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Explanation by induction?

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Abstract Philosophers of mathematics commonly distinguish between explanatory and non-explanatory proofs. An important subclass of mathematical proofs are proofs by induction. Are they explanatory? This paper addresses the question, based on general principles about explanation. First, a recent argument for a negative answer is discussed and rebutted. Second, a case is made for a qualified positive take on the issue.

Keywords Mathematical induction · Mathematical explanation · Logic of explanation · Grounding

In the philosophy of mathematics it is commonplace to distinguish between explanatory and non-explanatory proofs. The former explain why their conclusion is true, while the latter merely show that their conclusion is true.[1] An important subclass of mathematical proofs are proofs by induction. However, intuitions are divided about

[1] See, e.g., Bolzano (2004, §§13, 14), Kitcher (1975, p. 254), Steiner (1978, p. 135), and Mancosu (2001, p. 98f).

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which of them, if any, are explanatory.[2] This suggests that direct reliance on intuitions is a poor guide for deciding the question. An argument from more general principles about explanation is needed.

In a recent paper, Lange (2009) argues for a general negative take on the explanatory power of inductive proofs: according to him, no inductive proof is explanatory. Lange claims that his argument presupposes very little about mathematical explanations—in essence, only that they cannot run in a circle. His argument, if sound, would break the tie between conflicting intuitions. However, in Sect. 1 we argue that Lange's argument fails. We show that it has to rely on the unacknowledged assumption that universally quantified sentences explain their instances. But this assumption, we argue, is false.

In Sect. 2 we build a *qualified* positive case for the explanatory potential of inductive proofs: roughly put, some (but not all) inductive proofs are recipes for generating explanatory proofs of everything that explains their conclusion.

1 Against Lange

1.1 Lange's main argument

We will argue that Lange misses his goal of ending the 'fruitless exchange of intuitions' (2009, p. 205) amongst participants in the debate over the explanatory power of inductive proofs. Our argument proceeds in three steps. First, we show that Lange underplays the assumptions on which his argument rests. In addition to the asymmetry of explanation, Lange has to appeal to a principle which links universally quantified truths and their instances. Secondly, we show that once the most straightforward candidate principle is in place, a much simpler argument for the explanatory impotence of inductive proofs becomes available. Finally, we argue that this is cold comfort for the opponent of inductive explanations, since the needed principle is false.

To set the stage, a proof by mathematical induction is a deduction of a conclusion of the form '$\forall$nF(n)' that relies on the corresponding instance of the following schema:

Induction Schema

$$[\mathrm{F}(1)\&\forall n(\mathrm{F(n)} \rightarrow \mathrm{F(n+1)})] \rightarrow \forall \mathrm{nF(n)}$$

In such a proof, one first shows that the number 1 has a certain property P and that if some number has P, its successor does as well.[3] With the relevant instance of the *Induction Schema*, one then derives the universal claim that every number has P.

Now consider the following variation on the *Induction Schema*:

Upwards and downwards from 5 schema

$$[\mathrm{F}(5)\&\forall \mathrm{n}(\mathrm{F(n)} \rightarrow \mathrm{F(n+1)})\&\forall \mathrm{n}(\mathrm{F(n+1)} \rightarrow \mathrm{F(n)})] \rightarrow \forall \mathrm{nF(n)}$$

[2] While some authors assume that inductive proofs are typically explanatory, e.g. Kitcher (1975, p. 265) and Brown (1997, p. 177), others assume that they are generally *not* explanatory, e.g. Hanna (1990, p. 10) and Hafner and Mancosu (2005, p. 237).

[3] We follow Lange in assuming that 1 is the first natural number. Nothing hinges on this.

In a *5-proof*, one deduces a conclusion of the form '∀nF(n)', relying on the corresponding instance of the *Upwards and Downwards from 5 Schema*: one first shows (i) that the number 5 has a certain property P, (ii) that if some number has P, its successor has it, and (iii) that if some number's successor has P, the number itself has it. With the relevant instance of the *Upwards and Downwards from 5 Schema*, one then derives that every number has P.

Lange (2009, p. 209f.) can now be seen to argue as follows:

The main argument

P1 If there is an inductive proof to some conclusion C, then there is also a 5-proof of C.
P2 If an inductive proof to the conclusion C is explanatory and there is a 5-proof of C, then there is an explanatory 5-proof of C.
P3 An inductive proof and a 5-proof to the same conclusion cannot both be explanatory.
C1 Therefore, no inductive proof is explanatory.

The reasoning is clearly valid, (P1) is provably true,[4] and we simply grant Lange (P2) for the sake of argument.[5] Our critique concentrates on (P3).

1.2 The assumptions of the argument

Lange (2009, p. 206f.) takes his argument to rely only on the following constraints on explanations in mathematics:

Explanation A proof is only explanatory if each of its premisses at least partially explains its conclusion;
Asymmetry If x at least partially explains y, y does not even partially explain x. He justifies *Asymmetry* with the help of two further principles:
Irreflexivity Nothing explains itself (not even partially).
Transitivity If x at least partially explains y, and y at least partially explains z, then x at least partially explains z.

We accept the constraints: *Explanation* is basically an agreement about our use of the term 'explanatory' as applied to proofs, and *Asymmetry* and *Transitivity* are plausible structural principles of explanation. But Lange's contention that his argument requires only these principles is incorrect.[6] To see this, consider his justification for (P3):[7]

[4] *Sketch*: The *Induction Schema* is provable from the *Upwards and Downwards from 5 Schema* in conjunction with the other Peano axioms. Suppose that $F(1) \& \forall n(F(n) \rightarrow F(n+1))$. Let $G(n) \leftrightarrow_{\text{def}} \forall m(m \leq n \rightarrow F(m))$. Then it is easy to show with the help of the *Upwards and Downwards from 5 Schema* that $\forall n G(n)$, from which it follows that $\forall n F(n)$. Thanks to Nick Haverkamp for pointing the proof out to us.

[5] Baker (2010) argues that Lange's justification for (P2) is 'too quick' since there is a difference between inductive proofs and 5-proofs ("minimality") which Lange has not shown to be explanatorily irrelevant. Baker's conclusion is somewhat weak. Our argument will not merely point to an oversight in Lange's original argument, but to a principled reason why his argumentative strategy fails.

[6] Actually, Lange even claims that the argument only relies on *Explanation* and *Asymmetry*.

[7] Cp. Lange (2009, p. 209f.).

The argument for (P3)

P4 If an inductive proof to the conclusion that $\forall n F(n)$ is explanatory, its premiss that $F(1)$ partially explains why $F(5)$.[8]

P5 If a 5-proof to the conclusion that $\forall n F(n)$ is explanatory, its premiss that $F(5)$ partially explains why $F(1)$.

C2 Therefore, (P3): an inductive proof and a 5-proof to the same conclusion cannot both be explanatory.

(C2) follows from (P4) and (P5) by *Asymmetry*. But why believe (P4) and (P5)? Any satisfactory justification for either of them will carry over, *mutatis mutandis*, to the other. So, let us take a closer look at Lange's justification for (P4):

L1 [I]f the argument from mathematical induction uses $F(1)$ to explain why it is the case that for any n, $F(n)$, then for any $n \neq 1$, the argument uses $F(1)$ to explain why $F(n)$ is true—and so, in particular, uses $F(1)$ to explain why $F(5)$ is true (Lange 2009, p. 210).

Neither *Asymmetry* (which is clearly irrelevant here) nor *Explanation* can warrant the claim just quoted: the latter states only that the premisses of an explanatory proof explain its *conclusion*: a universal claim—not one of its instances. Of course, given *Explanation* and *Transitivity*, anything explained by the conclusion of an explanatory proof will in turn be explained by the proof's premisses. But for this consideration to be pertinent a further principle is needed. While Lange does not make it explicit, the following quotation hints at what he may have had in mind:

L2 In explaining why for any λ and r, $\mathrm{E} = 2\lambda/r$, Coulomb's law explains in particular why $\mathrm{E} = 4$ dyn/statcoulomb if $\lambda = 10$ statcoulombs/cm and $r = 5$ cm. By the same token, if $F(1)$ explains why for any n, $F(n)$, then $F(1)$ explains in particular why $F(5)$ (Lange 2009, p. 210).

Here, Lange seems to take the general truth that, for all λ and r, $\mathrm{E} = 2\lambda/r$ to explain the particular instance cited. This suggests that he relies on the following principle:

Case-By-Rule A universally quantified truth explains its instances.

Case-By-Rule would close the gap in Lange's argument. For, suppose that the proposition that $F(1)$ explains why $\forall n F(n)$. Since the proposition that $\forall n F(n)$ explains—by *Case-By-Rule*—why $F(5)$, *Transitivity* yields that the proposition that $F(1)$ also explains why $F(5)$.

But while Lange's argument thus goes through, it now appears to be unnecessarily convoluted. We can get to its conclusion by a much shorter route:

The direct argument

P6 If some inductive proof is explanatory, then—by *Explanation*—its premiss that $F(1)$ partially explains its conclusion that $\forall n F(n)$.

[8] For present purposes we follow Lange in taking 'F(1)' to be a *premiss* of an inductive proof. We discuss this issue in Sect. 2.1.

P7 Given *Case-By-Rule*, no instance I of a universal statement U can (partially) explain U, since I would also be explained by U, in violation of *Asymmetry*.
C1 Therefore, no inductive proof is explanatory.

So, Lange's argument has to rely on an unacknowledged constraint on explanation. The most straightforward candidate, *Case-By-Rule*, would validate his argument, but at the cost of making the bulk of it superfluous.

1.3 Alternatives to Case-By-Rule?

Perhaps, then, Lange should not be read as relying on *Case-By-Rule*. Indeed, in a recent paper Lange comments on his original argument: it is not, he writes, based 'on the premiss that if a fact helps to explain a given universal generalization, then it must help to explain every instance of that generalization' (Lange 2010, fn. 15). So, there is a strong reason not to ascribe *Case-By-Rule* to Lange. However, he fails to mention which other principle his argument is supposed to rely on. Since this is the crucial question for evaluating the argument, some further speculation is appropriate.

The beginning of quotation **L1** supports the conjecture that Lange merely has a *restricted* version of *Case-By-Rule* in mind:

Restricted Case-By-Rule A universally quantified truth explains its instances, except those that explain it.

Restricted Case-By-Rule certainly looks peculiar. If the principle is true, there is *always* an explanatory relationship between a universally quantified truth and any of its instances: either the former explains the latter or the latter explains the former. It is hard to see what could motivate such a principle without also motivating that the direction of the explanatory relation is uniform, i.e. without either motivating *Case-By-Rule* or

Rule-By-Case A universally quantified truth is explained by its instances.

However, the restriction of *Restricted Case-By-Rule* is redundant once we have *Case-By-Rule* (while *Rule-By-Case* would undermine Lange's argument; see below).

But suppose that *Restricted Case-By-Rule* could be independently motivated. While we cannot run *The Direct Argument* with *Restricted Case-By-Rule* in place of *Case-By-Rule*, Lange's original argument will not go through either. In *The Argument for (P3)*, Lange has to assume, for *reductio*, that both the inductive proof of the conclusion that $\forall n F(n)$ and the 5-proof of the same conclusion are explanatory. By *Explanation*, both the proposition that $F(1)$ and the proposition that $F(5)$ explain why $\forall n F(n)$. But then, *Restricted Case-By-Rule* neither allows us to conclude that the proposition that $\forall n F(n)$ explains why $F(5)$, nor to conclude that it explains why $F(1)$. Thus, we cannot use *Transitivity* and *Asymmetry* to derive a contradiction.

So, we are still in search of the missing link in Lange's argument. A final attempt: perhaps *Case-By-Rule* should not be restricted with respect to the *instances* of a universal generalisation it applies to, but with respect to the *universal generalisations* themselves. Perhaps a universal generalisation explains its instances only if it has a certain feature. In the example in **L2** it is Coulomb's *Law* that allegedly explains its instances. This points towards an influential tradition in the philosophy of science. On Hempel's account, any explanation must involve a law, where laws are taken to be universally quantified truths.[9] Although nothing in Hempel's account would commit its proponent to accepting anything as general as *Case-By-Rule* or as peculiar as *Restricted Case-By-Rule*, it seems to presuppose:

Case-By-Law A universally quantified truth which is a law explains its instances.

Case-By-Law would be of some help for Lange's argument. If the conclusion U of an inductive proof is a mathematical law then, by *Case-By-Law*, U explains all of its instances. By *Explanation* and *Transitivity*, the premisses of the inductive proof would explain all of U's instances if the proof were explanatory. The same goes, *mutatis mutandis*, for 5-proofs. Thus, *Case-By-Law* could sustain the argument for inductive proofs whose conclusions are mathematical laws.

However, our praise of *Case-By-Law* already points to its weakness. It could only support an argument to the effect that *no* inductive proof is explanatory, if *all* universal generalisations open to inductive proofs were mathematical laws. However, this seems rather implausible. It would moreover be strange for Lange to rely on this contention, since he devotes a whole paper to the phenomenon of mathematical *coincidences*. Whatever other merits *Case-By-Law* may have (cp. Sect. 1.5), it is unable to support Lange's argument in full generality.

A combination of *Restricted Case-By-Rule* and *Case-By-Law* is testament to the fact that two wrongs often don't make a *right* but a *very wrong*. This takes us back to *Case-By-Rule* as the only real candidate for closing the gap in Lange's argument. As we have seen, it would indeed suffice to achieve Lange's goal, though via a much shorter route.[10] But how innocent is the assumption on which the argument is built?

1.4 Evaluating Case-By-Rule

An argument resting on *Case-By-Rule* is ill suited as a quick and easy way of settling the dispute about the explanatory power of inductive proofs. For, even if the principle may not be obviously false, it is certainly far from uncontroversial. In fact, there is good reason to deny it. As a first step, note that it conflicts with *Rule-By-Case*: Take any universally quantified truth U and one of its instances I. Given *Case-By-Rule*, U explains I. Given *Rule-By-Case*, I explains U. But, by *Asymmetry*, this cannot both be the case. So, if *Rule-By-Case* is correct, *Case-By-Rule* is not.

Drawing on current work concerning purely formal and logical features of explanation, we now argue that *Rule-By-Case* receives strong motivation from independent general considerations. Several authors have recently explored the explanatory relationships that hold between sentences solely on the basis of their logical form and the nature of the logical constants involved in them. It is fruitful to bring their results to bear on the issue at hand.

[9] See Hempel and Oppenheim (1948).

[10] Incidentally, this shorter route would also be available in cases where *Case-By-Law* is applicable.

Consider a propositional language. There are general principles establishing explanatory links between true sentences of such a language solely on the basis of their truth-functional structure. Classical truth-functional compounds are true/false *because of* the truth-values of their component sentences. So, for instance, any truth explains any truth-functional disjunction of which it is a disjunct, any true conjunction is at least partially explained by any of its conjuncts, etc.[11] These explanatory relations reflect how the truth values of such sentences depend on each other. A similar dependence obtains between quantified sentences and their instances. So, we should expect there to be general explanatory principles for the quantifiers as well. The obvious candidates for the universal quantifier are *Case-By-Rule* and *Rule-By-Case*.

But which of the two is correct? We can take a hint from the truth-functional case. There is a close connection between universal quantification and conjunction. Restricting ourselves to finite domains and assuming a reasonably rich language, a universal quantification will be true just in case the conjunction of its instances is. As we have already noted, true conjunctions are explained by their conjuncts. Hence, the conjunction of instances of a true universally quantified sentence is partially explained by each of those instances. If there is a general explanatory link between quantified sentences and their instances at all, it should mirror this link. So, if there is an explanatory link between universally quantified truths and their instances, the latter partially explain the former.[12] Consequently, if there is any general explanatory principle for the universal quantifier, it is *Rule-By-Case*. Since we have already argued that it is very plausible that there is such a general principle, we should accept *Rule-By-Case* and deny *Case-By-Rule*. Not only is the latter principle false because it has *some* false instance, *Asymmetry* and *Rule-By-Case* imply that it *only* has false instances.

Admittedly, in a finitary language there is no analogue of the equivalence between universal quantification and conjunction once we consider infinite domains. Thus, the intimate link between universal quantifications and conjunction is severed for the simple reason that we lack the resources for formulating the relevant conjunction. We do not take this to weaken the argument for *Rule-By-Case*, though. First, in a sufficiently rich language that allows for infinite conjunctions, the strong link reemerges.[13] Second, given that the finite case is indicative of an explanatory link from the instances of a universal quantification to the quantification itself, we see no reason to suppose that this link is absent, let alone reversed, in the infinite case.

As an anonymous reviewer pointed out to us, in the infinite case every instance makes only an infinitesimal contribution to the truth of the universal statement. Nevertheless, an infinitesimal contribution *is* a contribution; and we see no harm

[11] Cp. Schnieder (2008, 2011), and also Correia (2010), and Fine (2010, forthcoming). Note that Fine and Correia are concerned with grounding which they take to be an explanatory relation expressible by 'because'-sentences; see Fine (2001, p. 15f.) and Correia (2010, p. 253f.).

[12] Cp. Fine (forthcoming), and Schnieder (2011). Starting from the equivalence of existentially quantified sentences and disjunctions, the same line or reasoning can be used to argue that existential generalizations are explained by any of their true instances. Cp. Lewis (1986, p. 223), Fine (2010, forthcoming), and Schnieder (2011).

[13] In fact, logicians such as Peirce, Zermelo and, at times, Hilbert have introduced the quantifiers via infinite conjunctions/disjunctions; for sources and discussion see Moore (1980).

in countenancing cases where an infinite number of instances each partially explain the corresponding universal statement. We thus endorse *Rule-By-Case*.

1.5 Explanation by laws

Objection: as noted in in Sect. 1.3, an influential tradition in the philosophy of science accepts a position which, together with the assumption that laws are universal quantifications, implies a restriction of *Case-By-Rule* to universal generalisations that are laws, namely *Case-By-Law*. Thus, philosophers working in this tradition might balk at the reason we gave for denying *Case-By-Rule*. For, *Case-By-Law* and the assumption that laws are universal quantifications entail that some universal generalisations explain their instances. But this in turn implies—given *Asymmetry*—that *Rule-By-Case* is false.

We have strong sympathies with the intuition underlying the current objection, namely that laws explain their instances. On the other hand, we take the case for *Rule-By-Case*—which seems to be in direct conflict with the intuition—to be compelling. So, something has to give.

There are two reasonable ways of reconciling explanations by laws with the acceptance of *Rule-By-Case*. Firstly, note that the clash between the two claims turns on Hempel's identification of laws with universally quantified truths. If that identification is rejected, the apparent conflict disappears. In fact, Hempel's view has been denied by many philosophers in the more recent debate about lawhood.[14]

Secondly, even if one sticks to Hempel's identification, there is a way of reconciling *Rule-by-Case* with the intuition that, sometimes, things can be explained by laws. For, when we say that, e.g., Coulomb's Law explains why the magnitude of a certain electrical field is such-and-so, we may convey two things (we use 'u' as a place-holder for a sentence expressing Coulomb's Law and 'i' as a place-holder for a sentence expressing one of its instances):

(1) i because u;
(2) i because it is a law that u.

(1) is in direct conflict with *Rule-By-Case*. On the other hand, nothing about (2) rules out the truth of *Rule-By-Case*. If both (2) and *Rule-By-Case* are correct then the proposition *that it is a law that u* explains why i, which in turn (partially) explains why u. Of course, given *Transitivity*, the proposition that it is a law that u will also explain why u. Consequently, given *Asymmetry*, the proposition that u cannot also explain why it is a law that u. But the latter candidate explanation is squarely implausible in the first place. Thus, *Rule-By-Case* is compatible with one natural understanding of the claim that laws explain their instances, while *in*compatible with another. Given the strong motivation for *Rule-By-Case*, we would attribute any plausibility sentences like (1) may have to their close proximity to sentences like (2), in particular to the

[14] See e.g. Armstrong (1983) and Dretske (1977, p. 262) who argue that laws should be conceived of as stating certain relationships between properties; for discussion see Lange (1992).

fact that they both may be conveyed by the claim that a particular law explains its instances.

1.6 Conclusion

Taking stock, we have shown that Lange's argument against the explanatory power of inductive proofs has to rely on an unacknowledged principle—*Case-By-Rule*. However, this principle conflicts with another candidate principle linking universally quantified truths and their instances, namely *Rule-By-Case*, which receives independent support from general principles concerning explanation—in particular, that true conjunctions are explained by their conjuncts. Thus, we conclude that Lange's argument fails. If there is anything to be said for or against the explanatory power of inductive proofs, a new argument is needed. We take a fresh look in the next section.

2 On the explanatory power of inductive proofs

2.1 Premisses of inductive proofs

We will argue for a qualified positive take on the explanatory status of inductive proofs. To begin with, we need to address an issue that we have brushed over so far. Recall that *Explanation* says that a proof is only explanatory if each of its *premisses* at least partially explains its conclusion. During our discussion, we followed Lange by pretending that inductive proofs have premisses of the form '$F(1)$' (the *inductive basis*) and '$\forall n(F(n) \rightarrow F(n+1))$' (the *inductive step*). With *Explanation* we concluded that an inductive proof is only explanatory if the basis as well as the inductive step partially explain its conclusion. However, *no* inductive proof you are likely to find in a mathematics textbook (or anywhere else, for that matter) has the basis and the inductive step as genuine *premisses*. Rather, the basis and the inductive step are typically themselves *proved* in the course of a proof by mathematical induction (only if one of them is self-evidently true, such a proof is omitted). Thus, whether a particular inductive proof is explanatory will also depend on what goes on *before* the *Induction Schema* is applied.

Consider for instance the following caricature of an inductive proof:

Successor

Premiss:	Every natural number has a successor.
Proof of Inductive basis:	Thus, by universal instantiation, (i) the number one has a successor.
Proof of Inductive Step:	Further, by conditional proof, (ii) if some number has a successor, so does its successor.
Induction:	Consequently, by an application of *modus ponens* on the conjunction of (i) and (ii) and the relevant instance of the *Induction Schema*, every natural number has a successor.

Clearly, this proof is not explanatory. In fact, our principles about explanation entail that it is not. By *Explanation, Successor* is only explanatory if its premiss (that every natural number has a successor) at least partially explains its conclusion (the very same proposition). But this is ruled out by *Asymmetry*. In consequence, there is no hope for the general thesis that *every* inductive proof is explanatory.

However, our principles also show that, in *Successor*, something has gone wrong *before* we applied the *Induction Schema*. For, consider the subproof which establishes the inductive basis, i.e. the subproof that has

(3) Every natural number has a successor;

as a premiss and

(4) The number one has a successor;

as a conclusion. This subproof is already not explanatory. By *Explanation*, it is explanatory only if (3) at least partially explains (4). But *Rule-By-Case* entails that (4) partially explains (3). Thus, by *Asymmetry*, (3) does not even partially explain (4). What is wrong with *Successor* has nothing in particular to do with the application of the *Induction Schema*.

This suggests that a more limited question may still receive a positive answer, namely: are those inductive proofs explanatory whose subproofs of the basis and the inductive step are explanatory? Put differently: does the application of the *Induction Schema* preserve explanatory power?

2.2 On the explanatory potential of the Induction Schema

We will argue that the answer in part depends on which *form* inductive proofs have.

(i) If we conceive of them as proofs by *modus ponens* from the conjunction

$$F(1)\&\forall n(F(n) \rightarrow F(n+1))$$

of the inductive basis and the inductive step, and the relevant instance

$$[F(1)\&\forall n(F(n) \rightarrow F(n+1))] \rightarrow \forall n F(n)$$

of the induction schema to the conclusion $\forall n F(n)$, then inductive proofs are in general *not* explanatory.

(ii) We might treat the induction schema rather as a kind of *inference rule*, which allows the move from the inductive basis $F(1)$ and the inductive step $\forall n(F(n) \rightarrow F(n+1))$ to the conclusion $\forall n F(n)$. Whether inductive proofs in this sense are explanatory will depend on highly intricate matters that cannot be settled here.

(iii) However, even if inductive proofs in this second sense also turn out not to be explanatory, we will argue that a strong explanatory tie can still be established between inductive basis and step on the one hand, and the conclusion of an inductive proof on the other hand: given an explanation of the basis and the step, we are in a position to know of each instance which partially explains the inductive conclusion not merely *that* it holds, but *why* it holds. (Note that we distinguish between knowing and the weaker condition of being in a position to know.) So, our overall position is that the question whether inductive proofs are explanatory cannot, in this generality, be answered by a simple 'yes' or 'no'. Some of those proofs are, some are not.

Re (i): Suppose that explanatory proofs have been given for $F(1)$ and $\forall n(F(n) \rightarrow F(n+1))$, and that we derive the conjunction $F(1)\&\forall n(F(n) \rightarrow F(n+1))$. Since a true conjunction is (partially) explained by any of its conjuncts, this constitutes an explanatory proof of the conjunction. Suppose we now add the relevant instance of the induction schema $[F(1)\&\forall n(F(n) \rightarrow F(n+1))] \rightarrow \forall n F(n)$ and derive its consequent $\forall n F(n)$ via modus ponens. By *Explanation*, this proof of $\forall n F(n)$ is explanatory only if both the conjunction and the instance of the induction schema at least partially explain the conclusion $\forall n F(n)$. But it is in general not the case that a truth-functional conditional (even partially) explains its consequent. On the contrary, any truth Q explains any conditional $P \rightarrow Q$ of which it is the consequent. For, a truth-functional conditional $P \rightarrow Q$ is not only logically equivalent to the disjunction $\neg \mathrm{P} \vee Q$, but should have the same grounds.[15] However, a disjunction is explained by a true disjunct. Hence, the instance of the induction schema cannot explain the conclusion of the inductive proof. By *Explanation*—the principle that the premisses of an explanatory proof explain its conclusion—the inductive proof cannot be explanatory. This holds quite generally: if we take an instance of the induction schema to be a proper premiss of the respective proof, such a proof will never be explanatory.

Re (ii): We might think of an inductive proof to the effect that $\forall n F(n)$ as resting solely on the inductive basis $F(1)$ and the inductive step $\forall n(F(n) \rightarrow F(n+1))$, while the induction schema does not supply a further premiss but licences the move to the conclusion $\forall n F(n)$ (think of the schema as a rule of inference). Then, by *Explanation*, the resulting proof of $\forall n F(n)$ is explanatory only if both $F(1)$ and $\forall n(F(n) \rightarrow F(n+1))$ at least partially explain the conclusion $\forall n F(n)$. *Rule-By-Case* tells us that with respect to the inductive basis $F(1)$, this is indeed the case. But what about the inductive step $\forall n(F(n) \rightarrow F(n+1))$? The principles introduced so far are silent on this matter, for they tell us nothing about the explanatory relationship between two universally quantified statements. Given that every number is F, is every number F *because* every successor of an F is itself an F? We take this general principle to be dubious, but the matter cannot be settled here.[16] It would require deciding intricate questions about the explanatory relationships between quantified statements that are beyond the scope of this paper.

Re (iii): According to *Rule-By-Case*, a universally quantified truth is explained by its instances. Let us call the set of instances of a universally quantified truth U the immediate *explanatory basis* of $U(B_U)$. Every element of B_U partially explains U. It is a further question whether all the elements of B_U taken together also provide a *complete* explanation of U, or whether this requires the addition of a 'totality fact', stating e.g. that these are *all* the instances of U.[17] We will remain neutral on this issue.

15 Cp. Schnieder (2008, 2011), Correia (2010), Fine (forthcoming). This consideration applies to the natural language connective 'if … then' only if its truth-functional analysis is correct, which may well be doubted. But in the present context, only the truth-functional arrow is relevant.

16 Armstrong (1983, p. 40f.), however, seems to have as clear an intuition about these cases as about *Case-By-Rule*.

17 See e.g. Armstrong (1997, p. 198), Fine (2010, p. 109), and Schaffer (unpublished).

Now assume we have an inductive proof of a statement U. The corresponding inductive basis, i.e. the statement that 1 has a certain property, is one of the members of U's explanatory basis B_U. Moreover, the subproof of the inductive step gives us a method to derive other members of B_U, e.g. that the number 2 has the said property as well. In fact, the combination of inductive basis and step yield a recipe which allows us to proof of any arbitrary member of B_U that it holds. So, the inductive proof as a whole can be seen as explanatory in the sense that it puts us in a position to know of every member of the explanatory basis B_U that it holds. But knowing of every member of B_U that it holds puts one in a position to know, on the basis of one's knowledge of *Rule-By-Case*, why U holds.

So far, this is a somewhat weak result. Knowing of each member of the explanatory basis of a universal statement U that it holds gives us *some* explanatory knowledge of why U holds. But once we acknowledge the general principle *Rule-By-Case*, this is a kind of knowledge we can easily have for *any* true universal statement, as soon as we know that it is true and what its instances are—the explanation is not a *deep* one. An explanation of more satisfactory depth would enable us to know of each member of B_U not merely *that* it holds, but also *why* it holds. Can inductive proofs put us in such an epistemic position?

We think they do if they involve explanatory subproofs of *both* inductive basis and step. Assume, you know not only *that* every human is mortal, but also *why* every human is mortal. Assume further that you know that Socrates is human. This, it seems, puts you in a position to know not merely *that* Socrates is mortal, but also *why* he is mortal. If the intuition underlying this example is sound, it carries over to inductive proofs: if you know not merely *that F(1)*, but also *why* this is the case, and if you also know not merely *that* $\forall n(F(n) \rightarrow F(n+1))$, but also *why* this is the case, then you are in a position to know not merely *that* $F(1+1)$, but also *why* this is the case. Thus, having an explanatory proof of both the inductive basis and of the inductive step puts us in a position to know why $F(1+1)$. Correspondingly, knowing why $F(1+1)$ and $\forall n(F(n) \rightarrow F(n+1))$ puts us in a position to know, not merely that $F(1+1+1)$, but also why this is the case; and so on. Having an explanation of the inductive basis and the inductive step in principle suffices to know of every element of the explanatory basis of the inductive conclusion why it holds. In other words: having an explanation of both the inductive basis and of the inductive step puts us in a position to know, for every number n, why $F(n)$. Granted, being in such a position is not quite the same as knowing why $\forall n F(n)$. But everyone in such a position has the means to generate explanations of all the grounds that in turn explain $\forall n F(n)$. Whether or not he thereby knows why $\forall n F(n)$, we have no definite opinion about; but in any case he possesses important explanatory knowledge regarding the inductive conclusion.

Our argument does not apply to inductive proofs involving non-explanatory subproofs of basis or step; and indeed, non-explanatory subproofs arguably undermine the explanatory power of the full proof. Nor does our argument bear directly on proofs which lack a proof of the basis and simply take it for granted. Can such proofs be explanatory? We have to distinguish two cases: first, there are proofs whose basis is provable in an explanatory way, while the proof is omitted since it is trivial. Strictly speaking, the inductive proof itself then fails to be explanatory, but only in a superficial way; it is a near miss and directly points to an explanatory inductive proof. Second, there can be proofs which lack an explanatory subproof of the basis simply because the basis is explanatorily *fundamental* (it is true, but not because of anything else). We do not think that in this case, the lack of an explanatory subproof counts against the explanatory power of the proof; but the issue deserves further discussion.

Summing up, we have argued that proofs by induction have serious explanatory *potential* in the following sense: *if* they involve explanatory subproofs (and perhaps also if their basis is a fundamental truth), they put us in a position to know of each member of the explanatory basis why it holds. In that way, the inductive nature of a proof *preserves* explanatory power.

Whether a *particular* inductive proof is explanatory then hinges on whether its subproofs are explanatory. Since the subproofs can be of any kind whatsoever, a perfectly general answer to the question of whether inductive proofs are explanatory turns on a general account of explanation in mathematics. Offering such an account was no aim of this paper. As, e.g., Hafner and Mancosu (2005, p. 215f.) stress, mathematical case studies will have an important role to play in this connection. We must leave this for future research.

2.3 Another look at 5-proofs

Let us briefly return to Lange's idea of a 5-proof. We have seen that whenever there is a proof of a universally quantified theorem by mathematical induction, there is also a 5-proof of the theorem. Lange argued that corresponding proofs of the two sorts cannot both be explanatory, but his argument relies on a false principle, *Case-By-Rule*. Rejecting his argument, however, leaves the question of whether such proofs can both be explanatory unanswered. But what we have said in the preceding section about inductive proofs directly carries over to 5-proofs: if such a proof is given for some theorem U, then the proof provides a recipe for reaching each member of the explanatory basis B_U. If the proof moreover establishes its basis and the required inductive steps by explanatory subproofs, the proof even seems to yield a recipe for showing of every member of B_U *why* it is true.

So, we agree with Lange that 5-proofs should in principle have the same explanatory potential as proofs by induction. But while Lange concludes that both kinds of proof are explanatorily inert, we have given a reason to attest such proofs explanatory potential since they preserve explanatory power in the qualified sense specified above.

3 Methodological remarks: explanation in mathematics[18]

It has been a guiding conviction in our discussion that general principles about explanation can be brought to bear on the question of the explanatory power of inductive proofs. Certainly, mathematics—just like physics or any other science—has its distinctive *subject matter*. Due to a difference in subject matter, explanations in different fields can have very different features. But we do not seem to be trading on an ambiguity

[18] We would like to thank an anonymous referee for pushing us to comment on the methodology we followed.

when talking about physical, mathematical, or even aesthetic *explanations*. There is a common conceptual core that unites them—that makes them all cases of *explanation*. This common core gives rise to principles that hold for explanations of any kind, *a fortiori* for mathematical ones.

Many share our universalistic stance. In fact, it appears to be a presumption of any sustained discussion of mathematical explanation as witnessed by Lange's appeal to *Asymmetry*. Let us give just three more examples. First, Bernard Bolzano famously argued that the foundations of mathematics have to be improved by finding proofs that mirror the explanatory order. For that purpose, he developed a covering theory of grounding meant to apply to explanations in general.[19] More recent discussions follow suit: Paolo Mancosu criticises Nagel's approach to mathematical explanation for making it symmetric—which 'certainly goes against our intuitive conception of explanation'[20]—and Resnik and Kushner appeal to van Fraassen's work on explanation when holding that explanations in mathematics must provide answers to why-questions.[21]

We can distinguish two kinds of principles that govern explanation in general: the principles of the pure and impure logic of explanation.[22] The former deal with the structural limits and interconnections of explanations. They contribute to answering questions like: which form can an explanation never take? given certain explanations, which other explanations must also hold, and which cannot? The latter principles concern the interplay of explanation and purely logical notions. They contribute to answering questions like: given that a statement of a certain logical form is true, which other statements (partially) explain it? and which are (partially) explained by it? *Irreflexivity*, *Transitivity*, and *Asymmetry* belong in the first category, *Rule-by-Case* belongs in the second.

Principles of both kinds exhibit what has classically been called *topic-neutrality*: they are not concerned with the specific content of the statements involved, but only with their logical-explanatory structure. These principles will thus be applicable to any field that employs logical and explanatory notions, *a fortiori* to mathematics.

4 Conclusion

Our aim in this paper was to throw new light on the explanatory power of mathematical proofs by induction. We share two important methodological convictions with Lange. First, we agree with him that the question cannot be settled by appeal to controversial intuitions concerning mathematical induction. Rather, an answer must flow from general principles about explanation. Second, we agree with Lange that the principles at issue are not ones about *mathematical* explanation in particular—they are not concerned with the non-logical *content* of explanans and explanandum. But here our agreement ends. *Pace* Lange, what is relevant are not only general *structural* principles that belong to the pure logic of explanation and completely abstract away

19 See, e.g., Bolzano (2004), and Bolzano (1837, vol. II, §§198–222) for his general theory of grounding.

20 Mancosu (2000, p. 104).

21 Resnik and Kushner (1987, p. 152).

22 Cp. Fine (forthcoming: §6).

from the sentences involved. What is equally crucial are principles from the impure logic of explanation, positing explanatory relations between sentences on the basis of their logical *form*, or, to put it in other words, principles about *logical* explanation.

The paper has a negative and a positive dimension. First, we rebutted Lange's argument against the explanatory power of inductive proofs. We have shown that his argument has to rely on a controversial principle, *Case-By-Rule*, according to which a universal truth explains its instances. We have argued that this principle is false because the converse principle *Rule-By-Case* holds; it is the instances that (partially) explain the universal truth. Hence, the argument put forward by Lange fails.

Secondly, we argued that all proofs by induction are explanatory in an admittedly weak sense: they provide a recipe for generating all the immediate explanatory grounds of their conclusion and thereby put us in a position to know each member of the explanatory basis of the conclusion. Moreover, *some* inductive proofs are explanatory in a much stronger sense: given that both inductive basis and step have been proven in an explanatory way, an inductive proof puts us in a position to understand of every instance which explains the universal statement why this instance holds. Inductive proofs therefore have the potential for a deeper explanation of the universal statement. Whether they exhaust their potential turns on how their subproofs were performed.

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