, its existential domain, that is, the collection of objects the theory *requires* to exist) if T,R is d-R-complete; i.e. given any sentence φ in the language of T, either T,R $\vdash \varphi_R$ or T,R $\vdash \neg \varphi_R$. In words, the theory, extended with axioms that fix an intended domain, can decide as to its "truth" or "falsity" any sentence of the language that "refers" to this domain.

In addition to Husserl's explicit definitions and comments (quotes below), I add the following in support of this interpretation. First, Husserl distinguishes between *relative* and *absolute* definiteness, implying that a theory can be definite relative to its domain and *not* be absolutely definite. Since for him, as I claim, an *absolutely definite* theory is one that is d-complete, then relatively definite theories need not be d-complete (and so, a fortiori, they need not be categorical; i.e. relatively definite theories may admit non-intended interpretations, which the criticized axiom of completeness of Hilbert tries to eliminate by means that Husserl deemed inappropriate). Secondly, For Husserl, despite possibly being d-incomplete, relatively definite theories must decide what is and what is not true *in their domains*. By forcing R to refer to the intended universe of discourse, via the new axioms, one forces the restriction of any arbitrary sentence to R to refer to the intended domain. So, by requiring T,R to be d-R-complete one grants that every sentence *that refers to the intended domain* is deductively decidable, which is precisely what Husserl required of theories that are definite relative to their domains.

This interpretation tries to capture Husserl's *intentions* in contemporary terms, as inferred from his *own words*. I, however, do not claim that this is a "better" logical solution than Dedekind's and Hilbert's selection axioms, even though Husserl himself thought so throughout his entire philosophical career. My only claim is that, despite of what one may think of it, this, or something very close to this, better translates Husserl original notions of definiteness into more modern terms. My goal is not to vindicate Husserl's conceptions, as I read them, as the absolutely correct or most convenient way of dealing with the problem he had on his hands, but to make the point convincingly that the solution to the problem which I interpret as his is the one that best suits Husserl's *philosophical* outlook.

For Husserl, a theory must be "master in its domain", that is, ideally, absolutely definite (d-complete). A theory with unintended interpretations can only fulfill this desideratum by being definite relative to its intended interpretation. If T is not absolutely definite (d-complete) there may be sentences that are undecidable in T. If, however, T is definite relative to its domain (d-R-complete) a decision can nonetheless be reached insofar as these sentences refer to the domain of the theory.

The d-incompleteness of a relatively definite theory T may even be of some heuristic interest. Suppose that a given sentence ψ is *false* in A but *true* in some interpretation

B of T extending A. This, of course, implies that $\neg\psi_R$ is derivable in T,R but that neither ψ nor $\neg\psi$ are derivable in T. Let $T' = T \cup \{\psi\}$, which, by the previous observation, is a logically consistent extension of T.[23] Suppose now that φ is derivable in T'. Is φ_R *true* in A? Since T' is a consistent extension of T, T cannot prove $\neg\varphi$, but this is not inconsistent with T,R proving $\neg\varphi_R$. So, since this cannot be granted, φ_R is not *necessarily* true in A. For instance, suppose that φ requires a certain object to exist. In all interpretations of T', that is, all interpretations of T *where* ψ *is true*, such an object must exist. In A, however, it may not. In this case, in all interpretations of T that extend A where ψ is true the object that φ requires to exist can*not* be in the universe of A. From the point of view of A this object is *imaginary*. But it may happen that φ_R *is* true in A. In this case the object that φ requires to exist does exist in A and the T-undecidable sentence ψ played an important *heuristic* role, revealing the existence of an object in A maybe *before* the theory expressly designed for the theoretical investigation of A could detect it. The extension of T into T' may prove to be an efficient heuristic instrument for the derivation of *conjectures* regarding the intended interpretation, which, given the relative definiteness of T, can be put to test (in T,R) and either confirmed or refuted.[24]

Now, more importantly, Husserl clearly saw how the notion of a theory that is *not* d-complete, but that is d-complete relative to its domain, can solve the important problem of "imaginaries". Suppose that φ_R, a sentence *referring* to the domain of T, is derived in the theory T' as defined above. Suppose $T,R \vdash \neg\varphi_R$, then $T,R,\psi \vdash \neg\varphi_R$; but $T, \psi \vdash \varphi_R$, by hypothesis, and so $T,R, \psi \vdash \varphi_R$. A contradiction, for the theory T,R, ψ has an interpretation, namely, the expansion of B, the interpretation of T where ψ is true and R is interpreted as the universe of A. Hence, $T,R \vdash \varphi_R$ and, consequently, φ_R is true in A. The conclusion is that the adjunction as extra axioms to T of assertions that are *false* in A, but *true* in some extension of A, provided that T is *definite relative to its domain*, can be used to derive assertions that, *if referring to the domain of T*, are *necessarily true* in A. This vindicates logically the appeal to "absurdities" as logical tools for the derivation of truths, thus solving what Husserl called "the problem of imaginaries".

Let us produce some textual evidence for my interpretation.

This comes from Husserl's 1901 talks at Göttingen (in the version of Elisabeth and Karl Schuhmann). These talks are the *locus classicus* of Husserl's treatment of the concept of definiteness (completeness). He says:[25]

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[23] I am supposing a logical context in which a theory can be consistently extended by the adjunction of sentences that are logically independent of the theory.

[24] I find this observation relevant to understand the *heuristic* role of mathematical manipulations in purely mathematical extensions of *empirical* theories, the famously called "unreasonableness" of mathematics in empirical science.

[25] "Eine axiomatisch definierte Mannigfaltigkeit kann die Eigenschaft haben, daβ jedes ihrer Objekte operativ bestimmbar ist, und zwar eindeutig. D. h. jedes Objekt, das für sie als existierend definiert ist (in die Sphäre der Existenz gehört, welche die Axiome umschreiben), ist durch die zugrunde liegenden oder eine endliche Zahl willkürlich anzunehmender bestimmter Existenzen unmitteelbar oder mittelbar zu bestimmen, und zwar eindeutig. Eine solche Mannigfaltigkeit ist eine mathematische und ist definit (d.h. ihr Axiomensystem ist definit). [...] Relativ definit ist ein Axiomensystem, wenn es zwar für sein

> An axiomatically defined manifold can have the property that each of its objects is operationally determinable univocally. This means that every object that is defined for it as existing (belongs to the sphere of existence circumscribed by the axioms) is to be directly or indirectly univocally determined by the underlying ones, or a finite number of existing ones to be assumed arbitrarily. Such a manifold is mathematical and definite (i.e. its axiom system is definite). [...] An axiomatic system is relatively definite if, of course, it does not admit any more axioms for its existential domain but allows that the same, and then naturally also new, axioms hold for another domain, new axioms because the old axioms alone really determine only the old domain. Relatively definite are the domains of integers, of fractions, of rational numbers, as well as the discrete double-row numbers (complex numbers). I call a manifold absolutely definite if there is no other manifold that has the same axioms as it (taken all together). Continuous numbers series, continuous double-row numbers.

In this quote Husserl defines definiteness for *both* a theory and its domain. He also characterizes *mathematical* domains, but this notion will not concern us here. A *domain* is *definite* when its axiom system is definite. Here, however, Husserl does not take "domain" to refer to the intended interpretation, or its universe, but to the collection of entities the axiom system requires to exist. For example, the arithmetic of natural numbers requires the existence of 0 and the successor of every number that exists; so, 1, the successor of 0, 2, the successor of 1, and so on, must all exist; geometry requires points, lines and planes to exist.[26] I believe that we can, without causing any serious damages, maintain the ambiguity and understand by "domain" either the totality of objects of the intended interpretation or those the system requires to exist, its *existential domain*.

An axiomatic system is *relatively* definite, he continues, when (1) it does not admit extra axioms for its domain but (2) admits extra axioms for non-intended interpretations. A few things are worthy of notice in this definition. First, a relatively definite axiom system can be consistently extended by the adjunction of new axioms, and so it is not *necessarily* d-complete. Second, since the narrower system T has an interpretation (its domain) that is *different* from interpretations of its extension T' by the adjunction of new axioms, and since these interpretations are also interpretations of T (for T' *extends* T), T has *non-intended* interpretations, and so cannot be categorical.[27]

Footnote 25 continued
Existentialgebiet keine Axiome mehr zuläβt, aber es zuläβt, daβ für ein weiteres Gebiet dieselben und dann natürlich auch neue Axiome gelten. Neue Axiome, denn die bloβ alten Axiome bestimmen ja nur das alte Gebiet. Relativ definit ist die Sphäre der ganzen, der gebrochenen Zahlen, der rationalen Zahlen, ebenso der diskreten Doppelreihenzahlen (komplexen Zahlen). Absolut definit nenne ich eine Mannigfaltigkeit, wenn es keine andere Mannigfaltigkeit gibt, welche dieselben Axiome hat wie sie (alle zusammen). Kontinuierliche Zahlenreihe, kontinuierliche Doppelzahlenreihe" (Schuhmann and Schuhmann 2001, pp. 101–102).

26 A *mathematical* domain is one in which all the elements that exist are obtained as the closure of a basis of given elements by some set of operations. The idea is close to our notion of a freely generated structure.

27 Gauthier (2004, p. 124) also denies that by definiteness Husserl meant categoricity. He believes, however, that what was meant was semantic completeness. From my point of view, he is not so off the mark, but since I think Husserl clearly operates with a syntactic notion of deduction, syntactic completeness is, I believe, the correct reading.

My interpretation of Husserl's notion of relative definiteness (that is, definiteness relative to the domain of the theory) has both properties 1 and 2 above. The quote closes with a definition of absolute definiteness *for manifolds*. A manifold is absolutely definite, Husserl says, if there is no other manifold with the same axioms. It is, I admit, tempting to read this as requiring that the theory of an absolutely definite manifold is categorical, and so, supposing that an absolutely definite manifold is one whose theory is absolutely definite, that absolutely definite theories are categorical theories. If, however, we interpret, as I propose and as Husserl *explicitly* says in other places, that an absolutely definite theory is a d-complete theory, then the set of sentences (of the language of the theory) that are true in two different interpretations of such a theory *must coincide*. Hence, *from the perspective of the relevant language*, that is, of what can be said, there is essentially only one interpretation of an absolutely definite theory. This may be what Husserl had in mind here, that all interpretations of an absolutely definite axiom system are *theoretically indistinguishable*.

But there is another, possibly better reading of this. Suppose T is absolutely definite, i.e. in my interpretation, d-complete. Let's now by "domain" understand "existential domain", that is the set of objects whose existence the theory implicitly or explicitly requires. Since no element of a domain can exist that is not required to exist by its axiom system, as Husserl seems to demand here (first part of the quote), and since every element an absolutely definite system can define has already been defined and introduced in its "existential domain" (d-completeness), this domain cannot be *enlarged* and so given rise to *another* domain with the *same* system of axioms. Hence, an absolutely definite axiom system can have *only one existential* domain. Absolute definiteness is, then, in this reading, categoricity of a sort, but only with regard to *existential domains*.

So, no matter which interpretation one favors (I believe the second accords better with the letter and spirit of the quote), one is relieved of the burden of having to force upon Husserl an interpretation that he never explicitly allowed or could have implicitly meant.

Still in the Gottingen talks [Husserliana version, Husserl (1970a, pp. 441–442)], Husserl says that a domain of things (*ein Sachgebiet*) can be delimited (*umgrenzt*) (the term "umgrenzt" indicates, I believe, that Husserl has *non-extendibility* in mind) by an axiom system in either a complete or incomplete (*vollständig und unvollständig*) manner. The axiom system is complete, he continues, if it contains an axiom of completeness stating that "by these and only these axioms the domain is determined and no other [*axiom, JJS*] is therein valid").[28] If no such axiom belongs to the system, he claims, new axioms can be added to it and, as a consequence "the objects of the domain can be formally defined by new determinations".[29] However, to grant non-extendibility of the domain directly by means of an axiom of completeness, as Hilbert does, is for Husserl an *inauthentic* (*unechte*) way of proceeding, for, as he says, *any* axiom system can be so completed. Although the *inauthentic* axiom of completeness can assure non-extendibility, and, more importantly d-completeness, it does so in a trivial manner that does not interest Husserl.

[28] "Durch die und die Axiome ist das Gebiet bestimmt und altere gelten nicht", Husserl (1970a, p. 442).

[29] "Die Objekte des Gebietes formal durch neue Bestimmungen definiert sein können", Husserl (1970a, p. 442).

He faced, then, a problem: "We ask, then, if there are axiom systems that do not contain the axiom of closure but can, for each proposition, decide whether it belongs to the domain of deduction as to its truth or falsity".[30] This problem *obviously* has to do with deductive decidability of every proposition *referring to the domain*. An *authentic* way of granting non-extendibility of a system's *existential domain* is by requiring its *apophantic domain* to be non-extendible, that is, the absolute completeness (d-completeness) of the system. But, even if this is a *sufficient* condition for non-extendibility, it is not a *necessary* one, a weaker notion can accomplish the task, and this is where relative definiteness comes in. An axiom system that is definite relative to its existential domain cannot be enlarged by the adjunction of *any* independent sentence *referring* to this domain. So, no *new* object can be introduced in the system's existential domain; it is non-extendible. The problem is solved. Authentic non-extendibility is achieved in the most economic way.

The explicit definitions of definiteness in the Gottingen talks are the following:[31]

Definite in a restricted manner or relatively definite = definite in the sense so far—absolutely definite:

(1) An axiom system is *relatively definite* if each assertion that has a sense with respect to it is decided relative to its domain. An axiom system is absolutely definite if any assertion that has a sense with respect to it is decided in general. So, absolutely definite = complete in Hilbert's sense.
(2) If not only no axiom can be added for the objects of the domain (which receives its sense from the axioms already in place), but if no axiom in general can be added.
(3) But this implies that the multiplicity (the domain) cannot be enlarged and such that for the enlarged domain the same axioms are valid that are valid for the old one.

In (1) of this quote Husserl *explicitly* characterizes relative definiteness of an axiom system as *decidability* of sentences referring to its domain and absolute definiteness as decidability *tout court*, just as I interpret him. The comment that follows identifies absolute definiteness with completeness "in Hilbert's sense". What Husserl seems to have in mind is completeness as non-extendibility as expressed in Hilbert's *axiom* of completeness. Obviously, no interpretation of a d-complete theory can be *extended* by the *adjunction* of *definable* elements.[32]

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[30] "Wir fragen nun also, ob es Axiomensysteme gibt, die keine Schliessungsaxiom enthalten und doch, nämlich aufgrund ihrer besonderen Natur, es jedem Satz ansehen lassen, ob er in die Sphäre ihrer Deduktion nach Wahrheit und Falschheit gehört".

[31] "Beschränk definit oder relativ definit = definit im bisherigen Sinn—absolut definit: (1) Relativ definit ist ein Axiomensystem, wenn jeder nach ihm sinnvolle Satz in Beschränkung auf sein Gebiet entschieden ist. Absolut definit ist ein Axiomensystem, wenn jeder nach ihm sinnvolle Satz überhaupt entschieden ist. Also ist absolut definit = vollständig in Hilbertschen Sinn. (2) Wenn nicht nur für die Objekte des Gebietes kein Axiom hinzugefügt werden kann (das durch schon gegebene Axiome Sinn erhält), sondern wenn überhaupt kein Axiom hinzugefügt werden kann. (3) Darin liegt aber, dass die Mannigfaltigkeit (das Gebiet) nicht so zu erweitern ist, dass für das erweiterte dasselbe Axiomensystem gilt wie für das alte." (Schuhmann and Schuhmann 2001, p. 103).

[32] Compare with the interpretation I advanced above of Husserl's claim that another manifold cannot exist with the same theory of an absolutely definite manifold.

In (2) Husserl gives the immediate *consequences* of both notions, namely, the impossibility of adding new axioms to a relatively definite theory that *refer to its domain*, which in the previous quote appears as the essential trait of relative definiteness, and the impossibility *tout court* of extending an absolutely definite system by the adjunction of new independent axioms.

In (3) Husserl makes explicit the association of absolute definiteness with non-extendibility: the domain of an absolutely definite axiom system cannot be enlarged and still satisfy the same axioms. Although an absolutely definite system cannot be enlarged at all, provided one preserves the language, a relatively definite system can be enlarged by the adjunction of new axioms. But, first, these axioms must be undecidable in the narrower theory and, second, any new element they introduce cannot belong to the original "existential domain". So, in both cases, relative or absolute definiteness, the original "existential domain" cannot be enlarged.

Let's now consider a later work, *Formal and Transcendental Logic* (Husserl 1969), particularly Sect. 31. In it, Husserl gives an unambiguous characterization of the notion of definiteness, and it is clear that it is still the same of his earlier days. He first defines a *multiplicity*, that is, a domain of being,[33] to be *definite* if the "whole infinite system" of assertions that are *true* in it can be derivable from a *finite* set of axioms. In other words, a multiplicity is definite if its theory (the set of all assertions true in it) is finitely axiomatizable. This axiom system is then necessarily d-complete. The definiteness of a domain of being, Husserl says, fulfills the Euclidean ideal of a domain whose truths are "fully disclosed in a theory".

The notion of definiteness can be extended to formal multiplicities, that is, multiplicities defined by non-interpreted axiom systems. A formal multiplicity is definite if the axiom system that characterizes it (defines it "implicitly") is d-complete; only thus the "whole science-form [...] can be derived by pure deduction". "'Theoretically explainable province"' Husserl says, "and 'definite system of axioms' are equivalent". The idea of a province *completely* mastered by a theory requires, for Husserl, that this theory be *complete*, that is, that it is able to derive all that is true in the province. *This* is the Euclidean ideal, which, if extended to formal domains, requires their theories to be d-complete.

After characterizing definiteness for multiplicities, Husserl says that the term "complete system of axioms" he uses in this section of *FTL* as a synonym for "definite system of axioms" derives from Hilbert, who, he says, "attempts [...] to complete a system of axioms by adding a separate 'axiom of completeness"' Since Hilbert axioms of completeness make the axiom systems to which they are added categorical, some may feel tempted to conclude that, for Husserl, definiteness meant categoricity. But in keeping with what I said earlier, I reaffirm that the *primary* goal of Hilbert completeness axioms is to ensure *non-extendibility*. Hilbert's axioms of completeness, moreover, ensured d-completeness of the systems (or so both Hilbert and Husserl believed), and it is *this* that Husserl found relevant and synonymous to his notion of definiteness.

The same ideas reappear in Husserl 1970b, especially section 9f, where Husserl says: "Among multiplicities those that are called '*definite multiplicities*' distinguish

[33] "Multiplicity means properly the *form-idea of an infinite object-province for which there exists the unity of a theoretical explanation* or, in other words, the unity of a *nomological science*" (Husserl 1969, p. 95).

themselves, whose definition by a *complete system of axioms* give the formal object-substrata therein contained, in all their deductive determinations, a totality of a particular nature, with which one constructs, so to speak, *the logic-formal idea of a world in general*". The idea is clear; the formal domain of a non-interpreted theory is *definite* when this theory is d-*complete* in the sense that it gives the domain of the theory the character of a "world", that is, a realm of being in which every assertion about it that has a sense is either true (deducible in the theory) or false (its negation is deducible).

The concepts of definiteness are, moreover, just what he needed to solve the "problem of imaginaries". In his words, the problem is this[34]: "The tendency towards formalization manifested in arithmetical algebra led to forms of operation [*Operationsformen*] that were arithmetically meaningless, but which showed the noteworthy [*merkwürdige*] property of being, nonetheless, capable of being employed in the calculus". In other words, one can write in the language of the arithmetic of positive integers meaningless terms such as (2–3) or $\sqrt{-1}$ (denoted, respectively, by -1 and i) that do not denote anything, or better, denote *imaginary* entities. The problem originated in the fact that by pretending these symbols denote numbers proper and manipulating them as equal to denoting symbols a *more powerful* calculus can be obtained. But, Husserl asks, and this is, for him, where the problem lies, if a meaningful assertion can be derived in this calculus, are we justified in taking it as true?

In Husserl's words[35]:

> The narrow domain D has axioms A_D, the ensemble of purely logical consequences C_D, the rest of the domain Γ for example $A_D + A' = A_\Gamma$; or $A_\Gamma \supset A_D$, so the consequence (C = consequence): $C_\Gamma = C_D + C_{A'} = C_{D+A'}$. If a proposition does not contain enlarged operation complexes [*Operationskomplexionen*] [*imaginary numbers, my note*], it is not self-evident that it belongs to C_D.

Husserl supposes than that a narrow domain D is enlarged, by the adjunction of imaginary elements, into a domain Γ *such that* the axioms of the extended domain constitute a *consistent extension* of the axioms of the narrow one. Now, he says, it is not self-evident that a sentence in the language of A_D proved in A_Γ can be proved in A_D. In contemporary terminology, it is not obvious that any consistent extension is *conservative*. (Husserl commits an error here, it is not always the case that $C_D + C_{A'} = C_{D+A'}$. In general $C_D + C_{A'} \subseteq C_{D+A'}$).

For Husserl, the solution of this problem is the following[36]:

> A passage through the imaginary is allowed: (1) if the imaginary can be formally defined in a larger consistent system of deduction and if (2) the formalized original domain of deduction has the property that any sentence belonging to

[34] Husserl (1970a, p. 432).

[35] Husserl (1970a, pp. 439–440).

[36] "Ein Durchgang durch die Imaginäre ist gestattet: (1) wenn das Imaginäre sich in einem konsistenten umfassenden Deduktionssystem formal definieren lässt und wenn, (2) das ursprüngliche Deduktionsgebiet formalisiert die Eigenschaft hat, dass jeder in dieses Gebiet fallende Satz entweder aufgrund der Axiome dieses Gebietes wahr oder aufgrund derselben falsch, d.i. mit den Axiomen widersprechend ist" (Husserl (1970a, p. 441)).

this domain is either true on the basis of the axioms of the domain or else, it is false on the same basis, i.e. is in contradiction with the axioms.

In other words, imaginaries are allowed provided the original axiom system is definite relative to its domain.

For both Husserl and Hilbert, axiomatic systems, designed to organize domains of knowledge according to the Euclidean (or Aristotelian) ideal of theory, must obey some criteria of adequacy. The question is which. There are different criteria of success for axiomatic system, categoricity, s-completeness, d-completeness, to name the most important, and different perspectives from where to judge the degree of success of a determinate axiomatization. Do all the truths referring to a *particular* domain of being (for example, the natural numbers) follow by logical necessity from the basic axioms of the domain (d-A-completeness)? Is this domain exhaustively characterized up to logical equivalence (d-completeness)? Is it exhaustively characterized up to formal or algebraic equivalence (categoricity)? Does the system characterize the intended domain *itself*, not merely a *sub-domain* of it that happens to satisfy the same axioms (non-extendibility)? These notions differ subtly one from another and, even when logically equivalent, they display different shades of meaning. To make things worse, they can all be given the same denomination: completeness. The word "complete", when referring to axiomatic systems, seems naturally interpretable as meaning that the system does not need further axioms, that it is complete ("to say that an axiomatization is complete is [...] to say that the axiomatization has achieved its goal, in particular that no further addition of 'new axioms' is called for"—Awodey and Reck 2002, p. 5). Misinterpretations are bound to arise, particularly in relation to texts written when these notions were not clearly distinguished and, some of them, not even clearly formulated (or even formulated at all).

By about 1899–1900, both Husserl and Hilbert knew that axiomatic systems could be desinterpreted and differently reinterpreted, but seemingly neither had a clear idea of the extent to which reinterpretations could be carried out. There was not at the time a clear distinction between syntax and semantics and, despite Frege, no very careful delimitations of valid logical axioms and rules of inference; neither Hilbert nor Husserl were particularly concerned about making explicit the logical context in which axiomatizations were to be carried out. There was no distinction made between first and higher-order languages and obviously no sign of the modern obsession with first-order logic (in fact, the traditional conception of logic as the theory of *concepts*, not first-order logic, naturally dominated). So, confusion and misinterpretations are almost inevitable. The interpretation I put forward here, however, is I believe not only the one that better accords with Husserl's words but, more importantly, the one that better accords with his *philosophy*.

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