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Reliability via synthetic a priori: Reichenbach's doctoral thesis on probability

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Abstract Hans Reichenbach is well known for his limiting frequency view of probability, with his most thorough account given in *The Theory of Probability* in 1935/1949. Perhaps less known are Reichenbach's early views on probability and its epistemology. In his doctoral thesis from 1915, Reichenbach espouses a Kantian view of probability, where the convergence limit of an empirical frequency distribution is guaranteed to exist thanks to the synthetic a priori principle of lawful distribution. Reichenbach claims to have given a purely objective account of probability, while integrating the concept into a more general philosophical and epistemological framework. A brief synopsis of Reichenbach's thesis and a critical analysis of the problematic steps of his argument will show that the roots of many of his most influential insights on probability and causality can be found in this early work.

1 Historical background

Hans Reichenbach wrote his thesis *Der Begriff der Wahrscheinlichkeit für die mathematische Darstellung der Wirklichkeit* (The Concept of Probability in the Mathematical Representation of Reality) largely independently in 1914. It was accepted in March 1915 by Paul Hensel and Max Noether at the University of Erlangen. Unlike his later views, the thesis was deeply influenced by the Kantian view dominant in philosophy and epistemology at the time. Reichenbach took synthetic a priori principles to form the foundation of empirical knowledge, and transcendental arguments to be the appropriate method to support such principles. Reichenbach had studied with Ernst

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Cassirer, and had hoped (unsuccessfully) to write his dissertation with the neo-Kantian Paul Natorp (Gerner 1997, p. 15).

In 1914 the mathematics of probability was already reasonably well developed but there was not yet an agreed upon axiomatization. First proposals were around (e.g. Bohlmann 1901), but Andrey Kolmogorov only published the now standard set of axioms in 1933, while Reichenbach published his own (structurally, but not semantically similar) axiomatization in a paper in 1932. In 1914 the discussion surrounding the formal definition of randomness (involving von Mises 1919; Church 1940; Ville 1936; Copeland 1928; Wald 1938, etc.), which is generally regarded as one of the main impediments in the early development of an axiomatization of probability, had not yet started. Emile Borel had published a few papers on this topic, (Borel 1909) but Reichenbach does not appear to have been familiar with them at the time. Without an axiomatization or any other widely accepted foundation, probability claims and their corresponding inferences supplied successful heuristics, but were without any epistemological grounding.

2 Thesis synopsis

Reichenbach's thesis set out to change this situation by giving a detailed account of the concept of probability as it was used in the sciences, and by tying this concept into a broader philosophical and epistemological framework. Reichenbach intended to provide a purely objective account of the meaning of probability claims, conditions for the assertibility of probability claims and a foundation for a rational expectation that is based on judgments of probability.

Reichenbach contrasted his view with those of von Kries (1886) and Stumpf (1892a, b), both of whom he appears to have regarded as representative of two different accounts of probability common at the time. Kries' account of probability was based on equi-probable events: The probability of a particular event E is determined by the proportion of equi-probable 'ur-events' it derives from. These ur-events can be found by tracing back the (causal) history of E and its compliment, tracking each of their causal ancestor events, until an event space is found for which no reason is available to consider one event in the space more likely than any other. This space then constitutes the space of equi-probable ur-events. The inference from events for which there is no reason to believe that one is more likely than the other, to the claim that these events are equi-probable is licensed by the *principle of insufficient reason*,[1] which states that if there is no reason to distinguish the probability of two events, then the two events have equal probability. By tracing the dependency on its ur-events, the probability of the event E can be determined.

It is not obvious how a space of ur-events should be determined practically, whether practical constraints on the tracing of the causal history limits judgments of probability, or whether there always is a unique space of ur-events. A failure of uniqueness would

[1] The principle of insufficient reason is now also referred to as the principle of indifference, a term coined by the economist John Maynard Keynes in 1921. While there may be minor differences in the precise usage of the two terms, those are not relevant here.

imply problems similar to those in the Bertrand paradox (Bertrand 1889), where the probability of events is undetermined because symmetry conditions can be applied in different ways to yield conflicting probability judgments for events. In modern terminology Kries could be described as an objective Bayesian. He believed that probabilities are objective and that there is in some sense one correct objective probability for any event, but that ultimately, a human judgment enters into the account when the space of equi-probable events is determined.[2]

Reichenbach took issue with this human, and therefore in his view subjective component of the principle of insufficient reason. He considered such a subjective element alien to the scientific use of probability. Consequently, Reichenbach sought to develop Kries' account of probability in such a way that the principle of insufficient reason became redundant and that probability claims could be couched in a purely objective framework.[3]

Both Kries' and Reichenbach's views contrasted with that of Carl Stumpf. Stumpf had a purely subjectivist view of probability. He took probability to represent degrees of belief. He did not present his view explicitly in terms of wagers, but it could have been framed in those terms. Stumpf took the realization that a die is biased to constitute a *change* in probability as opposed to a *correction*. He did not view the prior belief that all sides of a die have equal probability as false. Instead, probability only constitutes a summary of the current knowledge an individual has about the events under consideration—and that can be updated. Stumpf did not give a detailed account of belief update or the constraints that beliefs are subject to, but it is obvious that he only required that a probability claim about an event reflect the considered knowledge of an individual. Events, for which an agent has no reason to believe that their probabilities differ, are assigned equal probability—until there is evidence to the contrary. Unsurprisingly, Reichenbach rejected this account as unsuitable for a description of probability in science, since it would replace the aim for objectivity with what appears to be subjective whim and autobiography.

Reichenbach proposed a foundation of probability based on Henri Poincaré's argument of arbitrary functions (1912). In modern terms one would describe this argument as an analysis of strike ratios.[4] For illustration, Reichenbach considered a moving tape that is punctured (repeatedly) by a projectile shot from a cylinder fixed above the moving tape. The event space (possible locations of punctures on the tape) is divided into narrow equally wide alternating black and white stripes orthogonal to the movement

[2] Salmon (1979) describes Reichenbach as having attempted to be an 'objective Bayesian'. While this may be an accurate description of Reichenbach's mature views, I think it is a misleading description of Reichenbach's early views. Reichenbach rejected (or at least tried to reject) any component of human judgment in the foundation of probability, and consequently he struggled with an account of how we come to know probabilities. In contrast, an objective Bayesian admits a component of human judgment in the assessment of probability, as did Reichenbach with the introduction of posits in his mature view on probability. The *synthetic* (a priori) part of Reichenbach's early view of probability admittedly constitutes an introduction of human constraints into the concept of probability, but I consider this to be of a more necessary nature than an objective Bayesian would require.

[3] "In particular, we will strive to get rid of the principle of insufficient reason, which Kries could not avoid and which, since it is purely subjective, would preclude the objective validity of the laws of probability." Reichenbach (1916, p. 13) translation by author.

[4] See, for example, Michael Strevens, *Bigger than Chaos* (2003).

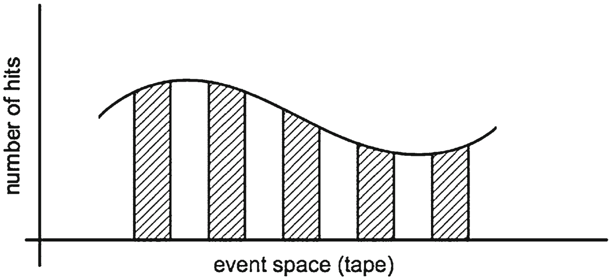

Fig. 1 Strike ratio: If the black and white stripes are equally wide, the probability of a white outcome is the same as the probability of a black outcome. (Figure taken from Reichenbach's thesis.)

of the tape. In a sequence of trials (shots at the moving tape) the number of punctures of the tape that fall within each stripe are counted and plotted as a histogram. As the number of outcomes increases and the width of the stripes decreases,[5] the histogram approximates a Riemann integrable function, as shown in Fig. 1.

Furthermore, the number of hits on white stripes is approximately equal to the number of hits on black stripes. That is, we find that the ratio of hits on white to hits on black stripes is approximately equal no matter what the Riemann integrable function is that the histogram converges towards. Reichenbach thereby was able to argue that it is not the *equi-probability* of the ur-events that is required to make sense of probability claims, but rather the *existence of a convergence limit* of the empirical frequency distribution to a continuous function. This result does not even depend on an equal partition of the event space: If the black stripes were twice as wide as the white ones, one would converge to a strike ratio of 2:1. As long as the empirical distribution converges to a continuous function, one is able to specify a probability distribution. Thus, Reichenbach could replace the principle of insufficient reason with an assumption about the existence of a convergence limit.[6]

Which conditions are necessary to justify the assumption that the empirical distribution has a convergence limit? Reichenbach identified two: causally independent and causally identical trials: Reichenbach deemed an assumption of convergence to a limiting distribution justified when (i) the shots of the projectile are independent of one another (that is, in particular, if the movement of the tape is independent of the shooting device) and (ii) if the repeated shots are generated by the same mechanism subject to the same forces. If the projectile were somehow attracted to the black stripes, or if the shooting mechanism varied between shots, one would not expect the distribution of hits to be equal or even stable.

Reichenbach then attempted to show that *causally* independent and *causally* identical trials imply *probabilistically* independent and identically distributed trials. Reichenbach did not provide a proof of this inference, but he did hint at an argument

[5] Reichenbach is not particularly precise about the nature of the limit. He explicitly states the decrease in stripe width, but like Poincaré he appears to assume that there are always enough events so that the histogram bins do not suddenly only contain one or no "hit".

[6] Thesis, pp. 21–26.

based on invariance constraints of a distribution generated by a causal structure that is subject to an intervention. He claimed that the marginal distribution of one of two causally independent variables is invariant when the other variable is subject to an intervention. This in turn implies the factorization of the joint probability into the two marginals over the variables, which constitutes probabilistic independence. The details of the argument are opaque, and its generalization to more complex scenarios is far from obvious.[7] We return to this point in the analysis.

Probabilistically independent and identically distributed trials provide a foundation for the weak law of large numbers. The weak law of large numbers guarantees that the sample average of a sequence of (probabilistically) independent and identically distributed trials converges to the distribution mean with probability 1, i.e. for all ε

$$\lim_{n\to\infty} P(|\overline{X} - \mu| < \varepsilon) = 1$$

In a sequence of Bernoulli trials the sample mean is an estimator of the probability of the event occurring, hence the weak law of large numbers would be of interest as a convergence guarantee towards a limiting distribution. Reichenbach did not follow this line of argument, nor did he discuss the relevance of the weak law of large numbers to his argument, even though there is reasonable evidence[8] that he was familiar with the law at the time. One explanation for this neglect is that the weak law of large numbers only guarantees convergence *in probability*. Since probability is exactly what Reichenbach was attempting to define, convergence in probability would have implied a circular definition. Instead, he attempted to provide a guarantee of convergence *with certainty*.

In retrospect it might be obvious that search for convergence with certainty is hopeless. But in his thesis Reichenbach argued that the existence of a convergence limit of the empirical frequency distribution is guaranteed by a synthetic a priori principle: the principle of lawful distribution. The argument for the synthetic a priori status of this

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7 "Consider the experiment in which two different variables x and y are simultaneously physically realized in repeating trials; then a vast variety of different combinations of x and y will be observed. Let each observed combination x, y be represented by a point in the x–y-plane. If the y variable is now forced to remain within an interval c–d, then all these points will lie in a band parallel to the x-axis. The distribution along this band is proportional to $\int_a^b f_1(x)dx$ if the interval from a to b varies arbitrarily. Each point on the band corresponds to an x-value and their distribution remains unchanged when the distribution of y-values is restricted: this is the condition of independence. Hence, if the limits c and d are fixed while a and b vary arbitrarily, the following equality must hold:

$$\int_a^b \int_c^d f(x, y)dxdy = k' \int_a^b f_1(x)dx$$

The same is true when a and b are fixed while c and d vary, i.e.

$$\int_a^b \int_c^d f(x, y)dxdy = k \int_a^b f_1(x)dx \int_c^d f_2(y)dy,$$

Reichenbach (1916, p. 33), translation by author. See also pages 33–36.

8 In notes pertaining to Reichenbach's thesis preserved in the Reichenbach Archives the weak law of large numbers is mentioned, but there is no elaboration.

principle is, in short, as follows. It is a transcendental argument in the spirit of Kant's argument for the synthetic a priori principle of causality.[9]

Reichenbach claims that our scientific knowledge is represented in the laws of nature. These laws, or at least some of them, are causal laws. In his view *at the time*, causal relations were assumed to be relations between individual token events, not between types of events—entirely in line with Kant's view of causality (or at least one of its interpretations). Hence, if our causal knowledge is restricted to token events, then in order to attain knowledge in terms of causal laws, one needs some procedure that aggregates token causal events into scientific laws. This aggregation is achieved by the calculus of probability. But we only ever have finitely many token causal events to aggregate. If we had no guarantee that the empirical frequency distribution of these finitely many token causal events converges, then we could not have the knowledge represented in the laws of nature. But we do have this knowledge and use it successfully, and hence we must have a guarantee of convergence. Hence, the principle of lawful distribution, stating that each empirical distribution has a convergence limit, is a necessary ingredient for the attainment of scientific knowledge; it is a synthetic a priori principle that complements Kant's principle of causality.[10]

Reichenbach claimed that if convergence did not occur (though it is not clear how or when that would be judged) then it is an indication that the conditions (causal independence of trials, causally identical trials) have not all been satisfied. However, such lack of convergence supposedly does not refute the principle of lawful distribution. Reichenbach admitted that his argument implies that the principle of lawful distribution is untestable, but he pointed out that the same criticism applies to Kant, whose principle of causality also fails to be testable. We apparently simply have to accept the lamentable nature of synthetic a priori principles.[11]

Reichenbach thus provided what he regarded to be an entirely objective account of probability. It is objective in the sense that it is only dependent on necessary constraints of experience, and it is not circular, since it builds on causally independent and causally identical trials. The principle of insufficient reason is replaced by the assumption of the existence of a convergence limit of the empirical frequency distribution, which in turn is guaranteed by the synthetic a priori principle of lawful distribution. On his view, the transcendental deduction of the principle of lawful distribution matches Kant's transcendental argument for the principle of causality. Kant had argued that Hume's

[9] Reichenbach (1916, ch. 3).

[10] "If there were a fully exact measurement of real things, then one should be able to claim that the value obtained once can be found again at any time at any place. Since we cannot claim that the value remains constant we have no choice but to assume that if it is not constant then there exists some law for its distribution in time and space. This is the synthetic a priori judgment which we must make. It should be added that the same judgment must hold for the combination of several processes. As before, we cannot claim that the calculated value remains constant, and we have to replace it with its lawful distribution in space and time. Hence we conclude that the principle of lawful connection of all events, which causality brings about, is insufficient for the mathematical representation of reality. A further principle has to be added, which connects the events—one could say—orthogonally; this is the principle of lawful distribution." (Reichenbach 1916, pp. 62–63); "Natural knowledge is only possible when this principle is added to the principle of causality, thereby, so to speak, connecting events orthogonally to the direction in which the relation of cause and effect connects them." (Reichenbach 1916, p. 74). (italics original, translation by author).

[11] Reichenbach (1916, p. 71).

skepticism towards causal knowledge disregarded the fact that causal knowledge is necessary for empirical knowledge, and hence that the principle of causality must be synthetic a priori. Reichenbach considered his argument to complete the missing link in Kant's account from token causal relations to causal relations of types, as they are represented in the (causal) laws of nature.

The meaning of (scientific) probability statements is thus given by a relative frequency of an event within a sequence of trials, and the justification of probability claims hinges on conditions concerning the relation between the trials that give rise to the frequencies. Reichenbach embedded his account in an epistemology that was regarded as standard at the time, and he claimed that this account provides grounds for a rational expectation: Given the convergence guarantee one could take the empirical distribution to be indicative of the limiting distribution. Though the empirical distribution may at any point diverge again before it converges, the guarantee of convergence at *some*—albeit unknown—finite point is (supposedly) sufficient to regard any expectation based on the empirical distribution as rational.[12] Convergence at some unknown point is (supposedly) better than no guarantee of convergence at all, and since this assurance is, so to speak, better than nothing, it is (supposedly) rational. Reichenbach's argument that belief in the accuracy of the empirical distribution is the *best* available strategy (and therefore rational) is not made explicit in the thesis; this line of argument only surfaces later as a justification of the straight rule (see below).

3 Analysis of thesis

Reichenbach successfully criticized accounts based on equi-probability and instead proposed a version of Poincaré's argument from strike ratios. He thereby avoided paradoxes resulting from the lack of uniqueness in determining the event space of equi-probable events (e.g. Bertrand's paradox). His new contribution was the synthetic a priori foundation with the principle of lawful distribution. It enabled him to abandon Kries' principle of insufficient reason and substitute a supposedly justified assumption about convergence. While the result might appear to successfully establish a more objective foundation of probability, it is not obvious that the account can hold what it promises. We will consider some of the problematic issues, and in many cases we will find that Reichenbach had much more to say about these points in his later works on probability.

In his thesis Reichenbach did not provide any guidance on how the trials that form the basis of his account of probability are supposed to be judged (i) causally independent, and (ii) causally identical. Reichenbach claimed that causal identity is satisfied when repeated trials are of the "same" process. Two processes are the same if they differ only in their position in space and time and all physically measurable variables have the same value. He must have assumed some caveat restricting the measurement to all *relevant* physically measurable variables, and we can guess that the judgment of relevance of variables was given by knowledge derived from the Kantian synthetic a priori principle of causality. None of this is elaborated in the thesis.

12 Reichenbach (1916, pp. 70–72).

The argument for causal independence had to be similar. Without recourse to probabilistic features (since those were to be defined in terms of these causal relations), it is unclear how causal independence can be judged. Unless this knowledge is supplied by some a priori principle, it would seem that a subjective—or at least somewhat arbitrary—assessment of causal independence would enter the supposedly objective foundation.

After publishing his thesis Reichenbach attended Albert Einstein's lectures in Berlin. Einstein's findings concerning the nature of space spelled trouble for the supposedly synthetic a priori assumption that space is Euclidean. As a result Reichenbach almost immediately became uneasy with the Kantian notion of causality as synthetic a priori. But as the above analysis shows, without the synthetic a priori principle of causality, Reichenbach loses the foundation of his account of probability in terms of causal independence and identical causal trials.

In his mature view on probability in *The Theory of Probability* (1935/1949), Reichenbach abandoned any attempt to build probability on token causal events. Instead, probability was defined in terms of properties of sequences. These sequences were supposed to correspond to sequences of trials, but the causal properties of the trials were no longer deemed relevant to the determination of probabilities. Instead, the sequences had to satisfy the mathematical conditions of *normality*. The class of normal sequences is more general than that of sequences generated from independent identically distributed trials (or random sequences), but excludes deterministic or patterned sequences. The motivation for recourse to normal sequences resulted from the difficulties that had been identified with a precise formal characterization of randomness in the early part of the twentieth century. For Reichenbach this generalization to normal sequences not only avoided the problems associated with randomness, but also seemed appropriate in light of sequences of trials in science. The condition of full independence and causal identity of trials is excessively strong, since, for example, a coin might be found to measurably wear down in a long sequence of flips. Nevertheless, it might still exhibit a stable probability of 1/2 for heads. Reichenbach considered the weakening to normal sequences to provide a more general foundation that would be appropriate even for such sequences.

While Reichenbach ultimately abandoned the causal foundation expounded in his thesis and concluded that probability could not be founded on causal notions, there is a sequence of papers published after his thesis in which he goes back and forth in taking causality or probability to be the more fundamental notion. (See Reichenbach 1925, 1929, 1930, 1932a.) The assessment of the connection between causality and probability is one of Reichenbach's greatest influences. In the thesis Reichenbach's derivation of probabilistically independent and identically distributed trials from causally independent and causally identical trials is far from complete. In order to complete the proof, Reichenbach would have needed a bridge principle that connects features of the causal structure to features of the probability distribution over that structure. Throughout his life there are indicators of the development of such a principle (Reichenbach 1920a,b, 1932b), but he only stated it explicitly as the *principle of common cause* in *The Direction of Time* (1956), published posthumously. By that time he regarded the scientific notion of causality to be a relation of event types rather than of token events. The principle of common cause links the causal feature of 'screening off' to probabilistic independence. Intuitively, a cause screens off its effects from any prior causal ancestors. This ‘screening off’ is reflected in the probability distribution generated by the causal structure, and so the principle of common cause states that a dependence between two variables is due to one causing the other, or the existence of a common cause of both variables. This contribution to the connection between probability and causality is widely regarded as one of Reichenbach’s most influential achievements. Reichenbach’s ideas were generalized to arbitrary causal structures in the causal Markov assumption,[13] first mentioned by Kiiveri and Speed (1982), which underlies most of the contemporary procedures of causal discovery. The basic issue already featured prominently in Reichenbach’s thesis.

The second aim of Reichenbach’s thesis was to supply a justification of why probability judgments determined in the way described supply a normative guide to action. The aim was to explain why probability claims supported a rational expectation for the occurrence of events. In this regard his conclusions are unsatisfying. Reichenbach unfortunately succumbed to the strong influence of the Kantian philosophy, which seems to have prevented him from presenting interesting results. He essentially claimed that we *must* assume that the strike ratios of the process under consideration will converge, since otherwise the knowledge represented in the laws of science would be impossible to attain. Reichenbach hinted at the weak law of large numbers, but did not lay out its relevance to the problem he was trying to tackle, nor did he discuss why he considered it to be inadequate. Instead, he argued that a guarantee of convergence could be given with certainty. But the claim is—even if one accepts the transcendental deduction—extremely weak and practically useless: Convergence is guaranteed at some point after some finite number of trials, but the actual point is unknown. This claim makes no headway into the actual question of how we are supposed to interpret the empirical distribution after a finite number of trials, what the empirical distribution tells us about future events or future distributions and how we could verify or falsify any probability claim. Furthermore, it is an extremely weak support for the basis of a rational expectation: Use the empirical distribution as basis for inferences because at some point the empirical distribution converges to the true distribution. No measure of confidence in the empirical distribution or measure of distance between the empirical and true distribution is provided.

A synthetic a priori assurance that the empirical distribution of a finite number of trials converges at some point begs the question of what assurance we have regarding probability claims based on empirical facts. Reichenbach does not deny this and admits that there is no way to disprove the principle of lawful distribution. But rather than admitting that he has provided an unsatisfactory argument, he argues that Kant’s argument for the principle of causality was no better. Sadly, reference to a poor argument of a greater authority does not make the present argument any better.

Later in his career, Reichenbach took several different approaches to address this problem. He developed a framework of higher-order probabilities that guarantee

[13] The causal Markov condition states that each variable in a causal structure (directed acyclic graph) is independent of its non-descendents given its parents (in the graph). Note that Reichenbach’s principle of common cause (Reichenbach 1956) is a special case of the causal Markov condition when taken to apply to distributional properties.

convergence (in higher-order probability)[14] and argued for what came to be known as the straight rule. The straight rule states that one should take the empirical distribution as representative of the limiting distribution. Although the empirical distribution might at any point be quite distinct from the limiting distribution or there might not actually be a limiting distribution, Reichenbach later argued that adherence to the straight rule is the *best* strategy available to find the truth, even if no guarantee of convergence is provided.[15]

This view is controversial, and within the sciences that use search procedures, there is a lively debate of how to handle situations in which one either has to accept such a weak convergence guarantee (known as pointwise convergence) or commit to stronger assumptions—whose support is dubious—to achieve a convergence guarantee that supports confidence intervals (so-called uniform convergence).

In *The Theory of Probability* Reichenbach claims that the probability of convergence can be estimated by integration of convergence results across different domains[16] using higher order probabilities combined with subjective posits. These posits, which basically amount to subjective guesses of probability values, seem like a peculiar reversal of Reichenbach's orginal aim at an objective foundation of probability. Reichenbach claimed that where possible, these guesses should be informed by available frequency information, but could otherwise just be blind guesses. He considered them to be innocuous, since their effect would "wash out" in the long run, as the probability assessment is updated in light of new data. Scientific objectivity would be reached in the limit. Needless to say, even this considered view was not spared from criticism (see, e.g. Nagel 1938), but it certainly addresses the problems of the thesis in more detail.

4 Conclusion

I have tried to argue that Reichenbach's thesis leaves us with a technical account of an (well, let us say) objective foundation of probability (strike ratios approximating a continuous function), but with no satisfactory meaning to our probability claims, as the convergence guarantee is bogus. The guarantees it provides, even if one accepts the Kantian spin, are useless for scientific inference. In that sense, Reichenbach failed to achieve his aim. But he achieved what a doctoral thesis should perhaps most importantly achieve: It furnished him with a lifetime's supply of interesting problems, to which he would make influential, though rarely uncontroversial, contributions.

I have not read Reichenbach as a limiting frequentist in 1915 (though he is obviously a frequentist), since he does not explicitly identify the probability with the *limit* of the relative frequency in an *infinite* series and he points out in later work that he

[14] Reichenbach (1932a, p. 614).

[15] Reichenbach (1949). For a development of Reichenbach's theory of induction see Reichenbach (1936, 1938, 1940)

[16] The domain of cross-integration was given by Reichenbach's theory of reference classes. We will not go into any detail of that theory here. Suffice it to say that Reichenbach's account of reference classes was problematic.

did not do so in his thesis.[17] The role of the limit is taken up by the synthetic a priori assurance given by the principle of lawful distribution. But since Reichenbach does require some kind of convergence the synthetic a priori principle seems very much like a limiting frequency wolf in the coat of some kind of sheep.

5 Epilogue

In unpublished autobiographical notes from August 6, 1927, Reichenbach gives a brief review of the main results of his thesis. He lists the following[18]:

(i) "The assumption of equi-probable events can be replaced by a continuity assumption.
(ii) The continuity assumption is essential to an understanding of causal claims.
(iii) An attempt to provide a guarantee of certain convergence.
(iv) An attempt to show that the principle of lawful distribution is a synthetic a priori principle and necessary for all knowledge."

In 1927 Reichenbach views points (iii) and (iv) as failures. His work in *The Theory of Relativity and a priori Knowledge* 1920, 1965, resulting from the lectures Reichenbach attended with Einstein after his doctoral thesis, convinced him of the impossibility of synthetic a priori principles. On (iii) he concedes that one can only guarantee convergence *in probability* (as is the case in the weak law of large numbers), rather than convergence *with certainty.* However, he considers (ii), the link between probability and causality, to be one of the most important discoveries since Hume.

The results of this importance that Reichenbach attached to this point can be found in several of his later works, and Reichenbach's insights on the relationship between probability and causality contributed crucially to the modern understanding of causality developed by Salmon (1984, 1998) and Suppes (1970), and the causal Bayes net representation in Spirtes et al. (2000).

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[17] Reichenbach (1932a, pp. 576–577).

[18] HR 044-06-21, Reichenbach Collection, Special Collections, University of Pittsburgh. (translation by author).

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