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Branching time, indeterminism and tense logic

Unveiling the Prior–Kripke letters

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Abstract This paper deals with the historical and philosophical background of the introduction of the notion of branching time in philosophical logic as it is revealed in the hitherto unpublished mail-correspondence between Saul Kripke and A.N. Prior in the late 1950s. The paper reveals that the idea was first suggested by Saul Kripke in a letter to A.N. Prior, dated September 3, 1958, and it is shown how the elaboration of the idea in the course of the correspondence was intimately intervowen with considerations of how to represent indeterminism and of the adequacy of tensed logic in light of special relativity. The correspondence underpins the point that Prior's later development of branching time may be understood as a crucial part of his attempt at the formulating a conceptual framework integrating basic human notions of time and free choice.

Keywords Branching time · Indeterminism · Tense logic · Free choice · Special relativity · Saul Kripke · A. N. Prior

1 The Diodorean modality and Kripke's branching time model

In the early 1950s Prior had become interested in the ideas on time and modality studied by the Megarian logician, Diodorus, who was a younger contemporary of Aristotle.

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Diodorus had argued in favour of determinism, and he had suggested that the concept of possibility should be understood in terms of temporal notions. In fact, according to Diodorus something is possible if and only if it is true now or at some time in the future. In Prior (1957a,b) discussed the properties of this Diodorean modality. In Prior (1957b) suggested that a proposition should be represented as an infinite sequence of 'true' and 'false'. For instance, a proposition, p, might correspond to the following sequence of 'true' and 'false': 131333333133133 and only 3's after this, where '1' is used for 'true' and '3' for 'false. This sequence represents the present, i.e. the first '1', and the future development. The past is not taken into account in this model. Given the proposition, p, we may form the proposition, $\Diamond p$, corresponding to 'possibly p'. (Prior himself symbolized this as Mp, and 'neccesarily p' as Lp.) The sequence for $\Diamond p$ can easily be constructed from the sequence corresponding to p. An element in the $\Diamond p$ sequence is 1, if and only if there is a 1 at the same place or at a place to the right (i.e. later) in the p sequence. Using this Diodorean notion of possibility we can easily find the sequence corresponding to the proposition, $\Diamond p$, where p is the proposition (sequence) mentioned above:

p	1	3	1	3	3	3	3	3	3	1	3	3	1	3	3	only 3's after this
$\Diamond p$	1	1	1	1	1	1	1	1	1	1	1	1	1	3	3	only 3's after this

Given this representation we may of course wonder to which axiom system the Diodorean modality would correspond. In a rather tentative chapter of his book, Prior suggested that the system in question is simply the modal system, S4. One of the first readers to react on the claims in Prior's book, *Time and Modality*, was Saul Kripke who was only 17 years old when he, in the letter to Prior reprinted below, pointed out that although fruitful Prior's considerations of modality were flawed by an error. Young Saul Kripke then continued his letter by explaining that the formula,

$$\Box\Diamond p \vee \Box\Diamond \sim p$$

can be verified using Prior's sequences and his definition of Diodorean modality but that this formula can be shown not to be provable in S4. However, even more important was the following passage from Saul Kripke's letter in which he suggested how the semantics of S4 could be visualized. Kripke's formulation of this very original idea in the letter makes it reasonable to classify the letter as one of the most important events in the history of logic during the twentieth century.

In this way Saul Kripke argued that S4 corresponds to a branching time system combined with the Diodorean notion of temporal modality. This presentation of branching time as a logical system is the first ever, and it was clearly recognised by Prior, who in later writings referred to it as "Kripke's branching time matrix for S4" (Prior, 1967, p. 27).

2 Branching time and indeterminism

As stated by Kripke, his notion of branching time may also be an appropriate way of representing indeterminism as understood by Prior. While recognising that freedom of choice is very limited, Prior in later writings professed that freedom of choice is real in the sense that the future is something we may to some extent make for ourselves (Prior, n.d., p. 48), published in Copeland (1996). Kripke's notion of branching time leaves room for this understanding of free choice by representing the present as having different possible, alternative futures—the content of the future is not fixed in such a way as to allow for only one possible progression of the world. From the present the world may take different paths into the future depending on, for instance, the choices of agents'. In the course of the correspondence the contours of Prior's later, more developed notion of indeterminism emerges.

Kripke's notion of branching time allows for an asymmetry between the past and future that in Prior's view was central to the notion of indeterminism. As Prior states in the fourth letter:

> For indeterminism asserts a certain difference between the future and the past (that one has always APnpPnNp, but not always AFnpFnNp) … a thing can only be undetermined if it is not-yet-past …. [L4: Oct. 27, 1958]

Translated from Polish notation into modern formalism Prior states that $P(n)p \vee P(n) \sim p$ is true in all possible cases, i.e. it is a theorem, whereas the same does not hold for $F(n)p \vee F(n) \sim p$. Here $P(n)$ means 'it was the case n time units ago', and $F(n)$ means 'it will be the case in n time units'. Now, Prior is here moving beyond Kripke's model of branching time and introduces some of the ideas to be further developed in his later writings.

On the face of it Kripke's model of branching time only justifies for the future operator, i.e. $F(n)$, to be interpreted as 'possibly, it will be the case in n time units'. The future consists of possibilities—and therefore any proposition about the future can only be true in the present when interpreted as a proposition about what may possibly be the case in the future. Consequently, given that $F(n)$ is to be interpreted as 'possibly, it will be the case in n time units' it follows that it may in some cases be true that $F(n)p \wedge F(n) \sim p$. Hence $\sim (F(n)p \wedge F(n) \sim p)$ is not a theorem in the system. However, in this interpretation it holds that $F(n)p \vee F(n) \sim p$.

By denying that it holds in all cases that $F(n)p \vee F(n) \sim p$, where $F(n)$ is defined as 'it will be the case in n time units', Prior is making a very illuminating claim in terms of defining indeterminism. Thus Prior equates indeterminism with the claim that some propositions p are of such a nature that neither $F(n)p$ nor $F(n) \sim p$ should be regarded as true in the present. Such propositions acquire their truth-value as and when the events they are about either occurs or not, i.e. in the present. Anticipating Prior's later insistence on the reality of freedom of choice the propositions that in this way acquire their truth-value in the present are those that depend on the free choice of some agent. Thus in Prior's opinion there is no truth about which future decision an agent will make until the agent has actually made his or her decision. However, regarding the past exactly one of the propositions, $P(n)p$ and $P(n) \sim p$, is true, since only one of the propositions, p and $\sim p$, corresponds with how things were n time units ago. In this way, indeterminism is precisely the asymmetry between past and future expressed here.

In later writings Prior contrasted this representation of indeterminism and the corresponding tense-logical system referred to as the Peircean system with the so-called

Ockhamistic system. In the latter system indeterminism would correspond to the rejection of the disjunction $\Box F(n)p \vee \Box F(n) \sim p$ as a theorem, whereas $F(n)p \vee F(n) \sim p$ would be accepted. Prior, however, preferred the Peircian system anticipated in his correspondence with Kripke.

3 Indeterminism and tense logic

Closely interwoven with Kripke and Prior's considerations of branching time and indeterminism are their reflections upon the adequacy of a logic based upon the tenses 'past', 'present, and 'future' instead of tenseless logic based upon the relations of 'earlier' or 'later than'. In his next letter to Prior dated October 13, 1958, Kripke expressed his doubts concerning a tensed logic in the following way:

> Is it your contention in Appendix A that a tenseless logic is really insufficient to represent the distinctions tense logic conveys? (…). I should think that, *for scientific discourse* a tenseless logic may be preferable. [L3: Oct. 13, 1958]

The Appendix A mentioned above is a separate part of Prior's *Time and Modality* (Prior, 1957b, pp. 102–122). In here, Prior referred to this statement made by Peirce in about 1903:

> I have thought that logic had not yet reached the state of development at which the introduction of temporal modifications of its forms would not result in great confusion; and I am much of that way of thinking yet. [C.S. Peirce 4.523]

Prior wanted to take up this Peircean challenge and to carry out the task by integrating tenses in logic. He argued that we should accept that "what is true at one time is in many cases false at another time, and vice versa" (Prior, 1957b, p. 104). If truth is tensed, i.e. if tensed propositions have tensed truth-conditions, as Prior seems to be arguing here, then clearly there is a case for accepting "tense distinctions as a proper subject of logical reflection"(Prior, 1957b, p. 104). Moreover, in his reply to Kripke dated October 27, 1958, Prior argued that only a tensed language can describe indeterminism. He wrote:

> You have a sentence beginning 'And if we accept indeterminism…', but I do not see how indeterminism can be expressed in a tenseless language at all. For indeterminism asserts a certain difference between the future and the past… [L4: Oct. 27, 1958]

Kripke's notion of branching time in combination with Prior's notion of indeterminism establishes an asymmetry between the past and the future. Certain expressions about the future are of such a nature that they remain possibilities until they acquire a truth-value in the present. In the quote, Prior is pointing out, it seems, that in a tenseless conception of the world there is no present—a 'now'—in which such sentences can acquire their truth-value. Without a present there is no way of upholding the asymmetry, and hence the suggested interpretation of indeterminism in branching time collapses.

Kripke, however, had independent reservations towards the existence of a present—a 'now'—on the grounds of relativistic physics. In his letter dated the 13th of October the above quote is continued:

> For example, in relativistic physics two events may be simultaneous to one observer but not to another, so that in tense logic one would say, ”It is the case that A, and it is the case that B,” while another (2nd) would say. ”It is the case that A, it is not now the case that B, although it will be the case that B.” And if we accept indeterminism the second might not even be sure that B will be the case. [L3: Oct. 13, 1958]

As introduced here, the relativity of simultaneity presents a problem for Prior's idea of an privileged moment—the present—at which the indeterminate becomes determinate. According to Prior this moment is objectively distinguishable (see below). However, if simultaneity is relative to a frame of reference, and thus in turn to the "nows" of observers, and none of these take precedence in constituting the privileged moment at which the determinate becomes indeterminate, then it seems as if there cannot be such a privileged, objective moment. Prior was certainly aware of the challenge from special relativity, and in his response he refers to Gödel's considerations of the implications of special relativity for the nature of time in his 1949-article "A Remark About the Relationship between Relativity Theory and Idealistic Philosophy":

> I cannot make any sense out of the idea of a thing being determined with respect to one observer and undetermined to another … And since a thing can only be undetermined if it is not-yet-past, this also must be something which it is in itself and not just with respect to some observer.—I would say about this, in fact, what Gödel says about existence in footnote 5 of his very important contribution to Schilpp's Einstein volume; though my conclusions are the opposite of his. [L4: Oct. 27, 1958]

In Footnote 5 of his article, Gödel makes the observation that the notion of a relative lapse of time cannot be tied to a change in the existing since existence cannot be relativized without loosing its meaning. Relativizing existence seems to imply the possibility for a thing to exist and not to exist which is plainly absurd. Similarly, Prior seems to be arguing, the notion of a relative lapse of time cannot be tied to the indetermined becoming determined since determinism cannot be relativized without loosing it's meaning. Relativizing determinism would imply that some event would both have occurred and not occurred which is plainly absurd. It follows, and this seems to be Prior's main point, that a proponent of the special theory of relativity is faced with the challenge of defining the notion of a relative lapse of time. At the same, however, it is also clear that Prior himself is still faced with the challenge of combining whatever notion of a relative lapse of time is meaningful with his own position implying an absolute time lapsing objectively (see Øhrstrøm and Hasle 1995, p. 197 ff).

Prior often revisited the questions regarding special relativity and tense logic. He considered Einstein's theory concerning the relativity of simultaneity as a challenge to Prior's own idea about the particular status of the Now. These questions have been further discussed by Müller and Strobach in this issue.

4 Branching time and the discreteness of time

As we have seen, it appears that Prior must have found Kripke's model of branching time rather attractive since it could reflect the asymmetry between past and future, which he regarded as fundamental for a satisfactory understanding of the time. Nevertheless, he also received Saul Kripke's idea with some reservation, since the idea—at least in Kripke's presentation—was based on the notion of discrete time:

> An odd point I notice about the tense-logical interpretation of your 'trees' is that the passage of time, represented by the movement to new 'levels', is discrete. I don't know whether this feature is eliminable; I have sometimes myself wondered whether the notion of 'alternative futures' presupposes the discreteness of time. [L4: Oct. 27, 1958]

It seems, however, that this hesitation regarding what limitations the idea of branching time may put on other aspects of the representation of time did not last long. It appears that he rather quickly came to the conclusion that the idea of branching can be combined with ideas of continuous time as well as with ideas of discrete time. In fact, as he reported it in *Past, Present and Future* the tense-logical axioms corresponding to the denseness of time and to the linearity of time are independent. This means that when we are formulating the axiomatic system corresponding to our view of time and reality, the question of linear versus branching time can be answered independently of how the question about denseness is answered.

Letters

L1. Kripke, Saul. Letter to A.N. Prior. Dated Sept. 3, 1958. The Prior Collection, Bodleian Library, Oxford.
L2. Prior, A.N. Letter to Saul Kripke. Dated Sept. 10, 1958. In Saul Kripke's possession.
L3. Kripke, Saul. Letter to A.N. Prior. Dated Oct. 13, 1958. The Prior Collection, Bodleian Library, Oxford.
L4. Prior, A.N. Letter to Saul Kripke. Dated Oct. 27, 1958. In Saul Kripke's possession.

The letters [L1–4] are published for the first time below.

L1. Kripke to Prior, September 3, 1958

119 North Happy Hollow Blvd.
Omaha 32, Nebraska
September 3, 1958

Professor A.N. Prior
Canterbury University College
New Zealand

I have been reading your book *Time and Modality* with considerable interest. The interpretations and discussions of modality contained in your lectures are indeed very fruitful and interesting. There is, however an error in the book which ought to be pointed out, if you have not learned of it already. Although I know of your paper on "Diodoran Modalities" only from the references in your book and from the review in *The Journal of Symbolic Logic*, it seems that the correction applies to this article as well. (I would greatly appreciate it if you could send me a copy of this article.)

The infinite matrix given for S4 (p. 23) verifies the formula ALMpLMNp. For if the sequence for p does not end with an infinite series of 3's, the sequence for LMp will contain nothing but 1's; and if the sequence for p *does* end with an infinite series of 3's, Np will not end with such a sequence, and hence LMNp will be all 1's. However, McKinsey and Tarski (J.S.L. XIII 1) have shown that ALαLβ cannot be an S4 thesis unless either Lα or Lβ is an S4 thesis. Hence ALMpLMNp is not a S4 thesis, and thus your matrix is not characteristic of S4. It or seems to me, however, that it probably ought to be an infinite characteristic matrix for a system intermediate between S4 and S5.

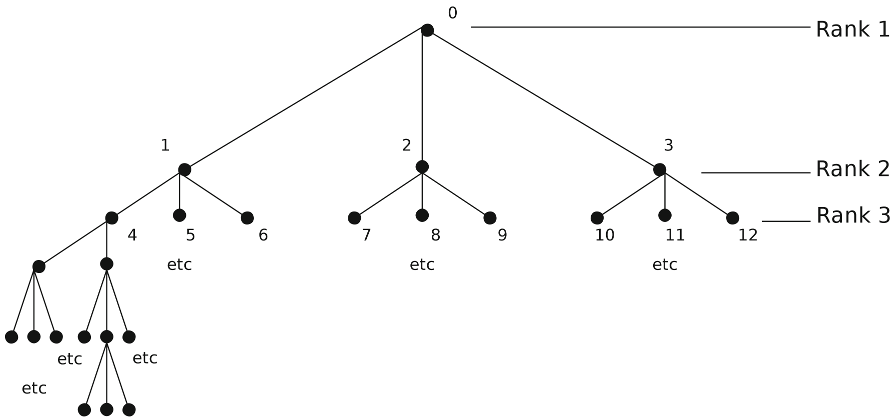
The axioms of the system, over and above the two valued base, are (with L primitive): CLpp, CLpLLp, CLCpqCLpLq, *and* CKMpMqAMKpMqMKqMp. The last axiom is not valid in S4, but it is verified by your infinite matrix. The rules of inference are detachment, substitution, and from a thesis α to infer Lα. I propose to call the resulting modal system "the Diodoran system", since it is presumably the system Diodorus would have favored.

Your infinite matrix for T (p. 24) suffers from similar difficulties; it also satisfies ALMpLMNp. In fact it contains the fiction that there are only two possible times, the present moment and the one immediately next; the moment after next is not possible, merely possibly possible. Thus the infinite matrix verifies CKMKpqMKpNqLp. I think we could axiomatize the system by CLpp, CLCpqCLpLq, and CKMKpqMKpNqLp, using the same rules as before. The resulting system is stronger than T, but neither stronger nor weaker than S3, S4, S5, Q, and the Diodoran system. It contains an infinity of modalities.

In view of my comments on S4, the "theorem" of footnote 1, p. 121, seems rather questionable, if conformity to S4 means not only containing all theorems of S4 but also lacking its non-theorems. Incidentally, your matrix for S5 seems indeed to be valid; in fact, it is in a sense a special case of a more general theorem of my own which will appear in J.S.L. sometime in the future. However, when I wrote the paper, I did not know about your fruitful interpretation in terms of an infinitely many valued logic. It does seems to me to be unnecessary to allow a non-denumerable infinity of sequences as you have done; if we restrict the sequences to those ending in infinite series of l's or 3's, we seem to lose no generality—and the number of truth values becomes denumerable, indeed effectively enumerable.

If the notions of your book are suitably modified, an infinite charac-matrix for S4 can also be obtained, as I will indicate below.

I have in fact obtained this infinite matrix on the basis of my own investigations on semantical completeness theorems for quantified extensions of S4 (with or without the Barcan axiom). However, I shall present it here from the point of view of your "tensed" interpretation. (I myself was working with ordinary modal logic.) The matrix seems related to the "indeterminism" discussed in your last chapters, although it probably cannot be identified with it. Now in an indetermined system, we perhaps should not regard time as a linear series, as you have done. Given the present moment, there are several possibilities for what the next moment may be like—and for each possible next moment, there are several possibilities for the next moment after that. Thus the situation takes the form, not of a linear sequence, but of a "tree":



The point 0 (or origin) is the present, and the points 1, 2, and 3 (of rank 2) are the possibilities for the next moment. If the point 1 actually does come to pass, 4, 5, and 6 are *its* possible successors, and so on. The whole tree then represents the entire set of possibilities for present and future; and every point determines a *subtree* consisting of its own present and future. Now if we let a tree sequence attach not three (as above) but a denumerable infinity of points to every point on the tree, we have a characteristic matrix for S4. An element of the matrix is a tree, with either 1 or 3 occupying each point; the designated tree contains only l's. If all points on the proper "subtree" determined by a point on the tree p are 1's, the corresponding point on Lp is a 1; otherwise, it is a 3. (In other words, a proposition is considered "necessary" if and only if it is and definitely always will be the case.) Analogously, for T, we can change the stipulation so that a point on Lp is a 1 if the corresponding point on p, and all the points immediately attached to it (i.e., possibilities for moments immediately following it), are l's; otherwise, it is a 3. As I have described the situation, the matrices for S4 and T are non-denumerable; but certain restrictions which I shall not describe here enable us to obtain a denumerably valued matrix for each system. (Because of the correspondence between S4 and the Heyting systerm, a denumerable matrix for the latter is also obtained, which should be compared to Jaskowski's. I have not yet carried out this comparison. McKinsey–Tarski (J.S.L. XIII l) also mention infinite matrices for these systems—S4 and Heyting's.)

Your system Q represents an ingenious piece of analysis, and I intend to investigate it by my own semantical methods. I think I should probably be able to obtain an axiomatization as a by-product. When I work such matters out, I shall write to you about them.

My corrections to your matrices for S4 and T may already be known to you. If they are not, I think a correction should be published somewhere, perhaps when J.S.L. reviews your book. If you know who the reviewer will be, you could write to him in my name or have me write to him. If not, you could write to Church, or try any other alternative you may desire. If you can reply to this letter to arrive before September 10th, write to the given return address; otherwise write to Harvard College, Weld House, Cambridge 38, Massachusetts, USA.

Yours very sincerely,
Saul A. Kripke

Note:
Added on the letter in Prior's hand:
"Passed on to Ivo and John L."

L2. Prior to Kripke, September 10, 1958

University of Canterbury
Christchurch, C.1, New Zealand
10/9/58

Dear Kripke

Many thanks for your letter. It was odd that you should write to me at just this time; having heard of you from John Lemmon, I sent you only a few days ago some material that I thought would interest you. I posted it, though, to the address given in the JSL Dec. 1957, and hope it eventually reaches you. The JSL address is Omaha 3, while that on your letter is Omaha 32. Apart from any results of this mix-up, surface mail from here to the US takes some time, so don't expect to see it for a couple of months anyhow. Airmail, on the other hand, isn't bad; it won't, indeed, go from there to here *and back* between Sept. 3 and Sept. 10 (so I shall send this not to Nebraska but to Harvard); still, it *has* got *here* today.

I had better, while I'm at it, give you my own movements. I shall be here until November 26 of this year; thereafter on the high seas (on the *Athenic*, via Panama to England) for about a month; and thereafter at the Philosophy Dept., University of Manchester, Manchester 13, England. When you're hopping across the pond, as I've no doubt you'll be doing from time to time, I hope you won't leave Manchester out of your itinerary.

It was pointed out to me last year that my matrix for S4 wouldn't do, and I have publicly recanted on this point in a note in *Philosophical Quarterly* for July 1958 (that note, though, itself contains at least one error, that Parry's S4.5 is a system between S4 and S5—Lemmon has now proved its equivalence to S5). This result has now been worked into a whole heap of results authored mainly by Lemmon, Dummett, Geach and Hintikka (whom you'll no doubt be seeing in Harvard). The main ones I have at present from Lemmon are these: if QS = CMLpLMp (Obviously equivalent to your own ALMpLMNp), L = ALCLpLqLCLqLp, H = CKMpMqAMKpMqMKqMp (also used by you), G = CKpMLpLp, then S3 + G = S4 + G→ (but not =) S3 + L = S4 + L = S3 + H = S4 + H→ (but not =) S3 + QS = S4 + QS. Matrix verifying S4 + QS but not L is 16-valued Boolean for C-N with this for L:

p	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Lp	1	16	11	16	13	16	15	16	9	16	11	16	13	16	15	16

(1 designated). There is a general suspicion that S4 + L = D, i.e. Diodorean modality (characterized by my pseudo-S4 matrix), but so far as I know of no proof as yet. I have

a proof in my note that if T is the Tarski–McKinsey transformation that correlates S4 with Heyting's calculus, then $\vdash\alpha$ by my matrix if and only if $\vdash$Tα by that Gödel matrix in his 1932 thing; and Dummett has proved that this matrix is characteristic for Heyting + $\vdash$ ACpqCqp (which is TL).

That my matrix for T is similarly faulty I did *not* know; nor did Lemmon when he pointed out the fault in my matrix for S4. He himself at that time suggested the following matrices for S2, his own E2, and S3: Use '$\alpha = 1$m (0m)' for 'α takes the value 1 (0) in the m'th place', it being understood that for m ≥ 1, $\alpha = 1$m or $\alpha = 0$m (and that $\alpha \neq 1_0$ and $\neq 0_0$). For this notation he goes

For S2: L$\alpha = 1$m if and only if $\alpha = 1$m and $\alpha = 1$(m − 1); *i, 0i
For E2: L$\alpha = 1$m if and only if $\alpha = 1$m and $\alpha = 1$(m − 1); *i,
For S3: L$\alpha = 0_1$;
L$\alpha = 1_2$ if and only if $(n)_{n>0}(\alpha = 1n)$
and for all m > 2, L$\alpha = 1$m if and only if $(n)_{n\geq m}(\alpha = 1n)$; * i, 0i

I don't know whether these have faults similar to those you've found in mine.—I notice that while if QS is added to S3 only it gives the reduction thesis CMLpLMLp, it doesn't when added to T.

About Q: The following are conjectures only, but may interest you:—Within QM for that portion 1Q which does not employ the operator L, Lemmon has found no law of QM that does not follow from p.c. + my rule M2 for S5 + the following modification of M1: $\vdash$C$\alpha\beta\rightarrow\vdash$CM$\alpha\beta$, if β is fully modalised *and contains no variables not in* α. And he has found no law of Q as a whole that doesn't follow from this lot together with the following further postulates involving L: the rule RL: $\vdash$C$\alpha\beta\rightarrow\vdash$CL$\alpha$L$\beta$, provided β has no variables not in α, and the axioms 1. CKLpLqLKpq, 2. CLpp, 3. CLpLLp, 4. CLNLNpMp, 5. CMqMCLpq. And I have proved these … (continued on other air-letter)

L3. Kripke to Prior, October 13, 1958

Oct. 13, 1958

Dear Professor Prior,

Your letters have been received, although thus far your printed material has not. Incidentally, my address at Harvard can now be given more exactly as Weld 42, Harvard, Cambridge 38, Mass. (The zone numberings in Omaha shouldn't cause any difficulty on your printed matter; this change in numbering was made recently.) I don't know how often I'll be "hopping across the pond"—after all, I am only a college freshman—but when I do, I'll try and put Manchester on my itinerary.

Lemmon's matrices, like all such linear attempts, are faulty. In fact, his matrices for E2 and S2 verify CLCppCKMKpqKMpNqLp. However, I have found matrices of my own for S2 and E2 which work, using trees as before, but now permitting *end-points*—sorts of Armageddons after which time stops. Then for E2 we evaluate every point except as in T; in an endpoint Lp is always "0" no matter what p is. (Note that two trees with endpoints are not necessarily alike in structure—in evaluating Apq, we consider only the common part of p and q as to structure.) S2 is like E2 except that the origin is evaluated as in T even if it is an endpoint.

Lemmon's matrix for S3 seems irreparable even if we use trees. It verifies MLLCpp; but what about S8, a consistent extension of S3?

You mention the L-modal system without functorial variables; did you know that this system should really be credited not to Lukasiewicz but to Curry—as far as priority goes? Curry first proposed this system in the last section of his *A Theory of Formal Deducibility*; there he had serious misconceptions about it, but he cleared these up in "The Elimination Theorem when Modality is Present", J.S.L. '52. There he gives the interpretation of Mp as CMp which you mention in your article on the British logicians' conference. I have succeeded in proving the elim. thm. for his Gentzen-like formulation, and hence the correctness of his other conjectures. As far as I know, no one seems to know about this—even Curry himself may not be aware of the connection with Lukasiewicz.

I think I ought to be able to prove the following to be an axiomatization of the strict implication part of S4, using your notation with respect to C and capitals—CPCqP, Cpp, CCpqCCpCqrCpr, subst, det. The crucial step in a proof of equivalence with S4 would be the deduction theorem: If $P_1, \ldots, P_n$, p[251C?] q, then $P_1, \ldots, P_n$[251C?] Cpq.[1] I haven't quite worked out the details.[2]

Undoubtedly "trees" will be of some interest from the point of view of tense logic, although I haven't yet given much thought along this line. However, I am a little uncertain as to your own beliefs concerning the importance of tense logic. Is it your contention in Appendix A that a tenseless logic is really insufficient to represent the distinctions tense logic conveys? Do you think a tensed logic is needed for scientific discourse? I should think that, *for scientific discourse* a tenseless logic may be preferable. For example, in relativistic physics two events may be simultaneous to one observer but not to another, so that in tense logic *one* would say, "It is the case that A, and it is the case that B," while *another* (2nd) would say. "It is the case that A, it is not now the case that B, although it will be the case that B." And if we accept indeterminism, the second might not even be sure that B will be the case. It thus seems that tenseless logic would be better equipped to handle these distinctions without contradiction, and without making propositions dangerously relative to the individual, rather than public property. (By saying that propositions are public property, I mean that they have the same meaning and truth-value to all observers. Maybe sameness of meaning is an exorbitant demand, but certainly sameness of truth-value is not.) And your arguments in the appendix by no means convince me that there are any real inadequacies in a tenseless logic. However, we do often use tense logic in ordinary discourse and it is of some interest to formalize it and investigate it, as you have done. The analogy with modality of course heightens our interest.

The phrase "two possible times" meant that only events occurring in two times, this one and the one that actually turns out to be next, are called "possible". In a tree matrix

1 Turnstyles are missing in the original letter.

2 Added in Prior's hand: "Barcan, Deduction Theorem etc., JSL XI (1946) p. 117".

for T, there are infinitely many “possible times”, this moment and all the possibilities for the next.

Thank you very much for your conjectural material on Q, etc. No comment as yet; when I have any, I will tell you.

Sincerely yours,
Saul Kripke

L4. Prior to Kripke, October 27, 1958

University of Canterbury
Christchurch, C.1
New Zealand
27/10/58

Dear Kripke

Thank you very much for your letter of the 13th.

Isn’t the deduction theorem ‘If $P_1, \ldots P_n, p \vdash q$, then $P_1, \ldots P_n \vdash$ Cpq’ implicitly proved for S4 in Barcan’s paper in JSL XI (1946) p. 117?—If your postulates are sufficient for C4, it follows easily that Lemmon’s CqCpp, CCpqCCqrCpr, CCpCpqCpq are also sufficient. Your axioms follow easily from these, and I presume you can do the converse (I, I must confess, can only do it with your third axiom replaced by its lemma, i.e. the usual Frege CCpCqrCCpqCpr. Lemmon once had this sort of variant of a conjectural C5 base).

Yes, it is certainly my contention that you can do things with a tensed logic that you can’t with a tenseless. I’d agree, though, that you get into trouble if you identify the earlier-later relation of relativistic physics with the earlier-later relation defined in the ordinary way in terms of tenses (‘p earlier than q’ as ‘it is, has been or will be that KKPpNpq’); and would conclude that they are not in fact the same relation, though there is some sort of connection between them. As for what kind of logic ‘quantified discourse’ requires, that seems to me to depend on how much this term covers. There are areas of science within which certain questions arishing out of ordinary tense-distinctions aren’t answerable, and in which ‘it is better to use a language in which such questions cannot even be stated; and I take it that the moral of Special Relativity is that the theory of light-propagation is such an area.—In your paragraph about this you have a sentence beginning ‘And if we accept indeterminism…’, but I do not see how indeterminism can be expressed in a tenseless language at all. For indeterminism asserts a certain difference between the future and the past (that one has always APnpPnNp, but not always AFnpFnNp), which is not at all the same thing as a difference between the earlier and the later. And it seems to me that if any particular matter is at any time undetermined, that that is something that it is in itself and not just with respect to some observer—I cannot make any sense out of the idea of a thing being determined to one observer and undetermined to another (it can, of course, be within the reach of one agent’s influence and not of another’s; but that is a different matter). And since a thing can only be undetermined if it is not-yet-past, this also must be something which it is in itself and not just with respect to some observer.—I would say about this, in fact, what Gödel says about existence in Footnote 5 of his very important contribution to the Schilpp Einstein volume; though my conclusions are the opposite of his.

Thank you very much for drawing my attention to this t-model material in Curry.

Yours sincerely
Arthur N. Prior

PS: That the McKinsey matrix is not characteristic for T was apparently noted a couple of years ago in T. Smiley's (Cambridge) doctoral thesis. But Smiley hadn't a matrix that was characteristic for T (or for S4). An odd point I notice about the tense-logical interpretation of your 'trees' is that the passage of time, represented by the movement to new 'levels', is discrete. I don't know whether this feature is eliminable; I have sometimes myself wondered whether the notion of 'alternative futures' presupposes the discreteness of time.

A. N. P.

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